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A.I. MARKUSHEVICH

COMPLEX NUMBERS AND CONFORMAL MAPPINGS

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

А. И. Маркушевич

КОМПЛЕКСНЫЕ ЧИСЛА И КОНФОРМНЫЕ ОТОБРАЖЕНИЯ

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COMPLEX NUMBERS AND CONFORMAL MAPPINGS

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FOREWORD

The book acquaints the reader with complex numbers and functions of a complex argument (including Zhukovsky's function as applied to the construction of a wing section). The material is presented in a geometric form. Complex numbers are considered as directed line segments and functions as mappings. To prepare the reader to such an understanding of complex numbers, we begin with a geometric interpretation of real numbers and operations on them. The book is based on a lecture delivered by the author to high-school students. To read the book, the reader need not be acquainted with complex numbers.

The author

1. To represent real numbers geometrically use is made of a *number axis*, i. e. a straight line on which are indicated a point A – the *origin* of coordinates – representing the number 0, and another point B representing the number $+1$ (Fig. 1).

The direction from A to B is taken as the positive direction of the number axis and the segment AB as a unit length. Any segment AC represents a real number x whose absolute value is equal to the length of that segment. When C does not coincide with A (i. e. when the number x is not equal to zero), then x is positive if the direction from A to C coincides with the positive direction of the axis and negative if that direction is opposite to the positive direction of the axis.

Fig. 1.

2. Let us consider arbitrary intervals of the number axis as directed segments, i. e. *vectors on a straight line.* We shall differentiate the beginning and the end of each vector taking the direction from the beginning to the end as the direction of the vector. Vectors will be designated by two letters: the first letter denoting the beginning and the second the end of the vector. Every vector, irrespective of where its beginning is located (not necessarily at A), will represent a certain real number whose absolute value is equal to the length of the vector. This number is positive when the direction of the vector coincides with the positive direction of the axis and negative when this direction is opposite to the positive direction of the axis. Thus, for instance, vector AB (A is the beginning and B is the end) represents the number $+1$, while vector BA (B is the beginning and A is the end) represents the number -1.

3. The direction of the vector can be defined by indicating the angle between that vector and the positive direction of the axis. If the direction of the vector coincides with the positive direction of the axis, the angle can be considered to be of 0°. If it is opposite to the positive direction of the axis, then the angle can be taken to equal 180° (or $-180°$). Let x be some real number; if $x \neq 0$, then the angle between the vector

representing that number and the positive direction of the number axis is called the *argument* of the number x. It is evident that the argument of a positive number is equal to 0°, and the argument of a negative number, to 180° (or to −180°). The argument of the number x is denoted as Arg x (Arg being the first three letters of the Latin word argumentum, which can be translated here as a sign, an indication). The number 0 is represented not by a vector but by a point. Although in further discussion we shall consider a point to be a special case of a vector, a vector of zero length, we shall not be able to define in that case either its direction or an angle it makes with the number axis; therefore we shall not assign any argument to the number 0.

4. Let us turn to the geometric interpretation of operations on real numbers. We shall begin with addition and multiplication from which we can easily pass to the inverse operations,

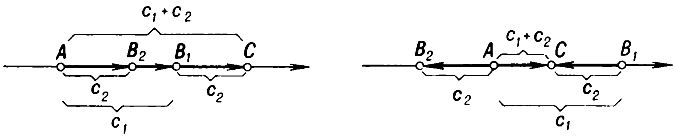


Fig. 2.

subtraction and division. Let c_1 and c_2 be two real numbers and AB_1 and AB_2, the vectors representing them. We need the rules which will make it possible, knowing vectors AB_1 and AB_2 to construct the vector representing the sum $c_1 + c_2$ or the product c_1c_2. We shall begin with addition. So, what should be done with the vector AB_1 representing the first summand to obtain the vector AC representing the sum?

It is easy to verify that to do this it is sufficient, in all cases, to mark off a vector B_1C, equal in length and direction to vector AB_2, from the end of vector AB_1; vector AC will be the required one (Fig. 2).

5. Now let us pass to multiplication. If one of the factors is equal to zero then the product is also zero; in that case, the vector representing the product reduces to a point. Now let us suppose that neither of the factors is equal to zero. Then the absolute value * of the product c_1c_2 will be equal to $|c_1| \cdot |c_2|$,

* The absolute value of a certain number c is written $|c|$. For instance, $|5| = 5$, $|-3| = 3$, $|0| = 0$.

that is to the product of the absolute values of c_1 and c_2. Therefore the length of vector AD representing the product will be equal to the product of the lengths of vectors AB_1 and AB_2 representing the factors. The sign of the product c_1c_2 will coincide with that of c_1 when $c_2 > 0$ and will be opposite to it when $c_2 < 0$. In other words, the direction of AD coincides with the direction of AB_1 when Arg $c_2 = 0$ (this means that $c_2 > 0$), and is opposite to the direction of AB_1 when Arg $c_2 = 180°$ (and this means that $c_2 < 0$). Now it is easy to answer the question of what should

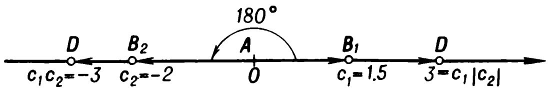


Fig. 3.

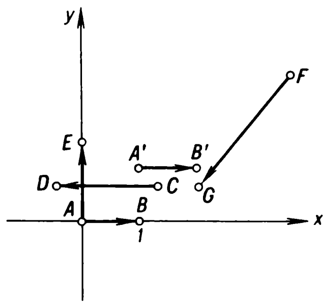


Fig. 4.

be done with the vector AB representing the multiplicand c_1 to obtain from it the vector AD representing the product c_1c_2 ($c_1 \neq 0$ and $c_2 \neq 0$). To do this we must multiply the length of the vector AB_1 by $|c_2|$ (retaining the direction) and then rotate the altered vector through an angle equal to the argument c_2 (i.e. through 0° if $c_2 > 0$ and through 180° if $c_2 < 0$); the resulting vector will represent the product. This rule is illustrated by Fig. 3 ($c_1 = 1.5$ and $c_2 = -2$).

6. Each vector on a straight line we associated with the number represented by that vector. Now we shall consider various vectors in a plane and each of them we shall associate with the number represented by the vector considered. The numbers we shall arrive at in this way, *complex numbers*, are of a different, more general character than real numbers. The latter will turn out to be a special case of complex numbers, in the same way as integers are a special case of rational numbers and rational numbers, in their turn, are a special case of real numbers.

We begin with drawing two mutually perpendicular straight lines, two number axes Ax and Ay with a common origin A, in a plane, and take a line segment AB as a unit length (Fig. 4). Then any vector lying on the axis Ax or parallel to it can be considered, as before, to be a geometric image (representation) of a real number. Thus, vectors AB and $A'B'$, the length of each of which is equal to unity, and the direction coincides with the positive direction of Ax, represent the number 1, while the vector CD of length 2 and of the opposite direction represents the number -2. Vectors not lying on Ax and not parallel to that axis, such as AE and FG, do not represent any real numbers. As regards such vectors, we shall say that they *represent imaginary numbers*. And it should be noted that vectors equal in length, parallel to each other and of the same sense represent the same number, while vectors differing either in length or in direction represent different imaginary numbers. Here we forestall the events a little, since not yet knowing what imaginary numbers are we speak of their images; but in real life as well it sometimes happens that the acquaintance with the portrait forestalls meeting the original.

Somewhat earlier we have shown that operations on real numbers can be replaced by operations on vectors representing these numbers. In the same fashion we shall replace operations on imaginary numbers by operations on vectors representing them. We shall not think of any new rules but shall retain in a geometric form those found for addition and multiplication of real numbers, the only difference being that the latter were represented by vectors on the straight line Ax (or by vectors parallel to that line) while imaginary numbers are represented by vectors in a plane which do not lie on Ax and are not parallel to Ax.

7. Before going on with our discussion we must note that *complex numbers* (the word "complex" means "compound" here) may be both real (already known to us) and imaginary (as yet known only by their "portraits"). For comparison we shall recall that both rational and irrational numbers considered together are also called by a single name, real numbers.

Let us proceed to addition of complex numbers. We have agreed to retain the rule formulated for the addition of real numbers. Let AB_1 and AB_2 be two vectors representing some complex numbers c_1 and c_2; to construct the vector representing their sum $c_1 + c_2$ we mark off, from the end of the vector AB_1, a vector B_1C equal in length and of the same direction as the vector AB_2; the vector AC connecting the beginning of AB_1 with the end of B_1C will be the required one (Fig. 5).

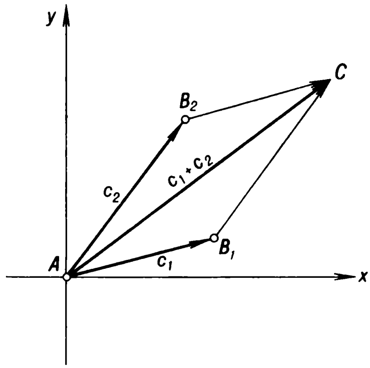


Fig. 5.

The only novelty here is that now we apply this rule to addition of complex numbers (represented by any vectors in a plane) while previously we used this rule only when we dealt with real numbers (represented by vectors on a straight line).

If we want to follow the same rule to construct the sum $c_2 + c_1$ (the summands have changed places), we have to mark off, from the end of the vector AB_2 representing c_2, a vector of equal length and of the same direction as the vector AB_1 (representing c_1). We shall evidently arrive at the same point C (in Fig. 5 AB_1CB_2 is a parallelogram) and, hence, the sum $c_2 + c_1$ is represented by the same vector AC as the sum $c_1 + c_2$. In other words, the rule of addition implies the validity of the commutative law:

$$c_2 + c_1 = c_1 + c_2.$$

The validity of the associative law can also be easily proved:

$$(c_1 + c_2) + c_3 = c_1 + (c_2 + c_3).$$

All the necessary constructions are shown in Fig. 6. It is evident that adding $(c_1 + c_2)$ (AC) with c_3 (CD) we obtain the same vector AD as we received while adding c_1 (AB_1) with $(c_2 + c_3)$ (B_1D).

8. Before turning to the discussion of multiplication, let us apply the concepts of absolute value and argument to complex numbers.

Suppose vector AB represents a complex number c. The abso-

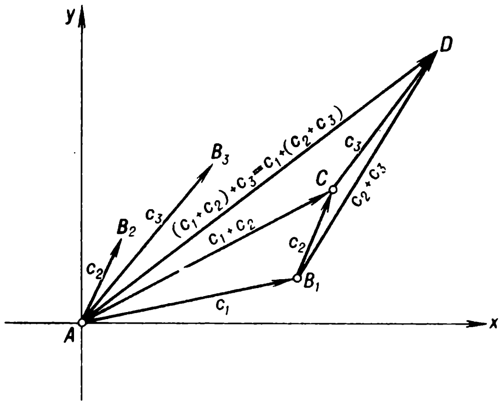


Fig. 6.

lute value of c is the length of the vector AB, and its argument c is the angle between the positive direction of the Ax axis and the vector AB. This angle can be reckoned counterclockwise, then it is positive, or clockwise, and in that case it is negative; besides, we can arbitrarily add to it any integer which is a multiple of 360°.

The designations of the absolute value and the argument of the number c are the same as those of real numbers: $|c|$ and Arg c. The only difference as compared to the case of real numbers is that the argument of an imaginary number is different from 0° and from $\pm 180°$, whereas the argument of a real number (different from zero) may be either 0° (when the number is positive) or $\pm 180°$ (when it is negative).

Figure 7 shows vectors AB, AB_1, AB_2 and AB_3 representing complex numbers c, c_1, c_2 and c_3. It is easy for the reader to verify the validity of the following assertions:

$$|c| = |c_1| = 1, \quad |c_2| = \sqrt{2}, \quad |c_3| = 2;$$

$$\operatorname{Arg} c = 0°, \ \operatorname{Arg} c_1 = 90°, \ \operatorname{Arg} c_2 = 45°, \ \operatorname{Arg} c_3 = -60° \text{ (or } 300°).$$

9. Having introduced the concepts of the absolute value and the argument of a complex number, it is a right time to state the rule of multiplication of complex numbers. It is precisely the same as the corresponding rule for multiplication of real numbers: to multiply a complex number c_1 by a complex number c_2 ($c_1 \neq 0$ and $c_2 \neq 0$), it is necessary to multiply by $|c_2|$

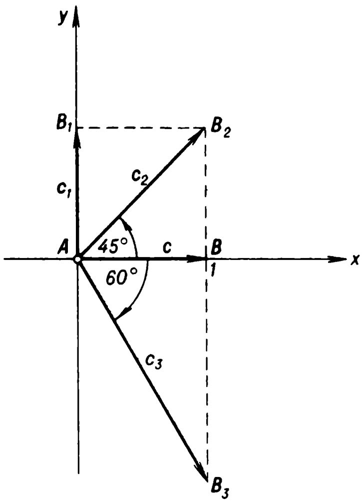


Fig. 7.

the length of the vector representing c_1 (without changing the direction) and then rotate the altered vector about the point A through an angle equal to the argument of c_2; the resulting vector will represent the product c_1c_2. For example, the product c_1c_2 is represented by vector AD (Fig. 8), and the product c_2c_3 by vector AE (Fig. 9).

One more rule of multiplication must be added: in the case when even one of the factors is equal to zero the product is also zero.

