COORDINATE GEOMETRY
WITH VECTORS AND TENSORS

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WITH VECTORS AND TENSORS

BY

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OXFORD
AT THE CLARENDON PRESS

Oxford University Press, Amen House, London E.C.4

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BOMBAY CALCUTTA MADRAS KARACHI LAHORE DACCA
CAPE TOWN SALISBURY NAIROBI IBADAN ACCRA
KUALA LUMPUR HONG KONG

FIRST PUBLISHED 1958
REPRINTED LITHOGRAPHICALLY IN GREAT BRITAIN
AT THE UNIVERSITY PRESS, OXFORD
FROM SHEETS OF THE FIRST EDITION
1962

PREFACE

I AM grateful for help received in the preparation of this book. Dr. W. L. Ferrar suggested a number of improvements which I gladly accepted, and my pupils D. H. Smith and P. E. Smith gave valuable help in the working of the Examples. Many of these Examples are taken from various papers set in the University of Cambridge; I am grateful for permission to use them.

My thanks are particularly due to the Staff of the Clarendon Press for their never-failing skill and care.

E. A. M.

2 June 1958

INTRODUCTION

THE detailed study of three-dimensional coordinate geometry is at present unfashionable. This book aims to give a course representing the minimum that a generally educated young mathematician will need if he is to handle problems as they arise later with any degree of understanding. The pruning has been rigorous, and not everyone will agree with the selection. The fact that more remains than others may approve reflects the natural preferences of the author.

I have tried to establish an understanding of coordinate methods *before* introducing vectors. I believe that the beginner is often confused when presented with vectors without any background to display their advantages, and 'ordinary' coordinate solid geometry is, in any case, a subject interesting in its own rights.

The chapter on tensors took me long to write, and I hope it will be found helpful. It can be delayed for some time if necessary, but experience seems to indicate that tensors are found hard at first and that an early introduction may remove some of the terrors which come when they are applied to physical problems in the later courses. The discussion on the general quadric, with which the book concludes, has been adapted to suit both those who study tensors at once and those who delay.

CONTENTS

I

COORDINATES, DIRECTION COSINES, PROJECTION

1. Coordinates

THE purpose of this book is to discuss, with the help of algebra, the properties of geometrical figures in space. The first step is to explain how algebraic symbols are used to specify the position of a point.

The diagram (Fig. 1) may be regarded for the moment as representing a corner of a room with the plane OXY as floor and the planes OXZ, OYZ as adjacent walls. The position of any object is determined when its distances from these three planes are known; conversely, its distances from the planes are determined when the position is known.

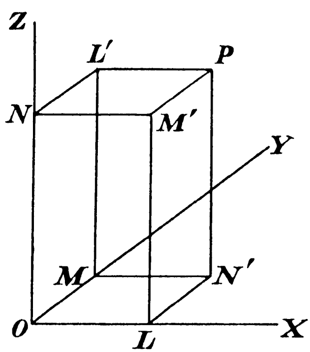

FIG. 1

Suppose, more abstractly, that OYZ, OZX, OXY are three mutually perpendicular planes meeting in pairs in the mutually perpendicular lines OX, OY, OZ. In the diagram, the plane XOY is conceived as 'horizontal' with OX running 'straight across the paper' after the manner familiar in plane coordinate geometry; OY, perpendicular to OX, is visualized as running 'into' the paper; and OZ, perpendicular to both, is 'vertical'.

The position of any point P is defined by the three distances, called COORDINATES:

x, the distance from the plane YOZ,

y, ,, ,, ,, ZOX,

z, ,, ,, ,, XOY.

The point is denoted by the symbol $P(x, y, z)$, or simply (x, y, z). The planes YOZ, ZOX, XOY are called the COORDINATE PLANES and their intersections OX, OY, OZ the COORDINATE AXES; the point O is called the ORIGIN.

ILLUSTRATION. The diagram (Fig. 2) represents a cube $OABCO'A'B'C'$ whose sides are all 4 units in length. Referred to coordinate axes OX, OY, OZ lying along the sides OA, OB, OC, the coordinates of the vertices are

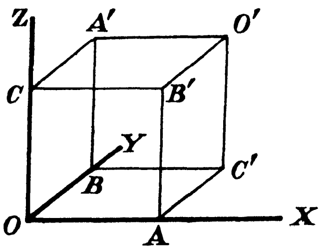

FIG. 2

O $(0, 0, 0)$,

A $(4, 0, 0)$, B $(0, 4, 0)$, C $(0, 0, 4)$,

A' $(0, 4, 4)$, B' $(4, 0, 4)$, C' $(4, 4, 0)$,

O' $(4, 4, 4)$.

EXAMPLES

The following examples all refer to the diagram of Fig. 1

1. The lengths of OL, OM, ON are 2, 3, 5 units respectively. Write down the coordinates of each of the points O, L, M, N, P, L', M', N'.

Write down also the coordinates of the middle points of OL, LM', $M'P$, PN', $N'M$, MO.

2. The coordinates of P are $(1, 4, 3)$. Write down the coordinates of L, M, N, L', M', N'.

3. The coordinates of the middle point of OP are $(1, 2, 4)$. Write down the coordinates of L', M', N'.

4. The point Q $(1, 5, 2)$ is taken on OP such that $OQ = \frac{1}{4}OP$. Write down the coordinates of P, L', M', N'.

5. The coordinates of P satisfy the equation

$$x + 2y + 3z = 1.$$

Prove that the coordinates of the middle point of OP satisfy the equation

$$2x + 4y + 6z = 1.$$

6. The coordinates of the middle point of OP satisfy the equation

$$x^2 + y^2 + z^2 = 8.$$

Prove that the coordinates of P satisfy the equation

$$x^2 + y^2 + z^2 = 32.$$

7. The coordinates of P are $(3, 2, 1)$. Write down the relation satisfied by the coordinates of any point in the plane $PM'N'$, and the two relations satisfied by the coordinates of any point on the line PL'.

