RATIO, PROPORTION AND MEASUREMENT IN THE ELEMENTS OF EUCLID.

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The following note is concerned with a few of the definitions and theorems of the fifth, sixth, seventh and tenth books of the Elements of Euclid, all of them relating to the theory of measurement, ratio and proportion. The fifth book, it may be explained, is devoted to the theory of ratio and proportion of magnitudes in general and the sixth to the applications of this theory in plane geometry, the seventh is the first of Euclid's three arithmetical books, and the tenth an elaborate treatise on incommensurable magnitudes.

According to tradition it was Pythagoras who first proved that the side and diagonal of a square are incommensurable. The early discovery of incommensurability had important consequences for the Greek mathematics. It revealed the fact that the theory of proportion of numbers and other commensurables already known to the Pythagoreans was inadequate for geometry and led to the invention of a general theory—that of Euclid's fifth book—which is independent of commensurability, and, as a consequence of this, to the complete exclusion of the notion of numerical measurement from the Greek geometry. But it also started investigations which resulted in the development of the doctrine of incommensurable magnitudes—chiefly line segments and rectangles corresponding to quadratic irrationalities—into a department of mathematics coördinate with arithmetic and geometry. The commentator Proclus attributes the general theory of proportion to Eudoxus and much of the doctrine of incommensurables to Theaetetus. Both lived in the century before Euclid (the fourth century B.C.).

Both of the above mentioned theories of proportion are contained in Euclid's Elements. The general theory is that presented in the fifth book and applied in the subsequent geometrical books. The older theory is developed in the seventh book as a theory for numbers but is used in the tenth book as applicable to all classes of commensurables. There is no intimation that this theory is included in the other. It is characteristic of both theories, as Euclid presents them, that they are based on a definition not of ratio but of proportion.

It may be added that the number system of Euclid is restricted to the cardinal numbers. He defines a number as a "multitude composed of units." In the proofs of the theorems in the arithmetical books numbers are represented by line segments in the same way that magnitudes in general are represented in the fifth and tenth books. The association of the notion of geometric magnitude with number is seen also in the characterization of numbers as plane, solid, similar, square, cube, and in modes of expression often used in proofs of theorems.

The seventh book begins by showing that if any two numbers be given they may be proved prime to one another, or their greatest common divisor be found, by the process which we now call the Euclidean method. From this result the conclusion is drawn that any given number is a definite determinable multiple, part, or "parts" (multiple of a part) of any other given number, and the theory of proportion of numbers is then developed on the basis of the definition:

Numbers are proportional when the first is the same multiple, part, or parts of the second that the third is of the fourth.

By thus basing the theory on the equality of ratios instead of ratio itself, the Greeks were enabled to discuss the fractional relation without the use of the fraction.

The tenth book opens with the theorem:

If from the greater of two given magnitudes of the same kind there be taken its half or more, from the remainder its half or more, and so on, a remainder will at length be reached which is less than the smaller of the two given magnitudes (X, 1).

Special interest attaches to this theorem as being that on which the method of exhaustions is based. The theorem itself and the assumption on which its proof is made to depend—the so-called "Axiom of Archimedes," that of the smaller of two given magnitudes of the same kind a multiple can be found which will exceed the greater—are probably both due to Eudoxus, for Archimedes attributes the method of exhaustions to this mathematician. Next comes the fundamental theorem:

Let A and B denote two unequal magnitudes of the same kind, of which A is the greater. Suppose B to be repeatedly subtracted from A until a remainder R_1 is found which is less than B, then R_1 to be repeatedly subtracted from B until a remainder R_2 is found which is less than R_1, and so on. If, however far this process may be carried, a remainder will never be found which is contained exactly in the remainder immediately preceding it, then the magnitudes A and B are incommensurable (X, 2).

The proof may be summarized thus: If A and B have any common measure, let it be E. Then E will also be a measure of all the remainders $R_1, R_2, \cdots$. But R_1 is less than half of A, R_3 less than half of R_1 and so on, and similarly R_2 is less than half of B, R_4 less than half of R_2, and so on. Hence, by the preceding theorem, all the remainders after a certain one will be less than E and cannot therefore be measured by E. The theorem follows from this contradiction.