If we need to apply the multiplication rule to the product c_2c_1 (the factors have changed places), we must multiply the length of the vector representing c_2 by $|c_1|$ and rotate the altered vector about point A through an angle equal to the argument of c_1. The result proves to be the same as in the case of the multiplica-

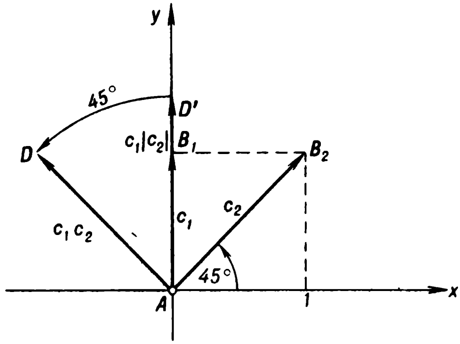


Fig. 8.

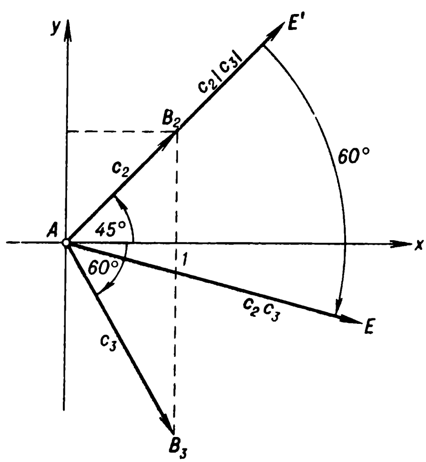


Fig. 9.

tion of c_1 by c_2: in both cases the length of the resulting vector is $|c_1||c_2|$, and the angle between Ax and this vector is equal to $\operatorname{Arg} c_1 + \operatorname{Arg} c_2$.

Thus we have

$$c_1c_2 = c_2c_1,$$

that is the commutative law is valid for the multiplication of complex numbers.

The associative law is valid as well:

$$(c_1c_2)\,c_3 = c_1\,(c_2c_3).$$

Indeed, each of the products being considered is represented by one and the same vector; its length is $|c_1|\cdot|c_2|\cdot|c_3|$, and the angle between the Ax axis and that vector is equal to $\operatorname{Arg} c_1 + \operatorname{Arg} c_2 + \operatorname{Arg} c_3$.

Let us now prove the validity of the distributive law:

$$(c_1 + c_2)\,c_3 = c_1c_3 + c_2c_3.$$

In Fig. 10 vector AB represents the sum of $c_1 + c_2$; if we retain the directions of AB_1 and AB_2 and multiply all the lengths

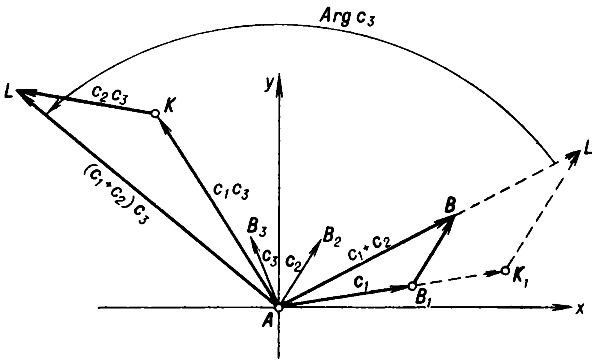


Fig. 10.

of the sides of the triangle AB_1B by $|c_3|$, we obtain the triangle AK_1L_1 which is similar to the triangle AB_1B. It is formed by the vectors AK_1, K_1L_1, AL_1 obtained from the vectors c_1, c_2 and $(c_1 + c_2)$ when all the lengths are increased $|c_3|$ times (the directions remain the same). Let us now turn the triangle AK_1L_1 about point A through the angle $\operatorname{Arg} c_3$; we shall receive the triangle AKL. According to the multiplication rule, the vector AK represents in it c_1c_3, the vector KL represents c_2c_3 and AL represents $(c_1 + c_2)\,c_3$. In accordance with the summation rule, we obtain from the same triangle

$$c_1c_3 + c_2c_3 = (c_1 + c_2)\,c_3,$$

and that is what we have to prove.

10. The operations of subtraction and division are defined as the inverse processes of addition and multiplication. In fact, we speak of the complex number d as the difference between the numbers c_1 and c_2 and write $d = c_1 - c_2$ if $c_1 = c_2 + d$, i.e. if c_1 is the sum of c_2 and d. Depicting this relation between c_2, d and c_1 in Fig. 11, we see that the vector representing the difference

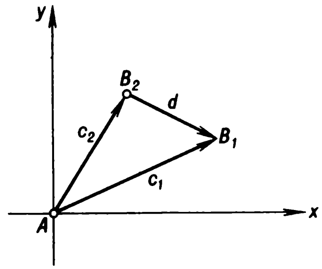


Fig. 11.

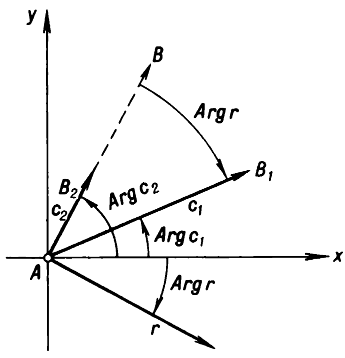


Fig. 12.

$c_1 - c_2$ is obtained if point B_2 (the end of the vector representing the subtrahend) is connected with point B_1 (the end of the vector representing the minuend) and then the former point is taken as the beginning of the vector and the latter as its end.

Analogously we call the complex number r the quotient of the numbers c_1 and c_2 ($c_2 \neq 0$) and write $r = c_1 : c_2$ or $r = \frac{c_1}{c_2}$ if $c_1 = c_2 r$, i.e. if c_1 is the product of c_2 by r (Fig. 12).

It follows that $|r|$, the length of the vector representing r, is $\frac{|c_1|}{|c_2|}$, and $\operatorname{Arg} r$ is equal to the angle B_2AB_1 reckoned in the direction from AB_2 to AB_1 (in Fig. 12 this direction is clockwise and therefore the angle must be assumed to be negative).

Let us consider particular cases. If c_1 and c_2 are represented by parallel vectors having the same direction, then the angle B_2AB_1 is equal to 0°, and, hence, $\operatorname{Arg} r = 0°$, i. e. r is a real positive number. Now if c_1 and c_2 are represented by parallel but oppositely directed vectors, then the angle B_2AB_1 is equal to 180° and the number r is real negative.

Summing up what was said above, we can say that addition and multiplication of complex numbers obey the same laws, commutative, associative and distributive, as in the case of real numbers, and subtraction and division, again as in the case of real numbers, are determined as operations inverse of addition and multiplication. Therefore, all the rules and formulas derived in algebra for real numbers must be valid for complex numbers as well, on the strength of the definition of the cited rules and operations. For instance,

$$(c_1 + c_2)(c_1 - c_2) = c_1^2 - c_2^2, \quad (c_1 + c_2)^2 = c_1^2 + 2c_1c_2 + c_2^2,$$

$$\frac{c_1}{c_2} + \frac{c_3}{c_4} = \frac{c_1c_4 + c_2c_3}{c_2c_4} \quad (c_2 \neq 0 \text{ and } c_4 \neq 0) \quad \text{and so on.}$$

11. While studying mathematics, the reader repeatedly comes across an expansion (or generalization) of the concept of a number: in arithmetic when fractions are introduced, and in algebra when negative numbers, and, later, irrational numbers are investigated. Each new expansion of the concept of a number makes it possible to solve problems which before that seemed insoluble or even meaningless. Thus the introduction of fractions makes possible the division of two numbers in all cases when the divisor is different from zero, for instance the division of 4 by 3 or 2 by 5; the introduction of negative numbers makes the operation of subtraction possible in all cases, for example it allows 5 to be subtracted from 2; the introduction of irrational numbers helps in the cases when it is necessary to express by a number the length of a line segment incommensurable with unity, for instance, the length of the diagonal of a square whose side is equal to unity. However, having only real numbers at our disposal we cannot extract a square root from a negative number. Let us prove that the introduction of complex numbers makes this problem solvable. The square root of the complex number c (we shall designate the root as $\sqrt{c}$) is naturally a complex number a whose square (i.e. the product of a by itself) is equal to c. In other words, $a = \sqrt{c}$ means that $aa = c$. Let c be a negative number, say, $c = -1$; to find $\sqrt{-1}$, we have to solve the equation $a^2 = -1$. To multiply a by a means, first, to multiply the length of the vector representing a by $|a|$, i.e. by the same length, retaining the direction, and then rotate the resulting vector about point A through an angle equal to Arg a. Evidently, the length of the vector obtained will then be equal to $|a|^2$. But the vector we have found must represent the number -1; hence, its length is equal to unity. Thus it follows that $|a|^2 = 1$ and

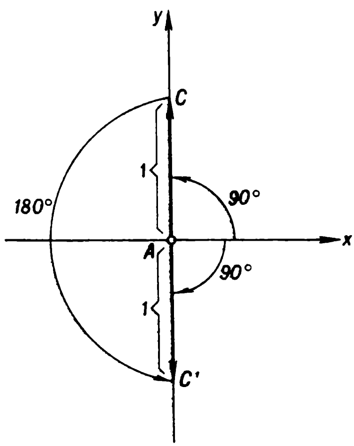


Fig. 13.

hence $|a| = 1$ (the length of a vector is always nonnegative). Further, the angle between the vector representing a^2 and the Ax axis is equal to $\operatorname{Arg} a + \operatorname{Arg} a = 2 \operatorname{Arg} a$; on the other hand, $a^2 = -1$, and so this angle must be equal to either $+180°$ or $-180°$. Therefore, $2 \operatorname{Arg} a = \pm 180°$, whence either $\operatorname{Arg} a = 90°$ or $\operatorname{Arg} a = -90°$. We have consequently obtained two different vectors AC and AC' representing two different values of $\sqrt{-1}$ (Fig. 13). The imaginary number represented by the vector AC is denoted by the letter i and is called an *imaginary unit*; we have $|i| = 1$, $\operatorname{Arg} i = 90°$. It readily follows that the imaginary number represented by the vector AC' can be obtained from i by multiplying i by -1. Indeed, in accordance with the multiplication rule, the length of AC must be multiplied by $|-1| = 1$ (this does not change the vector AC) and then rotate it about the point A through the angle $\operatorname{Arg}(-1) = 180°$; the resulting vector will be AC'. The imaginary number corresponding to that vector is $i(-1)$ or $-1 \cdot i$, in short, $-i$. Thus we have $\sqrt{-1} = \pm i$.

12. Consider an arbitrary vector AD lying on the Ay axis (or parallel to it) (Fig. 14). Assume that its length is l. If the direction of the vector coincides with the positive direction of the Ay axis (upwards from Ax), then the imaginary number c that the vector represents can be obtained from i by multiplying it by a positive number l, hence $c = li$.

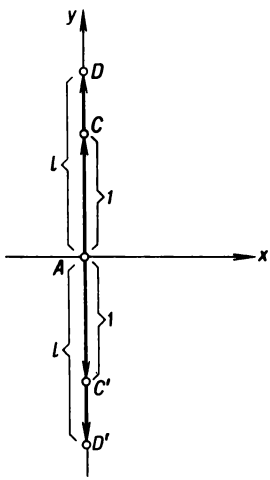


Fig. 14.

If the direction of AD is opposite to the positive direction of Ay the number c is obtained from i by means of multiplication by a negative number $-l$ (or from $-i$ multiplying it by l); hence, in this case $c = -li$.

Thus we have learned that any vector (of nonzero length) lying on the Ay axis (or parallel to it) represents an imaginary number of the form $\pm li$ with a plus or a minus sign depending on whether or not the direction of the vector coincides with the positive direction of Ay. The Ay axis is, therefore, called an *imaginary axis*. The Ax axis whose all vectors represent real numbers is called a *real axis*.

Let us consider an arbitrary vector $A'E'$ not lying on either of the axes and not parallel to them. By means of a construction shown in Fig. 15 we can express the number c represented by this vector as a sum of two other numbers: one number represented by the vector $A'B'$ parallel to Ax (or lying on Ax) and the other number represented by vector $B'E'$ parallel to Ay. But $A'B'$ represents a certain real number a while $B'E'$ is an imaginary number of the form bi, therefore $c = a + bi$.

And so we have expressed the imaginary number a by means of real numbers a and b and the number i. Since the vector $A'E'$ is not parallel to either of the axes, $a \neq 0$ and $b \neq 0$. It is easy

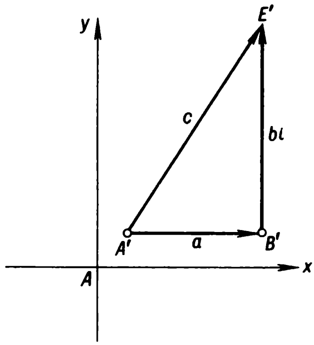


Fig. 15.

to realize that the numbers represented by vectors parallel to one of the axes can be written in an analogous form, namely, if the vector is parallel to the real axis it represents a number of the form $a + 0 \cdot i$ and if it is parallel to the imaginary axis then it represents a number of the form $0 + bi$.

Thus *every complex number c can be expressed as $c = a + bi$, where a and b are real numbers and i is the imaginary unit.*

13. Let us sum up what we have learned. We began with representation of real numbers by vectors lying on the same straight line, expressed the operation rules in geometric form, reducing these operations to the operations on vectors, and then began considering various vectors in a plane as vectors representing numbers of a more general kind, i. e. complex numbers which only in a special case (when vectors lie on the Ax axis or are parallel to it) reduce to real numbers. While extending to vectors in a plane the operations applied to vectors on a straight line, we introduced addition and multiplication (and then the inverse operations such as subtraction and division) and made sure that they obey the same laws as the operations on real numbers. The only fact we know about complex numbers is that they are all represented by vectors, and that any two vectors equal in length, parallel to each other and having the same direction represent one and the same complex number, while vectors differing either in length or direction represent different numbers. Now we know that complex numbers allow the square root to be extracted from -1 and we have introduced the number i as one of the two values of $\sqrt{-1}$ (the value of the root whose argument is $+90°$). Finally, proceeding from the rules of operations on complex numbers we have shown that every complex number c can be expressed as $c = a + bi$, where a and b are real numbers.