2. Sign

The extension to negative values of the coordinates is closely analogous to what is already familiar in plane coordinate geometry. In the diagram (Fig. 3) the coordinate axes OX, OY, OZ

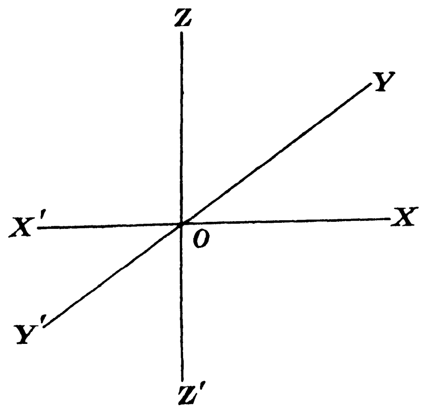

FIG. 3

are produced beyond O to X', Y', Z'. The coordinates of a point $P(x, y, z)$ are subjected to the rules:

x is positive (negative) if P is on the same side of the plane YOZ as X (X');

y is positive (negative) if P is on the same side of the plane ZOX as Y (Y');

z is positive (negative) if P is on the same side of the plane XOY as Z (Z').

3. Right-handed axes

We have incorporated in passing a fundamentally important convention about the axes themselves. Suppose that the lines $X'OX$, $Y'OY$ are drawn in a horizontal plane occupying the position customary in plane coordinate geometry (Fig. 4). There are then two possibilities for the line $Z'OZ$: it may be drawn with Z vertically 'upwards' or with Z vertically 'downwards'. We agree to adhere to the former. The axes then form what is called a RIGHT-HANDED SET. The reader should check that the axes shown in Fig. 3 (with OY regarded as running 'into' the paper) are subject to the convention:

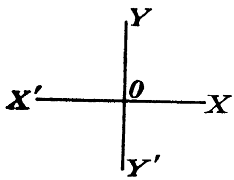

FIG. 4

A right-handed cork-screw turning from OY to OZ drives from O to X; turning from OZ to OX drives from O to Y; and turning from OX to OY drives from O to Z.

In contrast, the lines OX', OY, OZ form a LEFT-HANDED SET in the sense that a (right-handed) cork-screw turning from OY to OZ pulls from X' to O; turning from OZ to OX' pulls from Y to O; and turning from OX' to OY pulls from Z to O.

4. Sense on a line

It is often convenient to assign a direction of 'motion' along a line AB. To do so, we distinguish between the line regarded as described from A to B and the line regarded as described from B to A. When necessary, we use the notation:

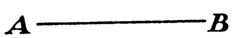

FIG. 5

$\overrightarrow{AB}$ for the line described from A to B,

$\overrightarrow{BA}$ for the line described from B to A.

One of these directions may be called *positive* and the other *negative*. In particular, the *positive* senses for lines parallel to the axes are defined to be $\overrightarrow{X'X}$, $\overrightarrow{Y'Y}$, $\overrightarrow{Z'Z}$.

The distinction in direction is called SENSE on the line.

The following theorem is elementary, but of basic importance:

If C is any point collinear with two points A, B, then

$$\overrightarrow{AB} = \overrightarrow{AC}+\overrightarrow{CB}.$$

The left-hand side is the distance from A to B in a definite sense; the right-hand side covers the same total distance in the same sense, but in two stages.

It is also worthy of remark that, if P is any point on the axis $X'OX$, then (with the convention $\overrightarrow{X'X}$ positive) the x-

coordinate of P satisfies the relation

$$x = \overrightarrow{OP}.$$

Analogous results hold for $Y'OY$ and $Z'OZ$.

5. Distance parallel to an axis

Let A, B be two given points whose join is parallel to, say, the axis $X'OX$. Suppose that their x-coordinates are x_1, x_2, and that the line AB meets the plane YOZ in U. Then

$$x_1 = \overrightarrow{UA}, \qquad x_2 = \overrightarrow{UB}.$$

Thus

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AU}+\overrightarrow{UB}\\ &= -\overrightarrow{UA}+\overrightarrow{UB}\\ &= -x_1+x_2,\end{aligned}$$

so that

$$\overrightarrow{AB} = x_2-x_1.$$

The corresponding (sensed) distances for directions parallel to the other axes are

$$y_2-y_1 \text{ for direction } Y'OY,$$

$$z_2-z_1 \quad \text{,,} \quad \text{,,} \quad Z'OZ.$$

6. The distance between two given points

To prove that *the distance between two points* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ *is given by the formula*

$$AB^2 = (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2.$$

Through A, B draw planes parallel to the coordinate planes so as to obtain the 'box' $APQRBP'Q'R'$ shown in the diagram (Fig. 6). The distance of P from the plane YOZ is equal to that of B, so that the x-coordinates of P, B are equal. Thus

$$\overrightarrow{AP} = x_2-x_1.$$

Similarly

$$\overrightarrow{AQ} = y_2-y_1, \qquad \overrightarrow{AR} = z_2-z_1.$$

By two applications of the theorem of Pythagoras,

$$\begin{aligned} AB^2 &= AR'^2+R'B^2 \\ &= AP^2+PR'^2+R'B^2 \\ &= AP^2+AQ^2+AR^2 \\ &= (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2. \end{aligned}$$

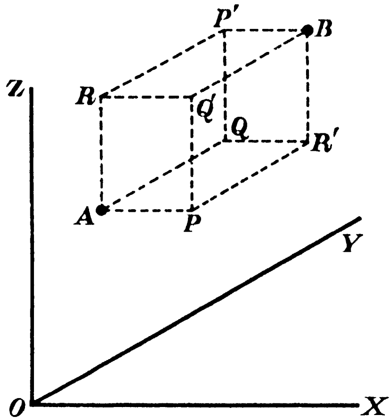

FIG. 6

NOTE. The result is true even when the line AB is parallel to a coordinate plane or to a coordinate axis. The modifications in the proof may easily be supplied.

