THE MATHEMATICAL THEORY OF RELATIVITY

DE DONDER

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BY

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FOREWORD

The reception accorded by the scientific public to Professor Born's "Problems of Atomic Dynamics," published by Technology in 1926, has made evident the value of extending to a wide circle of readers reports of lectures at the Institute by leading investigators in the several fields of modern physics. For the spring term of 1926 Professor T. De Donder of the University of Brussels was appointed special lecturer at the Department of Physics, where he delivered a course on the Mathematical Theory of Relativity. This book contains the text of these lectures and is the second of the series.

C. L. NORTON

Department of Physics
Massachusetts Institute of Technology
March, 1927

PREFACE

This book includes ten lectures on the Mathematical Theory of Relativity, as I have developed it during the last twelve years. These lectures were delivered at the Massachusetts Institute of Technology during the Spring Term of the academic year 1925–1926.

I desire to express to this Institute my deepest gratitude for the invitation which has been extended to me to give these lectures. It was very pleasant to think that I would be able to collaborate in the scientific research of this important and celebrated institution, and to renew the ties of friendship with the intellectual élite of the American people. I have not been disappointed in my hopes. The cordial reception extended to me by my colleagues in the Departments of Physics and Mathematics, the interesting conversations and exchanges of ideas I have had with them, have played no small part in creating this feeling. In the domain of thought we are all citizens of the same country. Barriers appear only where science ceases to cast its light; and these frontiers recede ever toward more remote regions. Thanks to science, the world becomes vaster and richer. The only revolutionists are scholars and artists, for they create new conditions and they change the aspect of the universe. To be sure, this evolution sometimes carries along with it sudden and terrible changes. Think of the Great War! The present adaptation towards the universality due to science is still painfully felt in many countries. In my fatherland, scientific reconstruction was greatly helped by the C. R. B. This Committee for Relief in Belgium was organized, during the war, for avoiding famine and, in collaboration with other American organizations, has helped Belgium very much in its new organization of teaching and scientific research. Indeed, Belgium will always remember with thankfulness and admiration that noble initiative of the United States of America.

Before closing this brief preface, I wish to express my sincere thanks to my colleague, Professor M. S. Vallarta, who has aided me, with his deep knowledge of the subject, in the final writing of these lectures.

TH. DE DONDER.

Massachusetts Institute of Technology
May, 1926

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THE MATHEMATICAL THEORY OF RELATIVITY

LECTURE 1

General Introduction

Space-time — Gravific field and the Γ-Map — Mass and electromagnetic fields — Restricted relativity — The Michelson-Miller experiments.

No science has excited the curiosity of the public at large as much as relativity. This is due, it is well to remind you, to the fact that the new conceptions of Lorentz, Minkowski and Einstein disrupt our former beliefs of space and time. It was necessary to destroy the rigid structure of Euclidean geometry, it was necessary to reject the universal time of Newton, in other words, it was necessary to assume that the standards of length and of time no longer have the same value for all observers. Space and time are now relative; they are united in a new conception in order to obtain a tool independent of the spectators using them. This new mathematical tool is called *space-time*.

In the first part of this course we shall make a systematical study of space-time, independent of any other physical conception such as electricity, mass, etc. We shall first consider the simplest space-time, that obtained by Minkowski, by expressing in one quadratic form the characteristic properties of Euclidean space and in which the propagation of light is isotropic and uniform. All the graphical representations will be constructed in a Euclidean space provided with clocks regulated by this light. We shall never use visualization in space-time itself. This space-time is only utilized as a purely mathematical tool particularly fitted to the needs of relativity and capable of giving us valuable information about the deformations of the ether when submitted to various physical actions. From the start we shall use this tool in all its generality. Thus on considering a Minkowskian field and in it a spectator having a uniform rectilinear motion, we shall obtain the Lorentz transformation with its physical interpretation of the contraction of the standard of length and the dilatation of the standard of time. This method of analysis is later extended to all types of motion in a Minkowski field.

It is now easy for us to develop the much more general concept of space-time which defines the Einsteinian *"gravific" field.* The graphical representation will be constructed as before, but will only have the significance of a *map,* i.e., of a picture drawn in Euclidean space using the earlier conception of a universal time. Thanks to space-time, or to the $(ds)^2$ of Einstein, it will be possible to pass from the numbers written on the map to the *physical* measurements obtained by various observers who explore the gravific field under consideration. This correspondence is made possible by a space-time which is independent of the spectator; it stays invariant. Our conceptions of space and time are only particular aspects of space-time.

The second part of this course will deal with the theory of gravific fields. The laws governing these fields will be obtained by writing that the variational covariant derivatives of a certain function called the *"phenomenal" function* are respectively equal to the variational covariant derivatives of another function called the *"gravific" function.* The latter is a linear function of the Gauss-Riemann curvature invariant. We thus introduce two arbitrary constants: Newton's gravific constant and Einstein's cosmic constant.

This phenomenal function will be called the *"mass"* function when the perturbations of the ether are produced by *matter* exclusively; it will be called *"electromagnetic"* function when the perturbations of the ether are caused by *electricity.* We have succeeded in writing the phenomenal function for the most general case when the gravific field is due to anisotropic, non-homogeneous bodies, electrically and magnetically polarized, at rest or in motion, which are the seat of convection currents. This function, on account of the fundamental principle of relativity, includes in itself celestial mechanics, dynamics of continuous media, Maxwell's electromagnetic field, Lorentz's electronic dynamics and Minkowski's electrodynamics of bodies in motion. Not only do we find in this ultimate synthesis all the classical results, but we obtain them with a greater degree of approximation.

Thus the Einsteinian relativity makes the physical world better known, it discloses unsuspected phenomena, it supplies the quantitative explanation of facts that previously had remained in the dark.

What is the source of this almost magic power of relativity? In my opinion we must look for it in its mathematical structure. Thanks to space-time and to tensor calculus, relativity brings to the fore what is *intrinsic* or *absolute*. When physical laws are expressed independently of the choice of space and time variables they leave what is unessential in the shade; all that which is particular to a given physical observer. The human spectator vanishes, making room for a single, absolute spectator. The latter tells us in mathematical language all that is essential in the laws of nature, and the laws he formulates have that perfect form, independent of time and space, characteristic of masterly works.

It seems that the different modifications sustained by the ether and which form the object of the study of the gravific field should cast some light on the nature of *electricity*. We might thus hope that this study of the ether from a novel standpoint would disclose the secret of the two electric fluids. That would unquestionably be the most important unification traceable to general relativity, even when compared with the deep correlation it has already established between matter and energy.

In spite of repeated attempts by Weyl, Eddington and Einstein, this is still an open question. Einstein, who on two occasions thought he had reached this long-sought result, was kind enough to write me in December 1925 that he had given up definitely that line of research, as it was bound to yield no result. This conclusion appears almost immediately from our own interpretation of the gravific field. The perturbations of the ether are given by the mathematical methods peculiar to general relativity only if we start from certain physical causes given "*a priori*." To give up these causes is to come back to the Minkowski field.

As early as 1914 we had the foundations for a theory of pure gravific electromagnetic fields. Researches in that direction have recovered all their importance since the failure of Weyl's theory. We have given a systematic and advanced description of these investigations in the second part of our new synthesis of relativity which has just been published in the "Mémorial des Sciences Mathématiques." Let us consider first the electromagnetic tensor of this field. It has the same form as Maxwell's tensor, but it contains an additional term depending essentially on masses in motion. This additional term replaces here the material connections which have to be taken into account in the Maxwellian theory. This electromagnetic tensor gives very remarkable and useful forms of the theorems of the momentum and of the energy of electric charges in motion. All these results are obtained with a minimum of hypotheses. If now we introduce a new fundamental electromagnetic function, in which the electromagnetic forces and the electric density appear, we obtain, besides the generalization of Maxwell's equations, some very important invariants concerning mass and electric densities, besides a very simple form of the above mentioned theorems of momentum and of energy. The relation enabling us to compute the ratio of mass to electric charge by means of the electromagnetic field appears to be worth mentioning. All equations of this electrodynamics can immediately be put in the Lagrangian form and next in the Hamiltonian or canonical form. These transformations are as successful in space-time as in space *and* time. By this it is meant that we can use as independent variables, either absolute time s or relative time t, at will.

If we immerse in the Minkowski field the electromagnetic field that we have studied we obtain the classical field of Maxwell-Lorentz. Here we find again the Maxwellian tensor with an additional term for masses and charges in motion. If we consider an observer having a rectilinear uniform motion in this field, by simply applying the Lorentz transformation, we obtain the *restricted relativity* of Maxwell's electromagnetic field. In short, everything evolves in perfect harmony.

When we formulate problems in general relativity we are somewhat astonished at the considerable amount of physical data that have to be chosen arbitrarily. The reason is essentially that relativity gives us a method for determining the deformation or the tension in the ether. We have at our disposal the ten components of the Einstein tensor. Ether adapts itself, in all points and at every instant, to the given physical data written on our map.

To formulate a problem in general relativity, we shall employ a map. This map will usually be a Minkowski map. We shall introduce arbitrary constants and functions, the latter permitting the map to be adapted to the particular physical problem under consideration. It will be permissible to proceed by successive approximations. We must not, however, lose sight of the fact that Einstein's gravific theory is, in fact, a first approximation of the perturbations produced in the ether by the bodies or phenomena which have been introduced there. We have seen that the Gauss-Riemann invariant of curvature is fundamental in Einsteinian relativity. Now there exists an infinity of other invariants. We justify this *very special* choice by noting that the Gauss-Riemann invariant is the only invariant which is *linear* with respect to the second derivatives of the Einstein tensor $g_{\alpha\beta}$. It is now clear that we retain in the Einstein method, only the principal part of ethereal deformations. Let us note, in passing, that it would be impossible to take into account all the infinity of curvature invariants of higher and higher order. Confronted with this impossibility, we could, it is true, adopt an opposite attitude, that of not working with any of the invariants of curvature. The attempts of Weyl, Eddington and quite recently Einstein, reduce themselves fundamentally to this attitude.

After this digression let us return to the map in which we formulated our problem. To fix our ideas let us consider, for example, the gravific field produced by a material sphere at rest, containing a perfect fluid of constant density. This is Schwarzschild's problem. The gravific field has spherical symmetry. Let us note that almost in spite of ourselves we have abandoned space-time in this formulation and have fallen back on the old habit of thinking of space *and* time.

If we wish to study the state of the ether in the case of a material sphere moving with uniform rectilinear motion we again formulate this problem on Minkowski's map. The gravific field has in this case axial symmetry. Let us note that if the mass of this sphere is negligible, we revert to the uniform rectilinear translation of a trihedron in the Minkowski field, in other words, to *restricted relativity*. This remark will allow us, in addition, to indicate the components $g_{\alpha\beta}$ which *cannot be annulled* in the $(ds)^2$ of the gravific field to be determined. The problem of uniform and rectilinear translation, on the map, of a material sphere leads us to wonder if it is possible, by optical experiments performed in such a gravific field, to detect this motion with respect to the Minkowski field at infinity. In other words, will it be possible to detect experimentally the *absolute* motion of matter with reference to the ether at *absolute* rest? The answer is *yes*. At first sight this result will surely appear paradoxical. It seems, indeed, to contradict the *opposite* affirmation of *restricted* relativity. But this apparent contradiction disappears if we remember that a gravific field differing from zero cannot be identified with a Minkowski field. A few months ago, I had occasion to develop these considerations before H. A. Lorentz, with respect to the recent experiments of D. C. Miller. Mr. Lorentz told me that in principle he was entirely in agreement with me on this point.

Let us consider Miller's results for a moment. We know that if these results were exact and if the gravific field in which the experiments were performed was *negligible*, it would be necessary to abandon the principle of isotropy of light propagation in the Minkowski field. In other words, physical interpretations would become impossible; general relativity would only furnish an analytical method allowing us to calculate the perturbations of the ether produced by the bodies and phenomena which we have placed in it, starting from a given map.

All the results would be in a certain sense totally contained *within the map*. Relativity would become the science of relative deformations.

Before concluding, it will be necessary to consider if the gravific field of the earth in its uniform rectilinear motion with respect to the ether at rest is sufficient to explain the displacements of the interference fringes observed by Miller. As this problem is not yet solved, it is desirable to ask whether the variations due to altitude, azimuth, right ascension and the various seasons of the year have the general trend indicated by the graphic results of Miller. This comparison has been made by Thirring[1] who concludes that: "we must assume an error of observation of the whole measured effect in order to have agreement between the observed and calculated curves. It follows from this that the effect observed on Mt. Wilson cannot be a real one; it has nothing to do with the anisotropy of light propagation due to the earth's motion, but must depend on unexplained disturbing influences."

Let us therefore await developments. Further experiments are necessary. One of my colleagues at the University of Brussels, Mr. Aug. Piccard, will attempt in the near future to repeat Michelson's experiment in a balloon[2]. At first sight, a balloon seems to be a laboratory short of stability for such a delicate experiment but it is hoped that these difficulties will eventually be overcome.

We shall see that general relativity applied to the most general type of electromagnetic bodies yields in a very simple, almost automatic way, the problem enunciated but not completely solved by Minkowski of the restricted relativity of such bodies.

In the same way we obtain the complete, rigorous expression for the electromagnetic tensor and we derive from it, as a particular illustration, the mechanical force in a Maxwellian field.

[1] Zeitschrift für Physik, Feb., 1926.

[2] Since this was written, Piccard and Stahel have repeated the Michelson experiment in a rotating balloon and in the laboratory, with negative results. See Piccard and Stahel, Comptes-Rendus, Aug. 17, 1926; Jan. 17, 1927; also Die Naturwissenschaften, Vol. 15, p. 121, 1927.

We then say a few words about the mysterious quantum. To throw some light on this obscure physical entity, we shall deduce at first from the relativistic electrodynamics expressed by means of points in space-time, the dynamics of an atomic or molecular system of any number of degrees of freedom. We shall then devise a general method of quantization in *space-time,* which we shall apply to the quantization of the point electron and to that of *continuous* systems: It will be shown that this quantization is a logical consequence of our gravific theory applied to *permanent* ensembles of continuous or point systems. Once more relativity unfolds the great physical drama of the universe clad in an immutable form bearing the stamp of eternal laws.

LECTURE 2

The Minkowskian Field

Physically empty space — The Minkowskian field — Distance — Time — Event — Simultaneity — Coincidence — Interval — Examples — Changes of variables performed by the physical observer $\overline{S}$ — Euclidean trihedron in Minkowski's field — Generalization of the Lorentz contraction and the Einstein dilatation — Restricted relativity — Michelson-Morley's experiment.

Experience teaches us that a portion of space without mass or electricity may be the seat of very different gravitational and electromagnetic fields. There exist, therefore, widely different physically empty spaces. In order to visualize better the idea of these physically empty spaces, we may appeal to a hypothetical fluid, "the ether," and to its perturbations. But, as in relativity neither its structure nor its way of acting plays any rôle, we shall avoid speaking of it.

An empty space will be called the seat of a Minkowski field when it is Euclidean and when the propagation of light rays occurs along Euclidean straight lines, with the same constant velocity in all directions. As an approximate example of a Minkowski field, we may cite the interstellar space in regions sufficiently remote from stars.

We shall now explain how an observer or physicist $\overline{S}$ will verify these properties by using standards of length and time, at rest in that space.

The *distance* between two points will be defined by its measure taken by means of the standard of length belonging to the observer $\overline{S}$ working in the Minkowski field. This measurement shall be made as follows:

Using a ray of light the observer $\overline{S}$ will trace the straight line connecting the two points and will then see how many times this straight segment contains the standard of length. In a similar way, he will construct a trirectangular trihedron $\overline{T}$

attached to this field and then will measure the rectangular coördinates of the two points with respect to the trihedron $\bar{T}$. Let $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$ be the values obtained thus for the coördinates of the first point P_1, and $\bar{x}_2$, $\bar{y}_2$, $\bar{z}_2$ the values found for the coördinates of the second point P_2. In agreement with the definition of Euclidean space, the observer $\bar{S}$ will find:

$$|P_1P_2| = \sqrt{(\bar{x}_1 - \bar{x}_2)^2 + (\bar{y}_1 - \bar{y}_2)^2 + (\bar{z}_1 - \bar{z}_2)^2} \qquad (1)$$

To define the measurement of time at different points in the Minkowskian field, the observer $\bar{S}$ will make use of the properties of light. He will define the mode of propagation of light by a positive number $\bar{c}$, arbitrarily chosen once for all. At the moment $\bar{t}_0$, arbitrarily chosen, $\bar{S}$ sends a light signal from P_0 towards P; he makes the number

$$\bar{t} = \bar{t}_0 + \frac{1}{\bar{c}}|P_0P| \qquad (2)$$

correspond with the point B and calls $(\bar{t}-\bar{t}_0)$ the *interval of time* taken by light to travel from P_0 to P. To keep track of these instants, he uses clocks fixed at the different points of the space considered. These clocks are regulated when the clock at P shows the time $\bar{t}$ at the instant when it receives the light signal. The number $\bar{t}_0$ will be called the initial instant, read by $\bar{S}$ on the clock fixed at P_0.

Owing to the introduction of the constant $\bar{c}$, the time $\bar{t}$ is thus determined by $\bar{S}$ at all points P of the field. In other words, the clocks attached to different points P are regulated by $\bar{S}$ by means of light signals.

An event will be defined for the observer $\bar{S}$ by four numbers $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\bar{x}_4$ or $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$. The first three determine the coördinates of the point where the event occurred, and the fourth determines the epoch of the occurrence of the event. These numbers are obtained by $\bar{S}$ in the manner shown above.

Two events defined respectively by the systems of numbers $(\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{t}_1)$, $(\bar{x}_2, \bar{y}_2, \bar{z}_2, \bar{t}_2)$ are called *simultaneous* for $\bar{S}$ when $\bar{t}_1 = \bar{t}_2$.

Two events are *coincident* for $\bar{S}$, or form a coincidence for $\bar{S}$, when

$$\bar{x}_1 = \bar{x}_2 \cdot \cdot \cdot \cdot \cdot \cdot \bar{t}_1 = \bar{t}_2.$$

Let us consider two points infinitely close together P_1, P_2, the coördinates of which with respect to the trihedron $\overline{T}$ are respectively $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ and $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$. Let us place $\bar{x}_2 - \bar{x}_1 = \delta\bar{x}$, $\bar{y}_2 - \bar{y}_1 = \delta\bar{y}$, $\bar{z}_2 - \bar{z}_1 = \delta\bar{z}$. The distance between the two points will be measured by

$$|P_1P_2| = |\sqrt{(\delta\bar{x})^2 + (\delta\bar{y})^2 + (\delta\bar{z})^2}.$$

Let us place $\delta\bar{\sigma} = |P_1P_2|$; we shall have

$$(\delta\bar{\sigma})^2 = (\delta\bar{x})^2 + (\delta\bar{y})^2 + (\delta\bar{z})^2. \tag{3}$$

From the definition of the measurement of time given above, we deduce that a light signal sent at the instant $\bar{t}$ from the point P_1 towards P_2, will reach the latter at the instant $\bar{t} + \delta\bar{t}$, defined by the equation,

$$\bar{t} + \delta\bar{t} = t + \frac{1}{\bar{c}}|P_1P_2| = \bar{t} + \frac{1}{\bar{c}}\,\delta\bar{\sigma}.$$

Consequently we have

$$d\bar{\sigma} = \bar{c}\,d\bar{t} \tag{4}$$

the differentials d (instead of δ) reminding us that we are dealing with the propagation of light.

It follows that the number $\bar{c}$ is the measure of the velocity of light in Minkowski's empty space, this measure being obtained by the observer $\overline{S}$. By introducing the coördinates, the preceding relation (4) may also be written:

$$-d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 + \bar{c}^2\,d\bar{t}^2 = 0. \tag{5}$$

Consider two infinitely close events, defined respectively for $\overline{S}$ by the numbers $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$, $(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z}, \bar{t} + \delta\bar{t})$. Let us place

$$\delta\bar{s}^2 = -\delta\bar{x}^2 - \delta\bar{y}^2 - \delta\bar{z}^2 + \bar{c}^2\delta\bar{t}^2. \tag{6}$$

We shall say that this $\delta\bar{s}^2$ *has the Minkowski*[1] *form.* We have thus (Equation (3)),

$$\delta\bar{s}^2 = -\delta\bar{\sigma}^2 + \bar{c}^2\delta\bar{t}^2. \tag{7}$$

[1] H. Minkowski "Raum und Zeit" Verh. d. Naturforsch. Ges. zu Köln. Lecture delivered on Sept. 21, 1908.

If in (7) we have

$$\bar{c}|\delta\bar{t}| \geq \delta\bar{\sigma}$$

we write, by definition,

$$\delta\bar{s} = |\sqrt{-(\delta\bar{\sigma})^2 + (\bar{c}\,\delta\bar{t})^2}|; \tag{8}$$

if, on the other hand, we have in (7)

$$\bar{c}|\delta\bar{t}| < \delta\bar{\sigma}$$

we write, by definition,

$$\delta\bar{s} = |\sqrt{(\delta\bar{\sigma})^2 - (\bar{c}\,\delta\bar{t})^2}|\sqrt{-1}. \tag{9}$$

The element $\delta\bar{s}$ defined in this way is called *the interval of the two infinitely close events* $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$, $(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z}, \bar{t} + \delta\bar{t})$ measured by $\bar{S}$ in the Minkowski field under consideration.

Let us consider with $\bar{S}$ a point P moving from $(\bar{x}, \bar{y}, \bar{z})$ to $(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z})$ and let $\delta\bar{t}$ be the lapse of time between the moment $\bar{t}$ when the point P is at $(\bar{x}, \bar{y}, \bar{z})$ and the moment $\bar{t} + \delta\bar{t}$ when it is at $(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z})$. The transport velocity $\bar{v}$ of P, measured by $\bar{S}$, is given by

$$(\bar{v})^2 = \left(\frac{\delta\bar{x}}{\delta\bar{t}}\right)^2 + \left(\frac{\delta\bar{y}}{\delta\bar{t}}\right)^2 + \left(\frac{\delta\bar{z}}{\delta\bar{t}}\right)^2 = \left(\frac{\delta\bar{\sigma}}{\delta\bar{t}}\right)^2. \tag{10}$$

The square of the interval between the two events, that is, between the passages (transits) of P at each of the two positions considered, is

$$(\delta\bar{s})^2 = [-(\bar{v})^2 + (\bar{c}\,\delta\bar{t})^2]. \tag{11}$$

This interval vanishes when $|\bar{v}| = \bar{c}$.

At the same point $(\bar{x}, \bar{y}, \bar{z})$ the observer $\bar{S}$ is reading off on a clock fixed at that point two infinitely close instants $\bar{t}$ and $\bar{t} + \delta\bar{t}$. We suppose $\delta\bar{t} > 0$. The interval between these two events measured by $\bar{S}$ is

$$\delta\bar{s} = \bar{c}\,\delta\bar{t};$$

hence

$$\delta\bar{t} = \frac{\delta\bar{s}}{\bar{c}}. \tag{12}$$

Thus $\delta\bar{s}/\bar{c}$ is precisely the interval of time read off by $\bar{S}$ on the clock fixed at $(\bar{x}, \bar{y}, \bar{z})$ and at rest with respect to him. For this reason, $\delta\bar{s}/\bar{c}$ is called, in the case considered, the *proper time* measured by $\bar{S}$.

Let us try to find in a general way for what cases the interval between two infinitely close events is zero. For this to be so, it is necessary and sufficient that we should have

$$|\delta\bar{\sigma}| = \bar{c}\,|\delta\bar{t}|. \tag{13}$$

If $\delta\bar{t} \neq 0$ it is necessary and sufficient that $(\delta\bar{\sigma})^2 = (\bar{c}\,\delta\bar{t})^2$. In case we have to deal with a *transfer* of the point P into another point infinitely close to it, the preceding condition is, according to (10), equivalent to $|\bar{v}| = \bar{c}$. This is the case in the first example above.

If $\delta\bar{t} = 0$ the condition (13) gives $\delta\bar{\sigma} = 0$, i.e. $\delta\bar{x} = \delta\bar{y} = \delta\bar{z} = 0$, thus for the observer $\bar{S}$ both events *coincide*.

If $\bar{S}$ considers, at the same instant $\bar{t}$, two different points infinitely close together $(\bar{x}, \bar{y}, \bar{z})$ and $(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z})$, he will find $\delta\bar{t} = 0$ and $\delta\bar{\sigma} > 0$; hence the interval between the two infinitely close events, *simultaneous* for S, defined by the numbers (x, y, z, t), $(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z}, \bar{t} + \delta\bar{t})$ will be, according to (9)

$$\delta\bar{s} = \delta\bar{\sigma}\sqrt{-1}. \tag{14}$$

We thus see that $\delta\bar{s}$ becomes imaginary.

The observer $\bar{S}$ has determined so far an event by the numbers $\bar{x}, \bar{y}, \bar{z}, \bar{t}$, defined by the *direct measurements* made by using his standards of length and his clocks. He has thus constructed a space-time reference system, in which space is divided into cubes, while at each vertex of this cubic net is fixed a clock regulated by $\bar{S}$. These synchronous clocks show intervals of time as small as we desire, and the dimensions of the cubic meshes may also be taken as small as we like.

To define the event $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ the observer S may sometimes find it convenient to use other numbers x', y', z', t' connected with $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ by given relations: in other words, he may perform on $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ a change of variables. It is important to notice that $\bar{S}$ makes this change of variables while *keeping his standards of length and his clocks.*

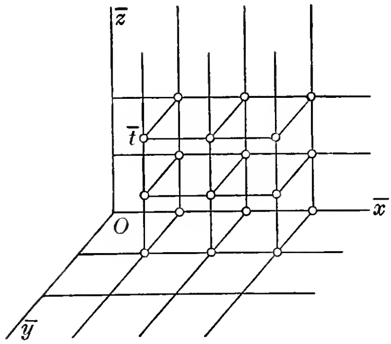

FIG. 1

Let us examine how the $(\delta\bar{s})^2$ may be written when we go over from the variables $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$, measured by $\bar{S}$, to the variables x', y', z', t' obtained through the relations:

$$\left.\begin{array}{ll} x' = x'(\bar{x}, \bar{y}, \bar{z}, \bar{t}) & z' = z'(\bar{x}, \bar{y}, \bar{z}, \bar{t}) \\ y' = y'(\bar{x}, \bar{y}, \bar{z}, \bar{t}) & t' = t'(\bar{x}, \bar{y}, \bar{z}, \bar{t}) \end{array}\right\} \tag{15}$$

Assuming that these equations define x', y', z', t' as uniform functions of $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$ and that they can be solved with respect to these variables, in such a way that they determine $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$ as uniform functions of x', y', z', t' we have

$$\begin{array}{ll} \bar{x} = \bar{x}(x', y', z', t'), & \bar{z} = \bar{z}(x', y', z', t'), \\ \bar{y} = \bar{y}(x', y', z', t'), & \bar{t} = \bar{t}(x', y', z', t'). \end{array} \tag{16}$$

For the sake of uniformity, we place

$$\bar{x} = \bar{x}_1, \quad \bar{y} = \bar{x}_2, \quad \bar{z} = \bar{x}_3, \quad \bar{t} = \bar{x}_4, \tag{17}$$

$$x' = x_1', \; y' = x_2', \; z' = x_3', \; t' = x_4'. \tag{18}$$

Using these notations and the former variables, the form (6) may be written

$$\delta\bar{s}^2 = -\delta\bar{x}_1^2 - \delta\bar{x}_2^2 - \delta\bar{x}_3^2 + \bar{c}^2\delta\bar{x}_4^2. \tag{19}$$

Comparing this with the general form

$$\delta \bar{s}^2 = \sum_{\alpha\beta}\sum g_{\alpha\beta}\, \delta \bar{x}_\alpha\, \delta \bar{x}_\beta \quad (\alpha, \beta = 1, 2, 3, 4)$$

we see that (19) determines the following values of the coefficients:

$$g_{11} = g_{22} = g_{23} = -1, \quad g_{44} = \bar{c}^2$$

while the *other* $g_{\alpha\beta}$*'s are zero.*

The coefficients of the form (19) are collected together in the following square array:

$$\left.\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \bar{c}^2 \end{matrix}\right\} \tag{20}$$

which, as will be seen later, defines a *tensor* $g_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3, 4)$.

Let us pass now to the accented variables. We have

$$\bar{x}_\alpha = \bar{x}_\alpha\,(x_1', \ x_2', \ x_3', \ x_4') \quad (\alpha = 1, 2, 3, 4).$$

We deduce therefrom

$$\delta \bar{x}_\alpha = \sum_{i=1}^{4} \frac{\partial \bar{x}_\alpha}{\partial x_i'}\, \delta x_i' \quad (\alpha = 1, 2, 3, 4)$$

the notations $\frac{\partial}{\partial x_1'}, \frac{\partial}{\partial x_2'}$ being symbols of partial derivatives. The $\delta \bar{x}_\alpha$ are therefore linear functions of $\delta x_\alpha'$, the coefficients of which are well-defined functions of the x_α'. Substituting the values of $\delta \bar{x}_1$, $\delta \bar{x}_2$, $\delta \bar{x}_3$, $\delta \bar{x}_4$ in the second member of (19), the latter becomes a quadratic form in the $\delta x_\alpha'$. The coefficients of this quadratic form are functions of x_α' which will be denoted by $g_{\alpha\beta}'$, so that the quadratic form under consideration becomes

$$\delta \bar{s}^2 = \sum_{\alpha\beta}\sum g_{\alpha\beta}'\, \delta x_\alpha'\, \delta x_\beta' \quad (\alpha, \beta = 1, 2, 3, 4). \tag{21}$$

Let us notice now that the interval $\delta \bar{s}$ is *invariant* with respect to all changes of variables $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\bar{x}_4$; the magnitude of this interval is evidently independent of the choice of the numbers $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$ or x', y', z', t', by means of which $\bar{S}$ chooses to define numerically all events. The left-hand member of equation (19) is not, therefore, affected by this change of variables. The coefficients of the quadratic form which now enter in the second member of (21) will in general be different from the coefficients $g_{\alpha\beta}$ exhibited in the array (20). The new coefficients $g_{\alpha\beta}'$ are collected in the following array:

$$\left.\begin{matrix} g_{11}', & g_{12}', & g_{13}', & g_{14}' \\ g_{21}', & g_{22}', & g_{23}', & g_{24}' \\ g_{31}', & g_{32}', & g_{33}', & g_{34}' \\ g_{41}', & g_{42}', & g_{43}', & g_{44}' \end{matrix}\right\} \qquad (22)$$

From the transformation just performed, we deduce that

$$g_{\beta\alpha}' = g_{\alpha\beta}'. \qquad (23)$$

Example: Suppose that $\bar{S}$ chooses Euclidean spherical coordinates $x_1' = \bar{r}$, $x_2' = \bar{\theta}$, $x_3' = \bar{\phi}$. Then formula (21) becomes:

$$\delta\bar{s}^2 = -\delta\bar{r}^2 - \bar{r}^2(\delta\bar{\theta}^2 + \sin^2\bar{\theta}\,\delta\bar{\phi}^2) + \bar{c}^2\delta\bar{t}^2. \qquad (24)$$

This is Minkowski's form in spherical coördinates.