Thus we see that c *consists* of two summands a and bi; one of them, a, is represented by a vector of the real axis and can be regarded as a product of the real number a by the real unity; the other number, bi, is represented by a vector of the imaginary axis and can be regarded as a product of the real number b by the number i. Such a structure of any complex number explains why all these numbers were termed complex (or compound) numbers.

Note that a is called the *real part* and b the *imaginary part* of the number c. For example, for the number $c = 3 - 2i$ the real part is equal to 3 and the imaginary part to -2.

14. If we represent complex numbers by vectors originating at the same point A, then unequal complex numbers will be associated with vectors that do not coincide with one another, and, conversely, noncoinciding vectors will be associated with different complex numbers. Take $c = a + bi$; then the end of the vector AE representing the number c will have an abscissa a and an ordinate b (Fig. 16).

Thus it follows that if the beginning of the vector representing the number $c = a + bi$ coincides with the origin of coordinates A, then the numbers a and b will be the coordinates of the end of that vector. Making use of this statement, we can represent complex numbers geometrically not only by vectors but also by points. In fact, every complex number $a + bi$ can be represented by a single point E with coordinates a and b, and, conversely, every point E' with coordinates a' and b' can be considered

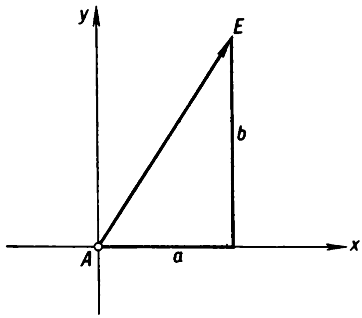


Fig. 16.

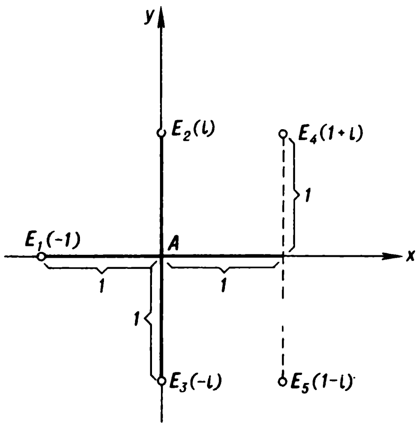


Fig. 17.

as the point representing the complex number $a' + ib'$. Shown in Fig. 17 are the points E_1, E_2, E_3, E_4 and E_5 representing (successively) the following numbers: -1, i, $-i$, $1 + i$, $1 - i$.

In what follows, for the sake of brevity, both the number z itself and the point E representing it will be called "point z". For instance, the expression "point $1 + i$" will represent the number $1 + i$ itself and the point E_4 representing it (Fig. 17). It will be clear from the context which of the two meanings is meant. Incidentally, it is better not to give much thought to this problem and consider both meanings to be equivalent.

15. Take z as an arbitrary point. If you add z to some number a you obtain a new point $z' = z + a$. It is evident that to pass from point z to point z', you must make a *translation by the vector* a, i.e. to displace point z in the direction of vector a by the distance equal to the length of that vector (Fig. 18). By choosing a requisite a you can obtain any displacement of point z. For instance, if you wish to displace point z in the

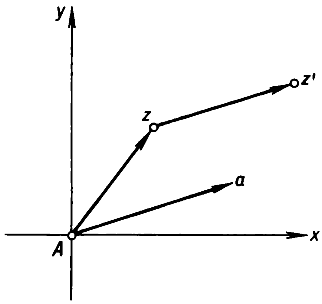


Fig. 18.

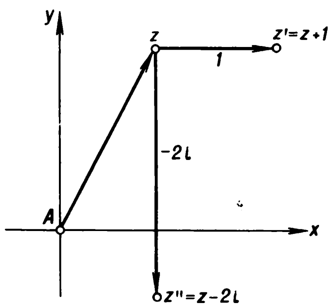


Fig. 19.

positive direction of the Ax axis by a unit, you take $a = 1$; point $z' = z + 1$ will be the required one. Now if you wish to displace z in the negative direction of the Ay axis by two units, you must take $a = -2i$; the required point will be $z'' = z + (-2i) = = z - 2i$ (Fig. 19).

Thus, the operation of addition $z' = z + a$ means geometrically the displacement of point z by the vector a.

16. Let us analyse multiplication of z by some number $c \neq 0$.

To multiply z by c, the length of the vector AE (i.e. the number $|z|$) must be multiplied by the number $|c|$, and the obtained vector AE_1 must be rotated through an angle equal to $\operatorname{Arg} c$ (Fig. 20). The first of these operations does not alter the direction of the vector AE, it can only change its length, namely, if $|c| < 1$, this length will decrease, if $|c| > 1$ it will increase,

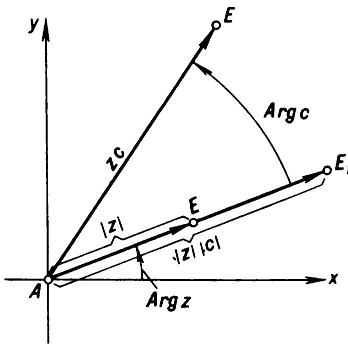


Fig. 20.

and, finally, if $c = 1$, it will remain as it was. We shall call this operation the *stretching* of the vector AE $|c|$ times. The word "stretching" should be understood here as a conventional term; a real stretching will occur only when $|c| > 1$, when the length of vector AE will increase in the process of multiplication $|c|$ times. However, we shall use this term even when $|c| = 1$ (the length of the vector AE does not change), and when $|c| < 1$ (the length of the vector AE decreases in multiplication).

If c is a positive real number, then $\operatorname{Arg} c = 0$.

In this case a rotation through the angle $\operatorname{Arg} c$ does not alter vector AE_1 found by means of stretching; hence the point E_1 represents the product zc. We may say that multiplication of z by the positive real number c means, in terms of geometry, stretching vector AE (representing z) c times. By varying c, it is possible to obtain various stretchings of the vector AE. Thus, to make a two-fold stretching, z must be multiplied by 2; to stretch it 2/3 times z should be multiplied by 2/3.

If the factor c is not a positive real number, then $\operatorname{Arg} c$ is not equal to zero. In that case the multiplication of z by c does not reduce to the stretching of vector AE alone but also requires that the stretched vector be turned about point A through the angle $\operatorname{Arg} c$. Consequently, in the general case the multiplication $z \cdot c$ means the *stretching* ($|c|$ times) followed by the *rotation* (through the angle $\operatorname{Arg} c$). In a special case, when the absolute value of c is equal to unity, the multiplication by c reduces to the only action of rotation of vector AE through the angle

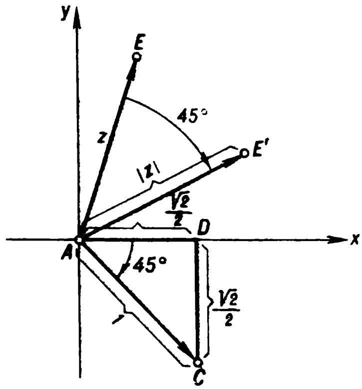


Fig. 21.

$\operatorname{Arg} c$ about point A. By choosing the proper values of c, we can rotate AE through any angle. Thus, for instance, if we want to rotate AE by 90° in a positive direction (counterclockwise) we must multiply z by i; indeed, $|i| = 1$ and $\operatorname{Arg} i = 90°$. To rotate AE by 45° in a negative direction (clockwise) z must be multiplied by the complex number c whose modulus is equal to unity and the argument, to $-45°$. It is easy to find this number by using Fig. 21 which contains point C representing the number c. The coordinates of point C are evident: $x = \frac{\sqrt{2}}{2}$, $y = -\frac{\sqrt{2}}{2}$, therefore $c = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. Thus, multiplying z by $c = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ is equivalent to turning the vector AE (representing z) through the angle of 45° about point A in the negative direction.

17. As we have seen, formulas $z' = z + a$ or $z' = cz$ transform

point z into point z'. Let us now consider not one but an infinite set of points z forming a geometric figure P (a triangle, for example, see Fig. 22). If the formula $z' = z + a$ is applied to each point z, then every previous point gives a new point z' translated by the vector a. All these translated points form a new figure P'. It can evidently be obtained if the whole figure P

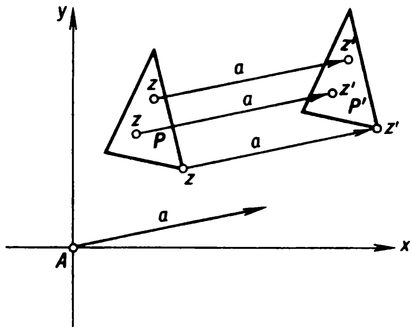


Fig. 22.

is translated, as a single whole, by the vector a. Thus, using formula $z' = z + a$, we can transform not only a single point, but a whole figure as well (a set of points). This transformation reduces to translation of the figure by the vector a. The new figure proves to be congruent to the original one.

18. We can also apply formula $z' = cz$ to each point z of the figure P. If c is a positive real number, then every point z of the figure P is transformed into a new point z' lying on the same ray, issuing from A, as the point z, the ratio $|z'|/|z|$ (the ratio of the distances of the points z' and z from A) being equal to c. In geometry such a transformation is called the *homothetic transformation* or *transformation of similitude* and the points z' and z are called *similar points*, point A being the *centre of similitude* and the number c, the *ratio of similitude*.

As a result of a homothetic transformation the set of all points of figure P passes into a certain new set of points forming the figure P' (Fig. 23). This figure is said to be *similar* to the given figure P. It is easy to see that in the case when P is a polygon (a triangle, for example) a similar figure P' is also a polygon similar to the polygon P. To prove this fact, it is sufficient to consider the homothetic transformation of the points lying on one of the sides BC of the polygon P (Fig. 23).

If B is transformed into B' and C into C' then, connecting B' and C' by a segment of a straight line, we find that the triangles ABC and $AB'C'$ are similar (the angle A is common and the sides forming it are proportional: $AB'/AB = AC'/AC = c$). It follows

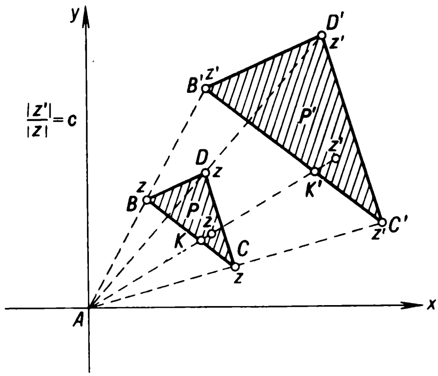


Fig. 23.

that the side $B'C'$ is parallel to BC and $B'C'/BC = c$. Let K be a point lying on BC; then the ray AK intersects $B'C'$ in a certain point K', the triangles AKC and $AK'C'$ are again similar and, consequently, $AK'/AK = AC'/AC = c$. Accordingly, the point K' is similar to the point K (with respect to the centre A, the ratio of the homothetic transformation being equal to c). Hence we conclude that all the points lying on the side BC pass, under the transformation of similitude, into points lying on the side $B'C'$; under this transformation, every point on $B'C'$ will be similar to one of the points lying on BC. Thus, the entire segment $B'C'$ will be similar to the segment BC. Repeating the reasoning for all the sides of the polygon P, we find that they are all transformed into the sides of a new polygon P', the respective sides being pairwise parallel, and the ratio of their lengths will be equal to one and the same number c:

$$B'C'/BC = C'D'/CD = D'B'/DB = c.$$

This proves the similarity of homothetic figures P and P'.

Thus, by using the formula $z' = cz$ (c being real positive), we can transform not only one point, but also a whole figure P. This is a homothetic transformation with centre in A and the ratio equal to c. In the case when P is a polygon, the transformed figure P' is also a polygon similar to P.

19. Let us now consider the case when the number c in the formula $z' = cz$ is not positive. First assume that $|c| = 1$. In this

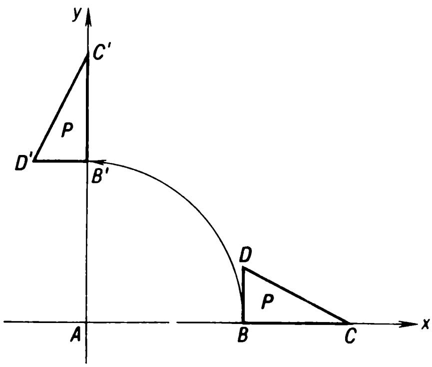


Fig. 24.

case the operation of multiplication reduces to rotating the vector Az about point A through an angle equal to the argument z. If this operation is applied to each point z of the figure P, then, as a result, the whole figure P will rotate through the angle $\operatorname{Arg} c$ about point A. Consequently, we see that by using formula $z' = cz$, where $|c| = 1$, we can transform any figure P into a figure P' obtained from P by means of rotation about the point A through the angle $\operatorname{Arg} c$. Let us take, for instance, $c = i$; since $\operatorname{Arg} i = 90°$, the transformation $z' = iz$ reduces to the rotation of the figure about the point A by 90°. Figure 24 shows the transformation of the triangle in the given case.

If we do not introduce the condition $|c| = 1$ in the formula $z' = cz$ and simply assume c to be some complex number (nonpositive and different from zero) then we can perform the corresponding transformation of the figure P in two stages. First we stretch it $|c|$ times which results in a homothetic transformation of the figure P into figure P_1 and then rotate P_1 about point A through the angle $\operatorname{Arg} c$.

Figure 25 shows the triangle P under the transformation $z' = \frac{i}{2} z$ $\left(\text{here } \left|\frac{i}{2}\right| = \frac{1}{2} \text{ and } \operatorname{Arg}\frac{i}{2} = 90°\right)$.