ILLUSTRATION. *To find the condition that the point* $P(x, y, z)$ *should be at a distance of* 13 *units from the point* $A(-3, 4, 12)$.

The formula gives the relation

$$AP^2 = (x+3)^2+(y-4)^2+(z-12)^2,$$

so that $$(x+3)^2+(y-4)^2+(z-12)^2 = 169,$$

or $$x^2+y^2+z^2+6x-8y-24z = 0.$$

EXAMPLES

1. Find the lengths of the six edges of the tetrahedron whose vertices are the points $(0, 0, 0)$, $(6, 8, 10)$, $(2, -3, 7)$, $(-5, 3, 0)$.

2. Find the equation of the locus of a point $P(x, y, z)$ which is equidistant from the two fixed points $A(3, 5, 7)$, $B(2, -4, 6)$.

3. Find the equation of the locus of a point $P(x, y, z)$ which moves

so that its distances from the fixed points $A(2,0,4)$, $B(1,-3,-5)$ are connected by the relations

$$\text{(i)}\quad PA^2+PB^2 = AB^2, \qquad \text{(ii)}\quad PA^2-PB^2 = 5,$$
$$\text{(iii)}\quad PA = 2PB.$$

7. Translation of the axes

When setting up a coordinate system we may, in the first instance, select the axes in many ways, but it often becomes convenient in the course of subsequent work to transfer calculations to some alternative system. Such a process is called a TRANSFORMATION.

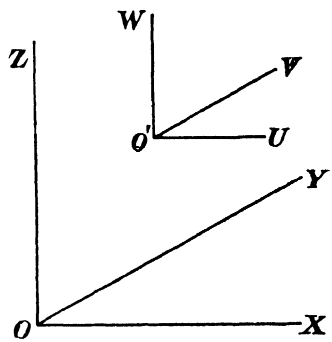

FIG. 7

Consider the special case in which the origin is transferred to a point O' while the new axes, denoted by $O'U$, $O'V$, $O'W$, remain parallel to OX, OY, OZ. The coordinates of O', referred to the original axes, may be denoted by (ξ, η, ζ).

Suppose that the coordinates of a point P are (x, y, z) referred to the axes OX, OY, OZ and (u, v, w) referred to $O'U$, $O'V$, $O'W$. Then, from the relations $\overrightarrow{AP} = x_2-x_1$, and so on, given in § 6 (p. 5),

$$\left.\begin{aligned} x &= u+\xi \\ y &= v+\eta \\ z &= w+\zeta \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} u &= x-\xi \\ v &= y-\eta \\ w &= z-\zeta \end{aligned}\right\}.$$

These relations are used to transfer the coordinates from either set of axes to the other.

8. Projection on a line

Let p be a given line. The PROJECTION of a point A on the line p is defined to be the foot of the perpendicular from A to p; call this point A'. If A, B are two given points, then the PROJECTION of the (sensed) segment $\overrightarrow{AB}$ on p is defined to be *the* *segment* $\overrightarrow{A'B'}$ *joining* A' *and* B', *the feet of the perpendiculars from* A *and* B *to* p.

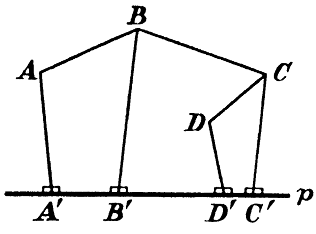

FIG. 8

Consider now a 'broken line' $ABCD$ (Fig. 8). Let A', B', C', D' be the feet of the perpendiculars from A, B, C, D to p. Then (p. 4)

$$\overrightarrow{A'D'} = \overrightarrow{A'B'}+\overrightarrow{B'C'}+\overrightarrow{C'D'},$$

so that *the projection of* $\overrightarrow{AD}$ *on* p *is the sum of the projections of* $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$ *on* p.

9. The angle between two skew lines

The angle between two SKEW (that is, non-intersecting) lines is the angle between two lines through a point, one parallel to each of the given lines. This conception is ambiguous in that it determines one or other of two supplementary angles, but the ambiguity is usually unimportant. For the sake of precision, however, we give an exact definition to be used if required:

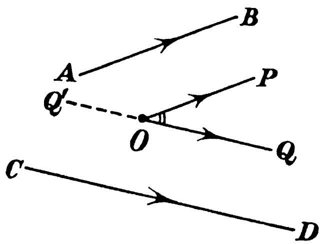

FIG. 9

Let $\overrightarrow{AB}$, $\overrightarrow{CD}$ be two *sensed* skew lines. *The angle between* $\overrightarrow{AB}$, $\overrightarrow{CD}$ is defined as follows:

Take any point O in space; through O draw the ray $\overrightarrow{OP}$ parallel to $\overrightarrow{AB}$ in the sense $\overrightarrow{AB}$, and the ray $\overrightarrow{OQ}$ parallel to $\overrightarrow{CD}$ in the sense $\overrightarrow{CD}$. *The angle between* $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$ *is defined to be the angle* POQ; this angle may be acute or obtuse, but we restrict it, for convenience, to be not greater than π. It is independent of the position of O.

Note that the angle between $\overrightarrow{AB}$ and $\overrightarrow{DC}$ (being the angle between OP and OQ', where $\overrightarrow{OQ'}$ is opposite in sense to $\overrightarrow{OQ}$) is the supplement of the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ (Fig. 9).