The observer $\bar{S}$ attached to the Euclidean trihedron $\bar{T}$ determines events by means of the numbers $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$, in the Minkowskian field. In order to study with $\bar{S}$ the physical effects on the standards of length and time, resulting from their motion in the field considered, we choose a system of reference T' moving in this field (Fig. 2). The observer $\bar{S}$ will define this motion by the equations

$$\begin{aligned} \bar{x} &= \bar{x}(x', y', z', t'), \quad \bar{z} = \bar{z}(x', y', z', t'), \\ \bar{y} &= \bar{y}(x', y', z', t'), \quad \bar{t} = t'. \end{aligned} \qquad (25)$$

We suppose that by this transformation there is established a one-to-one reciprocal correspondence between $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$ and x', y', z', t'. We propose to construct, with $\bar{S}$, a curvilinear trihedron (O'; x', y', z') in the following way: To fix our ideas we consider at the instant $\bar{t}$ or t', the point $x' = y' = z' = 0$; we thus obtain the *origin* O', at the time t' considered. To construct the x'-axis, we shall vary only x', keeping this same value of t' and taking $y' = z' = 0$. We shall proceed, in the same manner, to construct the y'-axis, and then the z'-axis. In an infinitesimal space region, it is permissible to replace these curvilinear axes by their tangents. For simplicity, we shall

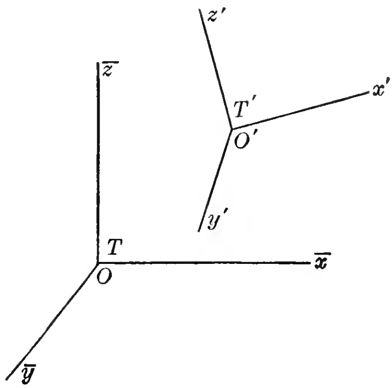

FIG. 2

suppose that the trihedron T' obtained in this way is trirectangular for $\bar{S}$.

To the trihedron T' we shall attach the observer S' who will use the variables x', y', z', t'. Attaching S' to the trihedron T' means that if a point is at *rest* with respect to T' during $\delta t' \neq 0$, this point is said to be at rest with respect to S'; on the contrary, if during $\delta t' \neq 0$ the point is moving with respect to T' it will be said to be moving with respect to S', during this same interval of time.

The observer $\bar{S}$ in the Minkowski field uses the quadratic form

$$\delta \bar{s}^2 = -\delta \bar{x}^2 - \delta \bar{y}^2 - \delta \bar{z}^2 + \bar{c}^2 \delta \bar{t}^2 \tag{26}$$

while S' expresses this same $\delta \bar{s}^2$ as follows:

$$\delta \bar{s}^2 = \sum_{\alpha}\sum_{\beta} g_{\alpha\beta}' \, \delta x_\alpha' \, \delta x_\beta' \quad (\alpha, \beta = 1, 2, 3, 4). \tag{27}$$

Let us attach to T' and to S' a *physical observer* $\bar{S}'$ using standards of length and time fixed with respect to himself. We agree that these are the standards of $\bar{S}$ taken over by $\bar{S}'$.

Let us *admit* with Einstein that in a certain infinitesimal space and time region, physical measurements made by $\bar{S}'$ give numbers $\bar{x}'$, $\bar{y}'$, $\bar{z}'$, $\bar{t}'$ which are such that the quadratic form (27) may be written:

$$\delta\bar{s}^2 = -\delta\bar{x}'^2 - \delta\bar{y}'^2 - \delta\bar{z}'^2 + \bar{c}^2\delta\bar{t}'^2. \tag{28}$$

Suppose further with Einstein that $\bar{c}' = \bar{c}$, in other words that both observers $\bar{S}$ and $\bar{S}'$ find the *same* velocity of light.

In the infinitesimal space and time region considered, let us expand $g_{\alpha\beta}'$ in a Taylor series starting with the event $(x_1')_0$, $(x_2')_0$, $(x_3')_0$, $(x_4')_0$. We get:

$$g_{\alpha\beta}' = (g_{\alpha\beta}')_0 + \sum_{\gamma=1}^{4} \left(\frac{\partial g_{\alpha\beta}'}{\partial x_\gamma}\right)_0 \delta x_\gamma' \tag{29}$$

where $(g_{\alpha\beta}')_0$ is the value of the function $g_{\alpha\beta}'$ for the event $(x_1')_0$, $(x_2')_0$, $(x_3')_0$, $(x_4')_0$ and where the $\delta x_\gamma'$'s are infinitesimal quantities in the region considered. Neglecting in (27) infinitesimal quantities of order higher than the second, we may consider the $g_{\alpha\beta}'$'s entering in this quadratic form as constants, i.e., the $(g_{\alpha\beta}')_0$'s.

To pass from (27) to the quadratic form (28), it will be sufficient to establish a *linear* correspondence between $\delta x_1'$, $\delta x_2'$, $\delta x_3'$, $\delta x_4'$ on the one hand, and $\delta\bar{x}_1'$, $\delta\bar{x}_2'$, $\delta\bar{x}_3'$, $\delta\bar{x}_4'$ on the other hand. From the theory of algebraic forms, we know that this correspondence can be established in an infinite number of ways. But it is essential to notice here, that S' and $\bar{S}'$ are fixed with respect to each other; in other words if $\delta x' = \delta y' = \delta z' = 0$ and $\delta t' \neq 0$ we consequently have $\delta\bar{x}' = \delta\bar{y}' = \delta\bar{z}' = 0$ and $\delta\bar{t}' \neq 0$ and conversely. As a result $\delta x'$, $\delta y'$, $\delta z'$ *must* be expressed as a linear function of $\delta\bar{x}'$, $\delta\bar{y}'$, $\delta\bar{z}'$ *only* ($\delta\bar{t}'$ excluded). Starting from the quadratic form (27), we have to proceed as follows:[1]

$$\delta\bar{s}^2 = \sum_i\sum_j g_{ij}\,\delta x_i'\delta x_j' - \left(\frac{\sum_i g_{i4}'\delta x_i'}{\sqrt{g_{44}'}}\right)^2 + \left(\frac{\sum_\alpha g_{\alpha 4}'\delta x_\alpha'}{\sqrt{g_{44}'}}\right)^2 \quad \begin{matrix}(\alpha = 1,2,3,4,)\\ (i,j = 1, 2, 3)\end{matrix} \tag{30}$$

[1] Th. De Donder, Académie Royale de Belgique, Bulletin, Dec., 1922, Feb., Mar., 1923.

We may notice that in (30) we formed the perfect square

$$\left(\frac{\sum_\alpha g'_{\alpha 4}\delta x_\alpha'}{\sqrt{g_{44}'}}\right)^2 \tag{31}$$

and in an infinitesimal region of space:

$$-\delta\bar{x}'^2 - \delta\bar{y}'^2 - \delta\bar{z}'^2 = \sum_i\sum_j\left(g_{ij}' - \frac{g_{i4}'g_{j4}'}{g_{44}'}\right)\delta x_i'\delta x_j'. \tag{33}$$

The linear correspondence between $\delta x'$, $\delta y'$, $\delta z'$ on the one hand and $\delta\bar{x}'$, $\delta\bar{y}'$, $\delta\bar{z}'$ on the other hand, is easily established through the theory of quadratic forms or analytic geometry (case of the ellipsoid). Before studying this correspondence, let us consider for a moment the relation (31).

Let us place $\delta x_1' = \delta x_2' = \delta x_3' = 0$, then (33) gives $\delta\bar{x}_1' = \delta\bar{x}_2' = \delta\bar{x}_3' = 0$ and the relation (32) becomes

$$\delta\bar{t}' = \frac{1}{\bar{c}'}\sqrt{g_{44}'}\,\delta t'. \tag{34}$$

The correspondence between the time $\bar{t}'$ of $\bar{S}'$, and the time t' of S' is given by

$$\bar{t}' - \bar{t}_0' = \frac{1}{\bar{c}'}\int_{t_0'}^{t'}\sqrt{g_{44}'}\delta t', \tag{35}$$

t_0' and t' being two instants chosen by S', and $\bar{t}_0'$, $\bar{t}'$ the corresponding instants read by $\bar{S}'$ on a clock at rest with respect to him. Integral (35) has a definite value, because x', y', z' are fixed; hence the relation between t' and $\bar{t}'$ is a one-to-one correspondence.

Let us notice also that at the same point for S' and $\bar{S}'$, two simultaneous events for S' are also simultaneous for $\bar{S}'$. In fact, formula (34) shows that $\delta t' = 0$ involves $\delta\bar{t}' = 0$. We already knew that any coincidence for S' must be a coincidence for $\bar{S}'$ and conversely. On the other hand, *at two different points for* S', simultaneity for S' does not involve simultaneity for $\bar{S}'$. If we consider two infinitely close points, where two events are simultaneous for $S'(\delta t' = 0)$, these events will be separated for $\bar{S}'$, by an interval of time

$$\delta \bar{t}' = \frac{\sum_{i=1}^{3} g_{i4}' \delta x_i'}{\bar{c}' \sqrt{g_{44}'}} \cdot \qquad (36)$$

Conversely, to the simultaneity at two different points for $\bar{S}'(\delta \bar{t}' = 0)$, corresponds a non-simultaneity for S'. We have, by (32):

$$\sum_{\alpha=1}^{4} g_{\alpha 4}' \, \delta x_\alpha' = 0, \qquad (37)$$

or

$$\delta t' = -\frac{\sum_{i=1}^{3} g_{i4}' \delta x_i'}{g_{44}'} \cdot \qquad (38)$$

Let us remember that by (25) $\bar{S}$ and S' use the same variable $\bar{t} = t'$.

Let us return now to the reduction problem formulated by (33) and write

$$c_{ij}' = -g_{ij}' + \frac{g_{i4}' g_{j4}'}{g_{44}'} \; (i, j = 1, 2, 3, 4) \qquad (39)$$

We have $c_{ij}' = c_{ji}'$. In the infinitesimal space-region of S', and during an infinitely small interval of time, all the c_{ij}''s may be treated as constants to a first approximation.

The quadratic form in the right-hand member of (33) will be written

$$\sum_i \sum_j c_{ij}' \delta x_i' \delta x_j' \; (i, j = 1, 2, 3). \qquad (40)$$

In the infinitely small region about O', let us draw another trirectangular Euclidean trihedron x^*, y^*, z^* having O' as origin. Let us place $x^* = x^*_1$, $y^* = x^*_2$, $z^* = x^*_3$. We perform on (40) the orthogonal transformation

$$x_i' = \sum_j x_j^* \cos (x_i', x_j^*) \; (i, j = 1, 2, 3) \qquad (41)$$

which transforms (40) into a quadratic form in δx_1^*, δx_2^*, δx_3^*. Let us choose the trihedron O' (x^*, y^*, z^*) in such a way that this quadratic form is independent of any cross terms; we shall have then, in the infinitely small region about O' and at the time t' (or $\bar{t}$):

$$(\delta\bar{x}')^2 + (\delta\bar{y}')^2 + (\delta\bar{z}')^2 = s_1^*(\delta x^*)^2 + s_2^*(\delta y^*)^2 + s_3^*(\delta z^*)^2 \quad (42)$$

s_1^*, s_2^*, s_3^*, being the roots of the equation,

$$\begin{vmatrix} c_{11}' - s^* & c_{12}' & c_{13}' \\ c_{21}' & c_{22}' - s^* & c_{23}' \\ c_{31}' & c_{32}' & c_{33}' - s^* \end{vmatrix} = 0. \quad (43)$$

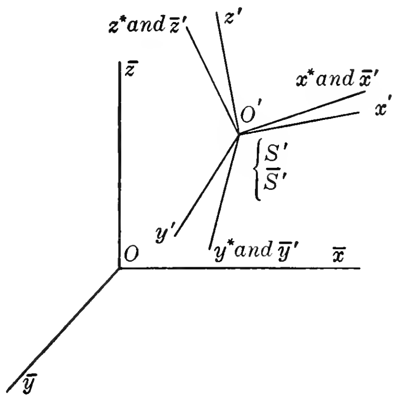

Fig. 3

As we know, they are all real and positive, because the form (40) is symmetrical and positive definite. From (42) we have

$$\delta\bar{x}' = \sqrt{s_1^*}\,\delta x^*,\quad \delta\bar{y}' = \sqrt{s_2^*}\,\delta y^*,\quad \delta\bar{z}' = \sqrt{s_3^*}\,\delta z^*. \quad (44)$$

Integrating under the conditions that x^* and $\bar{x}'$ vanish together, and the same for y^* and $\bar{y}'$, z^* and $\bar{z}'$, we shall have, keeping to the infinitesimal space region considered and at the time t',

$$\bar{x}' = x^*\sqrt{s_1^*},\quad \bar{y}' = y^*\sqrt{s_2^*},\quad \bar{z}' = z^*\sqrt{s_3^*}. \quad (45)$$

Summing up: in the infinitesimal space and time region considered, we have established a one-to-one reciprocal *correspondence* between the numbers (x', y', z', t') used by S' and the numbers $(\bar{x}', \bar{y}', \bar{z}', \bar{t}')$ used by $\bar{S}'$. Let us remember that between the numbers $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ used by $\bar{S}$ and the numbers (x', y', z', t') used by S', there exists, by (25), a one-to-one reciprocal *correspondence*, in the same infinitesimal region about (x', y', z', t'). Thus, finally, there is a one-to-one reciprocal correspondence between the numbers $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ used by $\bar{S}$ and the numbers $(\bar{x}', \bar{y}', \bar{z}', \bar{t}')$ used by $\bar{S}'$.

To fix the ideas let us consider with $\bar{S}$ or S', at the time $\bar{t}$ or t', an infinitesimal vector located on the x^* or x'-axis. The first formula (45) shows that the observer $\bar{S}'$ will obtain for the modulus of this vector a number $\sqrt{s_1^*}$ times larger than the one found by the observer S', and consequently also by the observer $\bar{S}$. Therefore we infer that $\bar{S}$ will say that the standard of length of $\bar{S}'$ is $\sqrt{s_1^*}$ times smaller than his. This is, by definition, the *generalized Lorentz contraction*, the magnitude of which is in general different according to the orientation of the standard of length with respect to the system of reference.

If in his infinitesimal region, the observer $\bar{S}'$ considers a sphere of radius $\bar{r}'$ and center A, he will write the equation of the sphere as follows:

$$(\bar{x}')^2 + (\bar{y}')^2 + (\bar{z}')^2 = (\bar{r}')^2. \tag{46}$$

Let us translate this equation into the language of $\bar{S}$. By means of relations (45) it becomes

$$s_1^*(x^*)^2 + s_2^*(y^*)^2 + s_3^*(z^*)^2 = (\bar{r}')^2 \tag{47}$$

and $\bar{S}$ will say that he is observing an infinitesimal ellipsoid.

Let us go back to Equation (33). This has been built up on the hypothesis that $\delta x' = \delta y' = \delta z' = 0$ to which corresponds, as we have seen, $\delta\bar{x}' = \delta\bar{y}' = \delta\bar{z}' = 0$. Hence $\delta t'$ is an infinitesimal time interval determined by S' at a point at rest with respect to his reference system, and $\delta\bar{t}'$ the corresponding interval of time read by $\bar{S}'$ on a clock which is also at rest with respect to that system. Supposing $\sqrt{g_{44}'} < \bar{c}$, then from

$$\delta\bar{t} = \frac{\bar{c}}{\sqrt{g_{44}'}}\,\delta\bar{t}' \tag{48}$$

it follows that $\delta\bar{t}' < \delta\bar{t}$. Hence the observer $\bar{S}$ will say that the standard of time used by $\bar{S}'$ is $\bar{c}/\sqrt{g_{44}'}$ times larger than his. This is the *generalized Einstein dilatation*. It can also be expressed by saying that $\bar{S}$ observes the clocks of $\bar{S}'$ to slow down with respect to his: this effect is ascribed by $\bar{S}$ to the motion of the clocks of $\bar{S}'$ in the Minkowski field which he is exploring.

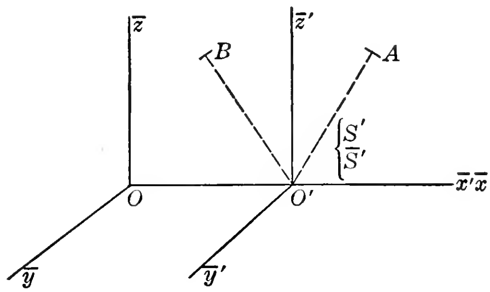

Fig. 4

Let us suppose that the system of reference $O'(x', y', z')$ be a Euclidean trihedron T' having with respect to T a *uniform rectilinear* motion parallel to the Ox'-axis. Let $\bar{v}$ be the velocity of this motion, measured by $\bar{S}$ and considered positive in the direction of increasing $\bar{x}$. Supposing that the $O'x'$ and $O'y'$-axes of T' are respectively parallel to $O\bar{x}$ and $O\bar{y}$ of T and that the origin O' is on the $O\bar{x}$-axis, then Equations (25) give here

$$\left.\begin{aligned} \bar{x} &= x' + \bar{v}t, & \bar{z} &= z', \\ \bar{y} &= y', & \bar{t} &= t'. \end{aligned}\right\} \tag{49}$$

Substituting these values in (26) we obtain

$$\delta\bar{s}^2 = -\delta x'^2 - \delta y'^2 - \delta z'^2 + (\bar{c}^2 - \bar{v}^2)\delta t'^2 - 2\,\bar{v}\,\delta x'\delta t'. \tag{50}$$

This form corresponds to (27) for the following values of $g_{\alpha\beta}'$:

$$g_{11}' = g_{22}' = g_{33}' = -1,\; g_{44}' = \bar{c}^2 - \bar{v}^2,\; g_{14}' = -\bar{v} \tag{51}$$

while the other $g_{\alpha\beta}'$ vanish. We may notice in passing that the velocity of light v' calculated by S' is given by $d\bar{s}^2 = 0$, that is, by (50),

$$\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2 + \left(\frac{dz'}{dt'}\right)^2 + 2\,\bar{v}\left(\frac{dx'}{dt'}\right) - (\bar{c}^2 - \bar{v}^2) = 0$$

or further if we call α the angle between the direction of propagation and the direction of translation,

$$v'^2 + 2\,\bar{v}v' \cos\alpha - (\bar{c}^2 - \bar{v}^2) = 0. \tag{52}$$

In the direction of translation, we have $v' = \bar{c} \pm \bar{v}$; in the directions perpendicular to the direction of translation, we have

$$v' = |\sqrt{\bar{c}^2 - \bar{v}^2}|.$$

Placing

$$\beta = \frac{1}{|\sqrt{1 - (\bar{v}/\bar{c})^2}|} \tag{53}$$

formulas (31) and (32) give

$$\delta\bar{x}' = \beta\delta x',\ \delta\bar{y}' = \delta y',\ \delta\bar{z}' = \delta z',\ \delta\bar{t}' = \frac{\delta t'}{\beta} - \frac{\beta\bar{v}}{\bar{c}^2}\,\delta x'. \tag{54}$$

Therefore under the condition that $\bar{x}'$ and x' vanish together, and also $\bar{y}'$ and y', $\bar{z}'$ and z', $\bar{t}'$ and t' (for $x' = 0$)

$$\bar{x}' = \beta x',\ \bar{y}' = y',\ \bar{z}' = z',\ \bar{t}' = \frac{t'}{\beta} - \frac{\beta\bar{v}}{\bar{c}^2}\,x'. \tag{55}$$

We may notice here that the correspondence established in formulas (55) is not limited to an infinitesimal space region. Hence, formulas (55) are rigorously valid in the whole Euclidean space x', y', z' and for all times t' of S'. In this particular case, we deduce immediately from (55) the Lorentz contraction and the Einstein dilatation.

By Equation (49), Equations (55) may be written

$$\bar{x}' = \beta(\bar{x} - \bar{v}\bar{t}),\ \bar{y}' = \bar{y},\ \bar{z}' = \bar{z},\ \bar{t}' = \beta\left(\bar{t} - \frac{\bar{v}\bar{x}}{\bar{c}^2}\right) \tag{56}$$

and conversely

$$\bar{x} = \beta(\bar{x}' + \bar{v}\bar{t}'),\ \bar{y} = \bar{y}',\ \bar{z} = \bar{z}',\ \bar{t} = \beta\left(\bar{t}' + \frac{\bar{v}\bar{x}'}{\bar{c}^2}\right). \tag{57}$$

Equations (56) and (57) define the Lorentz transformation.[1]

[1] J. J. Larmor, Æther and Matter, pp. 167–177, Cambridge, 1900; H. A. Lorentz, Versl. kon. Ak. van Wet. Amsterdam, 12, p. 986, 1904; H. Poincaré, Comptes-Rendus Acad. Sc. de Paris, 140, p. 1504, 1905; A. Einstein, Ann. d. Physik, 17, p. 891, 1905; see also Voigt, Göttingen Nachrichten, 1897.

The general method used here to obtain these equations of transformation shows that they form a *group*,[1] a statement which can be verified by calculation.

By Equations (56) or (57), we have anywhere in space and for any time

$$d\bar{s}^2 = -d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 + \bar{c}^2 d\bar{t}^2 \tag{58}$$

with $\bar{c} = \bar{c}'$. It follows that the observer $\bar{S}'$, according to his experiments, will be able to make the statement, as reasonably as $\bar{S}$, that the empty space in which he has made his measurements is a Minkowski field. In summing up we may say that neither of the observers $\bar{S}'$, $\bar{S}$ ··· having with respect to each other a uniform rectilinear translatory motion, will be able to state, according to his own measurements of length and time, that only *his* empty space is a Minkowski field; we therefore say that none of the trihedrons T, T' ··· are privileged in any way. This is the philosophical meaning of the expression *restricted relativity.*

Let us consider, with $\bar{S}$, two points on the $O\bar{x}$-axis, with the abscissas $\bar{x}_1$ and $\bar{x}_2$, *at the same time* $\bar{t}$. From the first of Equations (56) we have

$$\bar{x}_1' - \bar{x}_2' = \beta(\bar{x}_1 - \bar{x}_2). \tag{59}$$

If the two points considered are *at rest* with respect to $\bar{S}'$, then the observer $\bar{S}$ will say that $\bar{S}'$ finds a number β-times *larger* than the one he has found himself, because the standard of length used by $\bar{S}'$ has become β-times *smaller* than his. $\bar{S}$ will also say that the standard carried along in the system of $\bar{S}'$ has undergone the Lorentz contraction.

Consider with $\bar{S}'$ a point with abscissa $\bar{x}'$, at two different instants $\bar{t}_1'$ and $\bar{t}_2'$. From the last Equation (57) we have

$$\bar{t}_1' - \bar{t}_2' = \frac{1}{\beta}(\bar{t}_1 - \bar{t}_2). \tag{60}$$

[1] H. Poincaré, "Sur la Dynamique de l'Electron," Rend. d. Circ. Mat. di Palermo, July 23, 1905; pp. 129–175.

The observer $\overline{S}$ will say that the standard of time carried along in the system of $\overline{S}'$ has undergone the Einstein dilatation[1] or otherwise, that the clock used by $\overline{S}'$ runs β-times slower than his.

The Michelson-Morley Experiment.[2] Let $O'A$ and $O'B$ (Fig. 4) represent, according to $\overline{S}'$, the two equal arms of the apparatus used by Michelson and Morley. If we call $\overline{a}'$ and $\overline{b}'$ the measures of these arms, obtained by $\overline{S}'$, we have $\overline{a}' = \overline{b}'$. But according to Einstein's hypothesis $\overline{c}' = \overline{c}$, whatever be the direction of the light ray considered by $\overline{S}'$. Hence $2\,\overline{a}'/\overline{c}' = 2\,\overline{b}'/\overline{c}'$; in other words, *the time intervals required by the rays $O'AO'$ and $O'BO'$ are the same for $\overline{S}'$.* It follows that the Michelson experiment cannot detect the uniform rectilinear translation of $\overline{S}'$ with respect to $\overline{S}$. We have to keep in mind that these consequences result essentially from the fact that the field considered is identifiable with a Minkowski field. In the same way, it is possible to explain the Fizeau experiment and in the case of uniform rotation in the Minkowski field, our method yields an explanation of Sagnac's experiment.[3]

[1] A. Einstein, Annalen der Physik, Vol. 17, 1905, par. 4.

[2] A. A. Michelson, Amer. Jour. of Science, III series, 22,p. 1 20, 1881; Michelson and Morley, ibid. 34, p. 333, 1887; Phil. Mag., Vol. 24, p. 449, 1887; H. A. Lorentz, "Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern" ¶ 89–92, Leyden, 1895.

[3] T. De Donder, Mémorial des Sciences Mathématiques, Fasc. VIII, pp. 22–32, Paris, 1925.

LECTURE 3

The Einstein Field

Definition — The mathematical observer S and the physical observer $\overline{S}$ — The moving mathematical observer S' and the moving physical observer $\overline{S}'$ — The direct passage from S to $\overline{S}'$ — The theorem of parallel displacement — Motion of the reference trihedron of $\overline{S}'$ with respect to that of S.

So far we have considered only a Minkowski field explored by the observer $\overline{S}$. Let us remember that the latter used a Euclidean mesh system provided with clocks regulated by light signals (Fig. 1).

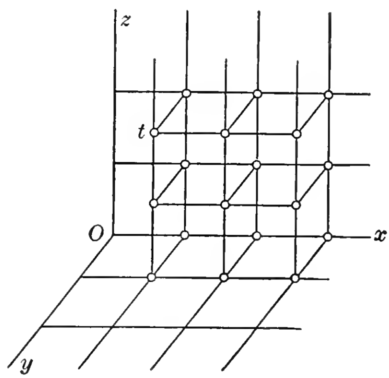

Fig. 5

Einstein supposes that there exist fields in which such a mesh system cannot be physically constructed. These fields will be called *Einstein fields.* Together with an observer or mathematician S, let us consider the Euclidean mesh system, provided with clocks and used by $\overline{S}$ to explore the Minkowski field. This mesh will have here only a representative value, such as a series of meridians and parallels on a geographic map. We shall call this Euclidean mesh including all its temporal, geometrical and physical indications a Γ-*map* (Fig. 5). The mesh system of this Γ-map being Euclidean, the measure $\delta\sigma$ of the distance on the Γ-map between the points (x, y, z) and $(x + \delta x, y + \delta y, z + \delta z)$ will be given by

$$\delta\sigma^2 = \delta x^2 + \delta y^2 + \delta z^2.$$

Using a fictitious light with the velocity $c = \bar{c}$, we may write as above,

$$\delta s^2 = -\delta x^2 - \delta y^2 - \delta z^2 + c^2 \delta t^2,$$

where δs is the interval on the Γ-map between the events (x, y, z, t) and $(x + \delta x, y + \delta y, z + \delta z, t + \delta t)$. Except in the Minkowski field these expressions correspond only approximately to *physical* or *real* measurements. In order to obtain the latter in a rigorous manner, let us generalize, with Einstein, the last quadratic form by placing

$$S \cdot \cdot \cdot \cdot \delta \bar{s}^2 = \sum_{\alpha}\sum_{\beta} g_{\alpha\beta}\, \delta x_\alpha\, \delta x_\beta \qquad (\alpha, \beta = 1, 2, 3, 4) \qquad (1)$$

where $g_{\alpha\beta} = g_{\beta\alpha}$ are ten functions of x_1, x_2, x_3, x_4. These are the *gravitational potentials* of Einstein. We shall say that they define the Einstein field considered.[1] The determination of these ten functions $g_{\alpha\beta}$ of x_1, x_2, x_3, x_4 constitutes the fundamental problem of gravific theory. In the particular case where these functions have the values shown in table (20), Lecture 2, we find again the particular form (5), Lecture 2, which $\bar{S}$ has associated with the Minowski field. Thus we see that $\bar{S}$ is identical with S. Let us remember that the numbers x_1, x_2, x_3, x_4 in the quadratic form (1) are just the same as those used in our Γ-map. We may say, with Einstein, that they are the *parameters* of the *space-time* defined by (1), but we wish to avoid any visualization in space-time. The quadratic form (1) is used by the observer S attached to the Euclidean trihedron $(O;\ x_1, x_2, x_3)$. An *event* will be defined by S by means of the four parameters x_1, x_2, x_3, x_4 or x, y, z, t. Two events (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) will be simultaneous for S, if $t_1 = t_2$.

[1] A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie." Annalen der Physik, 49, 1916.

Two events (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) will *coincide* for S, when we have at the same time $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$ and $t_1 = t_2$. For S the *square of the interval between two events* (x, y, z, t) *and* $(x + \delta x, y + \delta y, z + \delta z, t + \delta t)$ *is given by* (1).