From the proof of this theorem it follows that: *If on the contrary the magnitudes A and B are such that the process above described when applied to them will ultimately yield a remainder R_n which is exactly contained in that immediately preceding it, then A and B are commensurable and R_n is their greatest common measure* (X, 3).

A basis is thus secured for the proof of the following theorem, which in the rest of the tenth book plays the rôle of a working definition of commensurable magnitudes:

Commensurable magnitudes have to one another the ratio which a number has to a number (X, 5).

The proof is this: If A and B denote the magnitudes and D their greatest common measure, we shall have $A = aD$ and $B = bD$, where a and b are integers. Therefore, by the definition of proportion above given (here assumed by Euclid as a definition for commensurable magnitudes in general), $A : D :: a : 1$ and $D : B :: 1 : b$. Hence (VII, 14) $A : B :: a : b$.

One more proposition of this book, a consequence of that just proved, may be cited, the substance of which is that: *The squares on commensurable line segments have to one another the ratio which a square number has to a square number; and therefore line segments the squares on which have not such a ratio are incommensurable* (X, 9). It is attributed to Theaetetus and is of interest as being the generalization of the theorem that the side and diagonal of a square are incommensurable, in which this entire doctrine had its origin, and as being the first proof of the actual existence of incommensurables which occurs in Euclid's Elements.

One who examines Euclid's presentation of this theory of proportion can but be impressed by the care he takes before applying his definition to provide a general test—the method of greatest common measure—for actually determining for any given numbers or other commensurables whether or not they satisfy its conditions. One is reminded of the dictum of Kronecker that no mathematical definition can be regarded as resting on a secure foundation unless means are at hand for determining by a finite number of steps whether any given object conforms to it or not. This was the Greek doctrine of mathematical definition. Aristotle says: "In geometry the existence of points and lines is assumed, the existence of everything else must be proved, and nothing may be legitimately used whose existence has not been proved." It is for this reason also that

Euclid was at such pains to avoid hypothetical constructions in his geometric books.

Let us now turn to the general theory of proportion of the fifth book. The significant definitions of the book are these: *

DEFINITION 3. *A ratio is a sort of relation in respect of size between two magnitudes of the same kind.*

DEFINITION 4. *Magnitudes are said to have a ratio to one another which are capable when multiplied of exceeding one another.*

DEFINITION 5. *Let A, B, X, Y be four magnitudes, A of the same kind as B, X of the same kind as Y. If for all integral values of m and n t be the case that according as $mA \gtreqless nB$, so also is $mX \gtreqless nY$, then A is said to be in the same ratio to B as X to Y.*

DEFINITION 6. *Four magnitudes A, B, X, Y related as in Definition 5 are called proportional.*

DEFINITION 7. *If* (in the notation of Definition 5) *m and n can be found such that $mA > nB$ but $mX \ngtr nY$, then A is said to have a greater ratio to B than X has to Y.*

Here, as in the case of the earlier or arithmetical theory just considered, there is no explicit definition of ratio. Definition 3 is merely a vague indication of the sort of thing that ratio is and Definition 4 is equivalent to a statement that ratio exists between any two "magnitudes" for which the axiom of Archimedes (Eudoxus) above cited holds good. But a very definite notion of ratio is implied in the definition of proportion. For according to Definition 5 the condition that A, B, X, Y be proportional is this: If the multiples A, $2A$, $3A$, $\cdots$ and B, $2B$, $3B$, $\cdots$ be supposed arranged in a single sequence in the order of size, and so likewise the multiples X, $2X$, $3X$, $\cdots$ and Y, $2Y$, $3Y$, $\cdots$, the law of distribution of the multiples of A among those of B must be the same as that of the multiples of X among those of Y. Hence the "sameness" of the ratios $A : B$ and $X : Y$ means the sameness of these two laws of distribution, and the ratio $A : B$ itself means that size relation between A and B which is indicated by the manner in which the multiples of A are distributed among those of B. A single ratio does not admit of representation in finite terms. But for a theory of proportion and an abstract theory of ratio in general the representation of single ratios is not requisite. Sound and applicable definitions of the relations of equality and greater and lesser inequality for ratios are all that is needed. These are provided in Definitions 5 and 7. That in brief is the point of view which this scheme of definitions implies.