10. Direction cosines

The direction of a given line p is the same as that of a parallel line p' through the origin. In order to describe the latter, take an arbitrary point P on p', and complete the box shown in the diagram (Fig. 10) by drawing planes through P parallel to the coordinate planes. The angles between OP and the axes are POX, POY, POZ, and the cosines of these angles are called the DIRECTION COSINES of p. It is customary to use the letters l, m, n (or the Greek letters λ, μ, ν) for direction cosines, so that

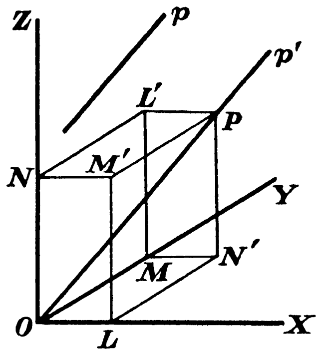

FIG. 10

$$l = \cos POX = \frac{OL}{OP},$$

$$m = \cos POY = \frac{OM}{OP},$$

$$n = \cos POZ = \frac{ON}{OP}.$$

These definitions have contained the implicit, and simplifying, assumption that the angles POX, POY, POZ are all acute. More generally, suppose that a *sense* is assigned along the line OP. It is usually immaterial which of the two available senses is selected, but, for precision, let $\overrightarrow{OP}$ be positive when P is on the same side of the plane XOY as Z. Let L be the projection (p. 7) of P on OX. The DIRECTION COSINE l is now *defined* by the relation

$$l = \overrightarrow{OL}/\overrightarrow{OP}.$$

Similarly,

$$m = \overrightarrow{OM}/\overrightarrow{OP},$$

$$n = \overrightarrow{ON}/\overrightarrow{OP}.$$

These three relations give complete precision.

The convention that $\overrightarrow{OP}$ is 'upwards' implies that n is always positive during its operation; we shall, indeed, sometimes refer to 'the n-positive convention'.

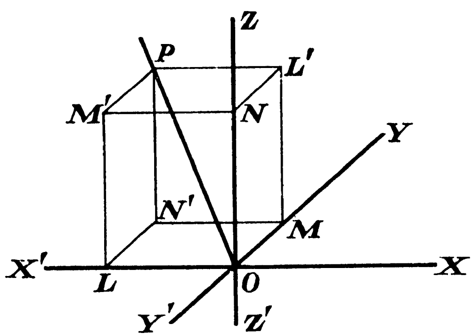

FIG. 11

NOTE. Too much emphasis should not be placed on the convention determining the sense of $\overrightarrow{OP}$; the agreement is merely one that can be invoked to settle doubtful cases. If, however, it becomes necessary, the two further considerations must also be kept in mind:

(i) If $n = 0$, so that the line is parallel to the plane XOY, take P on the same side of the plane ZOX as Y. Then m is always positive.

(ii) If $m = n = 0$, so that the line is parallel to the axis $X'OX$, take its sense to be that of $\overrightarrow{X'OX}$. Then l is always positive; in fact, $l = +1$.

COROLLARIES. (i) *If $P(x, y, z)$ is the point such that $\overrightarrow{OP} = r$ and the direction cosines of $\overrightarrow{OP}$ are (l, m, n), then*

$$x = lr, \quad y = mr, \quad z = nr.$$

For $x = \overrightarrow{OL} = l.\overrightarrow{OP} = lr$; similarly y, z.

(ii) *If* $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ *are two points such that* $\overrightarrow{PQ} = r$ *and the direction cosines of* $\overrightarrow{PQ}$ *are* (l, m, n), *then*

$$x_2 = x_1+lr, \quad y_2 = y_1+mr, \quad z_2 = z_1+nr.$$

This is merely the result of applying a translation of the axes (p. 7) to Corollary (i).

11. Given the direction cosines, to determine the direction

A line is uniquely determined in direction when its direction cosines are given. It is, in fact, possible *to construct the line through the origin having given direction cosines* l, m, n:

On the coordinate axes mark off $\overrightarrow{OL} = l$, $\overrightarrow{OM} = m$, $\overrightarrow{ON} = n$, paying attention to signs. The diagram (Fig. 11) illustrates the case when l is negative and m, n are positive. Complete the box as shown, by drawing planes through L, M, N parallel to YOZ, ZOX, XOY. The diagonal OP of this box is in the direction (l, m, n).

12. The formula $l^2+m^2+n^2 = 1$

It is an immediate consequence of the theorem of Pythagoras that, for the box $OLMNPL'M'N'$ defined in § 10 (p. 9),

$$OP^2 = OL^2+OM^2+ON^2,$$

whatever the senses of $\overrightarrow{OL}$, $\overrightarrow{OM}$, $\overrightarrow{ON}$. Hence

$$l^2+m^2+n^2 = \frac{OL^2}{OP^2}+\frac{OM^2}{OP^2}+\frac{ON^2}{OP^2}$$
$$= 1.$$

This formula is very important. It solves the problem, *to find actual direction cosines when their ratios are known.* Suppose, for example, that they are proportional to three numbers a, b, c, so that they are equal to

$$ka, \quad kb, \quad kc$$

for some value of k. Since then

$$k^2a^2+k^2b^2+k^2c^2 = 1,$$

it follows that $$k = \frac{\pm 1}{\sqrt{(a^2+b^2+c^2)}},$$

and the actual direction cosines are thus

$$\frac{\pm a}{\sqrt{(a^2+b^2+c^2)}}, \qquad \frac{\pm b}{\sqrt{(a^2+b^2+c^2)}}, \qquad \frac{\pm c}{\sqrt{(a^2+b^2+c^2)}}.$$

In problems where the 'n-positive' convention (p. 10) is being used, that choice of sign must be taken which makes $\pm c/\sqrt{(a^2+b^2+c^2)}$ positive; for example, if $a = 3$, $b = -4$, $c = -12$, then $k = \pm\frac{1}{13}$, and so $l = -\frac{3}{13}$, $m = \frac{4}{13}$, $n = \frac{12}{13}$.

When the direction of a line is specified by three numbers a, b, c the sum of whose squares is not equal to unity, the numbers are called DIRECTION RATIOS of the line.

13. The length of a projection

Let $\overrightarrow{AB}$ be a given segment of a straight line, and $\overrightarrow{p}$ some other given (sensed) line. Draw AA', BB' perpendicular to $\overrightarrow{p}$.