Let us attach to S an observer $\bar{S}$ equipped to take physical measurements. For this purpose we consider an infinitesimal space and time region, about the event (x_1, x_2, x_3, x_4), and try to write (1) in the Minkowski form (Eq. 6, Lecture 2). In the last lecture we have already explained how this passage is effected. Formulas (26) to (45) still hold after *suppression of the accents* of x', y', z' and t'. For example, Equations (30), (32) and (33) now become

$$\delta\bar{s}^2 = \sum_i\sum_j g_{ij}\delta x_i \delta x_j - \left(\frac{\sum_i g_{i4}\delta x_i}{\sqrt{g_{44}}}\right)^2 + \left(\frac{\sum_\alpha g_{\alpha 4}\delta x_\alpha}{\sqrt{g_{44}}}\right)^2 \quad \begin{matrix}(\alpha = 1, 2, 3, 4)\\ (i, j = 1, 2, 3)\end{matrix} \tag{2}$$

$$\bar{c}\,\delta\bar{t} = \sum_{\alpha=1}^{4} (g_{\alpha 4}/\sqrt{g_{44}})\,\delta x_\alpha \tag{3}$$

$$-\delta\bar{x}^2 - \delta\bar{y}^2 - \delta\bar{z}^2 = \sum_i\sum_j\left(g_{ij} - \frac{g_{i4}g_{j4}}{g_{44}}\right)\delta x_i\,\delta x_j. \tag{4}$$

A clock at rest for $\bar{S}$ will also be at rest for S; this can be expressed by saying that if $\delta\bar{x}_1 = \delta\bar{x}_2 = \delta\bar{x}_3 = 0$, then $\delta x_1 = \delta x_2 = \delta x_3 = 0$. The time shown by the clock used by $\bar{S}$ will be given by

$$\delta\bar{t} = \frac{\sqrt{g_{44}}}{\bar{c}}\,\delta t. \tag{5}$$

The time shown by $\bar{S}$'s clock will be expressed by

$$\bar{t} - \bar{t}^0 = \frac{1}{\bar{c}}\int_{t^0}^{t}\sqrt{g_{44}}\,\delta t. \tag{6}$$

We remember also that at the *same point* for S and $\bar{S}$, two simultaneous events for S will also be simultaneous for $\bar{S}$. On the other hand, at two different points for S, simultaneity for S does not involve simultaneity for $\bar{S}$. Conversely, to simultaneity at two different points for $\bar{S}$, that is to say, to $\delta\bar{t} = 0$, corresponds non-simultaneity for S. In order to obtain the correspondence between the space parameters x_1, x_2, x_3 used in the Γ-map by S, and the space measurements obtained by the physical observer $\bar{S}$, we shall proceed as in the last lecture (see Equations (39) to (45), Lecture 2), taking care to remove the accents from x_1', x_2', x_3'. For example, formula (45) becomes:

$$\bar{x} = x^* \sqrt{s_1^*}, \quad \bar{y} = y^* \sqrt{s_2^*}, \quad \bar{z} = z^* \sqrt{s_3^*}. \qquad (7)$$

Similarly we may extend to any Einstein field the considerations developed in the last lecture concerning the generalized Lorentz contraction and the Einstein dilatation. We only have to consider two observers $\bar{S}_1$ and $\bar{S}_2$ at rest with respect to S.

For example, let us suppose that S adopts on the Γ-map Euclidean spherical coördinates r, θ, ϕ and the fictitious time t. As an example, we consider, with S,

$$\delta\bar{s}^2 = -A\,\delta r^2 - r^2(\delta\theta^2 + \sin^2\theta\,\delta\phi^2) + c^2\,B\delta t^2 \qquad (8)$$

where A and B are functions of r only. We intend now to write the $\delta\bar{s}^2$ in Minkowski's form (24):

$$\delta\bar{s}^2 = -\delta\bar{r}^2 - \bar{r}^2(\delta\bar{\theta}^2 + \sin^2\theta\delta\bar{\phi}^2) + \bar{c}^2\,\delta\bar{t}^2. \qquad (9)$$

The numbers $\bar{r}$, $\bar{\theta}$, $\bar{\phi}$, $\bar{t}$ are thus obtained by the physical observer $\bar{S}$ performing his measurements in the *infinitesimal space and time region considered.* We may suppose that $\theta = \bar{\theta}$ and $\phi = \bar{\phi}$. Equation (5) gives immediately the correspondence between the number t used by the mathematical observer S and the number $\bar{t}$ used by the physical observer $\bar{S}$, namely, $\delta\bar{t} = \sqrt{B}\,\delta t$, or integrating, as B does not depend on t, $\bar{t} - \bar{t}^0 = \sqrt{B}(t - t^0)$. To find the correspondence between the numbers r and $\bar{r}$ used by the observers S and $\bar{S}$ respectively, let us place $\delta\theta = \delta\bar{\theta} = 0$, $\delta\phi = \delta\bar{\phi} = 0$; in this way these two observers are considering the same radial direction. Suppose further that $\delta t = 0$; hence, from (3) and (8), $\delta\bar{t} = 0$. We have $\delta\bar{r} = \sqrt{A}\,\delta r$, and integrating $\bar{r} - \bar{r}_0 = \int_{r^0}^{r} \sqrt{A}\,dr$.

As in the last lecture we may draw on the Γ-map a Euclidean trirectangular trihedron T' $(O';\ x',\ y',\ z')$ having a definite motion with respect to the trihedron T $(O;\ x,\ y,\ z)$ (Fig. 6).

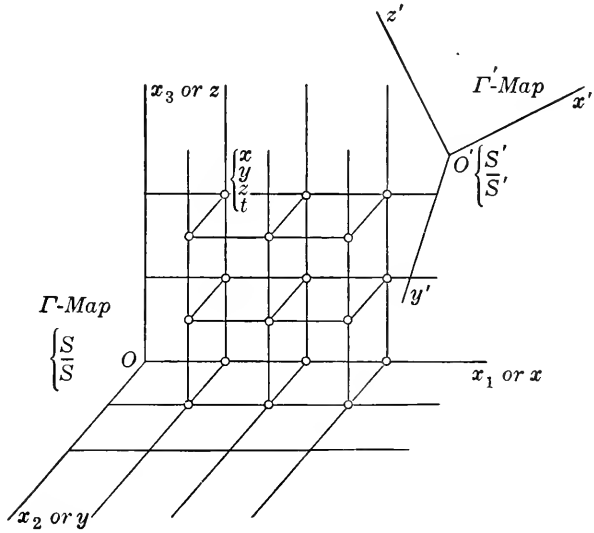

Fig. 6

The observer S will attach to this trihedron T' an observer S' who shall make use of the variables x', y', z', $t' = t$, and will construct in this way a Γ'-map (Fig. 6). The quadratic form (1) becomes now:

$$S' \cdots \ \delta \bar{s}^2 = \sum_{\alpha} \sum_{\beta} g_{\alpha\beta}' \ \delta x_\alpha' \ \delta x_\beta' \tag{10}$$

where we have placed

$$g_{\alpha\beta}' = \sum_{a} \sum_{b} g_{ab} \frac{\partial x_a}{\partial x_\alpha'} \frac{\partial x_b}{\partial x_\beta'} . \tag{11}$$

Let us now attach to S' an observer $\bar{S}'$, performing physical measurements in an infinitesimal space and time domain. Carrying through the same calculations indicated in the preceding lecture, we have:

$$\bar{S}' \cdots \ \delta \bar{s}^2 = -(\delta \bar{x}')^2 - (\delta \bar{y}')^2 - (\delta \bar{z}')^2 + (\bar{c}')^2 (\delta \bar{t}')^2. \tag{12}$$

We place, with Einstein,

$$\bar{c}' = \bar{c} = c \tag{13}$$

and we may extend to the Einstein fields all the results obtained above.

The results just outlined may be summed up in the following condensed table:

$$\left\{\begin{array}{l} S \cdots \delta \bar{s}^2 = \sum\limits_{\alpha}\sum\limits_{\beta} g_{\alpha\beta}\, \delta x_\alpha\, \delta x_\beta \\ \bar{S} \cdots \delta \bar{s}^2 \backsim -\delta \bar{x}^2 - \delta \bar{y}^2 - \delta \bar{z}^2 + \bar{c}^2\, \delta \bar{t}^2 \end{array}\right\} \tag{14}$$

$$\left\{\begin{array}{l} S' \cdots \delta \bar{s}^2 = \sum\limits_{\alpha}\sum\limits_{\beta} g_{\alpha\beta}'\, \delta x_\alpha'\, \delta x_\beta' \\ \bar{S}' \cdots \delta \bar{s}^2 \backsim -(\delta \bar{x}')^2 - (\delta \bar{y}')^2 - (\delta \bar{z}')^2 + (\bar{c}')^2(\delta \bar{t}')^2 \end{array}\right\} \tag{15}$$

In the preceding developments the motion of S' with respect to S was supposed known; we next passed from S' to $\bar{S}'$ by a change of variables from x', y', z', t' or t to $\bar{x}', \bar{y}', \bar{z}', \bar{t}'$. We now search for a change of variables such that the quadratic form (1) takes the Minkowski form, to the closest possible approximation, in an infinitesimal space and time region. We write the change of variables:

$$\begin{array}{ll} x = x(\bar{x}', \bar{y}', \bar{z}', \bar{t}'), & z = z(\bar{x}', \bar{y}', \bar{z}', \bar{t}'), \\ y = y(\bar{x}', \bar{y}', \bar{z}', \bar{t}'), & t = t(\bar{x}', \bar{y}', \bar{z}', \bar{t}'), \end{array} \tag{16}$$

such that

$$\delta \bar{s}^2 = \sum_{\alpha}\sum_{\beta} \bar{g}_{\alpha\beta}'\, \delta \bar{x}_\alpha'\, \delta \bar{x}_\beta' \tag{17}$$

where

$$\bar{g}_{\alpha\beta}' = \sum_{a}\sum_{b} g_{ab} \frac{\partial x_a}{\partial \bar{x}_\alpha'} \frac{\partial x_b}{\partial \bar{x}_\beta'} \tag{18}$$

takes to the closest approximation *the Minkowski form* in the infinitesimal space and time region considered. To fix our ideas, we suppose that the variable $\bar{x}_4'$ has the same dimensions as the time $t = x_4$ used by S. Let us construct with $\bar{S}'$ (Fig. 7) a curvilinear trihedron T' $(O'; \bar{x}', \bar{y}', \bar{z}')$ in the following manner: At a given instant $\bar{t}' = \bar{x}_4'$ we take as origin of this trihedron the point O' having the coördinates $\bar{x}_1' = \bar{x}_2' = \bar{x}_3' = 0$. To construct the $\bar{x}'$-axis, we vary only $\bar{x}'$ keeping the preceding value of $\bar{t}'$ and taking $\bar{y}' = \bar{z}' = 0$. We proceed in the same manner to construct the $\bar{y}'$-axis and then the $\bar{z}'$-axis. The trihedron T' thus obtained will be in general moving with respect to the trihedron T used by S.

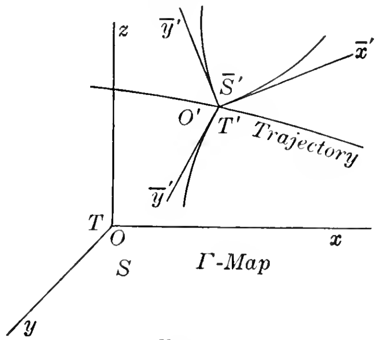

Fig. 7

In order to find the change of variables (16) which solves the problem in question, we assume that the x_i's can be developed in a Taylor series in the neighborhood of the point $\bar{x}_1' = \bar{x}_2' = \bar{x}_3' = \bar{x}_4' = 0$, thus,

$$x_i = x_i^0 + \sum_k \left(\frac{\partial x_i}{\partial \bar{x}_k'}\right)_0 \bar{x}_k' + \frac{1}{2!}\sum_k\sum_l \left(\frac{\partial^2 x_i}{\partial \bar{x}_k'\, \partial \bar{x}_l'}\right)_0 \bar{x}_k'\bar{x}_l' + \cdots$$
$$(i, k, l = 1, 2, 3, 4). \quad (19)$$

The index 0 denotes that the symbols so affected refer to $\bar{x}_1' = \bar{x}_2' = \bar{x}_3' = \bar{x}_4' = 0$. The $(\bar{x}_\alpha')$'s in (19) will be considered as infinitesimal quantities of the first order. Let us determine the coefficients of the series (19); we have first $x_i^0 = x_i(0, 0, 0, 0)$. These numbers are hence the coördinates and the time used by S and corresponding to the origin O' considered at the initial time $\bar{t}' = 0$. The four numbers x_i^0 are given by the data of the problem under consideration.

We assume that the functions $\bar{g}_{\alpha\beta}'$ may also be expanded in a series of integral positive powers in a domain of the first order about the point $\bar{x}_1' = \bar{x}_2' = \bar{x}_3' = \bar{x}_4' = 0$,

$$\bar{g}_{\alpha\beta}' = (\bar{g}_{\alpha\beta}')_0 + \sum_k (\bar{g}_{\alpha\beta,k}')\bar{x}_k' + \frac{1}{2!}\sum_k\sum_l (\bar{g}_{\alpha\beta,kl}')\bar{x}_k'\bar{x}_l' + \cdots \quad (20)$$

The problem in question is now to determine the quantities $\bar{g}_{\alpha\beta}'$ so that they are the same, to the closest possible approximation, as the components $\delta_{\alpha\beta}$ of the fundamental tensor of Minkowski,

$$\left.\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & (\bar{c}')^2 \end{matrix}\right\} \tag{21}$$

We suppose that $\bar{S}'$ is using rectangular coördinates $\bar{x}'$, $\bar{y}'$, $\bar{z}'$, and that he defines the time $\bar{t}'$ by means of the velocity $\bar{c}'$ of the observed light.

In order that this identification of the $\bar{g}_{\alpha\beta}'$'s with the $\delta_{\alpha\beta}$'s be satisfied in (20) to a first approximation, that is to infinitesimals of the first order, it is necessary and sufficient that in (20) the ten conditions $(\bar{g}_{\alpha\beta}')_0 = \delta_{\alpha\beta}$ be satisfied. The covariancy (18) furnishes then *ten* equations

$$\delta_{\alpha\beta} = (\bar{g}_{\alpha\beta}')_0 = \sum_i\sum_j (g_{ij})_0 \left(\frac{\partial x_i}{\partial \bar{x}_\alpha'}\right)_0 \left(\frac{\partial x_j}{\partial \bar{x}_\beta'}\right)_0 \tag{22}$$

which enable us to calculate ten out of the *sixteen* coefficients $(\partial x_i/\partial \bar{x}_k')_0$ entering in the terms of the first order in the series (19).

In order that the $\bar{g}_{\alpha\beta}'$'s coincide with the $\delta_{\alpha\beta}$'s not only to a first order of approximation but also to a second order of approximation, that is to infinitesimals of the second order, it is necessary and sufficient, by Equation (20) that forty equalities $(\bar{g}_{\alpha\beta,k}') = 0$ be satisfied. But, from (18), we obtain, by differentiation,

$$\bar{g}'_{\alpha\beta,k} = \sum_i\sum_j \left[\sum_m g_{ij,m} \frac{\partial x_m}{\partial \bar{x}_k'} \frac{\partial x_i}{\partial \bar{x}_\alpha'} \frac{\partial x_j}{\partial \bar{x}_\beta'} + g_{ij}\left(\frac{\partial^2 x_i}{\partial \bar{x}_\alpha' \partial \bar{x}_k'} \frac{\partial x_j}{\partial \bar{x}_\beta'} + \frac{\partial^2 x_j}{\partial \bar{x}_\beta' \partial \bar{x}_k'} \frac{\partial x_i}{\partial \bar{x}_\alpha'}\right)\right]$$
$$(i, j, m = 1, 2, 3, 4). \tag{23}$$

Placing in these equalities $\bar{x}_1' = \bar{x}_2' = \bar{x}_3' = \bar{x}_4' = 0$ and taking into account the conditions $(\bar{g}_{\alpha\beta,k}')_0 = 0$ we get forty equations determining the forty coefficients $\left(\frac{\partial^2 x_i}{\partial \bar{x}_k' \partial \bar{x}_l'}\right)_0$ of the terms of the second order in the series (19), after all the

$(\partial x_i/\partial \bar{x}_k')_0$'s have been calculated, or chosen, as we have just explained. To obtain explicitly the expressions of the coefficients $\left(\frac{\partial^2 x_i}{\partial \bar{x}_k' \, \partial \bar{x}_l'}\right)_0$ we notice that according to the equations $(\bar{g}_{\alpha\beta,k}') = 0$, we have

$$\left\{ \begin{matrix} \alpha\beta \\ k \end{matrix} \right\}' = 0.$$

The definition of this symbol will be given in the next lecture. By a simple calculation we may find

$$\left\{ \begin{matrix} \alpha\beta \\ k \end{matrix} \right\}' = \sum_i \left[\sum_j \sum_\nu \left\{ \begin{matrix} ij \\ \nu \end{matrix} \right\} \frac{\partial x_i}{\partial \bar{x}_\alpha'} \frac{\partial x_j}{\partial \bar{x}_\beta'} \frac{\partial \bar{x}_k'}{\partial x^\nu} + \frac{\partial^2 x_i}{\partial \bar{x}_\alpha' \, \partial \bar{x}_\beta'} \frac{\partial \bar{x}_k'}{\partial x_i} \right] \tag{24}$$

and we obtain

$$\left(\frac{\partial^2 x_i}{\partial \bar{x}_k' \, \partial \bar{x}_l'}\right)_0 = -\sum_\alpha \sum_\beta \left\{ \begin{matrix} \alpha\beta \\ i \end{matrix} \right\}_0 \left(\frac{\partial x_\alpha}{\partial \bar{x}_k'}\right)_0 \left(\frac{\partial x_\beta}{\partial \bar{x}_l'}\right)_0 \tag{25}$$

$$\bar{g}_{\alpha\beta}' = \delta_{\alpha\beta} + \tfrac{1}{2} \sum_k \sum_l (\bar{g}'_{\alpha\beta,kl})_0 \, \bar{x}_k' \, \bar{x}_l' + \cdots. \tag{26}$$

In order that the $\bar{g}_{\alpha\beta}'$'s be identical with the $\delta_{\alpha\beta}$'s to a third order of approximation, that is, to infinitesimals of the third order, one hundred relations $(\bar{g}'_{\alpha\beta,kl}) = 0$ must be satisfied. Now from (23) we deduce by differentiation the expressions $\bar{g}'_{\alpha\beta,kl}$ as functions of $\left(\frac{\partial^3 x_i}{\partial \bar{x}_k' \, \partial \bar{x}_l' \, \partial \bar{x}_m'}\right)$; by placing $(\bar{g}'_{\alpha\beta,kl}) = 0$ we obtain one hundred relations for determining the eighty coefficients $\left(\frac{\partial^3 x_i}{\partial \bar{x}_k' \, \partial \bar{x}_l' \, \partial \bar{x}_m'}\right)_0$ of the terms of the third order in the series (19), so that in general it is *impossible* to identify completely to a third order of approximation, the $\bar{g}_{\alpha\beta}'$'s with the $\delta_{\alpha\beta}$'s. The determination of these eighty coefficients can be made in many ways.[1]

For a clock at rest with respect to $\bar{S}'$ or at rest at the point

[1] See for example A. S. Eddington's "The Mathematical Theory of Relativity," pp. 78–81, Cambridge, 1923; A. D. Fokker, Versl. kong. Akad. van Wet. te Amsterdam, Oct. 30, 1920, pp. 614–616; G. Darmois, Annales de Physique, 1924.

$(\bar{x}_1', \bar{x}_2', \bar{x}_3')$ we have $\delta\bar{x}_1' = \delta\bar{x}_2' = \delta\bar{x}_3' = 0$ whence by (12) $\delta\bar{s}'/\bar{c}' = \delta\bar{t}'$. We have $\delta\bar{s} = \delta\bar{s}'$. Integrating, we obtain by (1)

$$\bar{t}' - \bar{t}'^0 = \frac{1}{\bar{c}'}\int_{t^0}^{t} \sqrt{\sum_{\alpha\beta}\sum g_{\alpha\beta}\delta x_\alpha \delta x_\beta} \tag{27}$$

where the integral is taken along a line λ provided with time indications t, drawn on the Γ-map. We have obtained the physical meaning of $\bar{s}'$: except for the factor $\bar{c}'$, it is the *interval* of the proper time of $\bar{S}'$. It is also said that it is the *time lived* by $\bar{S}'$.

Let us differentiate (19) with respect to the independent parameter $\bar{s}$; we obtain

$$u^\alpha = \sum_\beta \left(\frac{\partial x_\alpha}{\partial \bar{x}_\beta'}\right)_0 \bar{u}'^\beta + \sum_{\beta\gamma}\sum \left(\frac{\partial^2 x_\alpha}{\partial \bar{x}_\beta' \,\partial \bar{x}_\gamma'}\right)_0 \bar{u}'^\beta \bar{x}'^\gamma \quad (\alpha, \beta, \gamma = 1, 2, 3, 4) \tag{28}$$

where we have placed

$$u^\alpha = \frac{dx_\alpha}{d\bar{s}}, \qquad \bar{u}'^\alpha = \frac{d\bar{x}_\alpha'}{d\bar{s}}. \tag{29}$$

By (19) we have the series expansion

$$\frac{\partial x_\alpha}{\partial \bar{x}_\beta'} = \left(\frac{\partial x_\alpha}{\partial \bar{x}_\beta'}\right)_0 + \sum_\gamma \left(\frac{\partial^2 x_\alpha}{\partial \bar{x}_\beta' \,\partial \bar{x}_\gamma'}\right) \bar{x}_\gamma' + \cdots . \tag{30}$$

Let us differentiate these equations with respect to $\bar{s}$. We obtain

$$\frac{d}{d\bar{s}}\left(\frac{\partial x_\alpha}{\partial \bar{x}_\beta'}\right) = \sum_\gamma \left(\frac{\partial^2 x_\alpha}{\partial \bar{x}_\beta' \,\partial \bar{x}_\gamma'}\right)_0 \bar{u}'^\gamma + \cdots . \tag{31}$$

From (10) we have $\sum_{a}\sum_{b} \bar{g}_{ab}' \bar{u}'^a \bar{u}'^b = 1$, hence for any point *at rest* with respect to the trihedron $\bar{T}'$, $\bar{u}'^1 = \bar{u}'^2 = \bar{u}'^3 = 0$, $\bar{u}'^4 = 1/\sqrt{\bar{g}_{44}'}$.

Let us return to Equation (31). At the initial event $\bar{x}' = \bar{y}' = \bar{z}' = \bar{t}' = 0$ we have, by (26),

$$\left[\frac{d}{d\bar{s}}\left(\frac{\partial x_\alpha}{\partial \bar{x}_\beta'}\right)\right]_0 = \frac{1}{\bar{c}'}\left(\frac{\partial^2 x_\alpha}{\partial \bar{x}_\beta' \partial \,\bar{x}_4'}\right)_0. \tag{32}$$

Using Equations (25) the preceding relations become

$$\left[\frac{d}{ds}\left(\frac{\partial x_\alpha}{\partial \bar{x}_\beta'}\right)\right]_0 = -\frac{1}{\bar{c}'}\sum_a\sum_b \left\{ \begin{matrix} ab \\ \alpha \end{matrix} \right\}_0 \left(\frac{\partial x_a}{\partial \bar{x}_\beta'}\right)_0 \left(\frac{\partial x_b}{\partial \bar{x}_4'}\right)_0. \tag{33}$$

But by (26), (28) and since we are considering a point at rest with respect to $\bar{T}'$,

$$(u^b)_0 = \frac{1}{\bar{c}'}\left(\frac{\partial x_b}{\partial \bar{x}_4'}\right)_0 \quad (b = 1, 2, 3, 4). \tag{34}$$

Substituting (34) in (33), we get finally

$$\left[\frac{d}{ds}\left(\frac{\partial x_\alpha}{\partial \bar{x}_\beta'}\right)\right]_0 = -\sum_a\sum_b \left\{ \begin{matrix} ab \\ \alpha \end{matrix} \right\}_0 \left(\frac{\partial x_a}{\partial \bar{x}_\beta'}\right)_0 (u^b)_0. \tag{35}$$

We shall say that these relations express the *theorem of parallel displacement* or of geodesic translation, for the origin O' and at the initial time $\bar{t}' = 0$. This theorem may easily be correlated with the theory of parallel displacement due to Levi-Civita.[1]

If, at the point O', the observer $\bar{S}'$ draws, at the time $\bar{t}'$, the tangents to the curvilinear $\bar{x}'$-, $\bar{y}'$-, $\bar{z}'$-axes, respectively, a trirectangular trihedron which will still be called $\bar{T}'$ is obtained. We now investigate the motion of the trihedron $\bar{T}'$ with respect to the trirectangular trihedron T (Fig. 7).

For this purpose we consider, with S, the trajectory $x_i^0 = x_i^0(t)$ $(i = 1, 2, 3)$ described by the origin O' of $\bar{T}'$. To each of the points O' of this curve, we associate a trirectangular trihedron on the Γ-map formed by the tangent, the principal normal and the binormal, at O', to this trajectory. We call this trihedron $(O';\ X, Y, Z)$. We have now to perform the transformation of variables $x_i = x_i^0 + \sum_j (x_i, X_j)X_j$ $(i, j = 1, 2, 3)$. The direction cosines (x_i, X_j) are *known* functions of t.

Next, let us consider the points infinitely close to O' and attached to the trihedron $\bar{T}'$, and investigate this motion with respect to the trihedron (O', X, Y, Z). Using the foregoing transformation we have finally their motion with respect to the trihedron T. The problem is hence solved.[2]

[1] Rendiconti del Circolo Matematico di Palermo, 1917.

[2] See an example of this method in a note by T. De Donder, Bull. Ac. Roy. de Belgique, Feb. 13, 1923.

LECTURE 4

Kinematics

Velocity — Composition of velocities — Acceleration — Composition of accelerations.

The *contravariant* velocity is defined by cu^α ($\alpha = 1, 2, 3, 4$) where

$$u^\alpha = \frac{dx_\alpha}{ds}. \tag{1}$$

The u^α's are functions of the four parameters x_1, x_2, x_3, x_4, only. Let us place

$$u_\alpha = \sum_\beta g_{\alpha\beta} u^\beta \quad (\alpha, \beta = 1, 2, 3, 4). \tag{2}$$

These four functions define the *covariant* velocity cu_α ($\alpha = 1, 2, 3, 4$). We define $g^{\alpha\beta}$ as the cofactor of $g_{\alpha\beta}$ in the determinant g, divided by the determinant itself. From the identity

$$\sum_\alpha g_{\alpha\beta} g^{\alpha\gamma} = \epsilon_\beta{}^\gamma \tag{3}$$

where $\epsilon_\beta{}^\gamma$ is equal to 0 or 1 according to whether the indices are different or equal, we may write, by (2)

$$u^\alpha = \sum_\beta g^{\alpha\beta} u_\beta. \tag{4}$$

Let us place

$$W^2 = \sum_\alpha u_\alpha u^\alpha; \tag{5}$$

we have

$$W^2 = \sum_\alpha \sum_\beta g_{\alpha\beta} u^\alpha u^\beta = 1. \tag{6}$$

The preceding definitions may easily be translated in terms of space *and* time. We understand by this that we ascribe to x_4 a privileged rôle with respect to the three other parameters

x_1, x_2, x_3. We shall say that the latter refer to space, whereas x_4 represents the time t. Let us place

$$v^i = \frac{dx_i}{dt} \quad (i = 1, 2, 3), \qquad v^4 = 1 \tag{7}$$

and

$$\frac{ds}{dt} \quad \text{or} \quad \sqrt{\sum_{\alpha}\sum_{\beta} g_{\alpha\beta}v^\alpha v^\beta} = V \quad (\alpha, \beta = 1, 2, 3, 4) \tag{8}$$

whence

$$u^\alpha = v^\alpha V^{-1}. \tag{9}$$

These relations enable us to express the generalized velocity cu^α as a function of v^α. Let us notice that by (8)

$$V = \frac{1}{u^4}; \tag{10}$$

it follows from (9)

$$v^\alpha = \frac{u^\alpha}{u^4} \quad (\alpha = 1, 2, 3, 4). \tag{11}$$

These relations enable us to pass from v^α to u^α.

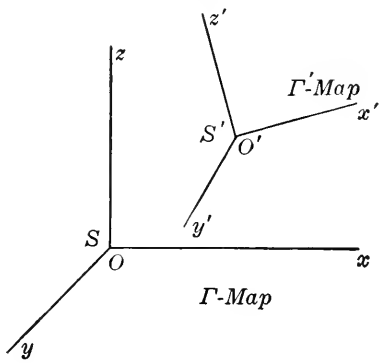

FIG. 8

The passage (Fig. 8) from the trihedron $(O;\ x, y, z)$ to the moving trihedron $(O';\ x', y', z')$ is performed through the orthogonal transformation,

$$\left.\begin{aligned} x_i &= x_i^0 + \sum_{j=1}^{3} x_j' \cos(x_j', x_i) \quad (i, j = 1, 2, 3) \\ t &= t' \end{aligned}\right\} \tag{12}$$

Taking the derivative with respect to $\bar{s}$, we have

$$u^i = \sum_{j=1}^{3} u'^j \cos(x_i, x_j') + \frac{\partial x_i}{\partial t} u'^4 \quad (i, j = 1, 2, 3) \tag{13}$$

placing

$$\frac{\partial x_i}{\partial t} = \frac{dx_i^0}{dt} + \sum_{j=1}^{3} x_j' \frac{d}{dt} \cos(x_j', x_i). \tag{14}$$

Let us go back to *space and time*, noticing that, by (12) and the invariance of ds, we have

$$V = V'. \tag{15}$$

Equation (13) now becomes

$$v^i = \sum_{j=1}^{3} v'^j \cos(x_i, x_j') + \frac{\partial x_i}{\partial t} \tag{16}$$

a result which may be stated as follows: The absolute velocity is equal to the resultant of the drag velocity $\partial x_i/\partial t$ and of the relative velocity, the components of which are v'^1, v'^2, v'^3.

Let us assume a Minkowski field and replace in (8) the quantities $g_{\alpha\beta}$ by their values taken from table (20), Lecture 2. We obtain, with the observer $\bar{S}$,

$$\bar{V} = \frac{1}{\bar{u}^4} = \sqrt{\bar{v}^2 + \bar{c}^2} = \bar{c}\sqrt{1 - (\bar{v}/\bar{c})^2} \tag{17}$$

where we have placed

$$\bar{v}^2 = \left(\frac{d\bar{x}}{d\bar{t}}\right)^2 + \left(\frac{d\bar{y}}{d\bar{t}}\right)^2 + \left(\frac{d\bar{z}}{d\bar{t}}\right)^2. \tag{18}$$

We shall have, by (17),

$$\left.\begin{aligned} \bar{u}^i &= \frac{\bar{v}^i}{\bar{c}\sqrt{1 - (\bar{v}/\bar{c})^2}} & \qquad & \bar{u}_i = -\bar{u}^i \\ & & \text{and} & \\ \bar{u}^4 &= \frac{1}{\bar{c}\sqrt{1 - (\bar{v}/\bar{c})^2}} & & \bar{u}_4 = (\bar{c})^2\bar{u}^4 \end{aligned}\right\} \tag{19}$$

If we neglect v as compared with $\bar{c}$ in (17) we have, approximately,

$$\bar{V} = \frac{1}{\bar{u}^4} \backsim \bar{c} \tag{20}$$

and by (11)

$$\bar{c}\bar{u}^1 \backsim \bar{v}^1, \quad \bar{c}\bar{u}^2 \backsim \bar{v}^2, \quad \bar{c}\bar{u}^3 \backsim \bar{v}^3. \tag{21}$$

The last two expressions furnish the physical interpretation given by $\bar{S}$ to the symbols $\bar{c}\bar{u}^\alpha$ and $\bar{V}$ and their order of magnitude.