20. In formulas $z' = z + a$ and $z' = cz$, z can be regarded as an *independent variable* and z' as a *function.* These are the simplest

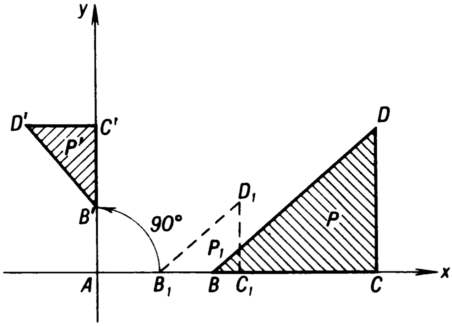


Fig. 25.

functions of a complex variable z. Subjecting z and some constant complex numbers to operations of addition, subtraction, multiplication and division as well as to raising to a power (regarded as a repeated multiplication), we shall obtain various other functions of z, for instance

$$z' = 1/z, \quad \text{or} \quad z' = z^2 + cz + d, \quad \text{or} \quad z' = \frac{z-a}{z-b} \quad \text{etc.}$$

Such functions of a complex variable are called *rational*; the term is due to the fact that the operations employed to define the functions (addition, subtraction, multiplication and division) are called *rational.* But rational functions are not the only functions of a complex variable; it is possible, for instance, to define and analyse functions of the form $z' = \sqrt[n]{z}$, $z' = a^z$, $z' = \sin z$ and others. In this book, however, we shall confine ourselves to rational functions, the simplest.

21. We have seen that functions $z' = z + a$ or $z'' = cz$ are associated with definite geometrical transformations of figures in a plane. In fact, if the variable z runs through the points of figure P then the function $z' = z + a$ runs through the points of figure P' obtained from P when the latter is translated by the vector a and the function $z'' = cz$ runs through the points of figure P'' obtained from P by means of a homothetic transformation with the ratio $|c|$ and a rotation about the point A through the angle $\operatorname{Arg} c$. Consequently, we can say that the function $z' = z + a$ itself performs a translation and the function $z'' = cz$ performs a homothetic transformation and a rotation (if c is a positive real number, then only a homothetic transformation is performed and if $|c| = 1$ but $c \neq 1$, then the only action is a rotation). Now what can be said about transformations performed by other functions of a complex variable, rational functions in particular? That is the question we shall try to answer in what follows. To assure the reader that this is not an idle pas-time, we inform him already at this stage that transformations performed by rational functions of a complex variable, while being remarkably versatile and possessing a wealth of geometric properties, also have some properties in common. This common property boils down to the following: while the size and the appearance of the figure are altered in the general case, the angles between any lines belonging to the figure under consideration are preserved. *

In special cases of functions $z' = z + a$ or $z' = cz$ the preservation of angles in the figures being transformed directly follows from the fact that here we mean translation, homothetic transformation or rotation. It is remarkable that the same thing is observed in transformations by means of any rational functions of a complex variable as well as by many other more general and more complex functions of a complex variable called analytic functions. But the scope of the book does not allow us to consider the latter.

22. A geometric transformation under which the angles between any two lines of the figure being transformed are preserved is called a *conformal transformation*, or, more often a *conformal mapping*.

A translation, homothetic transformation and rotation considered above may serve as examples of conformal mapping. Other examples will follow. For the time being, we shall clarify the requirement, contained in the definition of a conformal

* Strictly speaking, some particular points here may be such that the angles with vertices at these particular points alter, increasing two, three, or, in general, an integral number of times. But such points are an exception to the general rule.

mapping, that the angles between *any* two lines belonging to the figure under consideration be preserved. Let us consider the square *ABCD* constructed on the axes *Ax* and *Ay* (Fig. 26). We shall transform it into some other figure in such a way that the abscissa *x* of each point remains unchanged and the ordinate *y* doubles its length. Then the point *K*, for instance, passes into *K'*

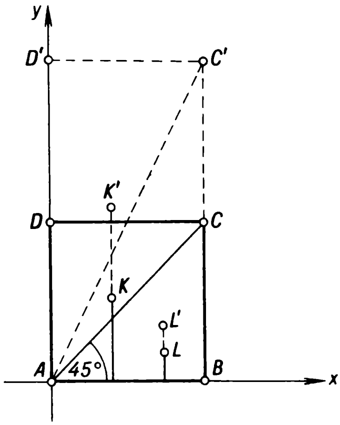


Fig. 26.

and *L* into *L'*. When all the points of the square are transformed in such a way then, evidently, the square *ABCD* is transformed into a rectangle *ABC'D'* with the same base and with the altitude twice as large. Under this transformation the side *AB* passes into itself (all the points remain as they are since their ordinates equal zero and will remain so after doubling), *AD* is transformed into *AD'*, *DC* into *D'C'* and *BC* into *BC'*. Naturally, the angles between the sides will remain right as before, they do not change. Now let us take the angle *BAC* between the side *AB* and the diagonal *AC* of the square considered (Fig. 26); the angle is equal to 45°. As a result of transformation, the side *AB* will not change its place but the line *AC* will pass into *AC'* (why?). Consequently, the angle *BAC* is transformed into another (larger) angle *BAC'*, that is it does not remain the same. If we take the angle *PQC*, instead of the angle *BAC*, with the vertex in some other point *Q* of the square *ABCD* (Fig. 27), then it is easy to show that this angle, too, will change under the transformation being performed.

Proceeding from our reasoning, we come to the conclusion that although the angles of the rectangle *ABCD* do not alter under the transformation considered (they remain right as before), the transformation is not conformal since for *any point* belonging

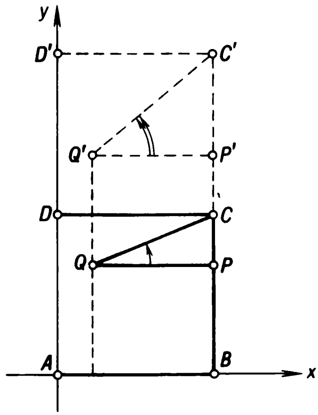


Fig. 27.

to *ABCD* there exists an angle with the vertex in this point which *changes* (increases) under the transformation being considered.

23. Before proceeding to consider our next geometric figure it is necessary to explain to the reader what we mean by an angle between two curves QR and QP intersecting at some point Q (Fig. 28).

Let us take, on the curve QP, an arbitrary point Q_1 differing from Q and draw a secant QQ_1. In just the same manner take a point Q_2 differing from Q on the curve QR and draw a secant QQ_2. The magnitude of the angle Q_1QQ_2 can be treated as an approximate value of the curvilinear angle PQR. The nearer the points Q_1 and Q_2 to the point Q, the closer will be the secants to the curves QP and QR near the point Q. In that case, the angle Q_1QQ_2 can be considered as a closer and closer approximation of the magnitude of the angle formed by our curves at the point Q. If Q_1 moves along the curve QP and Q_2 along QR, approaching Q indefinitely, then the secants QQ_1 and QQ_2 will rotate about the point Q approaching the limiting positions

QT_1 and QT_2. The rays QT_1 and QT_2 adjoin our curves near the point Q more closely than any other rays passing through that point. They are known as *tangents to the curves QP and QR* and the angle T_1QT_2 between them is taken as a measure of the angle at the point Q between the curves QP and QR. Thus, *the angle between two curves intersecting at some point is the angle between the tangents to the curves drawn at that point.*

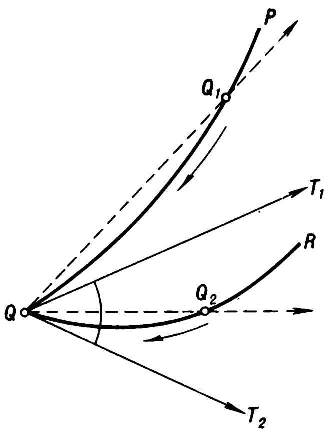


Fig. 28.

This definition is also valid in the case of an angle formed at the point Q by an arbitrary curve QP and a straight line QR (Fig. 29). Let QT_1 be a tangent to QP at the point Q. To use the definition cited above it is necessary to replace the straight line QR by a tangent to that line. But it is easy to see that the tangent to the straight line QR coincides with that same line. Indeed, to obtain a secant, we must take on QR a point Q_1 differing from Q and draw a straight line through Q and Q_1. It will evidently be the same line QR. When Q_1 approaches Q, the secant obtained remains unchanged. Therefore, the tangent, being the extreme position of the secant, is again the line QR. Consequently, the angle between the curve QP and the straight line QR must be understood as the angle between the tangent QT_1 to the curve QP at the point Q and the line QR itself. It may happen that the line QR is precisely the tangent to QP (i. e. QR coincides with QT_1); then the angle between QR and QP

vanishes. Consequently, the angle at the point Q between a curve and a tangent to it drawn at that point is equal to zero.

24. Conformal mappings find extensive application. They can be applied, for instance, in cartography when geographic maps are made.

Each geographic map represents a part of the earth's surface

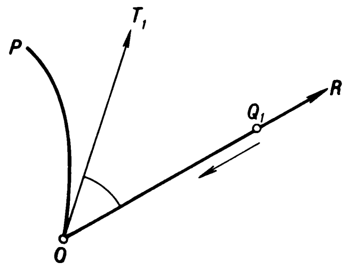


Fig. 29.

in a plane (on a sheat of paper). With such a representation the shapes of the continents, seas and oceans are more or less deformed. No special explanation is needed to assure the reader that it is impossible to spread and put on a plane without extension or compression, without discontinuities or folds a sphere, say, a part of a broken tennis ball. For the same reason, we can never depict a part of the earth's surface in a plane, i.e. make a map, without changing its proportions, and, consequently, without distorting its shape (that part of the surface can be treated as a sphere). It turns out, however, that it is possible to make a map *without changing the magnitude of the angles between various lines on the earth's surface.*

Suppose we wish to make a map of the Northern hemisphere in which all the angles between various directions on the earth's surface should remain full-sized. To visualize the process of such a construction, imagine a large terrestrial globe made of some transparent material, glass, for instance, covered with nontransparent paint so that only the contours of the continents, countries and seas in the Northern hemisphere as well as the net of the meridians and parallels are left unpainted and are, consequently, transparent. If we fix a small but very bright electric bulb on the South pole of the globe and put a screen in front of the globe at right angles to its axis, then in a dark room we shall see on the screen a contour map of the Northern hemisphere. It can be proved in terms of geometry that on such a map (it is called a map in a *stereographic projection*) all the angles between any lines on a globe in the Southern hemisphere are represented full-sized.

If we leave unpainted the (curvilinear) sides of some angle *PQR* with vertex at an arbitrary point of the Northern hemisphere then in the stereographic projection the angle will be represented full-sized (Fig. 30).

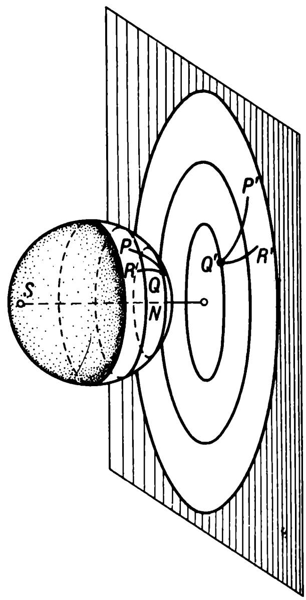


Fig. 30.

25. We have shown above how to obtain a map of the Northern hemisphere retaining the full size of all the angles. If we put a light source (a bulb) emitting the projecting rays not on the South but on the North Pole of the globe, we can use the same procedure to make a map of the Southern hemisphere with all the angles remaining full-sized. Each map obtained this way constitutes a plane figure which, when subjected to conformal mapping, will pass into a new figure which can also be considered as a geographic map. Since conformal mapping preserves the angles the new map will show the full-sized angles between the directions on the earth's surface. The right-hand picture in Fig. 31 represents a map of Greenland in stereographic projection and the left-hand picture shows the map which will be obtained from the latter if all its points are transformed according to the formula:

$$z' = \log_e |z| + i \operatorname{Arg} z.$$

Here the Napierian $e = 2.71828...$ serves as the base of the logarithms and $\operatorname{Arg} z$ is measured not in degrees but in radians.

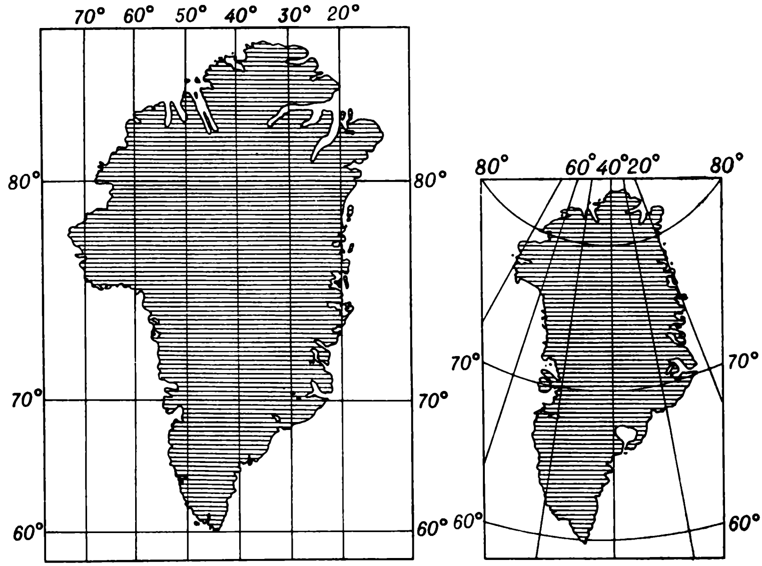
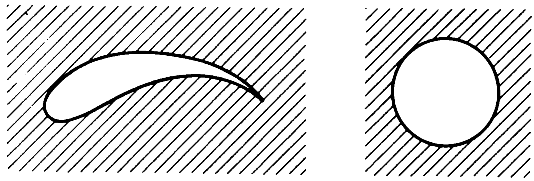


Fig. 31.