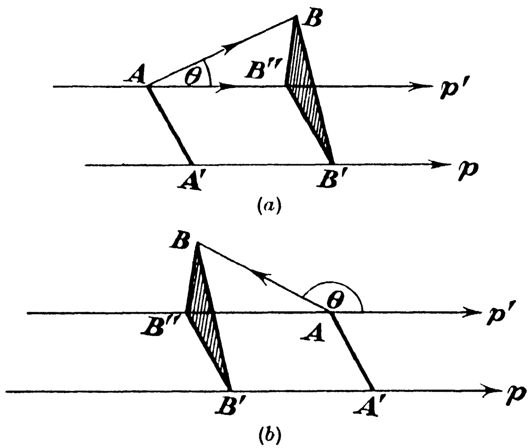

FIG. 12

Then the projection of $\overrightarrow{AB}$ on $\overrightarrow{p}$ is $\overrightarrow{A'B'}$. The angle θ between $\overrightarrow{AB}$ and its projection is found (p. 8) by drawing through A the line $\overrightarrow{p'}$ parallel to $\overrightarrow{p}$. [In Fig. 12 (a), θ is acute; in Fig. 12 (b), it is obtuse.]

Draw $B'B''$ perpendicular to p'. Then $A'B'$ is perpendicular to both $B'B''$ and $B'B$, so that p, which is $A'B'$, is perpendicular to the plane $BB'B''$. But p' is parallel to p, so that p', which is AB'', is also perpendicular to the plane $BB'B''$; in particular, AB'' is perpendicular to BB''.

In Fig. 12 (a), it therefore follows that $\cos\theta = AB''/AB$, and in Fig. 12 (b), that $\cos\theta = -AB''/AB$. Hence in *either* figure, with sensed lines,

$$\cos\theta = \overrightarrow{AB''}/\overrightarrow{AB}$$

$$= \overrightarrow{A'B'}/\overrightarrow{AB}.$$

Thus *the projection* $\overrightarrow{A'B'}$ *of the segment* $\overrightarrow{AB}$ *on the line* $\overrightarrow{p}$ *is given by the relation*

$$\overrightarrow{A'B'} = \overrightarrow{AB}\cos\theta,$$

where θ *is the angle between* $\overrightarrow{AB}$ *and* $\overrightarrow{p}$.

COROLLARIES. (i) *The numerical value of* $A'B'$ *is equal to the numerical value of* $AB\cos\theta$. This result is often all that is wanted.

(ii) Since $A'B' = AB''$, it follows that *the projections of* AB *on all parallel lines are equal.*

14. Projection of a segment of a coordinate axis upon the direction (l, m, n)

Let OP be the line through the origin with given direction cosines (l, m, n). Take two points A, B on the axis $X'OX$ so named, as we can, that $\overrightarrow{AB}$ is positive, and let their projections on OP be A', B'. Then (p. 10), with the 'n-positive' convention,

$$\overrightarrow{OA'} = \overrightarrow{OA}\cos A'OX = l\,\overrightarrow{OA},$$

$$\overrightarrow{OB'} = \overrightarrow{OB}\cos B'OX = l\,\overrightarrow{OB},$$

so that, on subtracting,

$$\overrightarrow{A'B'} = l\,\overrightarrow{AB}.$$

If l is positive (negative), then $\overrightarrow{A'B'}$ is positive (negative) so that $\overrightarrow{A'B'}$ is 'upwards' ('downwards').

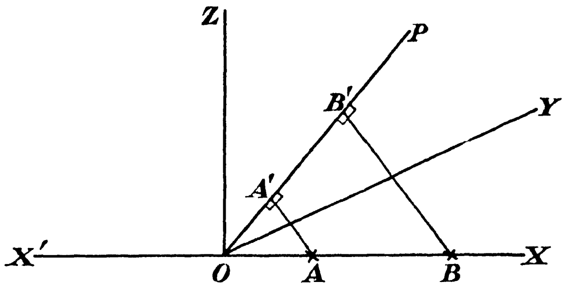

FIG. 13

Similar results hold for the other axes. The 'n-positive' convention implies that, if C, D are two points on $Z'OZ$ such that $\overrightarrow{CD}$ is positive, then the projection $\overrightarrow{C'D'} \equiv n\,\overrightarrow{CD}$ is necessarily 'upwards'.

15. The projection formula

To prove that *the length of the projection of the segment joining the points* $A(x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$ *upon a line with direction*

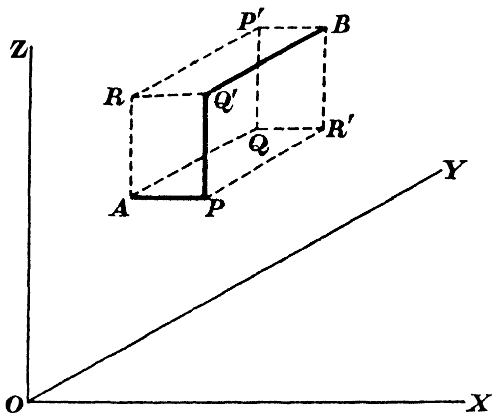

FIG. 14

cosines (l, m, n) *is*

$$l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1).$$

Through A, B draw planes parallel to the coordinate planes to obtain the 'box' $APQRBP'Q'R'$ shown in the diagram (Fig. 14). The projection of $\overrightarrow{AB}$ on the given line (not shown in the diagram) is (p. 8) the sum of the projections on it of

$$\overrightarrow{AP}+\overrightarrow{PQ'}+\overrightarrow{Q'B}.$$

Also, by § 14, the projections of $\overrightarrow{AP}$, $\overrightarrow{PQ'}$, $\overrightarrow{Q'B}$ are respectively

$$l(x_2-x_1), \quad m(y_2-y_1), \quad n(z_2-z_1).$$

Hence *the projection of* $\overrightarrow{AB}$ *is*

$$l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1).$$

16. The angle between two lines

To prove that *the angle between two lines with direction cosines* (l, m, n), (λ, μ, ν) *is* θ, *where*

$$\cos\theta = l\lambda+m\mu+n\nu.$$

Let OP, OQ be the lines through the origin with direction cosines (l, m, n), (λ, μ, ν), and let $\overrightarrow{OP} \equiv r$. Then (p. 10), if P is the point (x_1, y_1, z_1),