The *contravariant* acceleration c^2A^σ ($\sigma = 1, 2, 3, 4$) is defined by

$$A^\sigma = \frac{du^\sigma}{ds} + \sum_\alpha\sum_\beta \left\{ \begin{matrix} \alpha\beta \\ \sigma \end{matrix} \right\} \frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} \tag{22}$$

where we have introduced the Christoffel three-index symbols

$$\left\{ \begin{matrix} \alpha\beta \\ \sigma \end{matrix} \right\} = \sum_\tau g^{\sigma\tau} \begin{bmatrix} \alpha\beta \\ \tau \end{bmatrix} \tag{23}$$

and

$$\begin{bmatrix} \alpha\beta \\ \tau \end{bmatrix} = \tfrac{1}{2} \left(g_{\alpha\tau,\beta} + g_{\beta\tau,\alpha} - g_{\alpha\beta,\tau}\right). \tag{24}$$

By analogy to (2) we define the *covariant* acceleration c^2A_σ ($\sigma = 1, 2, 3, 4$), placing

$$A_\sigma = \sum_\tau g_{\sigma\tau}A^\tau. \tag{25}$$

Finally by a simple calculation we obtain the new expression,

$$A_\sigma = \frac{du_\sigma}{ds} - \tfrac{1}{2} \sum_\alpha\sum_\beta g_{\alpha\beta,\sigma}u^\alpha u^\beta. \tag{26}$$

By taking the derivative of W^2, defined by (6), with respect to $\bar{s}$ we get

$$\frac{dW^2}{ds} = 2\sum_\alpha A_\alpha u^\alpha = 0 \tag{27}$$

which may also be written

$$\sum_\alpha A_\alpha u^\alpha = 0. \tag{28}$$

We note that (26) may also be written

$$A_\sigma = \frac{d}{d\bar{s}}\left(\frac{\partial W}{\partial u^\sigma}\right) - \left(\frac{\partial W}{\partial x_\sigma}\right) \tag{29}$$

the parenthesis indicating that the variables u^σ and x_σ are to be considered as independent variables. To establish this relation, it is sufficient to notice that

$$\left(\frac{\partial W}{\partial u^\sigma}\right) = \frac{1}{W}\sum_\alpha g_{\sigma\alpha}u^\alpha = \frac{u_\sigma}{W}, \quad \left(\frac{\partial W}{\partial x_\sigma}\right) = \frac{\sum_\alpha\sum_\beta g_{\alpha\beta\,\sigma}u^\alpha u^\beta}{2\,W}. \tag{30}$$

It is important to point out that the covariant components A_σ may be derived from the invariant W by taking the variational or Lagrangian derivatives of W with respect to the variables x_σ and noting that $u^\sigma = dx_\sigma/d\bar{s}$. The covariancy (29) might give the starting point for obtaining all the formulas of the so-called absolute calculus.

In space and time we find, similarly,

$$VA_\sigma = \frac{d}{dt}\left(\frac{\partial V}{\partial v^\sigma}\right) - \left(\frac{\partial V}{\partial x_\sigma}\right) \tag{31}$$

where

$$\left(\frac{\partial V}{\partial v^\sigma}\right) = \frac{1}{V}\sum_\alpha g_{\sigma\alpha}v^\alpha, \quad \left(\frac{\partial V}{\partial x_\sigma}\right) = \frac{\sum_\alpha\sum_\beta g_{\alpha\beta,\sigma}v^\alpha v^\beta}{2\,V} \tag{32}$$

the parenthesis indicating that the variables v^σ and x_σ are to be considered as independent variables.

Consider an Einstein field, defined by

$$d\bar{s}^2 = \sum_\alpha\sum_\beta g_{\alpha\beta}\,\delta x_\alpha\,\delta x_\beta \tag{33}$$

and let x_1, x_2, x_3, x_4 be the coördinates used by the observer S who plots the Γ-map of this field. We introduce in the Γ-map (Fig. 8) another observer S' moving with respect to S and using the parameters x_1', x_2', x_3', x_4'. The problem of the composition of the contravariant accelerations is then reduced to the investigation of the contravariant relation

$$A'^\sigma = \sum_\alpha \frac{\partial x_\sigma'}{\partial x_\alpha}A^\alpha \tag{34}$$

where the new variables x_1', x_2', x_3', x_4' are given by the general transformation,

$$x_\alpha' = x_\alpha'(x_1, x_2, x_3, x_4) \quad (\alpha = 1, 2, 3, 4) \tag{35}$$

and conversely. Let us place, for simplicity,

$$(\alpha) = \sum_\beta \sum_\gamma \begin{Bmatrix} \beta\gamma \\ \alpha \end{Bmatrix} u^\beta u^\gamma. \tag{36}$$

We obtain, finally,

$$A'^\sigma = \frac{d^2x_\sigma'}{ds^2} + \sum_\alpha \sum_\beta \sum_\gamma \frac{\partial^2 x_\alpha}{\partial x_\beta' \partial x_\gamma'} \frac{\partial x_\sigma'}{\partial x_\alpha} u'^\beta u'^\gamma + \sum_\alpha (\alpha) \frac{\partial x_\sigma'}{\partial x_\alpha}. \tag{37}$$

This relation expresses in the most general way the analogue of *Coriolis' theorem.* We now show what is understood by this analogy in a particular case.

In order to define the change of variables, let us choose the Euclidean relations.

$$\left.\begin{aligned} x &= x_0 + \alpha_1 x' + \alpha_2 y' + \alpha_3 z' \\ y &= y_0 + \beta_1 x' + \beta_2 y' + \beta_3 z' \\ z &= z_0 + \gamma_1 x' + \gamma_2 y' + \gamma_3 z' \\ t &= t' \end{aligned}\right\} \tag{38}$$

In these formulas x_0, y_0, z_0, are the coördinates of the origin O' with respect to the fixed trihedron $(O;\ x, y, z)$. The direction cosines α_i, β_i, γ_i are given functions of t and are related by the conditions of orthogonality,

$$\left.\begin{aligned} &\alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1 && (i = 1, 2, 3) \\ &\alpha_i\alpha_j + \beta_i\beta_j + \gamma_i\gamma_j = 0 && (i \neq j;\ i, j = 1, 2, 3) \end{aligned}\right\} \tag{39}$$

Equations (38) and (39) are given by S and written down on the Γ-map by this observer (Fig. 8). In this case, the contravariant acceleration for S' has the components,

$$\left.\begin{aligned} A'^1 &= \frac{d^2x'}{ds^2} + 2\left(q'\frac{dz'}{ds} - r'\frac{dy'}{ds}\right)u'^4 + a_{x'}'(u'^4)^2 + \sum_\alpha \frac{\partial x_1'}{\partial x_\alpha}(\alpha) \\ A'^2 &= \cdots, \quad A'^3 = \cdots, \quad A'^4 = \frac{d^2t}{ds^2} + (4) \end{aligned}\right\} \tag{40}$$

In these formulas p', q', r' are the components of the instantaneous rotation of the moving trihedron $(O';\ x', y', z')$ with respect to the fixed trihedron $(O;\ x,\ y,\ z)$. This rotation is defined in the same way as in classical kinematics, that is, by

$$\left.\begin{aligned} p' &= \alpha_3\dot{\alpha}_2 + \beta_3\dot{\beta}_2 + \gamma_3\dot{\gamma}_2 = -(\alpha_2\dot{\alpha}_3 + \beta_2\dot{\beta}_3 + \gamma_3\dot{\gamma}_2) \\ q' &= \cdots, \quad r' = \cdots. \end{aligned}\right\} \tag{41}$$

On the other hand, we place as in ordinary kinematics

$$\left.\begin{aligned} a_{x'}' &= \alpha_1(\ddot{x}_0 + \ddot{\alpha}_1 x' + \ddot{\alpha}_2 y' + \ddot{\alpha}_3 z') \\ &\quad + \beta_1(\ddot{y}_0 + \ddot{\beta}_1 x' + \ddot{\beta}_2 y' + \ddot{\beta}_3 z') \\ &\quad + \gamma_1(\ddot{z}_0 + \ddot{\gamma}_1 x' + \ddot{\gamma}_2 y' + \ddot{\gamma}_3 z') \\ a_{y'}' &= \cdots, \quad a_{z'}' = \cdots. \end{aligned}\right\} \tag{42}$$

The last two components are obtained by replacing successively $\alpha_1,\ \beta_1,\ \gamma_1$, which are outside the parenthesis, by $\alpha_2,\ \beta_2,\ \gamma_2$ and $\alpha_3,\ \beta_3,\ \gamma_3$. The components given by (39) may be written in space *and* time as follows:

$$\left.\begin{aligned} A'^1 &= V^{-2}\left[\ddot{x}' + 2(q'\dot{z}' - r'\dot{y}') + a'_{x'} - \dot{x}'\frac{d}{dt}\log V' + \sum_\alpha [\alpha]\frac{\partial x_\sigma'}{\partial x_\alpha}\right] \\ A'^2 &= \cdots, \qquad A'^3 = \cdots, \qquad A'^4 = V^{-2}\left(\frac{d}{dt}\log V' + \sum_\alpha [\alpha]\frac{\partial x_\sigma'}{\partial x_\alpha}\right) \end{aligned}\right\} \tag{43}$$

where $p',\ q',\ r',\ a_{x'}' \cdots$ have the same meaning as before (Equations (41) and (42)), and where we have placed, as in (35),

$$[\alpha] = \sum_\beta\sum_\gamma \begin{Bmatrix} \beta\gamma \\ \alpha \end{Bmatrix} v^\beta v^\gamma. \tag{44}$$

Let us return for a moment to the definitions of the contravariant and covariant accelerations and inquire what becomes of these expressions in the Minkowski field. In this case we obtain

$$\bar{A}^\sigma = \frac{d^2\bar{x}_\sigma}{d\bar{s}^2} \tag{45}$$

or, in space *and* time,

$$\bar{A}^\sigma = \bar{V}^{-2}\left[\frac{d^2\bar{x}_\sigma}{dt^2} - \bar{v}^\sigma\frac{d}{dt}\log \bar{V}\right]. \tag{46}$$

Let us notice that these expressions are similar to those of classical mechanics and further that for all motions ordinarily considered in mechanics, we have approximately

$$\bar{c}^2\bar{A}^\sigma \backsim \frac{d^2\bar{x}_\sigma}{dt^2}. \tag{47}$$

Let us investigate now what becomes of Coriolis' theorem in the Minkowski field. All the Christoffel symbols being zero, the symbols (α) and $[\alpha]$ vanish and Equations (40) or (43) give the answer to this problem.[1]

[1] The detail of the calculations in this lecture may be found in Chap. 3 of the author's "Introduction à la gravifique einsteinienne." Mémorial des sciences mathématiques, Paris, Gauthier-Villars, 1925.

LECTURE 5

THE FUNDAMENTAL EQUATIONS OF THE GRAVIFIC FIELD

Variational derivative — The gravific equations — Theorem of the phenomenal tensor — Theorem of the gravific pseudo-tensor.

Let $\xi_1,\ \xi_2 \cdots$ be *arbitrary* functions of $(x_1 \cdots x_4)$. Let us denote the derivatives of one of these functions ξ_a with respect to x_k by the symbol $\xi_{a,k}$, similarly by $\xi_{a,kl}$ the second derivative of ξ_a with respect to x_k and to x_l. We consider an *arbitrary* function $F(\xi_1,\ \xi_2,\ \cdots,\ \xi_{1,1},\ \cdots)$ of the functions ξ_a and of their derivatives $\xi_{a,k} \cdots$. For purposes of simplification we denote the function $F(\xi_a,\ \cdots\ \xi_{a,k}\ \cdots)$ by the symbol F.

Let us place

$$\frac{\delta F}{\delta \xi_c} = \frac{\partial F}{\partial \xi_c} - \sum_j \frac{d}{dx_j}\left(\frac{\partial F}{\partial \xi_{c,j}}\right) + \sum_j \sum_k \frac{d^2}{dx_j dx_k}\left(\frac{\partial F}{\partial \xi_{c,jk}}\right) - \cdots. \tag{1}$$

This operation is called the *variational derivative of F with respect to ξ_c*. It has the following remarkable property: The variational derivative with respect to ξ_c of the partial derivative

$$\frac{\partial F}{\partial x_i} = \sum_a \left(\frac{\partial F}{\partial \xi_a} \xi_{a,i} + \sum_k \frac{\partial F}{\partial \xi_{a,k}} \xi_{a,ik} + \cdots\right) \tag{2}$$

is identically zero.

Let us consider now a function depending only on the $g^{\alpha\beta}$, $g^{\alpha\beta,i}$, $\cdots$. Its variational with respect to an arbitrary change of variables x_1, x_2, x_3, x_4 is that of a multiplier or density factor, that is

$$\left.\begin{aligned} \mathbf{M}^g &= \mathbf{M}^g(g^{\alpha\beta},\ g^{\alpha\beta,i},\ g^{\alpha\beta,ik},\ \cdots) \\ &= \mathbf{M}^g(g'^{\alpha\beta},\ g'^{\alpha\beta,i},\ g'^{\alpha\beta,ik},\ \cdots)\frac{\partial(x')}{\partial(x)} \\ &= \mathbf{M}'^g \frac{\partial(x')}{\partial(x)} \end{aligned}\right\} \tag{3}$$

where $x_1' \cdots x_4'$ are arbitrary functions of $x_1 \cdots x_4$. We call $\mathbf{M}^g$ the *characteristic gravific function.* Let us further consider a function $\mathbf{M}$, having the same variational but which, besides depending on $g^{\alpha\beta}$, $g^{\alpha\beta,i} \cdots$, may involve other functions, such as u^α, $\begin{Bmatrix} \alpha\beta \\ \gamma \end{Bmatrix}$, A_α, etc., already defined above. This function will be written explicitly in the following lectures and will then be called the *characteristic phenomenal function.*

Let us take the variational derivative of the sum

$$\mathbf{M}^g + \mathbf{M} \tag{4}$$

with respect to $g^{\alpha\beta}$. We have

$$\frac{\delta}{\delta g^{\alpha\beta}}(\mathbf{M}^g + \mathbf{M}) = \frac{\partial}{\partial g^{\alpha\beta}}(\mathbf{M}^g + \mathbf{M}) - \sum_j \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial g^{\alpha\beta,j}}(\mathbf{M}^g + \mathbf{M})\right] + \sum_j\sum_k \frac{\partial^2}{\partial x_j\, \partial x_k}\left[\frac{\partial}{\partial g^{\alpha\beta,jk}}(\mathbf{M}^g + \mathbf{M})\right] - \cdots. \tag{5}$$

The fundamental variational principle of the gravific theory consists in placing all ten variational derivatives with respect to the $g^{\alpha\beta}$'s equal to zero. We obtain in this way the ten fundamental equations of the gravific theory, i.e.,

$$\frac{\delta}{\delta g^{\alpha\beta}}(\mathbf{M}^g + \mathbf{M}) = 0. \tag{6}$$

Let us place also

$$\mathbf{T}^g{}_{\alpha\beta} = \frac{\delta \mathbf{M}^g}{\delta g^{\alpha\beta}}, \quad \mathbf{T}_{\alpha\beta} = -\frac{\delta \mathbf{M}}{\delta g^{\alpha\beta}}. \tag{7}$$

We call $\mathbf{T}^g{}_{\alpha\beta}$ the *symmetrical covariant gravific tensor,* $\mathbf{T}_{\alpha\beta}$ the *symmetrical covariant phenomenal tensor* or simply the *symmetrical tensor.* By (6) we have

$$\mathbf{T}^g{}_{\alpha\beta} = \mathbf{T}_{\alpha\beta}. \tag{8}$$

Let us denote by C the curvature invariant and by a and b two universal constants, i.e., the gravitational constant and the cosmological constant. We know that C is given by

$$C = \sum_\alpha\sum_\beta g^{\alpha\beta} C_{\alpha\beta} \tag{9}$$

where

$$C_{\alpha\beta} = \sum_{\sigma}\sum_{\tau}\left(\frac{\partial}{\partial x_\beta}\begin{Bmatrix}\alpha\sigma\\ \sigma\end{Bmatrix} - \frac{\partial}{\partial x_\sigma}\begin{Bmatrix}\alpha\beta\\ \sigma\end{Bmatrix} + \begin{Bmatrix}\beta\tau\\ \sigma\end{Bmatrix}\begin{Bmatrix}\alpha\sigma\\ \tau\end{Bmatrix} - \begin{Bmatrix}\sigma\tau\\ \sigma\end{Bmatrix}\begin{Bmatrix}\alpha\beta\\ \tau\end{Bmatrix}\right). \quad (10)$$

The Christoffel symbols $\begin{Bmatrix}\alpha\beta\\ \gamma\end{Bmatrix}$ have been defined above (Lecture 4, Equation (23)). If $\mathbf{M}^g$ is taken as

$$\mathbf{M}^g = (a + bC)\sqrt{-g} \quad (11)$$

we get by performing the operations indicated in (7)

$$-\tfrac{1}{2}(a + bC)g_{\alpha\beta} + bC_{\alpha\beta} = T_{\alpha\beta} \quad (12)$$

where

$$T_{\alpha\beta} = \frac{\mathbf{T}_{\alpha\beta}}{\sqrt{-g}}. \quad (13)$$

Let us multiply both sides of (12) by $g^{\alpha\beta}$ and sum over α and β. Then, by (9),

$$bC = -T - 2a \quad (14)$$

where we have placed

$$T_\alpha^\beta = \sum_i g^{\beta i} T_{\alpha i} \qquad (i = 1, 2, 3, 4) \quad (15)$$

and

$$T = \sum_i T_i^i \qquad (i = 1, 2, 3, 4). \quad (16)$$

By substituting (14) in (12) the ten fundamental equations of the gravific theory take the form

$$\frac{a}{2} g_{\alpha\beta} + bC_{\alpha\beta} = T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T \quad (\alpha, \beta = 1, 2, 3, 4). \quad (17)$$

The variational principle, as we have just presented it, is evidently a generalization of Hamilton's principle,[1] that is, equivalent to placing

$$\delta \int_\Omega (\mathbf{M}^g + \mathbf{M})\, dx_1\, dx_2\, dx_3\, dx_4 = 0. \quad (18)$$

[1] H. A. Lorentz, Versl. Akad. Amsterdam, February 12, 1915; David Hilbert, Göttinger Nachrichten, November, 1915; Th. De Donder, Versl. Akad. Amsterdam, May 27, 1916.

Ω being a region of space-time at the boundaries of which the variations must vanish. It is in order to avoid the use of a four-dimensional space that we have preferred the above presentation.

Applying to the function $\mathbf{M}^g$ theorems already proved[1] we have, because of the nature of the density factor of this function, the following four *identities:*

$$\sum_i\left[\frac{\partial}{\partial x_i}\sum_j g^{ij}\frac{\delta \mathbf{M}^g}{\delta g^{\alpha j}} + \frac{1}{2}\sum_j g^{ij,\alpha}\frac{\delta \mathbf{M}^g}{\delta g^{ij}}\right] = 0 \quad (i, j = 1, 2, 3, 4) \qquad (19)$$

or by (7)

$$\sum_i\left[\frac{\partial \mathbf{T}^{g}{}_{\alpha}^{i}}{\partial x_i} + \frac{1}{2}\sum_j g^{ij,\alpha}\mathbf{T}^g{}_{ij}\right] = 0 \quad (\alpha = 1, 2, 3, 4) \qquad (20)$$

where

$$\mathbf{T}^g{}_\alpha^i = \sum_j g^{ij}\mathbf{T}^g{}_{\alpha j}. \qquad (21)$$

The ten fundamental equations (6) together with the four identities (20) give us immediately the following four equations.

$$\sum_i\left[\frac{\partial \mathbf{T}_\alpha^i}{\partial x_i} + \frac{1}{2}\sum_j g^{ij,\alpha}\mathbf{T}_{ij}\right] = 0 \qquad (22)$$

where as in (21)

$$\mathbf{T}_\alpha^i = \sum_j g^{ij}\mathbf{T}_{\alpha j}. \qquad (23)$$

We say that the four equations (22) express the *theorem of the phenomenal tensor.* We note that Equations (22) may also be written

$$\sum_i\left[\frac{\partial \mathbf{T}_\alpha^i}{\partial x_i} - \frac{1}{2}\sum_j g_{ij,\alpha}\mathbf{T}^{ij}\right] = 0 \qquad (\alpha = 1, 2, 3, 4) \qquad (24)$$

or again,

$$\sum_i\left[\frac{\partial \mathbf{T}_\alpha^i}{\partial x_i} - \sum_j \left\{\begin{matrix}\alpha i\\ j\end{matrix}\right\}\mathbf{T}_j^i\right] = 0 \qquad (\alpha = 1, 2, 3, 4). \qquad (25)$$

[1] Th. De Donder "La synthèse de la gravifique," Comptes-Rendus de Paris, June 11, 1923, p. 1701; see also Bull. Acad. Roy. de Belgique, April, 1924.

We shall place

$$\mathbf{F}_\alpha = \sum_i \left[\frac{\partial \mathbf{T}_\alpha{}^i}{\partial x_i} - \frac{1}{2} \sum_j g_{ij,\alpha} \mathbf{T}^{ij} \right] \tag{26}$$

and shall say that $\mathbf{F}_\alpha$ ($\alpha = 1, 2, 3, 4$) are the components of *the total generalized force.* The four equations (24) may now be written

$$\mathbf{F}_\alpha = 0. \tag{27}$$

The theorem of the phenomenal tensor expresses therefore that the *four components of the total generalized force are zero.*

A *mixed gravific pseudo-tensor*[1] is, by definition, a set of sixteen functions $\binom{\beta}{\alpha}$, (α, $\beta = 1, 2, 3, 4$) satisfying the four equations

$$\sum_i \left[\frac{\partial}{\partial x_i} \binom{i}{\alpha} - \frac{1}{2} \sum_j g^{ij,\alpha} \mathbf{T}^g{}_{ij} \right] = 0. \tag{28}$$

Let us remember that the $\mathbf{T}^g{}_{ij}$'s depend only on the $g^{\alpha\beta}$'s and their derivatives with respect to x_1, x_2, x_3, x_4. On comparing (28) with the fundamental identities (20) we have

$$\sum_i \frac{\partial}{\partial x_i} \left[\binom{i}{\alpha} + \mathbf{T}^g{}_\alpha{}^i \right] = 0. \tag{29}$$

Using Equations (8) and Notations (23), we obtain four equations

$$\sum_i \frac{\partial}{\partial x_i} \left[\binom{i}{\alpha} + \mathbf{T}_\alpha{}^i \right] = 0 \tag{30}$$

which express *the theorem of the gravific pseudo-tensor.*

The theory of gravific waves and rays has been developed in the author's "Gravifique Einsteinienne" (Arts. 29 and 45) (Paris, 1921). Let us choose *new* variables x_1, x_2, x_3, x_4 such that the new $g_{\alpha\beta}$'s satisfy the four complementary equations:

$$\sum_\alpha \sum_\beta g^{\alpha\beta} (g_{\alpha\beta,\sigma} - 2\, g_{\sigma\alpha,\beta}) = 0. \tag{31}$$

[1] The first example of such a tensor was given by the author in Versl. Akad. Amsterdam, May, 1916; see also a note by T. Okaya, Japanese Journal of Physics, Vol. 3, pp. 95–115, 1924.

These relations may also be written

$$\sum_{\alpha}\sum_{\beta} g^{\alpha\beta}\begin{bmatrix}\alpha\beta \\ \sigma\end{bmatrix} = 0. \tag{32}$$

Using these new variables, the fundamental equations of the gravific field (17) become

$$\sum_{\alpha}\sum_{\beta} g^{\alpha\beta} g_{\sigma\tau,\alpha\beta} = (\sigma, \tau) \tag{33}$$

where the right-hand member (σ, τ) does *not* involve the second derivatives of the Einsteinian potentials. We further note that the left-hand member of each of these equations contains the second derivatives of only one of these potentials.

Following on the footsteps of J. Hadamard and E. Vessiot we consider the surfaces $f(x_1, x_2, x_3, x_4) = 0$ which satisfy the characteristic equation of (33), i.e.,

$$H = \sum_{\alpha}\sum_{\beta} g^{\alpha\beta} \frac{\partial f}{\partial x_\alpha} \frac{\partial f}{\partial x_\beta} = 0. \tag{34}$$

We consider also the characteristics of this equation, or the bicharacteristics of (33). These bicharacteristics are to be determined by the well-known Cauchy equations,

$$\frac{dx_\alpha}{dl} = \frac{\partial H}{\partial X_\alpha}, \qquad \frac{dX_\alpha}{dl} = -\frac{\partial H}{\partial x_\alpha} \tag{35}$$

where $X_\alpha = \dfrac{\partial f}{\partial x_\alpha}$ and we have introduced the parameter l. We shall say that the characteristic surfaces f determine the *gravific waves* and that the bicharacteristic lines determine the *gravific rays*. From their definition (35) it follows that the gravific rays are geodesics in the space-time (Equation (1), Lecture 3) and that they are lines of zero length, i.e. such that $\sum_{\alpha}\sum_{\beta} g_{\alpha\beta}\, dx_\alpha\, dx_\beta = 0$. This relation is independent of the choice of variables x_1, x_2, x_3, x_4. This fundamental property of gravific rays interpreted in space *and* time may be looked upon as a generalization of Fermat's theorem.

LECTURE 6

The Mass Gravific Field

Definition — The characteristic function — The fundamental equations — The mass tensor — Special cases — Dynamics in space and time — Examples — Physical measurement — Approximations.

Let us consider a gravific field due to masses. In order to describe the motion of these masses, we shall use, together with the observer S, the Euclidean coördinates x_1, x_2, x_3 and the time x_4, defined in Lecture 3. The motion of these masses will be described by S, on the Γ-map, by means of the covariant velocities u_α ($\alpha = 1, 2, 3, 4$) (Lecture 4), which should be considered as functions of x_1, x_2, x_3, x_4. The observer S will make use of the *mass density-factor* $\mathbf{N}$, which is a function of x_1, x_2, x_3, x_4. This observer will also make use of a *mass tensor*, the ten symmetrical components of which will be denoted by $\mathbf{P}_{\alpha\beta}$ and are also functions of x_1, x_2, x_3, x_4.

We place[1] in the case of the gravific mass field

$$\mathbf{M} = -\sum_{\alpha}\sum_{\beta} g^{\alpha\beta}(\mathbf{N}u_\alpha u_\beta + \mathbf{P}_{\alpha\beta}). \tag{1}$$

The characteristic function (1) enables us to compute the phenomenal or mass tensor $\mathbf{T}_{\alpha\beta}$ defined by (7), Lecture 5. We have, in this way,

$$\mathbf{T}_{\alpha\beta} = \mathbf{N}u_\alpha u_\beta + \mathbf{P}_{\alpha\beta}. \tag{2}$$

We place, as usual,

$$N = \frac{\mathbf{N}}{\sqrt{-g}}, \tag{3}$$

then Equations (2) become

$$T_{\alpha\beta} = Nu_\alpha u_\beta + P_{\alpha\beta}. \tag{4}$$

[1] Generalizing the results of H. A. Lorentz (Verslag. Ak. van. Wet. te Amsterdam, p. 1076, 1915), we first introduced a function $\mathbf{M}$ (Bull. Acad. Roy. de Belgique, p. 317, 1919) and then the present more general form (Bull. Acad. Royale de Belgique, p. 77, 1924).

We introduce now the mixed tensor

$$P_\alpha^\beta = \sum_i g^{\beta i} P_{\alpha i} \tag{5}$$

and the invariant

$$P = \sum_i P_i^i; \tag{6}$$

it follows that

$$T = N + P. \tag{7}$$

After these preliminaries, we find immediately that the ten Equations (17), Lecture 5, of the gravific field due to masses, may be written

$$\frac{a}{2} g_{\alpha\beta} + bC_{\alpha\beta} = N(u_\alpha u_\beta - \tfrac{1}{2} g_{\alpha\beta}) + P_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} P. \tag{8}$$

These are the fundamental equations of the gravific mass field.

Let us go back to the four Equations (22), Lecture 5, which express the theorem of the phenomenal tensor. We substitute in them the values of $\mathbf{T}_\alpha^\beta$ deduced from (2), i.e.,

$$\mathbf{T}_\alpha^\beta = \mathbf{N} u_\alpha u^\beta + \mathbf{P}_\alpha^\beta \tag{9}$$

whence

$$\sum_i \left[\frac{\partial}{\partial x_i} (\mathbf{N} u_\alpha u^i + \mathbf{P}_\alpha^i) - \tfrac{1}{2} \sum_j \sum_k g^{ij} g_{kj\,\alpha} (\mathbf{N} u_i u^k + \mathbf{P}_i^k) \right] = 0. \tag{10}$$

Let us place

$$\sum_i \frac{\partial}{\partial x_i} (\mathbf{N} u_\alpha u^i) - \frac{\mathbf{N}}{2} \sum_j \sum_k g_{kj,\alpha} u^j u^k = \mathbf{N}_\alpha \tag{11}$$

and

$$\sum_i \left[\frac{\partial \mathbf{P}_\alpha^i}{\partial x_i} - \tfrac{1}{2} \sum_j \sum_k g^{ij} g_{kj,\alpha} \mathbf{P}_i^k \right] = \mathbf{P}_\alpha. \tag{12}$$

Then the four Equations (1) can be abbreviated as follows:

$$\mathbf{F}_\alpha = \mathbf{N}_\alpha + \mathbf{P}_\alpha = 0 \quad (\alpha = 1, 2, 3, 4). \tag{13}$$

They express the theorem of the *total generalized force*. Let us notice that (11) may be written down immediately, using the covariant acceleration A_α introduced in Lecture 4,

$$\mathbf{N}_\alpha = \mathbf{N} A_\alpha + u_\alpha \sum_i \frac{\partial}{\partial x_i} (\mathbf{N} u^i). \tag{14}$$

Multiplying both sides of this identity by u^α and adding, we obtain

$$\sum_\alpha \frac{\partial}{\partial x_\alpha}(\mathbf{N}u^\alpha) + \sum_\alpha \mathbf{P}_\alpha u^\alpha = 0, \tag{15}$$

which is the *equation of continuity of mass.*

The theorem of the mass tensor (13) may also be written as follows by using the identities (14) and the relation (15),

$$\mathbf{F}_\alpha = \mathbf{N}A_\alpha - u_\alpha \sum_i \mathbf{P}_i u^i + \mathbf{P}_\alpha = 0. \tag{16}$$

Multiplying by $g^{\alpha\beta}$ and adding with respect to α we obtain

$$\mathbf{F}^\alpha = \mathbf{N}A^\alpha - u^\alpha \sum_i \mathbf{P}_i u^i + \mathbf{P}^\alpha = 0 \tag{17}$$

where

$$\mathbf{P}^\alpha = \sum_\beta g^{\alpha\beta}\mathbf{P}_\beta, \quad \mathbf{F}^\alpha = \sum_\beta g^{\alpha\beta}\mathbf{F}_\beta. \tag{18}$$

The *incoherent* mass fluid is by definition that for which $\mathbf{P}_{\alpha\beta} = 0$. The fundamental gravific Equations (18) become

$$\frac{a}{2} g_{\alpha\beta} + bC_{\alpha\beta} = N(u_\alpha u_\beta - \tfrac{1}{2} g_{\alpha\beta}) \tag{19}$$

and the theorem of the mixed tensor becomes, by Equations (10) to (12),

$$\mathbf{N}_\alpha = \sum_i \left[\frac{\partial}{\partial x_i}(\mathbf{N}u_\alpha u^i) - \frac{\mathbf{N}}{2}\sum_j g_{ij,\alpha}u^i u^j\right] = 0. \tag{20}$$

Equation (15) takes the very simple form

$$\sum_i \frac{\partial}{\partial x_i}(\mathbf{N}u^i) = 0. \tag{21}$$

This is the equation of continuity of the incoherent mass fluid. At every point where $\mathbf{N} \neq 0$ we have, by (16),

$$A_\alpha = \frac{du_\alpha}{ds} - \tfrac{1}{2}\sum_i\sum_j g_{ij,\alpha}u^i u^j = 0 \tag{22}$$

or, by (17),

$$A_\alpha = \frac{d^2x_\alpha}{ds^2} + \sum_i\sum_j \left\{ {ij \atop \alpha} \right\} u^i u^j = 0. \tag{23}$$

These equations can be given a very interesting interpretation: the tracks of any mass point and the mode of its motion may be obtained by taking the extremal

$$\delta \int \sqrt{\sum_{\alpha}\sum_{\beta} g_{\alpha\beta}\, dx_\alpha\, dx_\beta} = 0 \tag{24}$$

or

$$\delta \int ds = 0. \tag{25}$$

We take the variation with respect to x_1, x_2, x_3, x_4, assuming that the ends of the line along which the integration is taken are fixed. If we place

$$W = \sqrt{\sum_{\alpha}\sum_{\beta} g_{\alpha\beta} u^\alpha u^\beta} = 1 \tag{26}$$

the equations of the extremal (25) may be written

$$\left.\begin{aligned} \frac{dx_\alpha}{ds} &= u^\alpha \\ \frac{d}{ds}\left(\frac{\partial W}{\partial u^\alpha}\right) - \left(\frac{\partial W}{\partial x_\alpha}\right) &= 0 \end{aligned}\right\} \tag{27}$$

and we have seen in Lecture 4 that these equations are identical with Equations (23).