The formula is complicated and appears somewhat artificial. We shall not consider it here in detail and prove that the transformation performed by this formula is indeed conformal. We shall only say that a map obtained as a result of such a transformation was constructed by the Dutch scientist Gerhardus Mercator about 400 years ago. It is widely used in navigation. As compared to the map made in stereographic projection, it has some advantages: not only meridians but also parallels are represented here as straight lines; moreover, straight lines are used here to depict any ways on the earth's surface along which the magnetic needle retains its direction (so-called loxodromes).

26. The most important applications of conformal mapping refer to the problems of physics and mechanics. In many problems dealing with the electric potential at points in space surrounding a charged capacitor, for example, or with the temperature inside a heated body, with the velocity of particles in a fluid flux moving in a certain channel and streamlining some obstacles



(a) (b)

Fig. 32.

on its way, etc., it is necessary to compute the potential, temperature, velocity, etc. We can easily overcome all the difficulties in solving problems of this kind in the case when the bodies considered have a very simple shape (for example, a flat plate or a circular cylinder). But computations are necessary in many other, more complicated cases as well. For instance, to calculate the construction of an aircraft it is necessary to compute the velocity of the particles of the air flow streamlining the wing of the aircraft. *

A cross-section of an aircraft wing (its profile) is shown in Fig. 32*a*. Velocity computation is especially simple, however, when the cross-section of the streamlined body is a circle, i. e. when the body itself is a circular cylinder (see Fig. 32*b*).

It turns out that to reduce the problem of computing the velocity of the air particles in a flow streamlining the wing of an aircraft to a more simple problem of streamlining a circular cylinder, it is sufficient to make a conformal mapping of the figure hatched in Fig. 32*a* (*the exterior of a wing section*) onto the figure hatched in Fig. 32*b* (*the exterior of a circle*). Such a mapping is performed by means of a certain function of a complex variable.

* We realize, of course, that in flight both the air particles and the wing of the aircraft move. It is possible, however, proceeding from the laws of mechanics, to reduce the analysis to the case when the wing is stationary and is streamlined by an air flow.

Knowing this function, we can pass from the velocities in the flow streamlining a circular cylinder to those in a flow streamlining an aircraft wing and, consequently, to completely solve the problem on our hands.

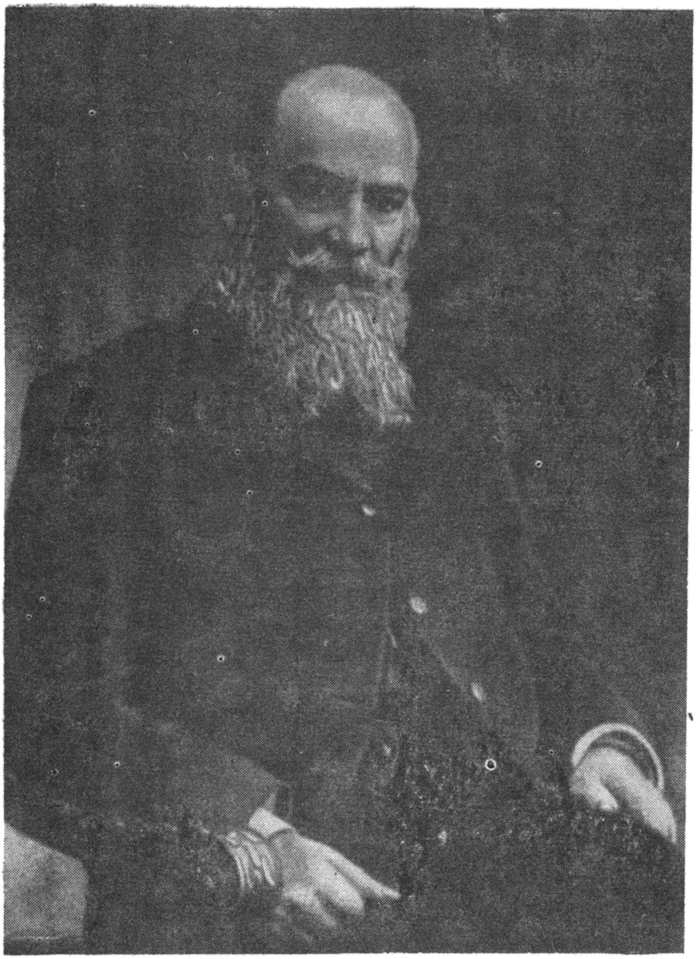
In the same way conformal mapping makes it possible to reduce the problems of computing the electric potential and the temperature of bodies of an arbitrary shape (any section profile) to the simple cases when the problem is already solved. The reverse passage to the space surrounding the originally given electrified (or heated) bodies is done by means of the same function of a complex variable which performs conformal mapping.

27. All that was said above about the application of conformal mapping to the problems of cartography, mechanics and physics was not followed by any proofs. Were they given, our reader would hardly understand them without special knowledge acquired at higher educational establishments.

From now on, to the end of the book, we shall deal with the simplest rational functions which should be used to perform certain conformal mappings. Here are the functions we shall speak of: (1) $z' = \frac{z-a}{z-b}$ (the so-called *linear-fractional* functions); (2) $z' = z^2$; (3) $z' = \frac{1}{2}\left(z + \frac{1}{z}\right)$. The latter function has been termed after the eminent Russian scientist Zhukovsky (1847-1921) whom V. I. Lenin referred to as "the father of Russian aviation". It is called Zhukovsky's function since Zhukovsky successfully employed it in solving certain problems in the theory of aircraft; in particular, he showed how this function can be used to obtain some profiles of the airplane wing which have both theoretical and practical significance.

28. We shall begin with the linear-fractional function $z' = \frac{z-a}{z-b}$. Here a and b are complex numbers (not equal to each other). Let us show that by means of this function each arc PLQ of the circle connecting the points a and b is transformed into a certain straight ray $P'L'$ emanating from the origin, the angle between the positive direction of the real axis and this ray being equal to the angle between the direction baN and the tangent to the arc of the circle at point a (Fig. 33).

Suppose point z lies on the arc PLQ (Fig. 33, left); we shall



Nikolai Egorovich Zhukovsky
(1847 – 1921)
widely used complex numbers and conformal mappings to calculate aircraft.

prove that its image (i. e. the point $z' = \frac{z-a}{z-b}$ corresponding to it) should lie on the ray $P'L'$ (Fig. 33, right). To construct vector z', we must know the length of this vector ($|z'|$) and the angle of inclination to the positive part of the real axis ($\operatorname{Arg} z'$). But z' is the quotient of the complex numbers $z-a$ and

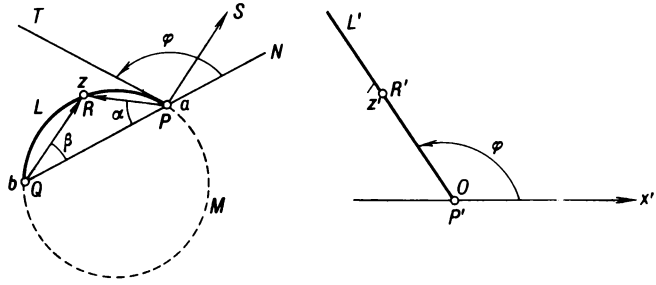


Fig. 33.

$z-b$ represented by the vectors PR and QR. Therefore, $|z'| = \frac{|z-a|}{|z-b|}$ and $\operatorname{Arg} z'$ is equal to the angle SPR (vectors PS and QR are equal) reckoned in the direction from PS to PR. It is evident that $\widehat{SPR} = \widehat{QRP}$ * and, hence, it is measured by half the arc QMP. The angle NPT is measured by half the same arc. Consequently, $\operatorname{Arg} z' = \widehat{SPR} = \widehat{QRP} = \widehat{NPT} = \varphi$. Thus we see that whatever the position of the points z on the arc PLQ, the corresponding points $z' = \frac{z-a}{z-b}$ have the same argument φ. And this means that all these points lie on one and the same ray $P'L'$ inclined to the positive part of the real axis through the angle φ.

This conclusion is also valid in the case when PLQ is not the arc of a circle but a rectilinear segment PQ. Then we should assume the angle $\varphi = 180°$ and the ray $P'L'$ to coincide with the negative part of the real axis (Fig. 34). Indeed, if z lies on the segment PQ, then the vectors representing $z-a$ and $z-b$ are of opposite directions. Hence it follows that the quotient

* The notation $\widehat{ABC}$ denotes the angle ABC.

$z' = \dfrac{z-a}{z-b}$ is a negative real number, i. e. z' lies on the negative part of the real axis.

We have proved that the images of the points of the arc PLQ lie on the ray $P'L'$. But do they fill the whole ray $P'L'$ or are there points on the latter which are not the images of the points

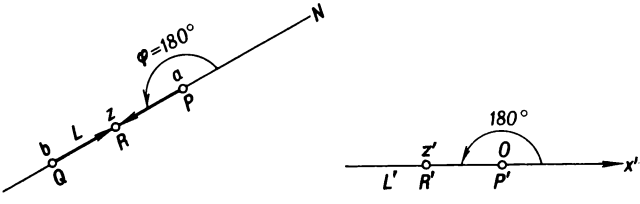


Fig. 34.

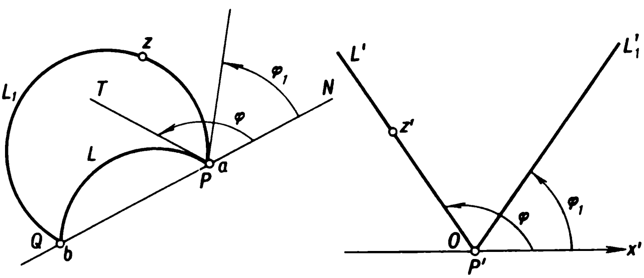


Fig. 35.

of the arc PLQ, not a single one? We shall now show that the images fill the whole ray.

Let us begin with the point P' (the origin); it is the image of the point P since $z' = \dfrac{z-a}{z-b}$ vanishes when $z = a$. We shall take an arbitrary point z' on the ray $P'L'$ (Fig. 35) differing from the point P' (i. e. $z' \neq 0$). It is evident that z' cannot be a positive real number since the ray $P'L'$ does not coincide with the positive real axis.

Considering z to be unknown, we solve the equation $z' = \dfrac{z-a}{z-b}$ for z and find $zz' - z'b = z - a$ whence $z = \dfrac{z'b - a}{z' - 1}$. Thus, for every point z' lying on $P'L'$ there exists one and only one value of z such that $z' = \frac{z-a}{z-b}$, i.e. such that z' is the image of z.

But where is that point z? Can it be possible that it does not lie on PLQ? Let us prove that it is impossible. First of all point z cannot lie on a straight line which is an extension of the segment PQ (outside of this segment), otherwise the numbers $z-a$ and $z-b$ would have the same arguments and $z' = \frac{z-a}{z-b}$ would be a positive number. Now if z does not lie on the indicated straight line outside the segment PQ, then P and Q can be connected by an arc of a circle so that the arc passes through z (if we assume that point z lies on the segment PQ, then instead of the arc we should take the segment itself).

We shall designate the arc by PL_1Q; since we have assumed it to be different from PLQ the tangent to it at point P forms with the direction baN an angle φ_1 not equal to φ (see Fig. 35). Therefore, the value of the function $z' = \frac{z-a}{z-b}$ at that point should be represented by a point of the ray $P'L'_1$ inclined to the positive part of the real axis at the angle φ_1 and, consequently, not coinciding with $P'L'$. We have come to a contradiction since in the case cited it turns out that the point z' differing from the point P' should be located both on the ray $P'L'$ and on the ray $P'L'_1$.

Thus, we have proved that every point z' located on $P'L'$ is an image of the single point z $\left(z = \frac{z'b-a}{z'-1}\right)$, the point z lying on PLQ. Hence it follows that if the point z' runs along the ray $P'L'$, then the corresponding point z determined by the equation $z' = \frac{z-a}{z-b}$ runs along the arc PLQ.

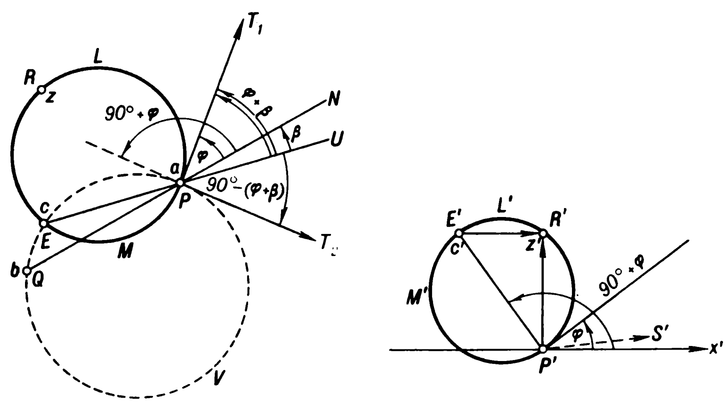
Let us finally show that when z describes the arc PLQ while moving in the same direction from point P to point Q all the time, then point z' traces out the ray $P'L'$ also in one and the same direction, receding from point P' indefinitely. To prove this it is sufficient to show that the distance $P'R' = |z'| = \frac{|z-a|}{|z-b|} = \frac{PR}{QR} = \frac{\sin\beta}{\sin\alpha}$ (see Fig. 33) increases, when point z moves in the indicated direction, acquiring infinitely large values. But $\varphi + \alpha + \beta = 180°$, whence $\beta = 180° - (\alpha + \varphi)$, $\sin\beta = \sin(\alpha +$

$+ \varphi) = \sin \alpha \cos \varphi + \cos \alpha \sin \varphi$ and, consequently,

$$P'R' = |z'| = \frac{\sin \alpha \cos \varphi + \cos \alpha \sin \varphi}{\sin \alpha} = \cos \varphi + \sin \varphi \cot \alpha.$$

When point z moves along PLQ from P to Q, the angle α decreases from the value $180° - \varphi$ to zero, while the angle φ remains unchanged. Therefore, $\cot \alpha$ increases from the value $-\cot \varphi$ to $+\infty$, and $|z'| = \cos \varphi + \cot \alpha \sin \varphi$ also increases (since the number $\sin \varphi$ is positive) from the value $\cos \varphi - \cot \varphi \sin \varphi = 0$ to $+\infty$.