$x_1 = lr, \quad y_1 = mr, \quad z_1 = nr.$

By the projection formula (p. 14), the length of $\overrightarrow{OP'}$, the projection of $\overrightarrow{OP}$ on OQ, is given by the relation

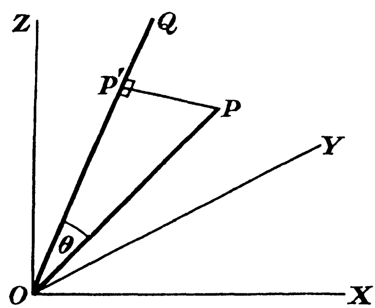

FIG. 15

$$\overrightarrow{OP'} = x_1\lambda+y_1\mu+z_1\nu$$
$$= r(l\lambda+m\mu+n\nu).$$

But (p. 13) $\qquad \overrightarrow{OP'} = \overrightarrow{OP}\cos\theta = r\cos\theta.$

Hence $\qquad \cos\theta = l\lambda+m\mu+n\nu.$

17. The condition for perpendicularity

It follows at once from the preceding paragraph that, *if the two directions* (l, m, n), (λ, μ, ν) *are perpendicular* (so that $\cos\theta = 0$), *then*

$$l\lambda + m\mu + n\nu = 0.$$

The converse result is also true.

EXAMPLES

1. If $A \equiv (1,0,0)$, $B \equiv (0,1,0)$, $C \equiv (0,0,1)$, find the direction cosines of the lines BC, CA, AB and the length of the projection of AB on AC.

2. Prove that the points $A \equiv (4,3,2)$, $B \equiv (8,7,6)$, $C \equiv (10,7,4)$, $D \equiv (6,3,0)$ are the vertices of a parallelogram $ABCD$, and find the angle between the sides AB, AD and the angle between the diagonals AC, BD.

3. The vertices of a triangle are

$$A \equiv (1,2,3), \quad B \equiv (5,6,5), \quad C \equiv (4,6,15).$$

Find the lengths of the sides AB, AC and the projection of AB on AC.

4. Find the acute angle between the lines whose direction ratios are $(-2, 2, -1)$ and $(12, -15, 16)$.

5. Find the coordinates of the two points at a distance of 5 units from the point $(3, 3, 3)$ along the line whose direction ratios are $(-9, 12, 20)$.

6. Prove that the line joining the points $(1, 1, 2)$, $(6, 5, -5)$ is perpendicular to the line joining the points $(0, 3, 2)$, $(4, 5, 6)$.

7. If $A \equiv (0,0,0)$ and $B \equiv (3,4,12)$, find the projection of AB on a line whose direction ratios are $(2, 2, -1)$.

8. Prove that, as θ and ϕ vary, the point

$$(1+3\sin\theta\cos\phi,\ 2+3\sin\theta\sin\phi,\ 3\cos\theta)$$

remains at a constant distance from the point $(1, 2, 0)$.

9. Prove that, if the line joining the points $(-2, -1, -2)$ and $(2, 1, 2)$ subtends a right angle at the point (x, y, z), then $x^2+y^2+z^2 = 9$.

10. If $A \equiv (a,b,c)$, $B \equiv (-a,-b,-c)$, $C \equiv (x,y,z)$, prove that

$$CA^2+CB^2 = 2OA^2+2OC^2.$$

11. The point $P(x,y,z)$ lies on the right circular cone, of angle $\frac{1}{6}\pi$ and with vertex the origin, whose axis is the line through the origin with direction cosines $(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$. Prove that

$$27(x^2+y^2+z^2) = 4(x+2y-2z)^2.$$

12. The point $P(x,y,z)$ lies on the right circular cone, of angle $\frac{1}{3}\pi$ and with vertex the origin, whose axis is the line through the origin with direction ratios $(2, -5, 4)$. Prove that

$$45(x^2+y^2+z^2) = 4(2x-5y+4z)^2.$$

II

THE STRAIGHT LINE AND THE PLANE

1. The distance formula

LET $A\,(a, b, c)$ be a given point and (l, m, n) a given direction; and let $P\,(x, y, z)$ be the point distant $r \equiv \overrightarrow{AP}$ from A along the line through A parallel to the given direction. Then (p. 11) *the coordinates of P may be expressed in terms of r as a parameter by means of the formulae*

$$x = a+lr, \quad y = b+mr, \quad z = c+nr.$$

These formulae express x, y, z by means of a *distance*. The next paragraph gives an alternative expression by means of a *ratio*.

2. The ratio formula

To prove that, *if $P\,(x, y, z)$ is the point on the straight line joining two points $A\,(x_1, y_1, z_1)$, $B\,(x_2, y_2, z_2)$ such that $\overrightarrow{AP}/\overrightarrow{PB} = k$, then*

$$x = \frac{x_1+kx_2}{1+k}, \qquad y = \frac{y_1+ky_2}{1+k}, \qquad z = \frac{z_1+kz_2}{1+k}.$$

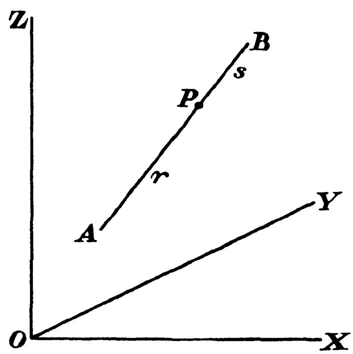

FIG. 16

Suppose that the direction cosines of the line AB are (l, m, n).