The mass fluid is called *perfect* when the tensor P_α^β has the particular form

$$P_\alpha^\beta = -\epsilon_\alpha^\beta p. \tag{28}$$

We recall that

$$P_\alpha^\beta = \frac{\mathbf{P}_\alpha^\beta}{\sqrt{-g}} \tag{29}$$

and that $\epsilon_\alpha^\beta = 1$ if $\beta = \alpha$ and $\epsilon_\alpha^\beta = 0$ if $\beta \neq \alpha$.

If a mass fluid is perfect in a system of coördinates x_1, x_2, x_3, x_4, *it is perfect in any other system* x_1', x_2', x_3', x_4'. This may be verified by the variance of the mixed tensor (28) and it is seen also that p is an invariant for all transformations of the variables x_1, x_2, x_3, x_4. The fundamental equations of the gravific field produced by a perfect mass fluid become

$$\frac{a}{2} g_{\alpha\beta} + bC_{\alpha\beta} = N(u_\alpha u_\beta - \tfrac{1}{2} g_{\alpha\beta}) + p g_{\alpha\beta}. \tag{30}$$

The theorem (13) of the phenomenal tensor becomes

$$\mathbf{N}_\alpha + \mathbf{P}_\alpha = 0 \tag{31}$$

where $\mathbf{P}_\alpha$ is given by (12), that is, by

$$\mathbf{P}_\alpha = -\sqrt{-g}\,\frac{\partial p}{\partial x_\alpha}. \tag{32}$$

This theorem may also be written

$$\mathrm{N}A^\alpha + \sqrt{-g}\left(u^\alpha \frac{dp}{ds} - \sum_\beta \frac{\partial p}{\partial x_\beta} g^{\alpha\beta}\right) = 0. \tag{33}$$

We have already established in Lecture 4 the relations enabling us to go over to space *and* time. Equations (10) then become

$$\mathbf{N}V^{-1}\left[\frac{d}{dt}\left(\frac{\partial V}{\partial v^\alpha}\right) - \left(\frac{\partial V}{\partial x_\alpha}\right)\right] - V^{-1}\left(\frac{\partial V}{\partial v^\alpha}\right)\sum_i \mathbf{P}_i v^i + \mathbf{P}_\alpha = 0. \tag{34}$$

The equation of continuity (15) may be written now,

$$\sum_i \frac{\partial}{\partial x_i}(\mathbf{N}V^{-1}v^i) + V^{-1}\sum_i \mathbf{P}_i v^i = 0. \tag{35}$$

Let us now use Equations (17). The formulas developed in Lecture 4 enable us to write

$$V^{-1}\mathbf{N}\left[\frac{d}{dt}(V^{-1}v^\alpha) + V^{-1}\sum_i\sum_k \begin{Bmatrix} ik \\ \alpha \end{Bmatrix} v^i v^k\right] - V^{-2}v^\alpha \sum_i \mathbf{P}_i v^i + \mathbf{P}^\alpha = 0 \tag{36}$$

or

$$\mathbf{N}V^{-2}\left[\frac{d^2 x_\alpha}{dt^2} + \sum_i\sum_j\left(\begin{Bmatrix} ij \\ \alpha \end{Bmatrix} - v^\alpha \begin{Bmatrix} ij \\ 4 \end{Bmatrix}\right)v^i v^j\right] + \mathbf{P}^\alpha - v^\alpha \mathbf{P}^4 = 0. \tag{37}$$

In the last equation if we place $\alpha = 4$ we obtain an identity.

A mass field is called *stationary* when all the $g_{\alpha\beta}$'s and all the v^α's are independent of $x_4 = t$. Then at any point where the velocity of the fluid is zero ($v^1 = v^2 = v^3 = 0$), Equations (36) become

$$\tfrac{1}{2}\,\mathbf{N}g_{44,\alpha} + g_{4\alpha}\mathbf{P}_4 - g_{44}\mathbf{P}_\alpha = 0 \quad (\alpha = 1, 2, 3). \tag{38}$$

If the fluid is incoherent and at rest, in a stationary mass field, the *potential* g_{44} *reduces to a constant.* This theorem results immediately from (38).

In the case of an incoherent mass fluid, we always have, by (23), at any point where **N** is different from zero,

$$\frac{d}{dt}(V^{-1}v^{\alpha}) + V^{-1}\sum_{i}\sum_{k}\begin{Bmatrix} ik \\ \alpha \end{Bmatrix} v^i v^k = 0 \quad (\alpha = 1, 2, 3, 4) \quad (39)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^1\frac{\partial}{\partial x} + v^2\frac{\partial}{\partial y} + v^3\frac{\partial}{\partial z}. \quad (40)$$

The equation of continuity (21) reduces to

$$\sum_i \frac{\partial}{\partial x_i}(\mathbf{N}V^{-1}v^i) = 0, \quad (41)$$

it follows that

$$\frac{d}{dt}\int \mathbf{N}V^{-1}\,\delta x_1\,\delta x_2\,\delta x_3 = 0 \quad (42)$$

where d/dt indicates a total derivative with respect to t, that is, *following the motion of the mass contained in the volume* $\delta x_1 \delta x_2 \delta x_3$, the visualization being made on the Γ-map. Let us remember that the point (x_1, x_2, x_3) has the velocity (v^1, v^2, v^3) on this Γ-map. Let us place[1] with S, in the case of the incoherent fluid,

$$c^2\,\delta m^* = \frac{\mathbf{N}}{V}\,\delta x_1 \delta x_2 \delta x_3. \quad (43)$$

The observer S will then write on the Γ-map that

$$\frac{d}{dt}\int \delta m^* = 0. \quad (44)$$

We multiply both sides of (39) by δm^*. By Equation (44) we have

$$\frac{d}{dt}\left(v^{\alpha}\frac{\delta m^*}{V}\right) + \frac{\delta m^*}{V}\sum_{i}\sum_{j}\begin{Bmatrix} ij \\ \alpha \end{Bmatrix} v^i v^j = 0. \quad (45)$$

The analogy between these equations written by S and those used in the classical dynamics of the mass-point is immediately apparent. The analogy with the first term in (45) is found

[1] T. De Donder, Bull. Acad. Roy. de Belgique, February, 1921, p. 101; also "La gravifique einsteinienne," Equations (185), (186).

in the Galilean expression for the force as a function of mass and acceleration; the analogy with the second term in (45), with opposite sign, is found in the force applied to the mass. If the observer S places

$$V\delta m = c\delta m^* \tag{46}$$

we may write Equations (45) as follows:

$$\frac{d}{dt}(v^\alpha \delta m) + \delta m \sum_i \sum_j \begin{Bmatrix} ij \\ \alpha \end{Bmatrix} v^i v^j = 0. \tag{47}$$

The parenthesis $(v^\alpha \delta m)$ is particularly interesting. Let us immerse a mass-point δm, considered as an exploring particle (i.e. such that its gravific action is completely negligible) in the gravific field defined by Minkowski's form,

$$\delta \bar{s}^2 = -\delta x^2 - \delta y^2 - \delta z^2 + c^2\, \delta t^2. \tag{48}$$

Then from (47) we obtain, placing $\alpha = 4$,

$$\frac{d}{dt}(\delta m) = 0. \tag{49}$$

Therefore, by introducing this result in formula (47) and placing now $\alpha = 1, 2, 3$, we find

$$\frac{dv^1}{dt} = \frac{dv^2}{dt} = \frac{dv^3}{dt} = 0. \tag{50}$$

The last equations define a uniform rectilinear motion. We thus have now a so-called *inertial* field. Let us remember that (Lecture 2) $S = \bar{S}$ in Minkowski's field and that we may set $x = \bar{x}$, $y = \bar{y}$, $z = \bar{z}$, $t = \bar{t}$. When the mass δm or $\delta \bar{m}$ is at rest with respect to $\bar{S}$, we see by (46) that $\delta \bar{m} = \delta m^*$. Thus, if we attach the observer $\bar{S}'$ to δm^* then this observer will obtain $\delta \bar{m}' = \delta m^*$.

Let us consider Equation (34); in the case of an incoherent mass fluid, we have

$$\frac{d}{dt}\left(\frac{\partial V}{\partial v^\alpha}\right) - \left(\frac{\partial V}{\partial x_\alpha}\right) = 0. \tag{51}$$

These relations may be condensed in the formula

$$\delta \int V dt = 0, \tag{52}$$

the variation being taken with respect to x_1, x_2, x_3, x_4 and vanishing at the limits of the integration. The tracks of the incoherent mass fluid are extremals defined by (52).

We have shown elsewhere (Théorie des champs gravifiques, Paris, Mémorial des Sciences mathématiques, 1926, p. 21) how these equations may be extended to the perfect mass fluid.

We shall show how we can pass from the numbers or parameters x_1, x_2, x_3, x_4, u^1, u^2, u^3, u^4, $\mathbf{N}$, written down by the observer S on the Γ-map to the *measurements* obtained by the observer $\bar{S}'$. To each of the mass particles we attach an observer $\bar{S}'$. All of these observers $\bar{S}'$ have taken along with them standards of length, time and mass which are identical when at rest in Minkowski's field. We have seen in Lecture 3 that it is always possible to find a transformation such that the array of the quantities $\bar{g}_{\alpha\beta}'$ reduces to a first approximation to

$$\begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & (\bar{c}')^2 \end{vmatrix} \tag{53}$$

We therefore have, to the same order of approximation, $\bar{g}' = -(\bar{c}')^2$.

Since $\mathbf{N}$ and $\sqrt{-g}$ are density-factors, we have

$$\frac{\mathbf{N}}{\overline{\mathbf{N}}'} = \frac{\sqrt{-g}}{\sqrt{-\bar{g}'}} = \frac{\partial(\bar{x}')}{\partial(x)} \tag{54}$$

where $\dfrac{\partial(\bar{x}')}{\partial(x)}$ denotes the Jacobian of $\bar{x}'$ with respect to x. Thus we have

$$\frac{\mathbf{N}}{\sqrt{-g}} = \frac{\overline{\mathbf{N}}'}{\bar{c}'}. \tag{55}$$

We may notice that from the invariance of the integral form $\mathbf{N}\ \delta x_1\ \delta x_2\ \delta x_3\ \delta x_4$ with respect to any change of the variables x_1, x_2, x_3, x_4 we have

$$\mathbf{N}\ \delta x_1\ \delta x_2\ \delta x_3\ \delta x_4 = \overline{\mathbf{N}}'\ \delta\bar{v}'\ \delta\bar{t}'. \tag{56}$$

In order to find the physical significance of the components of $T_\alpha{}^\beta$ we investigate their meaning for the observer $\bar{S}'$ at rest with respect to the volume $\delta\bar{v}'$. We suppose thus that the velocities $\bar{v}^1, \bar{v}^2, \bar{v}^3$ vanish and that the values of the $\bar{g}_{\alpha\beta}'$'s reduce to (53). We then get for the values of the $\overline{T}'^{\beta}_{\alpha}$'s, from (9)

$$\overline{T}'^{\beta}_{\alpha} = \left\| \begin{matrix} \overline{P}_1'^1 & \overline{P}_1'^2 & \overline{P}_1'^3 & \overline{P}_1'^4 \\ \overline{P}_2'^1 & \overline{P}_2'^2 & \overline{P}_2'^3 & \overline{P}_2'^4 \\ \overline{P}_3'^1 & \overline{P}_3'^2 & \overline{P}_3'^3 & \overline{P}_3'^4 \\ \overline{P}_4'^1 & \overline{P}_4'^2 & \overline{P}_4'^3 & \overline{N}' + \overline{P}_4'^4 \end{matrix} \right\| \tag{57}$$

The dimensions may be chosen in such a way that in this table the $\overline{P}'^{\beta}_{\alpha}$'s ($\alpha = 1, 2, 3$) have the dimensions of the classical elastic tensor of Cauchy. Then we also see that $\overline{P}_1'^4$, $\overline{P}_2'^4$, $\overline{P}_3'^4$ have the dimensions of momentum per unit volume and $\overline{P}_4'^1$, $\overline{P}_4'^2$, $\overline{P}_4'^3$ those of energy-flow per unit area and unit time. Finally $\overline{T}_4'^4$ has the dimensions of energy per unit volume; we place therefore

$$\overline{T}_4'^4 = \frac{\delta\bar{\epsilon}'}{\delta\bar{v}'} \tag{58}$$

$\delta\bar{\epsilon}'$ being the quantity of energy measured by $\overline{S}'$ and contained in the element of volume $\delta\bar{v}'$. Let us define the mass $\delta\overline{m}'$ measured by $\overline{S}'$ by means of

$$\delta\bar{\epsilon}' = (\bar{c}')^2\delta\overline{m}'. \tag{59}$$

The mass density $\overline{D}'$ measured by $\overline{S}'$ and defined by

$$\overline{D}' = \frac{\delta\overline{m}'}{\delta\bar{v}'} \tag{60}$$

will have the value

$$\overline{D}' = \frac{\overline{T}_4'^4}{(\bar{c}')^2} = \frac{\overline{N}' + \overline{P}_4'^4}{(\bar{c}')^2}. \tag{61}$$

It follows that $\overline{D}'$ has the dimensions $L^{-3}M$ where L is the dimension of length and M that of mass.

If the fluid is incoherent, all the P_{α}^{β}'s vanish. Thus all the $\overline{P}'^{\beta}_{\alpha}$'s also vanish on account of their variancy. Hence the array (57) reduces to

$$\overline{T}'^{\beta}_{\alpha} = \left\| \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{N}' \end{matrix} \right\| \tag{62}$$

and formula (61) becomes

$$\overline{D}' = \frac{\overline{N}'}{(\overline{c}')^2}. \tag{63}$$

The invariance of the N used by S gives $\overline{N}' = N$, i.e. $\overline{D}' = N/c^2$. We get in this way the physical meaning of N.

In the case of the perfect mass fluid, the $P_\alpha{}^\beta$'s are given by (28). It follows from the invariance of p that $p = \overline{p}'$, where $\overline{p}'$ denotes the mass pressure measured by $\overline{S}'$. The array (57) now becomes

$$\overline{T}'{}_\alpha{}^\beta = \left\| \begin{matrix} -\overline{p}' & 0 & 0 & 0 \\ 0 & -\overline{p}' & 0 & 0 \\ 0 & 0 & -\overline{p}' & 0 \\ 0 & 0 & 0 & -\overline{p}' + \overline{N}' \end{matrix} \right\| \tag{64}$$

Formula (61) gives

$$\overline{D}' = \frac{\overline{N}' - \overline{p}'}{(\overline{c}')^2}. \tag{65}$$

From the invariance of the N and of the p used by S it follows that

$$\overline{D}' = \frac{N - p}{c^2}. \tag{66}$$

We study with Einstein[1] the mass gravific field, to a first approximation. Therefore, we place

$$g_{\alpha\beta} = -\delta_{\alpha\beta} + \gamma_{\alpha\beta} \tag{67}$$

where $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$, $\delta_{11} = \delta_{22} = \delta_{33} = 1$ and $\delta_{44} = -c^2$. Einstein's hypothesis leads us to admit that the products of $\gamma_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$, $\gamma_{\alpha 4}/c$ $(\alpha = 1, 2, 3)$, γ_{44}/c^2 taken in pairs are negligible with respect to 1. Suppose, moreover, that we have taken new variables x_1, x_2, x_3, x_4, which satisfy, to a first approximation, the four relations

$$\sum_\sigma \sum_\tau g^{\sigma\tau}(g_{\sigma\tau,\alpha\beta} - g_{\alpha\sigma,\tau\beta} - g_{\beta\sigma,\alpha\tau}) = 0. \tag{68}$$

The components $C_{\alpha\beta}$ of the Riemann tensor reduce in this way, except for a factor, to the D'Alembertians,

$$C_{\alpha\beta} = -\tfrac{1}{2} \Box \gamma_{\alpha\beta}. \tag{69}$$

[1] A. Einstein, Sitzungsber. Akad. Berlin, pp. 688–696, 1916.

Let us go back now to the right-hand members of Equations (8). Besides the approximation which has just been introduced, let us assume that the velocities of the masses are so small that v^α/c is negligible compared with 1. We thus obtain,

$$\Box\gamma_{\alpha\beta} = -\frac{2}{b}T_{\alpha\beta}^{*} \tag{70}$$

where we have placed

$$T_{\alpha\beta}^{*} = \delta_{\beta\beta}\left[\delta_{\alpha\alpha}N\frac{v^\alpha}{c}\frac{v^\beta}{c} - P_\alpha^{\ \beta} + \tfrac{1}{2}\,\epsilon_\alpha^{\ \beta}(N + P + a)\right] \tag{71}$$

and where $\epsilon_\alpha^{\ \alpha} = 1$ and $\epsilon_\alpha^{\ \beta} = 0$ if $\beta \neq \alpha$. Integrating by the method of retarded potentials, we obtain

$$\gamma_{\alpha\beta} = \frac{1}{2\,\pi b}\int\frac{||T_{\alpha\beta}^{*}(x, y, z, t - r/c)||}{r}\,dv \tag{72}$$

where r is the distance between the point x, y, z and the element of volume dv. The symbol $||T_{\alpha\beta}^{*}||$ is used to denote that $T_{\alpha\beta}^{*}$ is taken at the time $(t - r/c)$.

In the *case of the incoherent fluid*, the left-hand member of Equations (45), expressing the theorem of the phenomenal tensor, may be written in vectorial form as follows,[1]

$$\frac{d}{dt}\left[\left(1+\frac{\sigma}{c^2}\right)\frac{v}{c}\,\delta m^{*}\right]-\left(\frac{1}{c}\frac{\partial A}{\partial t}+\frac{1}{c}\nabla\sigma+\frac{1}{c}(\nabla\times A)\times v\right)\delta m^{*}=0 \tag{73}$$

where we have placed, with Einstein, $\sigma = -\gamma_{44}/2$ and where A denotes a vector having the components $A_\alpha = \gamma_{\alpha 4}$ $(\alpha = 1, 2, 3)$. The element of mass δm^{*} has been defined before by (43). Equation (73) has been obtained by Einstein[2] in the case of a unit mass. The method followed here allows us to specify the meaning of the element δm^{*}.

[1] We have used here Gibbs' vector notation, where grad $\sigma = \nabla\sigma$, curl $A = \nabla \times A$, $[AB] = A \times B$.

[2] A. Einstein, "The Meaning of Relativity," Methuen, London, 1922; see Equation (116).

LECTURE 7

The Electromagnetic Gravific Field

Definition — The characteristic function — The fundamental equations — The electromagnetic tensor — Maxwell's equations — Electrodynamics — Dynamics of the electron.

Let us consider the case where the gravific field is due to electric charges. For this purpose we introduce the characteristic function (1)

$$\mathbf{M} = -\sum_{\alpha}\sum_{\beta} g^{\alpha\beta}(\mathbf{N}u_\alpha u_\beta - \tfrac{1}{2}\sqrt{-g}\sum_{i}\sum_{j} g^{ij}H_{\alpha i}H_{\beta j}) \tag{1}$$

where $\mathbf{N}$ is the generalized mass due to the electromagnetic field, u_α the covariant velocity of the electric charge and $H_{\alpha i}$ the covariant electromagnetic force. We place $H_{\alpha i} = -H_{i\alpha}$.

The characteristic function (1) enables us to calculate the phenomenal tensor $\mathbf{T}_{\alpha\beta}$ defined by Equation (7), Lecture 5, that is

$$\mathbf{T}_{\alpha\beta} = \mathbf{N}u_\alpha u_\beta + \tfrac{1}{4}\sqrt{-g}\;\; g_{\alpha\beta}\sum_{i}\sum_{j} H^{ij}H_{ij} - \sqrt{-g}\sum_{i} H_\alpha{}^i H_{\beta i} \tag{2}$$

where we have placed

$$H_\alpha{}^\beta = \sum_{i} g^{\beta i}H_{\alpha i}. \tag{3}$$

From (13) and (15), Lecture 5, we deduce immediately

$$T_\alpha{}^\beta = Nu_\alpha u^\beta + \tfrac{1}{4}\,\epsilon_\alpha{}^\beta\sum_{i}\sum_{j} H_{ij}H^{ij} + \sum_{i} H_\alpha{}^i H_i{}^\beta. \tag{4}$$

Explicitly the ten fundamental equations of the gravific theory become here:

$$\frac{a}{2}g_{\alpha\beta} + bC_{\alpha\beta} = N(u_\alpha u_\beta - \tfrac{1}{2}\,g_{\alpha\beta}) + \tfrac{1}{4}\,g_{\alpha\beta}\sum_{i}\sum_{j} H_{ij}H^{ij} - \sum_{i} H_\alpha{}^i H_{\beta i}. \tag{5}$$

The analogy between these Equations (5) and Equations (8), Lecture 6, which determine the gravific mass field, is worth noting. The factor N is now of *electromagnetic* origin, while in (1), Lecture 6, it was of *mass* origin.

The electromagnetic tensor (4) may also be written:

$$T_\alpha^\beta = Nu_\alpha u^\beta + \tfrac{1}{2} \sum_i (H_{\overline{\beta i}} H^{\overline{\alpha i}} - H_{\alpha i} H^{\beta i}) \tag{6}$$

where $H_{\overline{\alpha i}}$ denotes the symbol H provided with two lower indices which, with α and i, form an even permutation $\alpha i \overline{\alpha i}$ of the numbers 1, 2, 3, 4, that is,

$$\left.\begin{matrix} H_{\overline{12}} = H_{34}, & H_{\overline{13}} = -H_{24}, & H_{\overline{14}} = H_{23} \\ H_{\overline{23}} = H_{14}, & H_{\overline{24}} = -H_{13}, & H_{\overline{34}} = H_{12} \end{matrix}\right\} \tag{7}$$

Likewise $H^{\overline{\alpha i}}$ denotes H with two upper indices which form with α and i an even permutation, i.e.,

$$\left.\begin{matrix} H^{\overline{12}} = H^{34}, & H^{\overline{13}} = -H^{24}, & H^{\overline{14}} = H^{23} \\ H^{\overline{23}} = H^{14}, & H^{\overline{24}} = -H^{13}, & H^{\overline{34}} = H^{12} \end{matrix}\right\} \tag{8}$$

Writing the components of the mixed tensor (6) explicitly, we get the following complete table:

$$\left.\begin{matrix}
\left\{\begin{aligned}
T_1^1 &= Nu_1u^1 - \tfrac{1}{2}(H_{12}H^{12} - H_{34}H^{34} + H_{13}H^{13} \\
&\qquad - H_{42}H^{42} + H_{14}H^{14} - H_{23}H^{23}) \\
T_1^2 &= Nu_1u^2 - (H_{13}H^{23} + H_{14}H^{24}) \\
T_1^3 &= Nu_1u^3 - (H_{14}H^{34} + H_{12}H^{32}) \\
T_1^4 &= Nu_1u^4 - (H_{12}H^{42} + H_{13}H^{43})
\end{aligned}\right. \\
\left\{\begin{aligned}
T_2^1 &= Nu_2u^1 - (H_{23}H^{13} + H_{24}H^{14}) \\
T_2^2 &= Nu_2u^2 - \tfrac{1}{2}(H_{23}H^{23} - H_{41}H^{41} \\
&\qquad + H_{24}H^{24} - H_{13}H^{13} + H_{21}H^{21} - H_{34}H^{34}) \\
T_2^3 &= Nu_2u^3 - (H_{24}H^{34} + H_{21}H^{31}) \\
T_2^4 &= Nu_2u^4 - (H_{21}H^{41} + H_{23}H^{43})
\end{aligned}\right. \\
\left\{\begin{aligned}
T_3^1 &= Nu_3u^1 - (H_{32}H^{12} + H_{34}H^{14}) \\
T_3^2 &= Nu_3u^2 - (H_{34}H^{24} + H_{31}H^{21}) \\
T_3^3 &= Nu_3u^3 - \tfrac{1}{2}(H_{34}H^{34} - H_{12}H^{12} + H_{31}H^{31} \\
&\qquad - H_{24}H^{24} + H_{32}H^{32} - H_{41}H^{41}) \\
T_3^4 &= Nu_3u^4 - (H_{31}H^{41} + H_{32}H^{42})
\end{aligned}\right. \\
\left\{\begin{aligned}
T_4^1 &= Nu_4u^1 - (H_{42}H^{12} + H_{43}H^{13}) \\
T_4^2 &= Nu_4u^2 - (H_{43}H^{23} + H_{41}H^{21}) \\
T_4^3 &= Nu_4u^3 - (H_{41}H^{31} + H_{42}H^{32}) \\
T_4^4 &= Nu_4u^4 - \tfrac{1}{2}(H_{41}H^{41} - H_{23}H^{23} + H_{42}H^{42} \\
&\qquad - H_{31}H^{31} + H_{43}H^{43} - H_{12}H^{12})
\end{aligned}\right.
\end{matrix}\right\} \tag{9}$$

The symmetric tensor $\mathbf{T}_{\alpha\beta}$ may also be written

$$\mathbf{T}_{\alpha\beta} = \mathbf{N}u_\alpha u_\beta - \tfrac{1}{2}\sqrt{-g}\sum_i\sum_j g^{ij}\,(\sqrt{-g}H^{\overline{\alpha i}})\,(\sqrt{-g}H^{\overline{\beta j}}) + \sum_i\sum_j H_{\alpha i}H_{\beta j}\sqrt{-g}. \tag{10}$$

Let us go back now to the four Equations (22), Lecture 5, expressing the theorem of the phenomenal tensor. We obtain

$$\mathbf{F}_\alpha = \mathbf{N}A_\alpha + u_\alpha\sum_i\frac{\partial}{\partial x_i}(\mathbf{N}u^i) + \sum_i\sum_j\left[\sqrt{-g}\,H^{\overline{\alpha j}}\frac{\partial H_{\overline{ij}}}{\partial x_i} - H_{\alpha j}\frac{\partial}{\partial x_i}(\sqrt{-g}H^{ij})\right] = 0 \tag{11}$$

where A_α is given by (26), Lecture 4. Let us place in (11)

$$F_\alpha^{(e)} = \sum_i\sum_j\left[\sqrt{-g}H^{\overline{\alpha j}}\frac{\partial H_{\overline{ij}}}{\partial x_i} - H_{\alpha j}\frac{\partial}{\partial x_i}(\sqrt{-g}H^{ij})\right] \tag{12}$$

then the theorem of the electromagnetic tensor becomes

$$\mathbf{F}_\alpha = \mathbf{N}A_\alpha + u_\alpha\sum_i\frac{\partial}{\partial x_i}(\mathbf{N}u^i) + \mathbf{F}_\alpha^{(e)} = 0. \tag{13}$$

Multiply Equations (13) by u^α and sum over α. We obtain by (28), (5) and (6) of Lecture 4,

$$\mathbf{F}_\alpha = \mathbf{N}A_\alpha - u_\alpha\sum_i\mathbf{F}_i^{(e)}u^i + \mathbf{F}_\alpha^{(e)} = 0. \tag{14}$$

We place

$$\sum_\alpha A_\alpha A^\alpha = B. \tag{15}$$

Multiply (14) by A^α and sum over α; we get, by Equation (28), Lecture (4),

$$\mathbf{N} = -\frac{1}{B}\sum_\alpha A^\alpha\mathbf{F}_\alpha^{(e)}. \tag{16}$$

Let us now introduce *Maxwell's electromagnetic equations*

$$\sum_i\frac{\partial\mathbf{H}^{\alpha i}}{\partial x_i} = \sigma u^\alpha \tag{17}$$

and

$$\sum_i\frac{\partial\mathbf{H}_*^{\alpha i}}{\partial x_i} = 0 \tag{18}$$

where we have placed

$$H_{\overline{\alpha i}} = \mathbf{H}_*{}^{\alpha i}. \tag{19}$$

From (18) and (19) it follows immediately that we may write

$$H_{\alpha\beta} = \frac{\partial \phi_\alpha}{\partial x_\beta} - \frac{\partial \phi_\beta}{\partial x_\alpha} \tag{20}$$

where ϕ_α is the electromagnetic potential.