29. Let us now consider a circle PLM passing through a point a but leaving outside a point b (Fig. 36*a*). Suppose the angle between the tangent at the point a and the direction baN is equal



(a) (b)

Fig. 36.

to φ. Let us draw an auxiliary circle through the points a and b for which the tangent at point a forms an angle of $\varphi + 90°$ with the direction baN. This circle intersects the original circle at a certain point E; we shall designate by c the complex number represented by this point.

We shall now show that by means of the function $z' = \dfrac{z-a}{z-b}$ the circle PLM is transformed into a circle $P'L'M'$ (Fig. 36*b*) with the line segment $P'E'$ serving as the diameter, the point P' representing the number 0 and the point E' the number $c' = \dfrac{c-a}{c-b}$.

In this case the tangent to the circle $P'L'M'$ at point P' forms an angle φ with the positive direction of the real axis.

So we wish to prove that for every point z on PLM there exists a corresponding point $z' = \frac{z-a}{z-b}$ on the circle $P'L'M'$ for which the points 0 and $c' = \frac{c-a}{c-b}$ are the ends of the diameter. It is, evidently, sufficient to prove that from each point $z' = \frac{z-a}{z-b}$ (provided that z lies on PLM) the segment $P'E'$ can be seen at right angles, i. e. the angle $E'R'P'$ is the right angle *.

But the angle $E'R'P'$ is formed by the vectors $E'R'$ and $P'R'$ representing the numbers $z' - c'$ and z'; it is equal to the angle $S'P'R'$ (the vectors $P'S'$ and $E'R'$ being equal), reckoned in the direction from $P'S'$ to $P'R'$. The latter angle is equal to $\operatorname{Arg}\frac{z}{z'-c'}$, and therefore the angle $P'R'E'$ we are interested in also coincides with the argument of the number $\frac{z'}{z'-c'}$, i. e. we have $\widehat{P'R'E'} = \operatorname{Arg}\frac{z'}{z'-c'}$.

Let us transform the expression $\frac{z}{z'-c'}$ substituting $\frac{z-a}{z-b}$ for z' and $\frac{c-a}{c-b}$ for c'. We receive

$$\frac{z'}{z'-c'} = \frac{z-a}{z-b} : \left(\frac{z-a}{z-b} - \frac{c-a}{c-b}\right) =$$

$$= \frac{z-a}{z-b} : \frac{(z-c)(a-b)}{(z-b)(c-b)} = \frac{z-a}{z-c} : \frac{b-a}{b-c} = \frac{z''}{b''}.$$

We assume here $\frac{z-a}{z-c} = z''$ and $\frac{b-a}{b-c} = b''$.

It is evident that z'' is also a linear-fractional function of z. The only difference between this function $z'' = \frac{z-a}{z-c}$ and the

---

* Because the points of the plane from which the given segment is seen at a right angle lie on the circle constructed on that segment as the diameter.

original function $z' = \frac{z-a}{z-b}$ is that point b is replaced by point c. All that was proved in Sec. 28 is applicable to the new function, that is, if point z is located on the arc of a circle connecting a and c then point z'' must be located on a straight ray beginning at the origin. In that case, if a tangent to the arc of the circle at point a forms a certain angle α with the direction caU (see Fig. 36*a*), then the corresponding straight ray also forms an angle α with the positive direction of the real axis; in other words, the argument z'' is equal to α.

Since point z is on the arc of the circle PLE passing through points a and c and the angle between the tangent PT_1 to that circle and the direction caU is equal to $\beta + \varphi$ (see Fig. 36*a*), then $\alpha = \beta + \varphi$ and the argument of the number $z'' = \frac{z-a}{z-c}$ must be also equal to $\beta + \varphi$ for all the positions of z on the arc PLE.

On the other hand, point b is located on the arc PVE of the circle connecting points a and c. The tangent PT_2 to that arc at point a forms an angle $(\beta + \varphi) - 90°$ with the direction caU (the absolute value of this angle is equal to $90° - (\beta + \varphi)$ but it can be seen from Fig. 36*a* that in our case it is reckoned in the negative direction and, hence, should be taken with the negative sign). Therefore, the value of the linear-fractional function $\frac{z-a}{z-c}$ corresponding to $z = b$, that is the number $b'' = \frac{b-a}{b-c}$ must be represented by a point of the ray issuing from the origin at the angle $(\beta + \varphi) - 90°$ to the positive direction of the real axis, i.e. $\operatorname{Arg} b'' = (\beta + \varphi) - 90°$.

Recall that we wished to determine the angle

$$\widehat{P'R'E'} = \operatorname{Arg} \frac{z'}{z'-c'}.$$

We have found that $\frac{z}{z'-c'} = \frac{z''}{b''}$ and also that

$$\operatorname{Arg} z'' = \beta + \varphi, \ \operatorname{Arg} b'' = (\beta + \varphi) - 90°;$$

whence it follows that $\operatorname{Arg} \frac{z''}{b''} = 90°$ (Fig. 37) and

$$\widehat{P'R'E'} = \operatorname{Arg} \frac{z'}{z'-c'} = \operatorname{Arg} \frac{z''}{b''} = 90°.$$

Thus, the segment $P'E'$ is seen at the right angle from every point $z' = \frac{z-a}{z-b}$. This means that point z' is located on the circle $P'L'M'$ for which the segment $P'E$ serves as the diameter *.

The next thing we have to show is that a tangent to that circle at point P' forms an angle φ with the positive direction

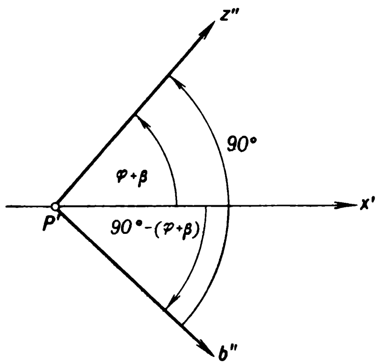


Fig. 37.

of the real axis. To do this, it is sufficient to prove that the angle between the diameter $P'E'$ and this direction of the axis is equal to $\varphi + 90°$. The latter angle coincides with $\operatorname{Arg} c' = \operatorname{Arg} \frac{c-a}{c-b}$. But point c is located on the arc PEQ of the circle connecting the points a and b. Since the tangent to this arc at point a forms an angle of $90° + \varphi$ with the direction baN, the point $c' = \frac{c-a}{c-b}$ must lie on the ray which also forms

* When we proved the case, we took point z on the arc PLE; then the corresponding point z' turned out to be on the semi-circle $P'L'E'$. But if we take point z on the arc EMP (see Fig. 36*a*), the proof will not be different, we only have to note in that case that the direction of the tangent to this arc at point a is directly opposite to PT_1. This means that Arg z'' will be equal, not to $\beta + \varphi$ but to $\beta + \varphi - 180°$. Therefore, for the angle $\widehat{P'R'E'} = \operatorname{Arg} \frac{z'}{z'-c'}$ we shall obtain the value $(\beta + \varphi - 180°) - (\beta + \varphi - 90°) = -90°$. This corresponds to the location of point z' on the semi-circle $E'M'P'$.

an angle of $90° + \varphi$ with the positive direction of the real axis, i.e. $\operatorname{Arg} c' = 90° + \varphi$, and that is what we had to prove.

30. Let us show, by way of example, how the hatched figure on the left-hand side of Fig. 38 is transformed when it is represented by the function $z' = \dfrac{z-1}{z+1}$. This function has the

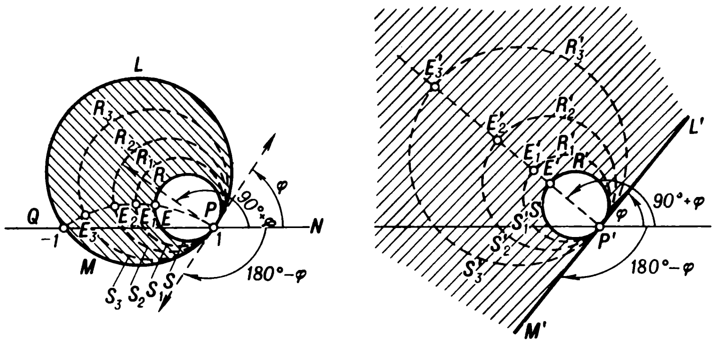


Fig. 38.

form $\dfrac{z-a}{z-b}$ with $a = 1$ and $b = -1$. Since the arc PLQ passes through the points 1 and -1 and forms at the point $a = 1$ an angle φ with the direction QPN, it is transformed, in accordance with what was said in Sec. 28, into the ray $P'L'$ issuing from the origin and also forming the angle φ with the positive direction of the real axis. The arc PMQ connects the same points 1 and -1 but it makes, at the point $a = 1$, an angle $\varphi - 180°$ with the direction QPN (in its absolute value this angle is equal to $180° - \varphi$; we have taken into account, however, that it is measured clockwise, i.e. in the negative direction). Therefore, the function $z' = \dfrac{z-1}{z+1}$ transforms the arc PMQ into a ray $P'M'$ issuing from the origin and forming an angle $\varphi - 180°$ with the positive direction of the real axis. It is evident that the rays $P'L'$ and $P'M'$ taken together constitute a single straight line; consequently, the function $z' = \dfrac{z-1}{z+1}$ transforms the entire circle $PLQM$ (consisting of the arcs PLQ and PMQ) into the whole line $M'P'L'$.

Let us draw through the points P and Q an arc of an auxiliary circle for which a tangent at the point P forms with QPN an angle $\varphi + 90°$. This arc intersects the circle PRS at point E. In accordance with the reasoning of Sec. 28, the arc PEQ is transformed by the function $z' = \frac{z-1}{z+1}$ into a ray issuing from the point P' and inclined at an angle $\varphi + 90°$ to the positive direction of the real axis. As this takes place, the point E is transformed into a certain point E' of that ray. In accordance with our reasoning in Sec. 29, the circle $PRES$ is transformed, by means of the function $z' = \frac{z-1}{z+1}$, into a circle $P'R'E'S'$ constructed on the segment $P'E'$ as on the diameter.

Thus, as a result of the transformation performed, the circle $PLQM$ becomes a straight line $M'P'L'$ and the circle $PRES$, contacting the latter from inside, passes into a circle $P'R'E'S'$ contacting the line $M'P'L'$ at point P'. Can we consider the problem of transforming the hatched figure by means of the function $z' = \frac{z-1}{z+1}$ to be solved? By no means, for the problem has not yet been completely solved: for the time being we have found what becomes with the contour of the figure and now we have to trace the transformation of the points of the figure located in the area between the circles $PRES$ and $PLQM$.

To elucidate this part of the problem, we shall note that we could have filled the entire hatched figure with circles which touch $PLQM$ at point P and are enclosed between $PRES$ and $PLQM$. They would intersect the arc PEQ at points lying between E and Q. In Fig. 38 dashed lines show three out of an infinite number of such circles; these three circles intersect the arc PEQ at points E_1, E_2 and E_3. If we consider the transformation of these circles by means of the function $z' = \frac{z-1}{z+1}$ and see into what lines they pass, we shall have an idea of the shape of the figure filled with all these lines. That will be precisely the transformed figure.

But taking into account the statements made in Sec. 29, we conclude that the circle $PR_1E_1S_1$ is transformed into the circle $P'R'_1E'_1S'_1$ and the circle $PR_2E_2S_2$ into the circle $P'R'_2E'_2S'_2$ etc.

We have shown at the end of Sec. 28 that as point z moves along the arc PQ, approaching Q, the corresponding point z' moves along the ray receding farther and farther from the initial point P'. It follows from this that if point E_2 is closer to Q than point E_1, then E'_2 is the image of the point E_2 and lies on the ray farther from P' than E'_1 which is the image of the point E_1. Therefore, the diameter $P'E'_2$ of the circle $P'R'_2E'_2S'_2$ which is the image of the circle $PR_2E_2S_2$ must be larger than the diameter $P'E'_1$ of the circle $P'R'_1E'_1S'_1$ which is the image of the circle $PR_1E_1S_1$ as it is shown in our drawing.

If we take the circle $PR_3E_3S_3$ intersecting PEQ sufficiently close to Q, we can succeed in obtaining its image $P'R'_3E'_3S'_3$ having an infinitely large diameter. Besides, we can show that any circle touching the line $M'L'$ at point P' and lying in the hatched area of the plane (Fig. 38, right) is associated with a certain circle contacting the circles $PRES$ and $PLQM$ at point P and lying in the hatched figure (see Fig. 38, left). It is evident that all the images of the circles such as $PR_1E_1S_1$, $PR_2E_2S_2$, $PR_3E_3S_3$ etc, filling in the hatched figure on the left-hand side of Fig. 38, will, in their turn, fill the figure hatched in the same drawing on the right. This figure is precisely the image of the initial figure when it is mapped by means of the function $z' = \frac{z-1}{z+1}$. Thus we see that the function $z' = \frac{z-1}{z+1}$ maps the figure bounded by two circles (see Fig. 38, left) onto the figure bounded by a straight line and a circle (Fig. 38, right).