Let $\overrightarrow{AP} = r$, $\overrightarrow{PB} = s$, so that $r/s = k$. Then (§ 1)

$$x-x_1 = lr,\quad y-y_1 = mr,\quad z-z_1 = nr,$$

$$x_2-x = ls,\quad y_2-y = ms,\quad z_2-z = ns.$$

Hence
$$\frac{x-x_1}{x_2-x} = \frac{r}{s} = k,$$

so that
$$x-x_1 = k(x_2-x),$$

or
$$x(1+k) = x_1+kx_2.$$

Hence
$$x = \frac{x_1+kx_2}{1+k}.$$

Similarly
$$y = \frac{y_1+ky_2}{1+k},$$

$$z = \frac{z_1+kz_2}{1+k}.$$

An alternative statement of the same result is that *the coordinates of the points of the line AB may be expressed in the form*

$$x = \lambda x_1+\mu x_2,\quad y = \lambda y_1+\mu y_2,\quad z = \lambda z_1+\mu z_2,$$

where the parameters λ, μ are connected by the relation $\lambda+\mu = 1$.

3. The equations of a straight line

The work of the preceding paragraphs establishes two PARAMETRIC FORMS for the coordinates of the points on a straight line:

(i) *If the line is through the point $A(a, b, c)$ with direction cosines (l, m, n), then*

$$x = a+lr,\quad y = b+mr,\quad z = c+nr,$$

where the parameter r is the length $\overrightarrow{AP}$.

(ii) *If the line joins the two points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, then*

$$x = \frac{x_1+kx_2}{1+k},\qquad y = \frac{y_1+ky_2}{1+k},\qquad z = \frac{z_1+kz_2}{1+k},$$

where the parameter k is the ratio $\overrightarrow{AP}/\overrightarrow{PB}$.

Note that this second parametric representation may also be expressed in the form

$$x = px_1+qx_2, \quad y = py_1+qy_2, \quad z = pz_1+qz_2,$$

where p, q vary from point to point of the line, subject to the condition

$$p+q = 1.$$

These representations may be re-cast to give EQUATIONS FOR THE STRAIGHT LINE; that is, equations connecting the coordinates of its points:

(i) From the first form, it follows that *the coordinates of the points of the straight line satisfy the* **two** *equations*

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}.$$

These equations are suitable for a line passing through a given point (a, b, c) in a given direction (l, m, n).

(ii) From the second form, we have, on returning to its derivation, the equations (p. 18)

$$x-x_1 = lr,$$

$$x_2-x = ls,$$

or, adding,

$$x_2-x_1 = l(r+s),$$

so that

$$\frac{x-x_1}{x_2-x_1} = \frac{r}{r+s},$$

with analogous results for the coordinates y and z. This slight variant gives EQUATIONS OF THE STRAIGHT LINE in the form in which they are usually exhibited:

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1},$$

the value of each ratio being $r/(r+s)$.

These equations are suitable for a line passing through two given points (x_1, y_1, z_1), (x_2, y_2, z_2).

NOTE. The two forms of equations are very important, but it is often better to use the *parametric* forms for the coordinates instead.

EXAMPLES

1. The coordinates of a point A are $(3, -1, 2)$ referred to axes OX, OY, OZ. Find the coordinates of the middle point Q of OA, and of the point R on OA such that $\overrightarrow{OR} = 5\overrightarrow{AO}$.

2. Find the coordinates of the points of trisection of the line joining the points $(2, 1, 5)$, $(-3, 2, -1)$.

3. Find the coordinates of the extremities of the line of which the points $(9, 6, 3)$ and $(-1, 2, -3)$ are the points of trisection.

4. The coordinates of the points A, B are $(2, 1, 0)$, $(-1, 3, 7)$ respectively. Find the coordinates of the point P such that $\overrightarrow{AP}/\overrightarrow{PB} = 5$, and of the point Q such that $\overrightarrow{AQ}/\overrightarrow{QB} = -5$.

5. Prove that the line joining the points $(2, 4, 3)$, $(4, 10, 7)$ meets the line joining the points $(2, -1, 5)$, $(5, -7, 17)$.

6. Find the coordinates of the points in which the line joining the points $(-2, 3, 7)$, $(6, -1, 2)$ meets the coordinate planes.

7. *Desargues' theorem.* The two triangles ABC, $A'B'C'$ are so related that AA', BB', CC' pass through the origin O. Prove that the points of intersection $(BC, B'C')$, $(CA, C'A')$, $(AB, A'B')$ are collinear.

4. The equation of a plane

Suppose that a given plane meets the axes of coordinates in points A, B, C, and denote by P the foot of the perpendicular from the origin to the plane.

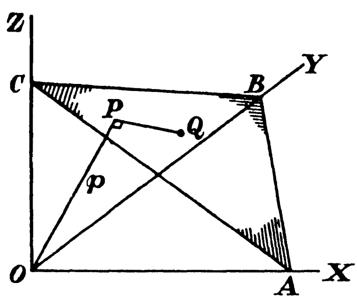

FIG. 17

Let $\overrightarrow{OP}$ be of length p, and let its direction cosines be (l, m, n). With the convention of signs that n is positive, the value of p will be positive (negative) if $\overrightarrow{OP}$ is 'upwards' ('downwards').

If Q (x, y, z) is an arbitrary point of the plane, the projection of $\overrightarrow{OQ}$ on the line OP is equal to $\overrightarrow{OP}$. Hence (p. 15)

$$lx+my+nz = p.$$

This equation, satisfied by the coordinates of all points lying in the plane, is called the EQUATION OF THE PLANE.

Conversely, if the coordinates of a point $Q\,(x,y,z)$ are subject to the relation

$$lx+my+nz = p,$$

then the projection of $\overrightarrow{OQ}$ on the fixed direction (l,m,n) has constant magnitude p, so that Q lies in a fixed plane to which the direction (l,m,n) is perpendicular.