It is important to notice that Maxwell's Equations (17) and (18) may be obtained from the fundamental electromagnetic function,

$$\mathbf{D}^{(e)} = \sum_\alpha [\sigma u^\alpha \phi_\alpha + \tfrac{1}{4}\sqrt{-g} \sum_\beta \sum_i \sum_j g^{\alpha\beta} g^{ij} H_{\alpha i} H_{\beta j}] \tag{21}$$

where σ is the electric density factor and $u^\alpha(\alpha = 1, \cdots, 4)$ are the contravariant components of the velocity of the electric charge, by taking the variational derivatives with respect to ϕ_α as shown in the author's "Théorie des champs gravifiques" (Paris, 1923).

From Equations (17) it follows immediately that

$$\sum_\alpha \frac{\partial}{\partial x_\alpha}(\sigma u^\alpha) = 0 \tag{22}$$

which expresses the *conservation of electric charge* in motion. Let us place

$$\delta\tau^{(e)} = \sigma\, \delta x_1\, \delta x_2\, \delta x_3\, \delta x_4 \tag{23}$$

whence

$$\frac{d}{ds}\int \delta\tau^{(e)} = 0. \tag{24}$$

Let us go back to (12); we have, by Equations (17) and (18)

$$\mathbf{F}_\alpha{}^{(e)} = \sigma \sum_i u^i H_{\alpha i}. \tag{25}$$

It follows, taking into account (20), that

$$\sum_\alpha \mathbf{F}_\alpha{}^{(e)} u^\alpha = 0. \tag{26}$$

Hence Equation (13) becomes

$$\sum_\alpha \frac{\partial}{\partial x_\alpha}(\mathbf{N} u^\alpha) = 0. \tag{27}$$

Let us place

$$\delta\tau^{(m)} = \mathbf{N}\,\delta x_1\,\delta x_2\,\delta x_3\,\delta x_4 \tag{28}$$

we obtain

$$\frac{d}{ds}\int \delta\tau^{(m)} = 0. \tag{29}$$

Equation (29) expresses the *conservation of mass of electromagnetic origin* in motion. From (22) and (27) it follows that

$$\frac{d}{ds}\left(\frac{\mathbf{N}}{\sigma}\right) = 0. \tag{30}$$

Hence the ratio $\mathbf{N}/\sigma$ remains constant during the motion of electric charge. The four Equations (14) of the theorem of the electromagnetic tensor become[1] by (26) and (25)

$$\mathbf{F}_\alpha = \mathbf{N}A_\alpha + \sigma\sum_\beta u^\beta H_{\alpha\beta} = 0. \tag{31}$$

We deduce immediately

$$\mathbf{F}^\alpha = \mathbf{N}A^\alpha - \sigma\sum_\beta u^\beta H_\beta{}^\alpha = 0 \tag{32}$$

and also

$$\sum_\alpha \mathbf{F}_\alpha u^\alpha = 0, \qquad \sum_\alpha \mathbf{F}^\alpha u_\alpha = 0. \tag{33}$$

By (25), relations (16) for $\mathbf{N}$ become

$$\mathbf{N} = -\frac{\sigma}{B}\sum_\alpha\sum_\beta A^\alpha u^\beta H_{\alpha\beta}. \tag{34}$$

Let us replace in (31) $\mathbf{N}$ by its value (34); we get, by (15)

$$\sigma\sum_j u^j \sum_i A^i[-A_\alpha H_{ij} + A_i H_{\alpha j}] = 0 \tag{35}$$

or, if $\sigma \neq 0$,

$$\sum_i\sum_j A^i u^j[-A_\alpha H_{ij} + A_i H_{\alpha j}] = 0. \tag{36}$$

We may not place in (31) *the mass density factor* $\mathbf{N}$ *of electromagnetic origin equal to zero,* since then Equations (31) would reduce for $\sigma \neq 0$, that is, at the electrified points, to a system

[1] T. De Donder, "La gravifique einsteinenne" Equation (350′).

of four linear homogeneous equations in u^1, u^2, u^3, u^4. The determinant of $H_{\alpha\beta}$ being in general different from zero, these equations admit only the solution $u^\alpha = 0$, which is absurd, owing to condition (6), Lecture 4.

The coefficients of σ in Equations (31) may be written in the Lagrangian form. In fact we have

$$\sum_\beta u^\beta H_{\alpha\beta} = \sum_\beta u^\beta(\phi_{\alpha,\beta} - \phi_{\beta,\alpha}) = \frac{d}{ds}\left(\frac{\partial U}{\partial u^\alpha}\right) - \left(\frac{\partial U}{\partial x_\alpha}\right) \tag{37}$$

where we have placed

$$U = \sum_\alpha u^\alpha \phi_\alpha. \tag{38}$$

We notice that the variables x_α, u^α are to be treated here as independent variables.

The theorem (31) of the electromagnetic tensor may also be written, by (37) of this Lecture and (29) of Lecture 4,

$$\mathbf{F}_\alpha = \mathbf{N}\left[\frac{d}{ds}\left(\frac{\partial W}{\partial u^\alpha}\right) - \left(\frac{\partial W}{\partial x_\alpha}\right)\right] + \sigma\left[\frac{d}{ds}\left(\frac{\partial U}{\partial u^\alpha}\right) - \left(\frac{\partial U}{\partial x_\alpha}\right)\right] = 0. \tag{39}$$

Let us multiply both sides of this equation by $\delta x_1\, \delta x_2\, \delta x_3\, \delta x_4$. We obtain by (24) and (29)

$$\int \delta\tau^{(m)}\left[\frac{d}{ds}\left(\frac{\partial W}{\partial u^\alpha}\right) - \left(\frac{\partial W}{\partial x_\alpha}\right)\right] + \int \delta\tau^{(e)}\left[\frac{d}{ds}\left(\frac{\partial U}{\partial u^\alpha}\right) - \left(\frac{\partial U}{\partial x_\alpha}\right)\right] = 0. \tag{40}$$

By definition the supplementary equation of Maxwell is

$$\psi = \frac{1}{\sqrt{-g}} \sum_\alpha \frac{\partial}{\partial x_\alpha}(\sqrt{-g}\, \phi^\alpha) = 0 \tag{41}$$

where we have placed $\phi^\alpha = \sum_\beta g^{\alpha\beta}\phi_\beta$. Maxwell's electromagnetic equations (17) may be simplified by using Equation (41) in such a way that each of them contains only the second derivatives of one single electromagnetic potential ϕ_α. After a few calculations, we get

$$\frac{\sigma u^\alpha}{\sqrt{-g}} = K_\alpha + \sum_i \sum_j g^{ij} \frac{\partial^2 \phi_\alpha}{\partial x_i\, \partial x_j} \tag{42}$$

where K_α does not include second derivatives of ϕ_1, ϕ_2, ϕ_3, ϕ_4.

We may also write all these electromagnetic equations in space and time. Let us recall Lecture 4 and place

$$\rho = \frac{\sigma}{V}. \tag{43}$$

The Maxwellian Equations (17) and (18) become

$$\left.\begin{array}{ll}\dfrac{\partial \mathbf{H}^{12}}{\partial x_2} - \dfrac{\partial \mathbf{H}^{31}}{\partial x_3} = \dfrac{\partial \mathbf{H}^{41}}{\partial t} + \rho v^1, & \dfrac{\partial \mathbf{H}^{23}}{\partial x_3} - \dfrac{\partial \mathbf{H}^{12}}{\partial x_1} = \dfrac{\partial \mathbf{H}^{42}}{\partial t} + \rho v^2 \\ \dfrac{\partial \mathbf{H}^{31}}{\partial x_1} - \dfrac{\partial \mathbf{H}^{23}}{\partial x_2} = \dfrac{\partial \mathbf{H}^{43}}{\partial t} + \rho v^3, & \dfrac{\partial \mathbf{H}^{41}}{\partial x_1} + \dfrac{\partial \mathbf{H}^{42}}{\partial x_2} + \dfrac{\partial \mathbf{H}^{43}}{\partial x_3} = \rho\end{array}\right\} \tag{44}$$

$$\left.\begin{array}{ll}\dfrac{\partial \mathbf{H}_*{}^{12}}{\partial x_2} - \dfrac{\partial \mathbf{H}_*{}^{31}}{\partial x_3} = \dfrac{\partial \mathbf{H}_*{}^{41}}{\partial t}, & \dfrac{\partial \mathbf{H}_*{}^{23}}{\partial x_3} - \dfrac{\partial \mathbf{H}_*{}^{12}}{\partial x_1} = \dfrac{\partial \mathbf{H}_*{}^{42}}{\partial t} \\ \dfrac{\partial \mathbf{H}_*{}^{31}}{\partial x_1} - \dfrac{\partial \mathbf{H}_*{}^{23}}{\partial x_2} = \dfrac{\partial \mathbf{H}_*{}^{43}}{\partial t}, & \dfrac{\partial \mathbf{H}_*{}^{41}}{\partial x_1} + \dfrac{\partial \mathbf{H}_*{}^{42}}{\partial x_2} + \dfrac{\partial \mathbf{H}_*{}^{43}}{\partial x_3} = 0\end{array}\right\} \tag{45}$$

Equation (22) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1}(\rho v^1) + \frac{\partial}{\partial x_2}(\rho v^2) + \frac{\partial}{\partial x_3}(\rho v^3) = 0 \tag{46}$$

which is equivalent to

$$\frac{d}{dt}\int \rho \, \delta x_1 \, \delta x_2 \, \delta x_3 = 0 \tag{47}$$

d/dt denoting the *total* derivative, and the integration being extended to a space region on the Γ-map. Let us place

$$\delta e^* = \rho \, \delta x_1 \, \delta x_2 \, \delta x_3. \tag{48}$$

Then the integral invariant (47) becomes

$$\frac{d}{dt}\int \delta e^* = 0. \tag{49}$$

The *electromagnetic force* (25) becomes in space and time

$$\mathbf{F}_\alpha{}^{(e)} = \rho(H_{\alpha 4} + \sum_{i=1}^{3} v^i H_{\alpha i}) \tag{50}$$

or explicitly

$$\left.\begin{array}{l}\mathbf{F}_1{}^{(e)} = \rho(v^2 H_{12} - v^3 H_{31}) - \rho H_{41} \\ \mathbf{F}_2{}^{(e)} = \rho(v^3 H_{23} - v^1 H_{12}) - \rho H_{42} \\ \mathbf{F}_3{}^{(e)} = \rho(v^1 H_{31} - v^2 H_{23}) - \rho H_{43} \\ \mathbf{F}_4{}^{(e)} = \rho(v^1 H_{41} + v^2 H_{42} + v^3 H_{43})\end{array}\right\} \tag{51}$$

The density factor **N** satisfies Equation (27). Thus, we have

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{N}}{V}\right)+\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{\mathbf{N}v^i}{V}\right)=0. \tag{52}$$

This equation is equivalent to

$$\frac{d}{dt}\int\left(\frac{\mathbf{N}}{V}\right)\delta x_1\,\delta x_2\,\delta x_3=0 \tag{53}$$

where d/dt denotes a total derivative and the integral is taken over a space region; we *follow* the motion of electric charge when we take the total derivative with respect to t. Let us place

$$c^2\,\delta m^*=\left(\frac{\mathbf{N}}{V}\right)\delta x_1\,\delta x_2\,\delta x_3, \tag{54}$$

the observer S will write on the Γ-map that

$$\frac{d}{dt}\int\delta m^*=0. \tag{55}$$

Let us go back to the theorem of the mixed tensor (32). We have

$$\mathbf{F}^\alpha=\frac{\mathbf{N}}{V}\left[\frac{d}{dt}\left(\frac{v^\alpha}{V}\right)+\frac{1}{V}\sum_{\beta}\sum_{\gamma}\begin{Bmatrix}\beta\gamma\\ \alpha\end{Bmatrix}v^\beta v^\gamma\right]-\rho\sum_{\beta}v^\beta H_\beta{}^\alpha=0. \tag{56}$$

Let us multiply both sides of (56) by $\delta x_1\,\delta x_2\,\delta x_3$. The last equation becomes, by (48), (54) and (55),

$$c^2\left[\frac{d}{dt}\left(\frac{v^\alpha}{V}\,\delta m^*\right)+\frac{\delta m^*}{V}\sum_{\beta}\sum_{\gamma}\begin{Bmatrix}\beta\gamma\\ \alpha\end{Bmatrix}v^\beta v^\gamma\right]-\delta e^*\sum_{\beta}v^\beta H_\beta{}^\alpha=0. \tag{57}$$

We may notice that the parenthesis entering in (57) is identical with the first member of (45), Lecture 6. If we put, with the observer S,

$$V\,\delta m=c\,\delta m^* \tag{58}$$

we may write (57) in the form

$$c\left[\frac{d}{dt}(v^\alpha\delta m)+\delta m\sum_{\beta}\sum_{\gamma}\begin{Bmatrix}\beta\gamma\\ \alpha\end{Bmatrix}v^\beta v^\gamma\right]-\delta e^*\sum_{\beta}v^\beta H_\beta{}^\alpha=0. \tag{59}$$

Let us place, by analogy with (38),

$$U^*=\sum_{\alpha}v^\alpha\phi_\alpha\qquad(\alpha=1,2,3,4). \tag{60}$$

We consider x_1, x_2, x_3, x_4 and v^1, v^2, v^3, v^4 as being two sets of independent variables; we have

$$\frac{d}{dt}\left(\frac{\partial U^*}{\partial v^\alpha}\right) - \left(\frac{\partial U^*}{\partial x_\alpha}\right) = \sum_\beta v^\beta(\phi_{\alpha,\beta} - \phi_{\beta,\alpha}) = V\sum_\beta u^\beta H_{\alpha\beta}. \quad (61)$$

Substituting in (31), the theorem of the electromagnetic tensor becomes

$$\left(\frac{N}{V}\right)\left[\frac{d}{dt}\left(\frac{\partial V}{\partial v^\alpha}\right) - \left(\frac{\partial V}{\partial x_\alpha}\right)\right] + \rho\left[\frac{d}{dt}\left(\frac{\partial U^*}{\partial v^\alpha}\right) - \left(\frac{\partial U^*}{\partial x_\alpha}\right)\right] = 0. \quad (62)$$

These relations have to be compared with (39) and also with Equation (51), Lecture 6, relating to the incoherent mass fluid. Multiply both sides of (62) by $\delta x_1\ \delta x_2\ \delta x_3$ and use (49) and (55); we obtain

$$\int c^2\delta m^*\left[\frac{d}{dt}\left(\frac{\partial V}{\partial v^\alpha}\right) - \left(\frac{\partial V}{\partial x_\alpha}\right)\right] + \int \delta e^*\left[\frac{d}{dt}\left(\frac{\partial U^*}{\partial v^\alpha}\right) - \left(\frac{\partial U^*}{\partial x_\alpha}\right)\right] = 0. \quad (63)$$

Let us introduce the canonical variables $p_\alpha^{(e)}$ and $p_\alpha^{(m)}$ by placing

$$p_\alpha^{(m)} = \left(\frac{\partial W}{\partial u^\alpha}\right), \quad p_\alpha^{(e)} = \left(\frac{\partial U^*}{\partial x_\alpha}\right) \qquad (\alpha = 1, 2, 3, 4). \quad (64)$$

Moreover, introduce the functions

$$\left.\begin{aligned} H^{(m)} &= -W + \sum_\alpha p_\alpha^{(m)} u^\alpha \\ H^{(e)} &= -U^* + \sum_\alpha p_\alpha^{(e)} u^\alpha \end{aligned}\right\} \quad (65)$$

where $H^{(m)}$ is a function of x_α and $p_\alpha^{(m)}$, and $H^{(e)}$ a function of x_α and $p_\alpha^{(e)}$. Equations (39) may be written in the canonical form (66)

$$\int\left(\frac{dp_\alpha^{(m)}}{ds} + \frac{\partial H^{(m)}}{\partial x_\alpha}\right)\delta\tau^{(m)} + \int\left(\frac{dp_\alpha^{(e)}}{ds} + \frac{\partial H^{(e)}}{\partial x_\alpha}\right)d\tau^{(e)} = 0. \quad (66)$$

We shall have by (64)

$$p_\alpha^{(m)} = u_\alpha, \quad p_\alpha^{(e)} = \phi_\alpha. \quad (67)$$

Substituting these values in (65) we see that we have the conditions

$$H^{(m)} = H^{(e)} = 0. \quad (68)$$

To study the dynamics of the electron in space-time, we introduce, by (24) and (29), the two constants $\tau^{(m)}$ and $\tau^{(e)}$ which characterize the electron from the points of view of mass and charge. Then Equation (40) becomes

$$\frac{d}{ds}\left(\frac{\partial L}{\partial u^\alpha}\right) - \left(\frac{\partial L}{\partial x_\alpha}\right) = 0 \tag{69}$$

where we have introduced the Lagrangian function,

$$L = W\tau^{(m)} + U\tau^{(e)}. \tag{70}$$

Using canonical variables and introducing the Hamiltonian function

$$H = -L + \sum_\alpha p_\alpha u^\alpha \quad (\alpha = 1, 2, 3, 4) \tag{71}$$

where

$$p_\alpha = \frac{\partial L}{\partial u^\alpha}, \tag{72}$$

we obtain the canonical equations of electronic dynamics

$$\frac{dx_\alpha}{ds} = \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{ds} = -\frac{\partial H}{\partial x_\alpha}, \quad (\alpha = 1, 2, 3, 4) \tag{73}$$

with the condition

$$H = 0. \tag{74}$$

This condition is equivalent to

$$W = 1. \tag{75}$$

Let us note that

$$p_\alpha = u_\alpha \tau^{(m)} + \phi_\alpha \tau^{(e)}; \tag{76}$$

thus (75) may be written in terms of canonical variables

$$\sum_{\alpha}\sum_{\beta} g^{\alpha\beta}(p_\alpha - \phi_\alpha\tau^{(e)})\,(p_\beta - \phi_\beta\tau^{(e)}) = (\tau^{(m)})^2. \tag{77}$$

From this follows Jacobi's equation which governs electronic dynamics

$$\sum_{\alpha}\sum_{\beta} g^{\alpha\beta}\left(\frac{\partial S}{\partial x_\alpha} - \phi_\alpha\tau^{(e)}\right)\left(\frac{\partial S}{\partial x_\beta} - \phi_\beta\tau^{(e)}\right) = (\tau^{(m)})^2. \tag{78}$$

It is known that if a complete integral $S(x_1, x_2, x_3, x_4; a_1, a_2, a_3, a_4)$ of this partial differential equation is found, then we may deduce immediately, through Jacobi's theorem, the general integral of the differential equations (73).

We shall now take up this problem in space *and* time. For this purpose we introduce the canonical variables

$$p_i^{(m)} = \left(\frac{\partial V}{\partial v^i}\right), \quad p_i^{(e)} = \left(\frac{\partial U^*}{\partial v^i}\right) \qquad (i = 1, 2, 3), \tag{79}$$

and introduce the Hamiltonian functions

$$\left.\begin{aligned} H_*^{(m)} &= -V + \sum_i p_i^{(m)} v^i \\ H_*^{(e)} &= -U^* + \sum_i p_i^{(e)} v^i \end{aligned}\right\} \qquad (i = 1, 2, 3). \tag{80}$$

Equation (63) furnishes the following three equations:

$$\int c^2\, \delta m^* \left[\frac{dp_i^{(m)}}{dt} + \frac{\partial H_*^{(m)}}{\partial x_i}\right] + \int \delta e^* \left[\frac{dp_i^{(e)}}{dt} + \frac{\partial H_*^{(e)}}{\partial x_i}\right] = 0. \tag{81}$$

We get immediately

$$\left.\begin{aligned} p_i^{(m)} &= \frac{1}{V} \sum_{\alpha=1}^{4} g_{\alpha i} v^\alpha \\ p_i^{(e)} &= \phi_i \end{aligned}\right\} \qquad (i = 1, 2, 3); \tag{82}$$

substituting these values in (80) we have, after a few reductions,

$$\left.\begin{aligned} H_*^{(m)} &= \frac{1}{V} \sum_{\alpha=1}^{4} g_{\alpha i} v^\alpha \\ H_*^{(e)} &= -\phi_4 \end{aligned}\right\} \tag{83}$$

We recall that $v^4 = 1$ and that we still have to calculate from (82) v^1, v^2, v^3 as functions of $p_1^{(m)}, p_2^{(m)}, p_3^{(m)}$ in order to substitute them in the right-hand member of the first Equation (83).

Let us extend the integrals appearing in (63) to an electron and replace δm^* and δe^* respectively by the *constants* m^* and e^* which characterize on the Γ-map the electron considered. We may then write (63) in the form

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial v^i}\right) - \left(\frac{\partial L^*}{\partial x_i}\right) = 0 \qquad (i = 1, 2, 3) \tag{84}$$

where we have introduced the Lagrangian function

$$L^* = c^2 m^* V + e^* U^*. \tag{85}$$

Let us now go over to canonical variables by introducing the Hamiltonian function

$$H^* = -L^* + \sum_{i=1}^{3} p_i v^i \tag{86}$$

where

$$p_i = \frac{\partial L^*}{\partial v^i} \qquad (i = 1, 2, 3). \tag{87}$$

We obtain immediately the canonical equations

$$\frac{dp_i}{dt} = -\frac{\partial H^*}{\partial x_i}, \qquad \frac{dx_i}{dt} = \frac{\partial H^*}{\partial p_i} \tag{88}$$

where

$$H^* = -c^2 m^* V^{-1} \sum_{\alpha=1}^{4} g_{\alpha 4} v^\alpha - e^* \phi_4. \tag{89}$$

By (82) and (89) we find that

$$H^* = -p_4 \tag{90}$$

where

$$p_4 = \frac{c^2 m^*}{V} \sum_{\alpha=1}^{4} g_{\alpha 4} v^\alpha + e^* \phi_4. \tag{91}$$

We have now, by (82), (87) and (91)

$$p_\alpha = \frac{c^2 m^*}{V} \sum_{\beta=1}^{4} g_{\alpha\beta} v^\beta + e^* \phi_\alpha \qquad (\alpha = 1, \cdots, 4) \tag{92}$$

from which we get immediately the v^α's as functions of p_1, p_2, p_3, p_4. Substituting in Equation (8) of Lecture (4), we obtain

$$\sum_\alpha \sum_\beta g^{\alpha\beta}(p_\alpha - e^*\phi_\alpha)(p_\beta - e^*\phi_\beta) = (c^2 m^*)^2. \tag{93}$$

From this equation we obtain p_4. It is easy to see that the radical has to be taken with a plus sign, if V is taken positive. By (90) the value of p_4 with opposite sign is nothing but the Hamiltonian function H^*. From this remark, it follows immediately that the Jacobi equation corresponding to the differential equations (88) may be obtained by replacing p_α by $\frac{\partial S}{\partial x_\alpha}$ ($\alpha =$ 1, 2, 3, 4) in Equation (93). Hence the Jacobi equation

$$\sum_\alpha \sum_\beta \left(\frac{\partial S}{\partial x_\alpha} - e^* \phi_\alpha\right)\left(\frac{\partial S}{\partial x_\beta} - e^* \phi_\beta\right) = (c^2 m^*)^2. \tag{94}$$

This equation may be written in several forms which are interesting from the standpoint of the applications to particular problems.[1]

[1] T. De Donder, Association française pour l'avancement des sciences. Congrès de Juillet. 1925.

LECTURE 8

The General Electromagnetic Mass Gravific Field

Definition — The characteristic function — The fundamental equations — The electromagnetic mass tensor — Generalized Maxwell equations — Case of perfect matter — Generalized electromagnetic potentials — Electrostriction.

Let us consider the general case where the gravific field is produced by anisotropic inhomogeneous bodies, electrically and magnetically polarizable. These bodies may be electrically and magnetically charged and be the seat of conduction currents. The gravific fields to be studied in this lecture include, as particular cases, those studied in the last two preceding lectures.

Let us introduce with Einstein[1] the force of electric polarization defined by six contravariant components,

$$P_{(e)}{}^{\alpha\beta} = P_{(e)}{}^{\alpha}u^{\beta} - P_{(e)}{}^{\beta}u^{\alpha}, \tag{1}$$

where $P_{(e)}{}^{\alpha}$ are the four contravariant components of the electric polarization. We recall that the u^{α}'s are the four contravariant components of the velocity of the masses.

In the same way we introduce the force of magnetic polarization,

$$P_{(\mu)}{}^{\alpha\beta} = P_{(\mu)}{}^{\alpha}u^{\beta} - P_{(\mu)}{}^{\beta}u^{\alpha}, \tag{2}$$

where the $P_{(\mu)}{}^{\alpha}$'s are the four contravariant components of the magnetic polarization.

We introduce also the electromagnetic force defined by six contravariant components,

$$K^{\alpha\beta} = H^{\alpha\beta} - P_{(e)}{}^{\alpha\beta} \tag{3}$$

and the adjoint electromagnetic force[2]

$$K_{*}{}^{\alpha\beta} = H_{*}{}^{\alpha\beta} + P_{(\mu)}{}^{\alpha\beta} \tag{4}$$

[1] A. Einstein, Sitzungsber. d. preuss. Akad. d. Wiss., p. 164, 1914.

[2] T. De Donder, Bull. Acad. Roy. de Belgique, April, 1924.

where we again recall that the meaning of the symbol $H_*^{\alpha\beta}$ is given in Equation (19), Lecture 7; that is, $H_{\overline{\alpha\beta}} = \mathbf{H}_*^{\alpha\beta} = H_*^{\alpha\beta}\sqrt{-g}$.

For the characteristic function of the field we are studying we place here

$$\mathbf{M} = \sum_{\alpha}\sum_{\beta} g^{\alpha\beta}[(-\mathbf{N}u_\alpha u_\beta - \mathbf{P}_{\alpha\beta}) + \tfrac{1}{2}\sqrt{-g}\sum_{i}\sum_{j} g^{ij}K_{\alpha i}\mathbf{K}_*^{\overline{\beta j}}] \quad (5)$$

where $\mathbf{K}_*^{\overline{\beta j}} = K_*^{\beta j}\sqrt{-g} = H_{\beta j} + \mathbf{P}_{(\mu)}^{\overline{\beta j}}$ and $\mathbf{N}$ is the mass density. The mass tensor is defined by $\mathbf{P}_{\alpha\beta}$. All these quantities as well as $K_{\alpha\beta}$ are to be considered as functions of x_1, x_2, x_3, x_4; that is, their variations with respect to $g^{\alpha\beta}$ are zero.

Let us now calculate the symmetric tensor $T_{\alpha\beta}$ defined by Equation (7), Lecture 5. To do this we must first write $\mathbf{M}$ in (5) in a symmetrical form with respect to α and β. For this purpose we permute α and β in (5) and take the half-sum. Thus we obtain finally

$$\begin{aligned} T_{\alpha\beta} = Nu_\alpha u_\beta + P_{\alpha\beta} - \tfrac{1}{2}\sum_{i}\sum_{j} g^{ij}(K_{\alpha i}\mathbf{K}_*^{\overline{\beta j}} + K_{\beta i}\mathbf{K}_*^{\overline{\alpha j}}) \\ + \tfrac{1}{4}\, g_{\alpha\beta}\sum_{i}\sum_{j}\sum_{k}\sum_{l} g^{kl}g^{ij}K_{ki}\mathbf{K}_*^{\overline{lj}}. \end{aligned} \quad (6)$$

From this we derive the mixed tensor

$$\begin{aligned} T_\alpha^{\ \beta} = Nu_\alpha u^\beta + P_\alpha^{\ \beta} - \tfrac{1}{2}\sum_{i}\sum_{j}\sum_{\nu} g^{ij}g^{\nu\beta}(K_{\alpha i}\mathbf{K}_*^{\overline{\nu j}} + K_{\nu i}\mathbf{K}_*^{\overline{\alpha j}}) \\ + \tfrac{1}{4}\,\epsilon_\alpha^{\ \beta}\sum_{i}\sum_{j}\sum_{k}\sum_{l} g^{kl}g^{ij}K_{ki}\mathbf{K}_*^{\overline{lj}} \end{aligned} \quad (7)$$

and obtain

$$T = \sum_{\alpha} T_\alpha^{\ \alpha} = N + P. \quad (8)$$

We may now write the fundamental equations of the electromagnetic mass field,

$$\frac{a}{2}g_{\alpha\beta} + bC_{\alpha\beta} = T_{\alpha\beta} - \tfrac{1}{2}\, g_{\alpha\beta}(N + P). \quad (9)$$

We shall study now the theorem of the electromagnetic mass tensor by using (22) of Lecture 5. We have

$$\mathbf{F}_\alpha = \mathbf{N}_\alpha + \mathbf{P}_\alpha + \mathbf{F}_\alpha^{(e)} = 0 \quad (10)$$

where

$$\left.\begin{aligned} \mathbf{N}_\alpha &= \mathbf{N}A_\alpha + u_\alpha \sum_i \frac{\partial}{\partial x_i}(\mathbf{N}u^i) \\ \mathbf{P}_\alpha &= \sum_i \left[\frac{\partial \mathbf{P}_\alpha{}^i}{\partial x_i} - \tfrac{1}{2}\sum_j\sum_k g^{ij} g_{kj,\alpha} \mathbf{P}_i{}^k\right] \end{aligned}\right\} \tag{11}$$

and

$$\mathbf{F}_\alpha{}^{(e)} = \sum_i \left[\frac{\partial}{\partial x_i}\mathbf{T}_\alpha{}^{(e)i} - \tfrac{1}{2}\sum_j\sum_k g^{ij} g_{kj,\alpha} \mathbf{T}_i{}^{(e)k}\right]. \tag{12}$$

Or, after some reductions,

$$\begin{aligned} \mathbf{F}_\alpha{}^{(e)} \doteq {} & -\tfrac{1}{2}\sum_i \frac{\partial}{\partial x_i}[\sqrt{-g}\sum_\epsilon\sum_\nu\sum_\tau g^{\epsilon\nu} g^{\tau i}(K_{\alpha\epsilon}\mathbf{K}_*{}^{\overline{\tau\nu}} + K_{\tau\epsilon}\mathbf{K}_*{}^{\overline{\alpha\nu}})] \\ & + \tfrac{1}{4}\sqrt{-g}\sum_k\sum_l\sum_\epsilon\sum_\nu g^{kl} g^{\epsilon\nu} \frac{\partial}{\partial x_\alpha}(K_{k\epsilon}\mathbf{K}_*{}^{\overline{l\nu}}). \end{aligned} \tag{13}$$

Multiplying both members of (10) by u^α and summing, we obtain

$$\sum_\alpha \mathbf{F}_\alpha u^\alpha = \sum_\alpha \frac{\partial}{\partial x_\alpha}(\mathbf{N}u^\alpha) + \sum_\alpha \mathbf{P}_\alpha u^\alpha + \sum_\alpha \mathbf{F}_\alpha{}^{(e)} u^\alpha = 0. \tag{14}$$

This relation expresses the *conservation of energy*. It may also be regarded as a generalization of the equation of continuity.