* I have quoted Definitions 3, 4 from Heath's Euclid, vol. 2, p. 114, and have paraphrased Definitions 5–7.

For synthetic geometry no simpler or more elegant definition of proportion than Definition 5 can be desired. Thus, consider Euclid's proof of the theorem that: *Triangles which have the same altitude are to one another as their bases* (*VI*, 1). If T_1, T_2 be the triangles and b_1, b_2 their bases, and any multiples of b_1, b_2 be taken, say mb_1, nb_2, and the corresponding multiples of T_1, T_2, namely mT_1, nT_2, it follows from the simplest geometrical considerations that according as $mb_1 \gtreqless nb_2$ so also is $mT_1 \gtreqless nT_2$. Hence $b_1 : b_2 :: T_1 : T_2$.

The mention of a few of the theorems of Book V will suffice to show how Euclid deduces the properties of proportions and the elements of an abstract theory of ratio from his definitions.*

THEOREMS 7–10. *According as* $A \gtreqless B$ *so also is* $A : C \gtreqless B : C$ *and* $C : A \lesseqgtr C : B$; *and conversely.*

Thus, if $A > B$, then $A : C > B : C$. For take m so that $m(A - B) > C$ and $mB > C$ (Definition 4). Next take n so that $(n - 1)C \leqq mB < nC$. Then $mA > nC$ but $mB < nC$. Hence $A : C > B : C$ (Definition 7).

THEOREM 11. *If* $A : B :: K : L$ *and* $K : L :: X : Y$, *then* $A : B :: X : Y$.

For, by Definition 5, according as $mA \gtreqless nB$, so also is $mK \gtreqless nL$ and therefore $mX \gtreqless nY$. Hence $A : B :: X : Y$.

THEOREM 13. *If* $A : B :: K : L$ *and* $K : L > X : Y$, *then* $A : B > X : Y$.

For, by Definitions 7, 5, m and n can be so chosen that $mK > nL$ but $mX \ngtr nY$, and therefore that $mA > nB$ but $mX \ngtr nY$. Hence $A : B > X : Y$.

THEOREM 16. *If* A, B, C, D *are of the same kind, and* $A : B :: C : D$, *then, "alternately,"* $A : C :: B : D$.

For it readily follows from Definition 5 that $mA : mB :: nC : nD$. By Theorems 7–10, 11, 13, according as $mA \gtreqless nC$, so also is $mA : mB \gtreqless nC : mB$, $\therefore\ nC : nD \gtreqless nC : mB$, $\therefore\ mB \gtreqless nD$. Hence $A : C :: B : D$ (Definition 5).

THEOREM 18. *If* $A : B :: X : Y$, *then* $A + B : B :: X + Y : Y$.

For if $m \geqq n$, then $m(A + B) > nB$ and $m(X + Y) > nY$. If $m < n$, then, according as $m(A + B) \gtreqless nB$, so also is $mA \gtreqless (n - m)B$, $\therefore\ mX \gtreqless (n - m)Y$, $\therefore\ m(X + Y) \gtreqless nY$. Hence, by Definition 5, $A + B : B :: X + Y : Y$.

THEOREM 22. *If* $A : B :: X : Y$ *and* $B : C :: Y : Z$, *then "ex aequali"* $A : C :: X : Z$.

For it readily follows from Definition 5 that $mA : nB :: mX : nY$ and $nB : pC :: nY : pZ$. According as $mA \gtreqless pC$, so also is $mA : nB \gtreqless pC : nB$, $\therefore\ mX : nY \gtreqless pZ : nY$, $\therefore\ mX \gtreqless pZ$. Hence $A : C :: X : Z$.

* Euclid represents the magnitudes and their multiples by line segments. In other respects the proofs as here stated, except that of Theorem 18, are substantially as Euclid gives them.

THEOREM 24. *If both* $A : C :: X : Z$ *and* $B : C :: Y : Z$, *then also* $A + B : C :: X + Y : Z$.
For since $A : C :: X : Z$ and $C : B :: Z : Y$ (Definition 5), we have $A : B :: X : Y$, $\therefore A + B : B :: X + Y : Y$, $\therefore A + B : C :: X + Y : Z$ (Theorems 18, 22).