31. Let us now consider a transformation by means of the function $z' = z^2$. In the note on p. 29 we warned the reader that exceptions are possible to the general rule of the preservation of angles in transformations by means of rational functions, namely, that the angles with vertices at certain singular points can increase or decrease several times over. In this particular case there is such a singular point, it is the origin of coordinates A. We will prove that all the angles with a vertex at A are doubled in size under the transformation $z' = z^2$.

Let us take a ray AM issuing from the point A and making angle φ with the positive part of the real axis (see Fig. 39). For every point z lying on this ray, $\operatorname{Arg} z = \varphi$. Since the vector $z' = z^2 = z \cdot z$ is obtained from the vector z by stretching it $|z|$ times and rotating the angle $\operatorname{Arg} z = \varphi$, we have

$|z'| = |z| \cdot |z| = |z|^2$, and $\operatorname{Arg} z' = \operatorname{Arg} z + \operatorname{Arg} z = 2\varphi$. Therefore point z' must lie on the ray $A'M'$ issuing from the point A' and forming an angle of 2φ with the positive part of the real axis.

If point z moves along AM, starting from the point A, receding from it indefinitely, then the corresponding point z' will

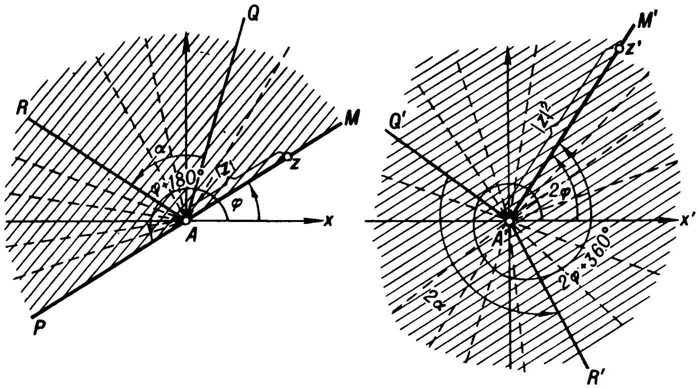


Fig. 39.

move along $A'M'$, starting from point A' and receding from it indefinitely; in this case the distance from z' to A' will always be equal to the square of the distance from z to A ($|z'| = |z|^2$).

It follows that the function $z' = z^2$ transforms the ray AM into the ray $A'M'$ inclined to the axis $A'x'$ at an angle twice as large as the initial angle.

It is easy to see that the function $z' = z^2$ transforms the ray AP forming with Ax an angle $\varphi + 180°$ (AM and AP lie on the same line) into the same ray $A'M'$. Indeed, if we double the angle $\varphi + 180°$, we shall obtain $2\varphi + 360°$; the ray inclined to $A'x'$ at this angle coincides with $A'M'$.

Let us see how the hatched figure on the left-hand side of Fig. 39 will be transformed by means of the function $z' = z^2$; the figure is called a half-plane. This half-plane can be regarded as being filled with an infinite number of rays issuing from A' and inclined to Ax at angles larger than φ but smaller than $\varphi + 180°$. The rays AM and AP constitute the boundary of the half-plane (one straight line); we shall not consider these rays to be a part of the half-plane. The function $z' = z^2$

transforms the rays belonging to the half-plane into various rays issuing from A' and inclined to $A'x'$ at angles larger than 2φ but smaller than $2\varphi + 360°$.

Hence it follows that the half-plane bounded by the rays AM and AP is transformed into a figure bounded by one ray $A'M'$ (Fig. 39, right). The latter figure can be characterized as a plane with the deleted (or excluded) ray $A'M'$. Stating this we wish to emphasize that this figure is constituted by all the points of the plane except those lying on $A'M'$.

If we take, in a half-plane, any two rays AQ and AR inclined to Ax at the angles φ_1 and φ_2 ($\varphi_2 > \varphi_1$), they will form an angle $\alpha = \varphi_2 - \varphi_1$. As a result of the transformation $z' = z^2$, these rays will pass into $A'Q'$ and $A'R'$ inclined to $A'x'$ at the angles $2\varphi_1$ and $2\varphi_2$. It is evident that the angle $Q'A'R'$ is equal to $2\varphi_2 - 2\varphi_1 = 2(\varphi_2 - \varphi_1) = 2\alpha$.

Thus we see that under the transformation $z' = z^2$ the angles with the vertex at A are doubled; in other words, the conformity of mapping is violated at point A.

32. We will show now that the angles with the vertex at any point $z_0 \neq 0$ do not change under the transformation $z' = z^2$. Hence it follows that the origin of coordinates is the only point at which the conformity under the given transformation is violated.

Let L be a curve issuing from the point z_0. If we take, on L, an arbitrary point z_1 different from z_0, then the direction of the secant connecting z_0 and z_1 will coincide with the direction of the vector Q_0Q_1 representing the difference $z_1 - z_0$ (Fig. 40, left). By means of the function $z' = z^2$ the curve L is transformed into a certain curve L' and the points z_0 and z_1 into new points $z'_0 = z_0^2$ and $z'_1 = z_1^2$, on the curve L'. The direction of the secant connecting z'_0 and z'_1 evidently coincides with the direction of the vector $Q'_0Q'_1$ representing the difference $z'_1 - z'_0$ (Fig. 40, right).

We shall now compare the directions of the two secants; to do this, it is sufficient to compare the directions of the vectors $z'_1 - z'_0$ and $z_1 - z_0$. Since the angle between them, reckoned from the vector $z_1 - z_0$ to the vector $z'_1 - z'_0$, coincides with the argument of the quotient $\dfrac{z'_1 - z'_0}{z_1 - z_0}$, the comparison reduces to the computation of $\operatorname{Arg}\dfrac{z'_1 - z'_0}{z_1 - z_0}$. The quotient $\dfrac{z'_1 - z'_0}{z_1 - z_0}$ can be transformed by replacing z'_1 and z'_0 by their expressions: $z'_1 = z_1^2$ and $z'_0 = z_0^2$.

We obtain

$$\frac{z'_1 - z'_0}{z_1 - z_0} = \frac{z_1^2 - z_0^2}{z_1 - z_0} = z_1 + z_0$$

and

$$\operatorname{Arg}\frac{z'_1 - z'_0}{z_1 - z_0} = \operatorname{Arg}(z_1 + z_0).$$

Consequently, the angle between the directions of the secants to the curves L' and L, drawn through the pairs of the

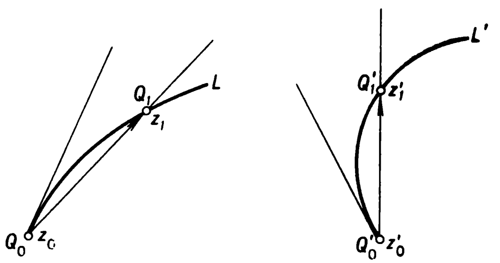


Fig. 40.

respective points z_0 and z_1 (on L) and $z'_0 = z_0^2$ and $z'_1 = z_1^2$ (on L'), is equal to $\operatorname{Arg}(z_1 + z_0)$. Now passing from secants to tangents, we shall make point z_1 approach indefinitely point z_0 along the curve L. Then the point $z'_1 = z_1^2$ will approach indefinitely the point $z'_0 = z_0^2$ along the curve L'. Therefore, the secants will also approach indefinitely the tangents drawn at the points z_0 and z'_0 and the angle between the secants will approach indefinitely the angle between the tangents. But the angle between the secants is equal to $\operatorname{Arg}(z_0 + z_1)$ and tends to $\operatorname{Arg}(2z_0)$ as z_1 tends to z_0, the latter, in its turn, coincides with $\operatorname{Arg} z_0$. Thus we have that the angle between the tangents to the curves L' and L drawn at the appropriate points $z'_0 = z_0^2$ and z_0 is equal to $\operatorname{Arg} z_0$.

If, for instance, $z_0 = 2$, then $\operatorname{Arg} z_0 = 0$, whence it follows that the direction of the tangent at the point $z_0 = 2$ to any curve L drawn through that point will coincide with the direction of the tangent at the point $z'_0 = z_0^2 = 4$ drawn to the curve L' into which the function $z' = z^2$ transforms the curve L. If $z_0 = i$, then

$\operatorname{Arg} z_0 = 90°$; consequently, the tangent at the point $z_0 = i$ to any curve L drawn through that point and the tangent at the point $z_0^2 = i^2 = -1$ to the image on the curve L' are mutually perpendicular.

Returning to the general case, we can say that the tangents rotate by the angle equal to $\operatorname{Arg} z_0$ when the curves passing through the point z_0 are transformed by means of the function $z' = z^2$.

It is easy to see now why the angles with the vertex at z_0 ($z_0 \neq 0$) remain unchanged under this transformation. If two curves L_1 and L_2 pass through the point z_0 and form an angle α at that point, this means that the tangents to the curves at that point form an angle α. As a result of the transformation, the point z_0 will pass into the point $z'_0 = z_0^2$ and the curves L_1 and L_2 will pass into the curves L'_1 and L'_2. The directions of the tangents, at point z_0, to the new curves are obtained from the initial directions of the tangents by rotating them through the same angle equal to $\operatorname{Arg} z_0$. The angle between the new tangents will evidently remain of the same magnitude α. And this means precisely that the angle between the curves with the vertex at any point $z_0 \neq 0$ does not change under the transformation $z' = z^2$.

We wish to note that the method we used to prove the conformity of mapping $z' = z^2$ is also applicable to other functions, for instance to the linear-fractional function $z' = \frac{z-a}{z-b}$ or to Zhukovsky's function $z' = \frac{1}{2}\left(z + \frac{1}{z}\right)$. But in this case we obtain some other expressions for the angle of rotation of the tangent. Thus, we shall have for a linear-fractional function that the tangents to the curves passing through the point z_0 rotate through an angle equal to $\operatorname{Arg} \frac{a-b}{(z_0-b)^2}$, and in the case of the Zhukovsky function, through an angle equal to $\operatorname{Arg}\left(1 - \frac{1}{z_0^2}\right)$. In the former case we must additionally assume that $z_0 \neq b$ (at that point the expression $\frac{z-a}{z-b}$ is meaningless); in the latter case we must additionally assume that $z_0 \neq 0$ (for the same reason), and, besides, that $z_0 \neq \pm 1$ (at these points

$1 - \frac{1}{z_0^2}$ vanishes and, hence, $\operatorname{Arg}\left(1 - \frac{1}{z_0^2}\right)$ becomes senseless). We could have verified that in the case of the Zhukovsky function the conformity is violated at points -1 and $+1$, since the angles with vertices at these points are doubled as a result of transformation.

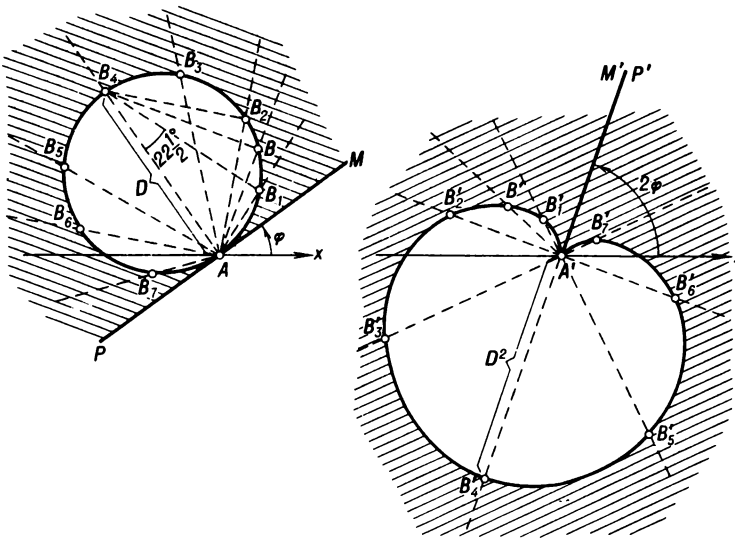


Fig. 41.

33. Let us see how a circle passing through the origin A will be transformed by means of the function $z' = z^2$. We shall assume that the tangent to the circle at that point forms an angle φ with Ax (Fig. 41). The circle is evidently located in a half-plane bounded by that tangent line. The function $z' = z^2$ transforms the half-plane into a plane with the ray $A'M'$ deleted. To find the image of the circle, let us draw from A arbitrary rays in the half-plane and mark off, on each of them, a point of intersection with the circle. For definiteness, we have seven rays in our drawing; all the angles MAB_1, B_1AB_2, B_2AB_3, ..., B_7AP are taken to be equal $\left(22\frac{1°}{2} \text{ each}\right)$. The

function $z' = z^2$ transforms them into rays forming between themselves the angles twice as large; each of the angles $M'A'B'_1$, $B'_1A'B'_2$, $B'_2A'B'_3$, ..., $B'_7A'P'$ is equal to 45°.

Let us calculate where the points B_1, B_2, B_3, ..., B_7 will pass to. The distances of their images B'_1, B'_2, B'_3, ..., B'_7 from the point A' will be equal to the squares of the distances AB_1, AB_2, AB_3, ..., AB_7. But it is seen from Fig. 41 that $AB_7 = AB_1 = AB_4 \sin 22\frac{1^\circ}{2} = D \sin 22\frac{1^\circ}{2}$ (D is the diameter of the circle); further, $AB_6 = AB_2 = D \sin 45^\circ$, $AB_5 = AB_3 = D \sin 67\frac{1^\circ}{2}$, $AB_4 = D$. It should also be noted that

$$\sin^2 22\frac{1^\circ}{2} = \frac{1 - \cos 45^\circ}{2} = \frac{2 - \sqrt{2}}{4} = \frac{2 - 1.4142...}{4} = 0.1464...,$$

$$\sin^2 45^\circ = 0.5000..., \quad \sin^2 67\frac{1^\circ}{2} = \cos^2 22\frac{1^\circ}{2} = 1 - \sin^2 22\frac{1^\circ}{2} = 0.8535....$$

Consequently, $A'B'_7 = A'B'_1 = 0.1464D^2$, $A'B'_6 = A'B'_2 = 0.5000D^2$, $A'B'_5 = A'B'_3 = 0.8535D^2$, $A'B'_4 = D^2$. Thus we see that a curve which is the image of the circle under the transformation $z' = z^2$ passes through the points A', B'_1, B'_2, B'_3, ..., B'_7.