Combining Equations (14) and (10) we may also write the theorem of the electromagnetic mass tensor as follows,

$$\mathbf{F}_\alpha = \mathbf{N}A_\alpha - u_\alpha \sum_i \mathbf{P}_i u^i - u_\alpha \sum_i \mathbf{F}_i{}^{(e)} u^i + \mathbf{P}_\alpha + \mathbf{F}_\alpha{}^{(e)} = 0. \tag{15}$$

Let us multiply this Equation by A^α and sum over α. We obtain, by (15) of Lecture 7 and (28) of Lecture 4,

$$N = -\frac{1}{B}\sum_\alpha A^\alpha(\mathbf{P}_\alpha + \mathbf{F}_\alpha{}^{(e)}). \tag{16}$$

By generalizing the equations given by Einstein in his paper just quoted, we write the electromagnetic equations as follows,

$$\sum_i \frac{\partial \mathbf{K}^{\alpha i}}{\partial x_i} = \sigma_{(e)} u^\alpha + \mathbf{L}_{(e)}{}^\alpha \tag{17}$$

and

$$\sum_i \frac{\partial \mathbf{K}_*^{\alpha i}}{\partial x_i} = \sigma_{(\mu)} u^\alpha + \mathbf{L}_{(\mu)}{}^\alpha \tag{18}$$

where the $\mathbf{L}_{(e)}{}^\alpha$'s are the contravariant tensor components of the electric conduction current, and $\sigma_{(e)} u^\alpha$ the components of the electric convection current. Likewise, $\mathbf{L}_{(\mu)}{}^\alpha$ and $\sigma_{(\mu)} u^\alpha$ are the magnetic conduction current and the magnetic convection current, respectively. Equations (17) and (18) give immediately,

$$\left.\begin{aligned} \sum_i \frac{\partial}{\partial x_i} (\sigma_{(e)} u^i + \mathbf{L}_{(e)}{}^i) &= 0 \\ \sum_i \frac{\partial}{\partial x_i} (\sigma_{(\mu)} u^i + \mathbf{L}_{(\mu)}{}^i) &= 0 \end{aligned}\right\} \tag{19}$$

expressing, respectively, the conservation of electric and of magnetic charge. Expression (13) for the force $\mathbf{F}_\alpha{}^{(e)}$ can be simplified by using (17) and (18).

Perfect matter[1] will be defined by the following relations,

$$\left.\begin{aligned} \mathbf{L}_{(e)}{}^\alpha &= -\sum_i\sum_j q_{(e)i}{}^\alpha \mathbf{H}^{ij} u_j, \quad \mathbf{L}_{(\mu)}{}^\alpha = \sum_i\sum_j q_{(\mu)i}{}^\alpha \mathbf{H}_*{}^{ij} u_j \\ \mathbf{P}_{(e)}{}^\alpha &= -\sum_i\sum_j p_{(e)i}{}^\alpha \mathbf{H}^{ij} u_j, \quad \mathbf{P}_{(\mu)}{}^\alpha = \sum_i\sum_j p_{(\mu)i}{}^\alpha \mathbf{H}_*{}^{ij} u_j \end{aligned}\right\} \tag{20}$$

where $q_{(e)i}{}^\alpha$ is a mixed tensor obtained by generalizing the electric conductivity (Ohm's law), $p_{(e)i}{}^\alpha$ the electric susceptibility (Poisson's law), $p_{(\mu)i}{}^\alpha$ the magnetic susceptibility, and $q_{(\mu)i}{}^\alpha$ the magnetic conductivity.

If we have

$$\sigma_{(\mu)} u^\alpha + \mathbf{L}_{(\mu)}{}^\alpha = 0 \tag{21}$$

then we may write, by (18),

$$\mathbf{K}_*^{\alpha i} = \phi_{\overline{\alpha, i}} - \phi_{\overline{i, \alpha}} \tag{22}$$

where the ϕ_α's are the components of the electromagnetic potential.

Let us now consider an example. We suppose that all the masses are *at rest* with respect to the observer S. It follows that at every point where mass is present we shall have $u^1 = u^2$

[1] T. De Donder, Comptes-Rendus, July 9, 1923; also "Théorie des champs gravifiques," 1926.

$= u^3 = 0$ and $u^4 = (g_{44})^{-\frac{1}{2}}$. From this it follows that

$$\left.\begin{aligned} p_{(e)}{}^{ij} &= 0 \qquad (i, j = 1, 2, 3) \\ p_{(e)}{}^{4} &= P_{(e)}{}^{i}u^4 \qquad (i = 1, 2, 3) \end{aligned}\right\} \tag{23}$$

and also

$$\left.\begin{aligned} p_{(\mu)}{}^{ij} &= 0 \qquad (i, j = 1, 2, 3) \\ p_{(\mu)i}{}^{4} &= P_{(\mu)}{}^{i}u^4 \qquad (i = 1, 2, 3) \end{aligned}\right\} \tag{24}$$

Denoting by H_x, H_y, H_z the components of the electric force, by B_x, B_y, B_z the electric induction, by $\mathbf{H}_x$, $\mathbf{H}_y$, $\mathbf{H}_z$ the magnetic force, by $\mathbf{B}_x$, $\mathbf{B}_y$, $\mathbf{B}_z$ the magnetic induction and by $H_x{}^a$, $H_y{}^a$, $H_z{}^a$ the so-called applied electric force, Equations (17) should reduce to the classical Maxwell electromagnetic equations for bodies at rest.[1] Thus we obtain the following expressions,

$$\left.\begin{aligned} &\mathbf{K}^{12} = c\mathbf{H}_z, \quad \mathbf{K}_*{}^{12} = c(H_z - H_z{}^a) \\ &\mathbf{K}^{13} = -c\mathbf{H}_y, \quad \mathbf{K}_*{}^{13} = -c(H_y - H_y{}^a) \\ &\mathbf{K}^{23} = c\mathbf{H}_x, \quad \mathbf{K}_*{}^{23} = c(H_x - H_x{}^a) \\ &\mathbf{K}^{14} = -B_x, \quad \mathbf{K}_*{}^{14} = \mathbf{B}_x \\ &\mathbf{K}^{24} = -B_y, \quad \mathbf{K}_*{}^{24} = \mathbf{B}_y \\ &\mathbf{K}^{34} = -B_z, \quad \mathbf{K}_*{}^{34} = \mathbf{B}_z \end{aligned}\right\} \tag{25}$$

Likewise we obtain the expressions

$$\left.\begin{aligned} &\mathbf{H}^{ij} = \mathbf{K}^{ij} \quad (i, j = 1, 2, 3) \quad \mathbf{H}_*{}^{ij} = \mathbf{K}_*{}^{ij} \\ &\mathbf{H}^{14} = -H_x \qquad \mathbf{H}_*{}^{14} = \mathbf{H}_x \\ &\mathbf{H}^{24} = -H_y \qquad \mathbf{H}_*{}^{24} = \mathbf{H}_y \\ &\mathbf{H}^{34} = -H_z \qquad \mathbf{H}_*{}^{34} = \mathbf{H}_z. \end{aligned}\right\} \tag{26}$$

To obtain the classical formulas

$$\left.\begin{aligned} B &= H + P \\ \mathbf{B} &= \mathbf{H} + \mathbf{P} \end{aligned}\right\} \tag{27}$$

where P is the electric polarization and $\mathbf{P}$ the magnetic polarization, we have to place

$$\left.\begin{aligned} &\mathbf{P}_{(e)}{}^{1}u^4 = P_x \qquad \mathbf{P}_{(\mu)}{}^{1}u^4 = \mathbf{P}_x \\ &\mathbf{P}_{(e)}{}^{2}u^4 = P_y \qquad \mathbf{P}_{(\mu)}{}^{2}u^4 = \mathbf{P}_y \\ &\mathbf{P}_{(e)}{}^{3}u^4 = P_z \qquad \mathbf{P}_{(\mu)}{}^{3}u^4 = \mathbf{P}_z \end{aligned}\right\} \tag{28}$$

[1] See, for example, T. De Donder, "Théorie mathématique de l'électricité," Eqs. 578, 579, 582, 587; Paris, Gauthier-Villars, 1925.

If we substitute these expressions in (12) or (13) we have the general expression for the *electromagnetic striction.* In order to deduce electrostriction from the gravific theory we have developed here, we split up the tensor $\mathbf{P}_\alpha^\beta$ which enters in (11) into two tensors, one of which is the electrostriction tensor and the other the mass tensor proper. In the study of electrostriction we shall only consider matter undergoing infinitely small deformations. The masses thus having infinitesimal velocities, we apply to this system the Maxwell equations governing the electromagnetic field for bodies at rest. Leaning upon the ideas of Maxwell, Hertz and Lorentz, we equate the elastic power of the electrostriction tensor to the power of electromagnetic hysteresis. We thus find again the classical expression for the electrostriction tensor as a function of the derivatives of the electromagnetic energy with respect to the deformations. Our relativistic method allows us to obtain these formulas in the case of any gravific field, while keeping, without modification or change, Maxwell's purely electromagnetic tensor.

To attain this result, let us go back to Equation (10). The values of $\mathbf{N}_\alpha$ and $\mathbf{P}_\alpha$ are given by (11) and $\mathbf{T}_\alpha^{(e)}$ by the last two terms of (7). Let us break up the tensor $\mathbf{P}_\alpha^\beta$ into two tensors, one of which, $\mathbf{P}_\alpha^{\beta(m)}$, is due to the mass field only, and the other, $\mathbf{P}_\alpha^{\beta(m,e)}$, is due to the combined action of the mass field and the electromagnetic field. We shall speak of the tensor $\mathbf{P}_\alpha^{\beta(m,e)}$ as the "electrostriction tensor." We shall have, by definition,

$$\mathbf{P}_\alpha^\beta = \mathbf{P}_\alpha^{\beta(m)} + \mathbf{P}_\alpha^{\beta(m,e)} \qquad (\alpha, \beta = 1, \cdots, 4). \tag{29}$$

We recall that, by (11),

$$\mathbf{P}_\alpha^{(m,e)} = \sum_\beta \left[\frac{\partial}{\partial x_\beta} \mathbf{P}_\alpha^{\beta(m,e)} - \tfrac{1}{2} \sum_j \sum_k g^{\beta j} g_{kj,\alpha} \mathbf{P}_\beta^{k(m,e)} \right] (\alpha, \beta, k, j = 1, \cdots, 4). \tag{30}$$

Let us now denote by v^i $(i = 1, 2, 3)$ the three rectangular components of the velocity v of a point (x, y, z) of the body, at the instant t. We calculate the work done by the electrostriction force vector $\mathbf{P}_i^{(m,e)}$ per unit time, during the motion of the body; this power is

$$\int_V \left(\sum_{i=1}^{3} \mathbf{P}_i^{(m,e)} v^i \right) \delta' v$$

where $\delta' v$ is an element of volume and $\delta' t = 0$. Integrating by parts by Green's theorem, we obtain immediately for this power

$$\left.\begin{array}{l} -\int_V \sum_i \sum_j \mathbf{P}_i^{j(m,e)} \frac{\partial v^i}{\partial x_j} \delta' v + \int_S (\mathbf{P}_{(n)i}^{(m,e)} v)\, \delta' s + \int_V \sum_i \frac{\partial \mathbf{P}_i^4}{\partial t} v^i\, \delta' v \\ -\frac{1}{2} \int_V \sum_i \sum_\beta \sum_k \sum_l g^{\beta k} g_{lk,i} \mathbf{P}_\beta^{l(m,e)} v^i \delta' v. \\ \qquad (i, j = 1, 2, 3;\ \beta, k, l = 1, 2, 3, 4;\ \mathbf{P}_i^{j(m,e)} = \mathbf{P}_j^{i(m,e)}) \end{array}\right\} \quad (31)$$

where $\delta' s$ is an element of the surface S which bounds the volume V. On the other hand the three rectangular components of $\mathbf{P}_{(n)i}^{(m,e)}$ are

$$\mathbf{P}_{(n)i}^{(m,e)} = \sum_j \mathbf{P}_i^j \cos(x_j, n) \quad (i, j = 1, 2, 3) \tag{32}$$

where n is the outside normal to S.

We note that the first integral in (31) contains derivatives with respect to the time t of the linear and surface deformations inside the body. To show this, we place

$$x_i = x_i^0 + \lambda_i(x_1^0, x_2^0, x_3^0, t) \qquad (i = 1, 2, 3) \tag{33}$$

where λ_i is the infinitesimal displacement starting from the initial point x_i^0 $(i = 1, 2, 3)$; for $t = 0$, $x_i = x_i^0$. It follows that

$$v^i = \frac{\partial \lambda_i}{\partial t}. \tag{34}$$

Let us place now $\dot{\lambda}_i = \frac{\partial \lambda_i}{\partial t}$, whence $v^i = \dot{\lambda}_i$ $(i = 1, 2, 3)$ and further write the classical notations

$$\left.\begin{array}{l} x_x = \frac{\partial \lambda_1}{\partial x^0}, \quad y_y = \frac{\partial \lambda_2}{\partial y^0}, \quad z_z = \frac{\partial \lambda_3}{\partial z^0}, \\ x_y = y_x = \frac{\partial \lambda_1}{\partial y^0} + \frac{\partial \lambda_2}{\partial x^0}, \; x_z = z_x = \frac{\partial \lambda_1}{\partial z^0} + \frac{\partial \lambda_3}{\partial x^0}, \; y_z = z_y = \frac{\partial \lambda_2}{\partial z^0} + \frac{\partial \lambda_3}{\partial y^0} \end{array}\right\}. \quad (35)$$

Except for an infinitesimal, we may write x_i instead of x_i^0 $(i = 1, 2, 3)$. We may therefore replace in (35) x_i^0 by x_i. It follows that (cf. (31))

$$\left.\begin{aligned}&\frac{\partial v^1}{\partial x_1} = \frac{\partial \dot{\lambda}_1}{\partial x_1} = \dot{x}_x, \quad \frac{\partial v^2}{\partial x_2} = \dot{y}_y, \quad \frac{\partial v^3}{\partial x_3} = \dot{z}_z \\ &\frac{\partial v^1}{\partial x_2} + \frac{\partial v^2}{\partial x_1} = \dot{x}_y, \quad \frac{\partial v^1}{\partial x_3} + \frac{\partial v^3}{\partial x_1} = \dot{x}_z, \quad \frac{\partial v^2}{\partial x_3} + \frac{\partial v^3}{\partial x_2} = \dot{y}_z\end{aligned}\right\} \tag{36}$$

the dot over x_x, y_y, $\cdots$ indicating a partial derivative with respect to t.

By definition we shall say that

$$-\int_V \sum_i \sum_j \mathbf{P}_i{}^{j(m,e)} \frac{\partial v^i}{\partial x_j} \delta' v \qquad (i, j = 1, 2, 3) \tag{37}$$

is the elastic power of electrostriction.

As the velocity v^i $(i = 1, 2, 3)$ of the system is assumed infinitesimal, we may, to a first approximation, apply Maxwell's equations governing the electromagnetic field for bodies at rest. We recall that formulas (25) give us a means of going over from Maxwell's field to ours supposed at rest. To attain the conservation of pure electromagnetic energy in our generalized Maxwell field it is enough to go back to Equation (609) of the author's "Théorie mathématique de l'électricité".[1] Among the different powers entering into this equation, we retain only the power of electromagnetic hysteresis.

$$\frac{1}{2}\int_V \left[\left(\frac{\partial B}{\partial t} \cdot H\right) - \left(B \cdot \frac{\partial H}{\partial t}\right) + \left(\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{H}\right) - \left(\mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial t}\right)\right] \delta' v. \tag{38}$$

Let us assume now that

$$H_i = \sum_j \epsilon_{ij}' B_j, \quad \mathbf{H}_i = \sum_j \mu_{ij}' \mathbf{B}_j \quad (i, j = 1, 2, 3). \tag{39}$$

If W denotes the density of localized electromagnetic energy, then we shall have for the localized energy of the whole system

$$\begin{aligned}\int_V W \delta' v &= \frac{1}{2}\int_V \left[(H \cdot B) + (\mathbf{H} \cdot \mathbf{B})\right] \delta' v \\ &= \frac{1}{2}\int_V \sum_i \sum_j (\epsilon_{ij}' B_i B_j + \mu_{ij}' \mathbf{B}_i \mathbf{B}_j)\, \delta' v \quad (i, j = 1, 2, 3).\end{aligned} \tag{40}$$

[1] Paris, Gauthier-Villars, 1925.

The integral (38) can now be written

$$-\int_V \left(\frac{\partial W}{\partial t}\right)_{B,\mathbf{B}} \delta' v \tag{41}$$

where the indices B, $\mathbf{B}$ denote that the partial derivative with respect to t is taken leaving B_i and $\mathbf{B}_i$ ($i = 1, 2, 3$) constant. We shall have

$$\int_V \left(\frac{\partial W}{\partial t}\right)_{B,\mathbf{B}} \delta' v = \tfrac{1}{2} \int_V \sum_i \sum_j \left(\frac{\partial \epsilon_{ij}'}{\partial t} B_i B_j + \frac{\partial \mu_{ij}'}{\partial t} \mathbf{B}_i \mathbf{B}_j\right) \delta' v. \tag{42}$$

Let us suppose, finally, that the density W is only a function of the angular and linear deformations $x_x, \cdots y_z$. The integral (38) may be written

$$\int_V \left(\frac{\partial W}{\partial t}\right)_{B,\mathbf{B}} \delta' v = \int_V \left[\left(\frac{\partial W}{\partial x_x}\right)_{B,\mathbf{B}} \dot{x}_x + \left(\frac{\partial W}{\partial y_y}\right)_{B,\mathbf{B}} \dot{y}_y + \left(\frac{\partial W}{\partial z_z}\right)_{B,\mathbf{B}} \dot{z}_z + \left(\frac{\partial W}{\partial x_y}\right)_{B,\mathbf{B}} \dot{x}_y + \left(\frac{\partial W}{\partial y_z}\right)_{B,\mathbf{B}} \dot{y}_z + \left(\frac{\partial W}{\partial z_x}\right)_{B,\mathbf{B}} \dot{z}_x\right] \delta' v. \tag{43}$$

Finally let us assume, as our fundamental hypothesis, that the elastic power of electrostriction (37) is equal to the power of electromagnetic hysteresis (38). We have just seen that this power may be expressed by (43); equating then (37) and (43) gives at once the electrostriction tensor $\mathbf{P}_i^{j(m,e)}$ ($i, j = 1, 2, 3$)

$$\left.\begin{aligned}
&\mathbf{P}_1^{1(m,e)} = \left(\frac{\partial W}{\partial x_x}\right)_{B,\mathbf{B}}, \quad \mathbf{P}_2^{2(m,e)} = \left(\frac{\partial W}{\partial y_y}\right)_{B,\mathbf{B}}, \quad \mathbf{P}_3^{3(m,e)} = \left(\frac{\partial W}{\partial z_z}\right)_{B,\mathbf{B}} \\
&\mathbf{P}_1^{2(m,e)} = \mathbf{P}_2^{1(m,e)} = \left(\frac{\partial W}{\partial x_y}\right)_{B,\mathbf{B}}, \quad \mathbf{P}_2^{3(m,e)} = \mathbf{P}_3^{2(m,e)} = \left(\frac{\partial W}{\partial y_z}\right)_{B,\mathbf{B}}, \\
&\mathbf{P}_3^{1(m,e)} = \mathbf{P}_1^{3(m,e)} = \left(\frac{\partial W}{\partial z_x}\right)_{B,\mathbf{B}}.
\end{aligned}\right\} \tag{44}$$

We thus obtain the classical formulas of Maxwell, Hertz, and Lorentz, extended to any gravific field. It will be noticed that we have not modified the purely electromagnetic tensor $\mathbf{T}_\alpha^{\beta(e)}$.

LECTURE 9

APPLICATIONS TO RESTRICTED RELATIVITY

Definition — Einstein's mass law — Restricted relativity for polarizable conductive matter in uniform rectilinear motion — The electromagnetic tensor and the mechanical forces in Maxwell's field.

Restricted relativity, so-called, is the theory obtained from the preceding considerations by neglecting the gravific field, that is, by assuming that the space-time is that of Minkowski. We have seen that the observer $\overline{S}$ will write Equation (6), Lecture 2,

$$\delta s^2 = -\delta x^2 - \delta y^2 - \delta z^2 + c^2\,\delta t^2 \tag{1}$$

where, for purposes of simplification, the bar over s, x, $\cdots$ has been dropped. We have also seen in Lecture 2 that by means of the Lorentz transformation given by Equations (56) and (57), Lecture 2, the observer $\overline{S}'$ will write

$$\delta s^2 = -\delta x'^2 - \delta y'^2 - \delta z'^2 + c^2\,\delta t'^2. \tag{2}$$

Restricted relativity enables us to go over from the measurements obtained by $\overline{S}$ to those obtained by $\overline{S}'$, or conversely. We have already studied in Lecture 2 this problem for the measurement of length and of time.

Einstein[1] has shown that the variancy of velocity gives a simple explanation of the rigorous Fizeau law which expresses the partial drag of waves in a moving medium. From the variancy of electric and magnetic forces in the Maxwell-Lorentz field he inferred Doppler's formula and stellar aberration and from the variancy of the mixed electromagnetic tensor in the same field he obtained the law of reflection of light on a mirror moving with uniform velocity in a straight line. In Lecture 6 we have shown that the observer $\overline{S}'$ will obtain δm^* as the

[1] A. Einstein "Annalen der Physik," Vol. 17, 1905.

measure of mass at rest with respect to himself. Thus we may place $\delta m^* = \delta \overline{m}'$. The observer $\overline{S}$ will obtain δm as the measure of this mass in uniform rectilinear motion at the velocity v with respect to him. By Equation (46), Lecture 6, we have here

$$V\,\delta m = c\,\delta m' \tag{3}$$

where

$$V = c\sqrt{1 - (v/c)^2} \tag{4}$$

and thus we obtain at once the law of variation of mass due to Einstein,

$$\delta m = \delta m'[1 - (v/c)^2]^{-\frac{1}{2}}. \tag{5}$$

Let us recall that $\delta m'$ is obtained by $\overline{S}'$ from a measurement of mass at rest with respect to himself.

We have shown[1] by the same synthetic method how it is possible to derive from Equations (69), Lecture 7, the dynamics of the electron and also the significance of the longitudinal and the transversal mass of the electron.

Starting from the general equations of the electromagnetic mass gravific field, Lecture 8, we shall give a synthetic theory of the electrodynamics of matter in uniform rectilinear motion in Minkowski's field. For this purpose we consider an observer $\overline{S}'$ in a Minkowski field who takes measurements on matter *at rest* with respect to himself. In Equations (23) to (28) of Lecture 8 we place $g_{11}' = g_{22}' = g_{33}' = -1$, $g_{44}' = c^2$ and all the other Einstein potentials equal to zero. Let us recall that, as all these measurements are obtained by $\overline{S}'$, we have to accent all the formulas (23) to (28) of the preceding lecture. By means of the Lorentz transformation and the variancy of these symbols we shall obtain at once the measurements taken by $\overline{S}$, with respect to whom matter is in rectilinear uniform motion at a velocity v. This is the general method. To illustrate the procedure we shall write explicitly the electromagnetic mixed tensor obtained by $\overline{S}$

[1] T. De Donder, "La gravifique einsteinienne," §84 and §85.

$$
\left.\begin{aligned}
T_1^{(e)1} &= \tfrac{1}{2}(H_xB_x - H_yB_y - H_zB_z + \mathbf{H}_x\mathbf{B}_x - \mathbf{H}_y\mathbf{B}_y - \mathbf{H}_z\mathbf{B}_z)\\
T_2^{(e)2} &= \tfrac{1}{2}(H_yB_y - H_zB_z - H_xB_x + \mathbf{H}_y\mathbf{B}_y - \mathbf{H}_z\mathbf{B}_z - \mathbf{H}_x\mathbf{B}_x)\\
T_3^{(e)3} &= \tfrac{1}{2}(H_zB_z - H_xB_x - H_yB_y + \mathbf{H}_z\mathbf{B}_z - \mathbf{H}_x\mathbf{B}_x - \mathbf{H}_y\mathbf{B}_y)\\
T_2^{(e)1} &= T_1^{(e)2} = \tfrac{1}{2}(H_xB_y + H_yB_x + \mathbf{H}_x\mathbf{B}_y + \mathbf{H}_y\mathbf{B}_x)\\
T_3^{(e)2} &= T_2^{(e)3} = \tfrac{1}{2}(H_yB_z + H_zB_y + \mathbf{H}_y\mathbf{B}_z + \mathbf{H}_z\mathbf{B}_y)\\
T_3^{(e)1} &= T_1^{(e)3} = \tfrac{1}{2}(H_zB_x + H_xB_z + \mathbf{H}_z\mathbf{B}_x + \mathbf{H}_x\mathbf{B}_z)\\
T_4^{(e)1} &= -c^2T_1^{(e)4} = \frac{c}{2}(H_y\mathbf{H}_z - H_z\mathbf{H}_y + B_y\mathbf{B}_z - B_z\mathbf{B}_y)\\
T_4^{(e)2} &= -c^2T_2^{(e)4} = \frac{c}{2}(H_z\mathbf{H}_x - H_x\mathbf{H}_z + B_z\mathbf{B}_x - B_x\mathbf{B}_z)\\
T_4^{(e)3} &= -c^2T_3^{(e)4} = \frac{c}{2}(H_x\mathbf{H}_y - H_y\mathbf{H}_x + B_x\mathbf{B}_y - B_y\mathbf{B}_x)\\
T_4^{(e)4} &= \tfrac{1}{2}(B \cdot H + \mathbf{B} \cdot \mathbf{H})
\end{aligned}\right\} \tag{6}
$$

If there is an *applied* force H^a then we must write in the preceding formulas (6) instead of H the vector $H - H^a$. It is interesting that in the Minkowski field the electromagnetic tensor T_i^j $(i, j = 1, 2, 3)$ of the most general Maxwellian field is always *symmetric*. To obtain the *mechanical* force due to electromagnetic forces in such a field we have only to substitute the preceding tensor (6) in Equation (12), Lecture 8. We obtain

$$
\mathbf{F}_\alpha^{(e)} = c\sum_i \frac{\partial T_\alpha^{(e)i}}{\partial x_i} \tag{7}
$$

because all the $g_{kj,\alpha}$'s vanish in the Minkowski field. We recall that in this field $\sqrt{-g} = c$. In the special case of the Maxwell-Lorentz field, that is, if there are neither polarization nor conduction currents, then the force per unit of volume given by (7) becomes

$$
\left.\begin{aligned}
\mathbf{F}_1^{(e)} &= c\rho\left[H_x + \frac{1}{c}(v_y\mathbf{H}_z - v_z\mathbf{H}_y)\right]\\
\mathbf{F}_2^{(e)} &= c\rho\left[H_y + \frac{1}{c}(v_z\mathbf{H}_x - v_x\mathbf{H}_z)\right]\\
\mathbf{F}_3^{(e)} &= c\rho\left[H_z + \frac{1}{c}(v_x\mathbf{H}_y - v_y\mathbf{H}_x)\right]
\end{aligned}\right\} \tag{8}
$$

These formulas may also be derived from (51), Lecture 7, bearing in mind the values given in Equations (25) and (26), Lecture 8. The complete calculation of the force by using (6) and (7) would be very interesting. Finally let us note that

$$\mathbf{F}_4{}^{(e)} = -c\rho(v_x H_x + v_y H_y + v_z H_z) \tag{9}$$

which may also be written

$$\mathbf{F}_4{}^{(e)} = -(\mathbf{F}_1{}^{(e)} v_x + \mathbf{F}_2{}^{(e)} v_y + \mathbf{F}_3{}^{(e)} v_z) \tag{10}$$

and may be interpreted physically as the work of the electromagnetic force per unit volume and unit time.

To pass from the measurements obtained by the observer $\overline{S'}$ with respect to whom matter is at rest, to the measurements of $\overline{S}$ with respect to whom matter is in uniform rectilinear motion, we shall use the following relations which are derived from the Lorentz transformation. For the mixed electromagnetic tensor we have

$$\left.\begin{aligned}
T_1{}^{(e)1} &= \beta^2\left(T_1{}^{(e)'1} - \frac{v^2}{c^2} T_4{}^{(e)'4} - \frac{2\,v}{c} T_4{}^{(e)'1}\right)\\
T_2{}^{(e)2} &= T_2{}^{(e)'2}, \quad T_3{}^{(e)3} = T_3{}^{(e)'3}\\
T_4{}^{(e)4} &= \beta^2\left(-\frac{v^2}{c^2} T_1{}^{(e)'1} + T_4{}^{(e)'4} + \frac{2\,v}{c^2} T_4{}^{(e)'1}\right)\\
T_2{}^{(e)1} &= T_1{}^{(e)2} = \beta\left(T_1{}^{(e)'2} - \frac{v}{c^2} T_4{}^{(e)'2}\right)\\
T_1{}^{(e)3} &= T_3{}^{(e)1} = \beta\left(T_1{}^{(e)'3} - \frac{v}{c^2} T_4{}^{(e)'3}\right)\\
T_2{}^{(e)3} &= T_3{}^{(e)2} = T_2{}^{(e)'3}\\
T_1{}^{(e)4} &= -\frac{1}{c^2} T_4{}^{(e)1} = \beta^2\left[T_1{}^{(e)'4}\left(1+\frac{v^2}{c^2}\right) - \frac{v}{c^2}\left(T_4{}^{(e)'4} - T_1{}^{(e)'1}\right)\right]\\
T_2{}^{(e)4} &= -\frac{1}{c^2} T_4{}^{(e)2} = \beta\left(T_2{}^{(e)'4} + \frac{v}{c^2} T_2{}^{(e)'1}\right)\\
T_3{}^{(e)4} &= -\frac{1}{c^2} T_4{}^{(e)3} = \beta\left(T_3{}^{(e)'4} + \frac{v}{c^2} T_3{}^{(e)'1}\right)
\end{aligned}\right\} \tag{11}$$

where we have placed

$$\beta = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}. \tag{12}$$

The same relations hold for all the mixed tensors which have been used in this field. For instance the coefficients which were introduced in Equation (20), Lecture 8, to generalize the electric and magnetic conductivities and the electric and magnetic susceptibilities change in the way indicated by relations (11) when we pass from the observer $\bar{S}'$ to the observer $\bar{S}$.

The foregoing examples are sufficient to show that the method is quite general and that we have obtained restricted relativity of the most general kind of matter.

LECTURE 10

Relativistic Quantization

Quantization of the point electron — Continuous systems — Quantization in space-time and in space and time.