Theorem 22 implies a definition of the product of the two ratios $A : B$ and $B : C$, and Theorem 24 a definition of the sum of the two ratios $A : C$ and $B : C$, in the same sense that Definition 5 implies a definition of ratio itself. The Greeks called $A : C$ the ratio "compounded" of the ratios $A : B$ and $B : C$ or of any two ratios equal to these. Thus in VI, 23, Euclid proves that: *Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.*

There is a close relationship between the notion of the ratio of two incommensurables implied in Definition 5 and the irrational number as defined by Dedekind. The essential element in Definition 5 is the recognition of the fact that when A and B are incommensurable a definition of the ratio $A : B$ involves the comparison of mA and nB for all pairs of positive integral values of m and n, and the separation of these pairs into two classes, the class (m_1, n_1) for which $m_1A > n_1B$ and the class (m_2, n_2) for which $m_2A < n_2B$. This separation of all integral pairs (m, n) into the two classes (m_1, n_1) and (m_2, n_2) is identical with the Dedekind cut defining the irrational number which expresses the ratio $A : B$. Moreover Euclid was in possession of everything needed to give the notion of this cut the same expression in terms of the ratios $n_1 : m_1$ and $n_2 : m_2$ that the Dedekind definition would give it in terms of the fractions n_1/m_1 and n_2/m_2. For since $m_1A > n_1B$ and $m_2A < n_2B$, we have $m_1n_2 > n_1m_2$ and therefore $n_1 : m_1 < n_2 : m_2$, by Definition 7. Furthermore, by what is contained in the proof of X, 2, cited above, if any particular pair m_1', n_1' be assigned, another pair m_1, n_1 can be found such that $m_1'A - n_1'B > m_1A - n_1B$ and $n_1 > n_1'$ and therefore such that $n_1(m_1'A - n_1'B) > n_1'(m_1A - n_1B)$, or $n_1m_1' > n_1'm_1$, or $n_1 : m_1 > n_1' : m_1'$ (Definition 7); hence there is no greatest $n_1 : m_1$. And similarly there is no least $n_2 : m_2$. Hence, had Euclid created numbers to correspond to his ratios $n_1 : m_1$ and $n_2 : m_2$, he would have been in a position to create an irrational number to correspond to the ratio $A : B$ and to define this number ordinally with respect to the numbers $n_1 : m_1$ and $n_2 : m_2$ quite after the manner of Dedekind.

Since the natural numbers are included among the ratios $n : m$, and since $n_1 : m_1 < A : B < n_2 : m_2$ (as readily follows from Definition 7), the system of ratios, commensurable and incommensurable, as defined by Euclid, is in fact a part of the system of real numbers as we define that system, a part which becomes identical with the whole when the postulate is made that for every Dedekind cut there exists a pair of magnitudes A and B which will yield this cut in the manner above described (Cantor's axiom). Of course, Euclid had no such conception of his ratios; even the concept of $n : m$ as a single number was foreign to him.

It was merely as useful symbols in algebraic reckoning and without definition that irrational numbers first made their way into mathematics. The irrational was not defined until the renascence in mathematics of the fine critical insight possessed by the Greeks. The basis of sound definition was then found in notions already familiar to Eudoxus and Euclid.

Euclid avoids the fraction and the irrational by basing his theory of proportion upon the equality of ratios instead of ratio itself, and his definition is in terms of the indeterminate integral multipliers m and n. One who has read the writings of Kronecker—as extreme a purist with respect to number as the Greeks themselves—will notice an interesting parallelism between this procedure of Euclid and the way of escape from algebraic numbers found by Kronecker in the use of congruences involving indeterminates.*

The importance of the rôle which this general theory of proportion played in the Greek mathematics is not easily overstated. It was the key to the solution of the problem set by the discovery of incommensurability, the creation of a theory of geometric magnitude independent of the notion of numerical measurement. But it was also the instrument which for the Greek geometers served the purposes which are served for us by an algebraic algorithm and the equation. It was an unwieldy instrument but they developed great power in the use of it, as is shown for example in Euclid's geometric treatment of the quadratic (VI, 27–29) and in his discussion and classification of binomial line segments corresponding to quadratic irrationalities in the tenth book, but more strikingly still in the Conics of Apollonius.

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* L. Kronecker *Ueber den Zahlbegriff*. Journal für die reine und ange wandte Mathematik Band 101.