To have a better idea of this curve, we should have taken a larger number of rays. This curve is termed a *cardioid* (heart-like from the Greek kardia meaning heart). It is easy to realize that the hatched figure on the left-hand side of Fig. 41 (it is obtained from the half-plane by deleting the circle) is transformed, by means of the function $z' = z^2$, into the figure hatched on the right-hand side of the same drawing. The latter is bounded by the cardioid and the ray $A'M'$ forming an angle of 2φ with the positive direction of the real axis.

It can be shown that the ray $A'M'$ is directed along the tangent to each of the two arcs of the cardioid emanating from the point A. Indeed, let us draw on the left-hand side of Fig. 41 an arbitrary ray AB and assume that B denotes the point of its intersection with the circle; if angle $MAB = \alpha$, then $AB = D \sin \alpha$. By means of the function $z' = z^2$ this ray is transformed into the ray $A'B'$ (Fig. 41, right); as a result, the point B', the image of the point B, gets on the cardioid.

From the properties of the transformation $z' = z^2$, known to us, we have: $\widehat{M'A'B'} = 2\alpha$ and $A'B' = AB^2 = D^2 \sin^2 \alpha$. Let us assume the angle α to be variable and make it approach zero indefinitely. Then the angle 2α between $A'B'$ and $A'M'$ will also approach zero indefinitely and the ray $A'B'$ itself, which is a secant for the cardioid, will rotate about the point A',

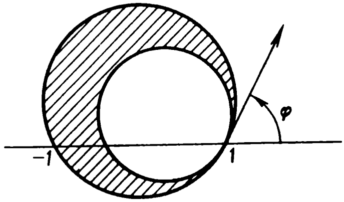


Fig. 42.

approaching indefinitely the limiting position $A'M'$. In the process, the point B', which is the nearest to A' point of intersection of the secant and the curve, will approach A' indefinitely, since the distance $A'B' = D^2 \sin^2 \alpha$ tends to zero as α tends to zero. It follows that $A'M'$, which is the limiting position of the secant, is the tangent to the arc $A'B'_1B'_2...$ at the point A'. It can also be proved that $A'M'$ is the tangent to the arc $A'B'_7B'_6...$ at the same point A'.

34. Now let us turn to Zhukovsky's function

$$z' = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

and apply it to the transformation of the figure bounded by two circles: one circle passing through the points -1 and $+1$, and the other circle contacting the first circle from within at the point 1; the figure is hatched in Fig. 42.

Let us first make sure that the transformation

$$z' = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

can be reduced to several more simple transformations of the kind known to us, which may be performed one after another. To this end let us consider the ratio

$$\frac{z'-1}{z'+1}.$$

Substituting in it, for z', the expression

$$\frac{1}{2}\left(z+\frac{1}{z}\right),$$

we shall find

$$\frac{z'-1}{z'+1}=\frac{\frac{1}{2}\left(z+\frac{1}{z}\right)-1}{\frac{1}{2}\left(z+\frac{1}{z}\right)+1}=\frac{z^2+1-2z}{z^2+1+2z}=\left(\frac{z-1}{z+1}\right)^2.$$

Thus, from the fact that

$$z'=\frac{1}{2}\left(z+\frac{1}{z}\right),$$

it follows that

$$\frac{z'-1}{z'+1}=\left(\frac{z-1}{z+1}\right)^2.$$

The converse is also true: the second leads to the first. Indeed, we obtain, from the second, the expression

$$z'-1=z'\left(\frac{z-1}{z+1}\right)^2+\left(\frac{z-1}{z+1}\right)^2,$$

whence

$$z'\left[1-\left(\frac{z-1}{z+1}\right)^2\right]=1+\left(\frac{z-1}{z+1}\right)^2,$$

and further:

$$z'=\frac{\left(1+\frac{z-1}{z+1}\right)^2}{1-\left(\frac{z-1}{z+1}\right)^2}=\frac{(z+1)^2+(z-1)^2}{(z+1)^2-(z-1)^2}=$$

$$=\frac{2z^2+2}{4z}=\frac{1}{2}\left(z+\frac{1}{z}\right).$$

Thus the relations

$$z' = \frac{1}{2}\left(z + \frac{1}{z}\right) \text{ and } \frac{z' - 1}{z' + 1} = \left(\frac{z - 1}{z + 1}\right)^2$$

are equivalent (one follows from the other).

Therefore the Zhukovsky transformation

$$z' = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

can be represented in the form

$$\frac{z' - 1}{z' + 1} = \left(\frac{z - 1}{z + 1}\right)^2.$$

The result must be the same. But now we can see that the transition from z to z' can be performed in three stages: first, perform the transition from z to an auxiliary variable z_1 by the formula

$$z_1 = \frac{z - 1}{z + 1}, \tag{1}$$

then pass from z_1 to z_2 according to the formula

$$z_2 = z_1^2, \tag{2}$$

and, finally, from z_2 to z' by the formula

$$\frac{z' - 1}{z' + 1} = z_2. \tag{3}$$

We can easily make sure that if we substitute the expression for z_1 from formula (1) into formula (2) and then insert the expression obtained for z_2 into formula (3), we shall have the transformation required:

$$\frac{z' - 1}{z' + 1} = \left(\frac{z - 1}{z + 1}\right)^2.$$

What is the sense in replacing one transformation of Zhukovsky by three transformations (1), (2) and (3) performed one after another? The fact is that each of them is simpler than Zhukovsky's transformation and is already known to us.

Thus, let us apply to the figure shown in Fig. 42 transformation (1), then apply transformation (2) to what we shall obtain and, finally, to the result of the second operation, we shall apply transformation (3).

Recall that we found, in Sec. 30, that the figure shown on the left-hand side of Fig. 38 (and it coincides with the figure in Fig. 42) is transformed by means of the function

$$z_1 = \frac{z-1}{z+1}$$

(i. e. the function (1)) into the figure shown in Fig. 38, right. The latter figure is bounded by a straight line passing through the point O and forming an angle φ with the positive direction of the real axis and a circle contacting this line at the point O. This figure can be characterized as a half-plane with the circle deleted.

Let us transform this figure by means of the function $z_2 = z_1^2$. It is sufficient to have a look at Fig. 41 to realize that this problem was already solved in Sec. 33. At the end of Sec. 33 we made a note to the point that we must obtain here a figure depicted on the right-hand side of Fig. 41; it is bounded by a ray and a cardioid.

Consequently, it remains to apply to the latter figure the transformation $\frac{z'-1}{z'+1} = z_2$. It follows from what was said in Sec. 28 (with the only difference that here z_2 is regarded as an independent variable and z' as a function) that when z_2 describes the ray $A'M'$ issuing from the origin and inclined

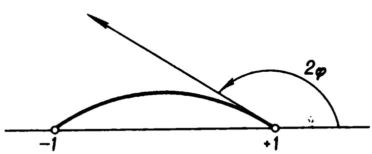


Fig. 43.

to the positive part of the real axis at an angle 2φ, the corresponding point z' describes an arc of a circle connecting the points $+1$ and -1; the tangent at the point $+1$ to this arc constitutes with the direction from point -1 to point $+1$, i. e. with the positive direction of the real axis, the angle 2φ as well (Fig. 43).

We have thus found the image of the ray $A'M'$ under the transformation $\frac{z'-1}{z'+1} = z_2$. To find the image of the cardioid, we could have traced how its points are transformed, for instance, the points $B'_1, B'_2, \ldots, B'_7$. We will not, however, perform cumbersome calculations, but will limit ourselves to depicting the transformed curve in its final form in Fig. 44.

The figure bounded by it has the shape of an aircraft wing section. Sections of this kind were first considered by Russian

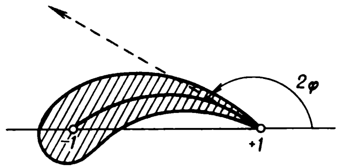


Fig. 44.

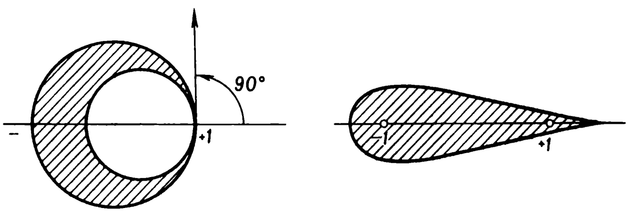


Fig. 45.

scientists N. E. Zhukovsky and S. A. Chaplygin for which reason they are called *Zhukovsky-Chaplygin sections*. By changing the angle of inclination φ of the tangent line to the circle at point 1 (Fig. 42) and the radius of the smaller circle, various sections can be obtained. In particular, if the angle φ is right, i. e. if the larger circle is constructed on the interval from -1 to $+1$ as on the diameter, the corresponding section is symmetric with respect to the real axis (Fig. 45). Such a section is sometimes called *Zhukovsky's vane*.

Zhukovsky-Chaplygin's sections are the principal sections employed in theoretical investigations of an aircraft wing.

EXERCISES AND PROBLEMS

1. Prove that if two complex numbers $c = a_1 + ib_1$ and $c_2 = a_2 + ib_2$ are equal, then their real parts are equal and imaginary parts are equal: $a_1 = a_2$ and $b_1 = b_2$.

Instruction. Proceed from the fact that equal complex numbers are represented by parallel vectors which are of equal length and of the same direction.

2. Using commutative, associative and distributive laws of addition and multiplication, perform the following operations on complex numbers:

(a) $(3 - 7i) + (-2 + i) + (-1 + 5i)$;

(b) $(3 - 7i)(3 + 7i)$;

(c) $(1 + i)(1 + i\sqrt{3})$;

(d) $(1 + i)^2 : (1 - i)^2$;

(e) $\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^4$.

Answers: (a) $-i$; (b) 58; (c) $1 - \sqrt{3} + i(1 + \sqrt{3})$; (d) -1; (e) -1.

3. Prove that any complex number $c = a + bi \neq 0$ whose modulus is equal to r and argument to α can be represented in the form

$$c = r(\cos\alpha + i\sin\alpha)$$

(*trigonometric form of a complex number*).

Instruction. Express a and b in terms of r and α with the aid of a drawing on which $c = a + bi$ is represented in vector form.

4. Prove that if

$$c_1 = r_1(\cos\alpha_1 + i\sin\alpha_1) \text{ and } c_2 = r_2(\cos\alpha_2 + i\sin\alpha_2),$$

then

$$c_1c_2 = r_1r_2[\cos(\alpha_1 + \alpha_2) + i\sin(\alpha_1 + \alpha_2)].$$

Instruction. Make use of the geometric statement of the rule of multiplication of complex numbers or multiply c_1 by c_2 using the rules of addition and multiplication and then apply the formulas for the cosine and sine of the sum.

5. Proceeding from the result of the previous problem, prove that if

$$c = r(\cos\alpha + i\sin\alpha)$$

(r is the absolute value of c and α is the argument of c) then

$$c^n = r^n(\cos n\alpha + i\sin n\alpha)$$

(n is a natural number). Derive from this that

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$$

(*Moivre's formula*).

6. Making use of Moivre's formula (see Problem 5), calculate:

$$\text{(a)}\ \left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^{100};\quad \text{(b)}\ \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{217}$$

Instruction. $\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}=\cos 45^\circ+i\sin 45^\circ$; $\frac{\sqrt{3}}{2}+\frac{i}{2}=\cos 30^\circ+$ $+i\sin 30^\circ$.

Answers:

$$\text{(a)}\ -1;\quad \text{(b)}\ \frac{\sqrt{3}}{2}+\frac{i}{2}.$$

7. Proceeding from Moivre's formula (see Problem 5), derive the formulas for $\cos n\alpha$ and $\sin n\alpha$ with $n = 2$, 3 and 4.

Instruction. In Moivre's formula $(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$ the term $\cos\alpha + i\sin\alpha$ should be raised to the power n by a direct multiplication (for example $(\cos\alpha + i\sin\alpha)^2 = \cos^2\alpha + 2i\sin\alpha\cos\alpha - \sin^2\alpha$) and then it should be written that the real and imaginary parts to the right and to the left of the equality sign in Moivre's formula are equal between themselves.

Answers: $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$; $\sin 2\alpha = 2\sin\alpha\cos\alpha$; $\cos 3\alpha =$ $= \cos^3\alpha - 3\cos\alpha\sin^2\alpha$; $\sin 3\alpha = 3\sin\alpha\cos^2\alpha - \sin^3\alpha$; $\cos 4\alpha = \cos^4\alpha -$ $- 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha$; $\sin 4\alpha = 4\sin\alpha\cos^3\alpha - 4\sin^3\alpha\cos\alpha$.

8. What will the triangle with its vertices at the points 0, $1 - i$, $1 + i$ pass to under the transformation

$$z' = \left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)z?$$

What is the geometrical meaning of this transformation?

Instruction. Begin with the elucidation of the geometrical meaning. But you can also begin with calculating the vertices of the transformed triangle.

9. What shall a semi-circle pass to as a result of the transformation $z' = \frac{z-1}{z+1}$ if it is located above the real axis and rests on the interval with the end points -1 and $+1$ as the diameter?

Answer: Into a right angle bounded by the upper part of the imaginary axis and the negative part of the real axis.

10. What will the angle α with the vertex at the origin pass to as a result of the transformation $z' = z^3$?

Answer: Into the angle 3α with the vertex at the origin.

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