Let us go back to the fundamental formula (40) of Lecture 7 and apply it to the case of a point electron defined by the two constants $\tau^{(m)}$ and $\tau^{(e)}$. This constancy along a trajectory is expressed, from Equations (24) and (29) of Lecture 7, by the following invariant relations:

$$\frac{d\tau^{(m)}}{ds} = 0, \qquad \frac{d\tau^{(e)}}{ds} = 0. \tag{1}$$

If we place

$$\mu = \frac{\tau^{(m)}}{\tau^{(e)}} \tag{2}$$

it follows that μ is also invariant. We note that

$$\frac{\partial}{\partial u^\alpha}\left(\tfrac{1}{2}\, W^2\right) = \frac{\partial W}{\partial u^\alpha}, \quad \frac{\partial}{\partial x_\alpha}\left(\tfrac{1}{2}\, W^2\right) = \frac{\partial W}{\partial x_\alpha} \tag{3}$$

because

$$W^2 = g_{\alpha\beta}u^\alpha u^\beta = 1. \tag{4}$$

Equation (40), Lecture 7, may now be written

$$\frac{d}{ds}\left[\frac{\partial}{\partial u^\alpha}\left(\tfrac{1}{2}\,\tau^{(m)}W^2\right)\right] - \frac{\partial}{\partial x_\alpha}\left(\tfrac{1}{2}\,\tau^{(m)}W^2\right) + \frac{d}{ds}\left[\frac{\partial}{\partial u^\alpha}\left(\tau^{(e)}U\right)\right] - \frac{\partial}{\partial x_\alpha}\left(\tau^{(e)}U\right) = 0. \tag{5}$$

We now introduce the Lagrangian function

$$L = \tfrac{1}{2}\,\mu W^2 + U \tag{6}$$

hence the Hamiltonian variables

$$p_\alpha = \frac{\partial L}{\partial u^\alpha} = \mu u_\alpha + \phi_\alpha \qquad (\alpha = 1, \cdots, 4). \tag{7}$$

Using x_α, p_α and s as variables, the Hamiltonian function

$$H = -L + \sum_\alpha p_\alpha u^\alpha \tag{8}$$

will be written

$$H = \tfrac{1}{2}\,\mu W^2. \tag{9}$$

Replace now the u_α's by their values from (7). By (4) we obtain Jacobi's equation

$$\frac{1}{2\,\mu}\sum_\alpha\sum_\beta g^{\alpha\beta}\left(\frac{\partial S}{\partial x_\alpha} - \phi_\alpha\right)\left(\frac{\partial S}{\partial x_\beta} - \phi_\beta\right) + \frac{\partial S}{\partial s} = 0 \tag{10}$$

where S is Jacobi's function. We know, besides, from Jacobi's classical theory that, by (9),

$$\frac{\partial S}{\partial s} = -H = -\tfrac{1}{2}\,\mu. \tag{11}$$

Substituting in (10) we have

$$\sum_\alpha\sum_\beta g^{\alpha\beta}\left(\frac{\partial S}{\partial x_\alpha} - \phi_\alpha\right)\left(\frac{\partial S}{\partial x_\beta} - \phi_\beta\right) - \mu^2 = 0. \tag{12}$$

We have just seen that μ is an invariant along a trajectory in space-time. Integrating (11) we obtain

$$S = -\tfrac{1}{2}\,\mu s + S_0(x_1, x_2, x_3, x_4). \tag{13}$$

Place

$$kS = \log \psi \tag{14}$$

by (13) we have

$$\psi = e^{-\frac{k}{2}\mu s}\psi_0 \tag{15}$$

where

$$\psi_0 = e^{kS_0(x_1, x_2, x_3, x_4)}. \tag{16}$$

From (14), (11) and (15) follows immediately that

$$\psi = -\frac{2}{k\mu}\frac{\partial \psi}{\partial s} \tag{17}$$

and

$$\frac{\partial S}{\partial x_\alpha} = -\frac{\mu}{2}\frac{\dfrac{\partial \psi}{\partial x_\alpha}}{\dfrac{\partial \psi}{\partial s}} \qquad (\alpha = 1, \cdots, 4). \tag{18}$$

Substituting in (12) we obtain

$$J = \sum_{\alpha}\sum_{\beta} g^{\alpha\beta}\left(\frac{\mu}{2}\frac{\partial\psi}{\partial x_\alpha} + \phi_\alpha \frac{\partial\psi}{\partial s}\right)\left(\frac{\mu}{2}\frac{\partial\psi}{\partial x_\beta} + \phi_\beta \frac{\partial\psi}{\partial s}\right) - \mu^2\left(\frac{\partial\psi}{\partial s}\right)^2 = 0. \quad (19)$$

Note that J is an invariant for all changes of variables $x_1, \cdots, x_4$ and that $s' = s$.

Now let us apply the following quantization rule:[1] The variational derivative of the left-hand member of the Jacobian Equation (19), with respect to ψ, shall vanish. Explicitly

$$\frac{\delta}{\delta\psi}(J\sqrt{-g}) = 0 \quad (20)$$

where

$$\frac{\delta}{\delta\psi} = \frac{\partial}{\partial\psi} - \sum_{\alpha=1}^{4}\frac{\partial}{\partial x_\alpha}\left(\frac{\partial}{\partial\left(\dfrac{\partial\psi}{\partial x_\alpha}\right)}\right) - \left(\frac{\partial}{\partial s}\frac{\partial}{\partial\left(\dfrac{\partial\psi}{\partial s}\right)}\right). \quad (21)$$

After performing the indicated partial differentiations we finally obtain

$$-\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\psi}(J\sqrt{-g}) = \tfrac{1}{2}\mu^2\Box\psi + 2\,\mu\sum_{\alpha}\sum_{\beta} g^{\alpha\beta}\phi_\alpha\frac{\partial^2\psi}{\partial s\partial x_\beta} + \mu D\frac{\partial\psi}{\partial s} + 2\,[F - \mu^2]\frac{\partial^2\psi}{\partial s^2} = 0 \quad (22)$$

where

$$\left.\begin{aligned} \Box\psi &= \frac{1}{\sqrt{-g}}\sum_{\alpha}\frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\sum_{\beta} g^{\alpha\beta}\frac{\partial\psi}{\partial x_\beta}\right) \\ D &= \frac{1}{\sqrt{-g}}\sum_{\alpha}\frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\sum_{\beta} g^{\alpha\beta}\phi_\beta\right) \\ F &= \sum_{\alpha}\sum_{\beta} g^{\alpha\beta}\phi_\alpha\phi_\beta \qquad (\alpha, \beta = 1, \cdots, 4) \end{aligned}\right\} \quad (23)$$

In (22) replace the derivatives of ψ with respect to s by their values,

$$\frac{\partial\psi}{\partial s} = -\tfrac{1}{2}\,k\mu\psi, \qquad \frac{\partial^2\psi}{\partial s^2} = \tfrac{1}{4}\,k^2\mu^2\psi \quad (24)$$

[1] T. De Donder and F. H. van den Dungen, "La quantification déduite de la gravifique einsteinienne," Comptes-Rendus, July 5, 1926.

whence

$$\Box\psi - 2k\sum_{\alpha}\sum_{\beta} g^{\alpha\beta}\phi_\alpha \frac{\partial\psi}{\partial x_\beta} - kD\psi + (F - \mu^2)k^2\psi = 0. \quad (25)$$

From the invariance of $\tau^{(m)}$ and $\tau^{(e)}$ with respect to every change of the variables $x_1, \cdots, x_4$, we have

$$\mu = \frac{\tau^{(m)}}{\tau^{(e)}} = \frac{m^*c^2}{e^*} \quad (26)$$

where m^* and e^* are respectively the mass and the charge of the *point* electron at rest in a Minkowski field.

Substituting in (25) we obtain the *fundamental quantization equation of the point electron:*

$$\Box\psi - 2k\sum_{\alpha}\sum_{\beta} g^{\alpha\beta}\phi_\alpha \frac{\partial\psi}{\partial x_\beta} - kD\psi + \left(F - \frac{m^{*2}c^4}{e^{*2}}\right)k^2\psi = 0. \quad (27)$$

We note that $D = 0$ is the generalization of Maxwell's complementary equation. By way of *example* we consider the case where H does not involve the time t explicitly. We then place

$$\frac{\partial S}{\partial t} = \frac{Ec}{e^*} \quad (28)$$

where E is the total energy of the electron[1] including its mass energy m^*c^2. By (14)

$$\frac{\partial\psi}{\partial t} = \frac{kcE}{e^*}\psi \quad (29)$$

and by (29) we may also eliminate the derivative of ψ with respect to t in the fundamental equation (27). We thus obtain

$$\left.\begin{aligned}
&\Delta\psi + \frac{2kc}{e^*}\left(E - \frac{e^*\phi_4}{c}\right)\sum_{i=1}^{3} g^{i4}\frac{\partial\psi}{\partial x_i} + \frac{1}{\sqrt{-g}}\sum_{i=1}^{3}\frac{\partial\psi}{\partial x_i}\frac{\partial}{\partial t}(g^{i4}\sqrt{-g})\\
&-2k\sum_{i=1}^{3}\sum_{j=1}^{3} g^{ij}\phi_i\frac{\partial\psi}{\partial x_j} - k\psi\left[D - kF - k\left(\frac{c^2g^{44}}{e^{*2}}\left(E - \frac{e^*\phi_4}{c}\right)^2 - \mu^2\right)\right.\\
&\left.+2\frac{kc}{e^*}\left(E - \frac{e^*\phi_4}{c}\right)\sum_{i=1}^{3} g^{i4}\phi_i - \frac{cE}{e^*}\frac{1}{\sqrt{-g}}\sum_{\alpha=1}^{4}\frac{\partial}{\partial x_\alpha}(g^{\alpha 4}\sqrt{-g})\right] = 0
\end{aligned}\right\} \quad (30)$$

[1] The value $(-Ec)$ is given by the second term of Equation (89), Lecture 7.

where

$$\Delta\psi = \frac{1}{\sqrt{-g}} \sum_i \frac{\partial}{\partial x_i}\left(\sum_j g^{ij}\sqrt{-g}\,\frac{\partial\psi}{\partial x_j}\right), \tag{31}$$

$$F = \sum_i\sum_j g^{ij}\phi_i\phi_j \quad (i, j = 1, 2, 3). \tag{32}$$

Let us now go over to the more special case of Minkowski's field and of the electrostatic field. We have then $\phi_1 = \phi_2 = \phi_3 = 0$, $\frac{\partial\phi_4}{\partial t} = 0$. Equation (30) becomes then

$$\Delta\psi + \left(\frac{k}{e^*}\right)^2 \psi\,[(E - V)^2 - (c^2m^*)^2] = 0 \tag{33}$$

where

$$V = \frac{e^*\phi_4}{c}. \tag{34}$$

We recall that we have here, by (31),

$$\Delta\psi = -\nabla^2\psi = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) \tag{35}$$

and Equation (33) may also be written[1]

$$\Delta\psi + \left(\frac{k}{e^*}\right)^2 \psi\,[(\epsilon - V)^2 + 2\,(\epsilon - V)c^2m^*] = 0 \tag{36}$$

where

$$\epsilon = E - m^*c^2. \tag{37}$$

To a first order of approximation (36) may be written

$$\Delta\psi + 2\left(\frac{k}{e^*}\right)^2 c^2m^*(\epsilon - V)\psi = 0. \tag{38}$$

For (38) to become identical with Schrödinger's equation[2], we merely have to place

$$\frac{k}{e^*} = \frac{2\,\pi i}{hc} \tag{39}$$

[1] This is the result obtained in our note in Comptes-Rendus, Oct. 11, 1926.

[2] E. Schrödinger, Ann. d. Physik, 79, 361/376, 1926; see Equations (5) (23) and (24). Also Ann. d. Physik, 79, 489 and 734; 80, 437, 1926.

where h is Planck's constant. Explicitly Schrödinger's equation (38) may be written in rectangular coördinates,

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+2\left(\frac{2\,\pi}{h}\right)^2 m^*(\epsilon - V)\psi = 0. \qquad (40)$$

We have thus shown that *the quantization of the point electron can be deduced from Einstein's gravific theory* by means of an absolute extremal.

This method may also be applied to continuous systems as follows:[1] Let us consider a holonomic system of f degrees of freedom and place

$$x_\alpha = x_\alpha(x_1', x_2', x_3', x_4'; q_1, \cdots, q_f) \qquad (41)$$

where x_α represents an event in a given reference system and x_α' the same event in another reference system such that x_α' stays constant when the proper time s increases by ds. We have therefore $dx_\alpha' = 0$, and finally $q_1, \cdots, q_f$ are functions of s. We place

$$\kappa^\phi = \frac{dq_\phi}{ds} \quad (\phi = 1, \cdots, f) \qquad (42)$$

for the generalized contravariant velocity, and besides, we introduce the variation δ' such that $\delta' q_1 = \cdots = \delta' q_f = 0$ and $\delta' s = 0$. The $\delta' x_1 \cdots \delta' x_4$ are, in general, non-vanishing. We have now,

$$W^2 = \sum_\phi\sum_\psi g_{\phi\psi}{}^*\kappa^\phi\kappa^\psi = 1 \qquad (\phi, \psi = 1, \cdots, f) \qquad (43)$$

where

$$g_{\phi\psi}{}^* = \sum_\alpha\sum_\beta \frac{\partial x_\alpha}{\partial q_\phi}\frac{\partial x_\beta}{\partial q_\psi} g_{\alpha\beta} \quad (\alpha, \beta = 1, \cdots, 4). \qquad (44)$$

We shall have, besides,

$$U = \sum_\phi \phi_\phi{}^*\kappa^\phi \qquad (45)$$

where

$$\phi_\phi{}^* = \sum_\alpha \phi_\alpha \frac{\partial x_\alpha}{\partial q_\phi}. \qquad (46)$$

[1] T. De Donder, Journal of Mathematics and Physics, Vol. V, No. 4, June, 1926; Comptes-Rendus, June 7, 1926.

Place

$$Q_\phi^m = \frac{d}{ds}\left(\frac{\partial W}{\partial \kappa^\phi}\right) - \left(\frac{\partial W}{\partial q_\phi}\right) \tag{47}$$

and

$$Q_\phi^e = \frac{d}{ds}\left(\frac{\partial U}{\partial \kappa^\phi}\right) - \left(\frac{\partial U}{\partial q_\phi}\right). \tag{48}$$

We readily show that

$$Q_\phi^m = \sum_\alpha \left[\frac{d}{ds}\left(\frac{\partial W}{\partial u^\alpha}\right) - \left(\frac{\partial W}{\partial x_\alpha}\right)\right]\frac{\partial x_\alpha}{\partial q_\phi} \tag{49}$$

and

$$Q_\phi^e = \sum_\alpha \left[\frac{d}{ds}\left(\frac{\partial U}{\partial u^\alpha}\right) - \left(\frac{\partial U}{\partial x_\alpha}\right)\right]\frac{\partial x_\alpha}{\partial q_\phi} \quad (\alpha = 1, \cdots, 4). \tag{50}$$

Our fundamental Equations (40), Lecture 7, furnish therefore the corresponding equations for a continuous system of f degrees of freedom,

$$\int Q_\phi^m \, \delta'\tau^{(m)} + \int Q_\phi^e \, \delta'\tau^{(e)} = 0 \tag{51}$$

where

$$\delta'\tau^{(m)} = \mathbf{N}\, \delta'x_1\, \delta'x_2\, \delta'x_3\, \delta'x_4 \text{ and } \delta'\tau^{(e)} = \sigma\, \delta'x_1\, \delta'x_2\, \delta'x_3\, \delta'x_4. \tag{25}$$

These notations have already been defined by (23) and (28) of Lecture 7. We note that as in (3)

$$\left.\begin{aligned} \frac{\partial}{\partial \kappa^\phi}\left(\tfrac{1}{2} W^2\right) &= \frac{\partial W}{\partial \kappa^\phi} \\ \frac{\partial}{\partial q_\phi}\left(\tfrac{1}{2} W^2\right) &= \frac{\partial W}{\partial q_\phi} \end{aligned}\right\} \tag{52'}$$

and (51) may also be written

$$\int \left[\frac{d}{ds}\left(\frac{\partial}{\partial \kappa^\phi}\tfrac{1}{2} W^2\right) - \left(\frac{\partial}{\partial q_\phi}\tfrac{1}{2} W^2\right)\right]\delta'\tau^{(m)} + \int \left[\frac{d}{ds}\left(\frac{\partial U}{\partial \kappa^\phi}\right) - \left(\frac{\partial U}{\partial q_\phi}\right)\right]\delta'\tau^{(e)} = 0. \tag{53}$$

From (41) follows that

$$\delta'\tau^{(m)} = \mathbf{N}\frac{\partial(x)}{\partial(x')}\, \delta'x_1' \cdots \delta'x_4' \quad \text{and} \quad \delta'\tau^{(e)} = \sigma\frac{\partial(x)}{\partial(x')}\, \delta'x_1' \cdots \delta'x_4'. \tag{54}$$

From the invariancies expressed by (24) and (29) of Lecture 7, we may assume that $\delta'\tau^{(m)}$ and $\delta'\tau^{(e)}$ depend on $x_1', \cdots, x_4'$ only. Then (53) may be written

$$\frac{d}{ds}\left(\frac{\partial L^*}{\partial \kappa^\phi}\right) - \left(\frac{\partial L^*}{\partial q_\phi}\right) = 0 \quad (\phi = 1, \cdots, f) \tag{55}$$

where

$$L^* = \int (\tfrac{1}{2} W^2 \,\delta'\tau^{(m)} + U\,\delta'\tau^{(e)}) \tag{56}$$

and the integral is taken over the whole continuous system. If we introduce the generalized momenta

$$p_\phi{}^* = \left(\frac{\partial L^*}{\partial \kappa^\phi}\right) \tag{57}$$

and the Hamiltonian function

$$H^* = -L^* - \sum_\phi p_\phi{}^* \kappa^\phi, \tag{58}$$

we shall have the $2f$ canonical equations of the continuous system

$$\frac{dq_\phi}{ds} = \frac{\partial H^*}{\partial p_\phi{}^*}, \qquad \frac{dp_\phi{}^*}{ds} = -\frac{\partial H^*}{\partial q_\phi} \qquad (\phi = 1, \cdots, f). \tag{59}$$

The quantization of continuous systems will be carried out by applying to the H^* given by (58) the same method as was applied at the beginning of this Lecture to the quantization of the point-electron. The only difficulty in given cases arises from the integration (56) and this owing to the four dimensional $\delta'\tau^{(m)}$ and $\delta'\tau^{(e)}$.

By the definitions already given, we have, as in (31),

$$\frac{\delta'\tau^{(m)}}{\delta'\tau^{(e)}} = \frac{c^2\,\delta m^*}{\delta e^*} \tag{60}$$

where the notations have already been defined by (48) and (54) of Lecture 7.

To a first approximation it might be *assumed* that $\delta'\tau^{(e)}$ is δe^* multiplied by c/ν, ν being a universal constant having the dimensions of frequency (time to the power minus one). This constant will thereafter disappear in the course of our calculations.

We may avoid the difficulty arising from the integration (56), extended over a *four* dimensional region, by reconsidering in space *and* time the problem of the continuous system. For this purpose we place

$$x_i = x_i(x_1', x_2', x_3'; q_1, q_2, \cdots, q_f) \quad (i = 1, 2, 3) \tag{61}$$

where ($x_1, x_2, x_3, x_4 = t$) define an event and x_1', x_2', x_3' are invariants with respect to the time t. We shall have then $dx_i'/dt = 0$ ($i = 1, 2, 3$). Finally $q_1, \cdots, q_f$ are functions of t defining the state of the system at the instant t. We place

$$\frac{dq_\phi}{dt} = \dot{q}^\phi \quad (\phi = 1, 2, \cdots, f). \tag{62}$$

The points of the system at the instant t are defined by x_1', x_2', x_3'. We thus have to consider the infinitesimals $\delta' x_i$ ($i = 1, 2, 3$); besides $\delta' t = \delta' q_1 = \cdots = \delta' q_f = 0$. It follows that

$$V^2 = \left(\frac{ds}{dt}\right)^2 = \sum_{\phi=1}^{f} \sum_{\psi=1}^{f} g_{\phi\psi}{}^* \dot{q}^\phi \dot{q}^\psi + 2 \sum_{\phi=1}^{f} g_\phi{}^* \dot{q}^\phi + g_{44} \tag{63}$$

where

$$g_{\phi\psi}{}^* = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} \frac{\partial x_i}{\partial q_\phi} \frac{\partial x_j}{\partial q_\psi}, \quad g_\phi{}^* = \sum_{i=1}^{3} g_{i4} \frac{\partial x_i}{\partial q_\phi}. \tag{64}$$

We may prove, as before, that Equation (81), Lecture 7, furnishes the fundamental equations of continuous systems in space *and* time

$$\int [A_\phi{}^m c^2\, \delta m^* + A_\phi{}^e\, \delta e^*] = 0 \quad (\phi = 1, \cdots, f) \tag{65}$$

where

$$A_\phi{}^m = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}^\phi}\right) - \left(\frac{\partial V}{\partial q_\phi}\right), \quad A_\phi{}^e = \frac{d}{dt}\left(\frac{\partial U^*}{\partial \dot{q}^\phi}\right) - \left(\frac{\partial U^*}{\partial q_\phi}\right). \tag{66}$$

The symbols δm^*, δe^* are defined by (54) and (48), Lecture 7; the invariancy of these equations is indicated by (55) and (49) of the same Lecture. By (61) we have

$$c^2\, \delta m^* = \frac{\mathbf{N}}{V} \frac{\partial(x)}{\partial(x')}\, \delta' x_1'\, \delta' x_2'\, \delta' x_3', \quad \delta e^* = \rho \frac{\partial(x)}{\partial(x')}\, \delta' x_1'\, \delta' x_2'\, \delta' x_3'. \tag{67}$$

These expressions being invariant with respect to t, we may assume that they can be expressed by means of the parameters x_1', x_2', x_3', only, independently of $q_1, \cdots, q_f$; $\dot{q}^1, \cdots, \dot{q}^f$. Equations (65) become then

$$\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{q}^\phi}\right) - \left(\frac{\partial \mathbf{L}}{\partial q_\phi}\right) = 0 \quad (\phi = 1, 2, \cdots, f) \tag{68}$$

where

$$\mathbf{L} = \int [Vc^2\, \delta m^* + U^*\, \delta e^*], \tag{69}$$

the integral being extended to the whole continuous system at the instant t considered. If we introduce the generalized momenta

$$p_\phi = \left(\frac{\partial \mathbf{L}}{\partial \dot{q}^\phi}\right) \tag{70}$$

the Hamiltonian function

$$\mathbf{H} = -\mathbf{L} + \sum_\phi p_\phi \dot{q}^\phi \tag{71}$$

and assume that **H** does not involve t explicitly, then **H** is an invariant of the canonical equations of motion of the system. By analogy with (89), Lecture 7, and with (28) of this Lecture, we place

$$-\mathbf{H} = cE = c(\epsilon + c^2 m^*). \tag{72}$$

By Jacobi's equation

$$\mathbf{H} + \frac{\partial \mathbf{S}}{\partial t} = 0 \tag{73}$$

we shall have

$$\frac{\partial \mathbf{S}}{\partial t} = cE = c(\epsilon + c^2 m^*) \tag{74}$$

whence

$$\mathbf{S} = cEt + S_0(x_1, x_2, x_3). \tag{75}$$

To quantize[1] these systems it is only necessary to follow the procedure indicated at the beginning of this Lecture. This

[1] The details will be found in our paper: "Contribution a la quantification relativistique," Bull. Ac. Roy. de Belgique, Oct. 9, 1926.

method leads to the solution of the quantization of the uncharged pure mass rotator, of the rigid polyatomic molecule and of the electron rotating uniformly about an axis.

Equation (25) may be generalized as follows for relativistic continuous (four dimensional) systems:

$$\Box\psi - 2k\sum_{\phi=1}^{f} Q^{\phi}\frac{\partial\psi}{\partial q_\phi} + k^2\left[Q - \frac{\tau^{(m)2}}{\tau^{(e)2}}\right]\psi = 0 \tag{76}$$

where

$$\tau^{(m)}P_{\phi\psi} = \int g^*{}_{\phi\psi}\,\delta'\tau^{(m)}, \quad \tau^{(e)}Q_\phi = \int \phi_\phi{}^*\,\delta'\tau^{(e)}, \quad (\phi, \psi = 1, \cdots, f) \tag{77}$$

$$\tau^{(m)} = \int \delta'\tau^{(m)}, \qquad \tau^{(e)} = \int \delta'\tau^{(e)}, \tag{78}$$

$$\Box\psi = \frac{1}{\sqrt{-P}}\sum_\phi \frac{\partial}{\partial q_\phi}\left[\sqrt{-P}\sum_\psi P^{\phi\psi}\frac{\partial\psi}{\partial q_\psi}\right], \tag{79}$$

$$Q = \sum_\phi Q_\phi Q^\phi, \quad Q^\phi = \sum_\psi P^{\phi\psi}Q_\psi, \tag{80}$$

P is the determinant of $P_{\phi\psi}$, k is Schrödinger's constant $2\,\pi i/ch$ and e^* the electronic charge. The universal constant e^* has been introduced so that in the limiting case of the electron (m^*, e^*), our generalized Equation (76) may reduce to Equation (25). It is enough to note that the q_ϕ's become the x_α's $(\alpha = 1, \cdots, 4)$, $P_{\phi\psi}$ becomes $g_{\alpha\beta}$, Q^ϕ goes over into ϕ^α, Q_ϕ into ϕ_α $(\alpha, \beta = 1, \cdots, 4)$, and, above all, that $\tau^{(m)}/\tau^{(e)}$ becomes m^*c^2/e^*.

The relativistic meaning of the generalized Schrödinger equation is then that this equation *alone* is equivalent to the system of *two* equations in S:

$$[S] = 0 \qquad \Box S = 0 \tag{81}$$

where S is a real Jacobian function and where we have placed

$$[S] = \sum_{\phi}\sum_{\psi} P^{\phi\psi}\left[\frac{\partial S}{\partial q_\phi} - e^*Q_\phi\right]\left[\frac{\partial S}{\partial q_\psi} - e^*Q_\psi\right] - \left(\frac{e^*\tau^{(m)}}{\tau^{(e)}}\right)^2 \tag{82}$$

$$\Box S = \frac{1}{\sqrt{-P}}\sum_\phi \frac{\partial}{\partial q_\phi}\left[\sqrt{-P}\sum_\psi P^{\phi\psi}\frac{\partial S}{\partial q_\psi}\right] \tag{83}$$

for Equations (83) are equivalent to the single imaginary equation

$$\Box S + iK[S] = 0 \tag{84}$$

where K is a real constant, $K = 2\,\pi e^*/ch$.

If now in (86) we introduce the function ψ defined by

$$S = \frac{1}{iK}\log\psi \tag{85}$$

we obtain immediately our generalized Equation (76). The magic power of Schrödinger's equation and its generalization arises, therefore, first from the fact that it merges both Equations (83) into a *single* equation, and, second, from its being *linear* in $\partial\psi/\partial q_\phi$, while the first of our equations is quadratic in $\partial\psi/\partial q_\phi$ ($\phi = 1, \cdots, f$).

The relation between statistical mechanics and our relativistic quantization is quite interesting. A *permanent* ensemble of continuous systems is one for which

$$\frac{d}{ds}\int\sqrt{-P}\,\delta q_1 \cdots \delta q_f = \text{const.} \tag{86}$$

the integral being taken over any configuration manifold $q_1, \cdots, q_f$ (with $\delta s = 0$). But since

$$\kappa^\phi = \frac{dq_\phi}{ds} \qquad (\phi = 1, \cdots, f) \tag{87}$$

the equation of " continuity " of configuration may be written

$$\sum_\phi \frac{\partial}{\partial q_\phi}(\sqrt{-P}\,\kappa^\phi) = 0. \tag{88}$$

But we have Maxwell's generalized complementary equation

$$\sum_\phi \frac{\partial}{\partial q_\phi}(\sqrt{-P}\,Q^\phi) = 0 \tag{89}$$

and since $\sigma_m \dfrac{\partial(x)}{\partial(x')}$ and $\sigma_e \dfrac{\partial(x)}{\partial(x')}$ are independent of $q_1, \cdots, q_f$; $\kappa^1, \cdots, \kappa^f$ for our continuous systems, we have

$$\frac{\partial\tau^{(e)}}{\partial q_\phi} = \frac{\partial\tau^{(m)}}{\partial q_\phi} = 0 \tag{90}$$

hence, adding together (92) and (93), we obtain

$$\sum_{\phi} \frac{\partial}{\partial q_{\phi}}\left[\frac{\sqrt{-P}}{\tau^{(e)}} \sum_{\psi} P^{\phi\psi}(\tau^{(m)}\kappa_{\psi} + \tau^{(e)}Q_{\psi})\right] = 0. \tag{91}$$

Since

$$p_{\psi} = \tau^{(m)}\kappa_{\psi} + \tau^{(e)}Q_{\psi}, \quad p_{\psi} = \frac{\partial S}{\partial q_{\psi}}, \tag{92}$$

we have, substituting in (95),

$$\Box S = \frac{1}{\sqrt{-P}} \sum_{\phi} \frac{\partial}{\partial q_{\phi}}\left[\sqrt{-P} \sum_{\psi} P^{\phi\psi} \frac{\partial S}{\partial q_{\psi}}\right] = 0 \tag{93}$$

which is our D'Alembertian wave equation.[1]

If to this equation we join the generalized Jacobian equation $[S] = 0$ (83), we have already shown above how it is possible to deduce our relativistic quantization from these two equations. Relativity is thus able, not only to furnish quantization, but even to show that it is a consequence of the condition of *permanence* of statistical ensembles.[2]

In (85) we have implicitly assumed that the modulus (or amplitude) of ψ is constant. In the general case where the modulus A of ψ is *variable* it is still possible to deduce the fundamental equation (76) from our theory applied to electronic or molecular systems having internal stresses which as a whole form a permanent luminous source. The modulus A is nothing but the potential A of the internal stresses $\mathbf{P}_{\alpha}$ which enter in Equation (11) of Lecture 8 by placing[3]

$$\mathbf{P}_{\alpha} = \mathbf{N}\frac{\partial A}{\partial x_{\alpha}}. \tag{94}$$

Einstein's gravific theory furnishes, therefore, the physical interpretation of the function ψ. Owing to its marvelous generality, our gravific theory embraces the whole quantization problem.

[1] T. De Donder, Comptes-Rendus, Feb. 21, 1927, Bull. Ac. Roy. de Belgique, March 7, 1927.

[2] The text of this lecture has been considerably altered since its original presentation on May 19, 1926. See reference on p. 95 for the original text of this lecture.

[3] For the details of the calculations see T. De Donder, Bull. de l'Ac. R. Belgique, April 2, 1927.