Any line perpendicular to a plane is said to be NORMAL to it. The derivation of the equation shows that *the normals to the plane* $lx+my+nz = p$ *have direction cosines* (l,m,n).

The equation may be cast into an alternative form:

The plane meets the line OX, given by $y = z = 0$, where

$$lx = p,$$

so that $$\overrightarrow{OA} = p/l.$$

Similarly, $$\overrightarrow{OB} = p/m, \qquad \overrightarrow{OC} = p/n.$$

The position of the plane is determined when $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are known; write

$$\overrightarrow{OA} = a, \quad \overrightarrow{OB} = b, \quad \overrightarrow{OC} = c,$$

so that $$l = p/a, \quad m = p/b, \quad n = p/c.$$

The equation of the plane is thus

$$\frac{px}{a}+\frac{py}{b}+\frac{pz}{c} = p,$$

or $$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1.$$

This alternative form of equation, in terms of the constants a, b, c, is known as INTERCEPT FORM. (Its use presupposes that *the plane does* **not** *pass through the origin.*)

5. The general linear equation

The form of the GENERAL LINEAR EQUATION in x, y, z is

$$ax+by+cz+d = 0,$$

where a, b, c, d are constants. For precision of statement suppose that c is positive, multiplying throughout by -1 to that end if necessary. Divide by $+\sqrt{(a^2+b^2+c^2)}$, and write

$$\frac{a}{\sqrt{(a^2+b^2+c^2)}} = l, \qquad \frac{b}{\sqrt{(a^2+b^2+c^2)}} = m,$$

$$\frac{c}{\sqrt{(a^2+b^2+c^2)}} = n, \qquad \frac{d}{\sqrt{(a^2+b^2+c^2)}} = -p.$$

The equation then assumes the form (with $l^2+m^2+n^2 = 1$)

$$lx+my+nz = p.$$

Thus (§ 4) *the general linear equation represents a* PLANE, whose normals have direction cosines (l, m, n) and whose distance $\overrightarrow{OP}$ from the origin is equal to p.

The constants a, b, c of this general form are the DIRECTION RATIOS (*p.* 12) *of the normals to the plane.*

ILLUSTRATION. *To indicate the position of the plane*

$$x+2y-2z-6 = 0.$$

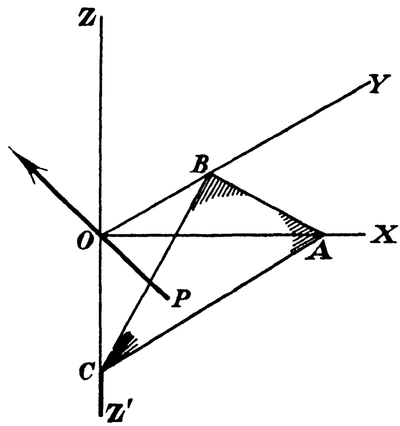

FIG. 18

With positive coefficient for z, the equation is

$$-x-2y+2z = -6.$$

Divide by $+\sqrt{(1^2+2^2+2^2)} = 3$. Thus

$$-\tfrac{1}{3}x-\tfrac{2}{3}y+\tfrac{2}{3}z = -2.$$

If OP is drawn perpendicular to the plane, the direction cosines of OP are $(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$. The length of the perpendicular from the origin is -2, so that P lies 'below' the origin.

The plane meets the axes OX, OY, OZ in points A, B, C, where $\overrightarrow{OA} = 6$, $\overrightarrow{OB} = 3$, $\overrightarrow{OC} = -3$.

6. The length of the perpendicular from a given point to a plane

To prove that, *if N is the foot of the perpendicular from the point $Q\,(x_1, y_1, z_1)$ to the plane*

$$lx+my+nz = p$$

(*where l, m, n are actual direction cosines, n positive*), *then*

$$\overrightarrow{NQ} = lx_1+my_1+nz_1-p.$$

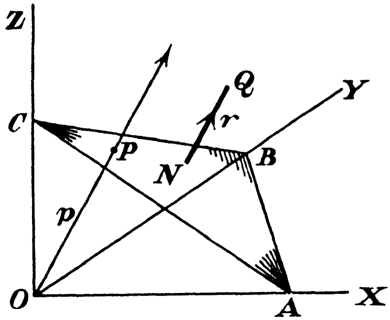

FIG. 19

Suppose that N is the point (u, v, w), where, since N is in the plane,

$$lu+mv+nw = p.$$

If $\overrightarrow{NQ} = r$, then (p. 17)

$$x_1-u = lr, \quad y_1-v = mr, \quad z_1-w = nr,$$

so that $$u = x_1-lr, \quad v = y_1-mr, \quad w = z_1-nr.$$

Hence $$l(x_1-lr)+m(y_1-mr)+n(z_1-nr) = p.$$

Also, since l, m, n are actual direction cosines,

$$l^2+m^2+n^2 = 1,$$

so that $$lx_1+my_1+nz_1-r = p,$$

or $$r = lx_1+my_1+nz_1-p.$$

The modifications to be made in the proof when $n = 0$, or when $m = n = 0$, are obvious. Compare p. 10.

ILLUSTRATION. *To find the lengths of the perpendiculars from the points* $A\,(1, 2, 3)$, $B\,(-6, 0, 0)$ *to the plane* $x+y+z = 0$.

The equation of the plane is

$$(1/\sqrt{3})x+(1/\sqrt{3})y+(1/\sqrt{3})z = 0.$$

Hence the length of the perpendicular from A is

$$(1/\sqrt{3}).1+(1/\sqrt{3}).2+(1/\sqrt{3}).3 = +2\sqrt{3}.$$

The point A is therefore 'above' the plane, distant $2\sqrt{3}$ units from it.

The length of the perpendicular from B is

$$(1/\sqrt{3})(-6)+(1/\sqrt{3}).0+(1/\sqrt{3}).0 = -2\sqrt{3}.$$

The point B is therefore 'below' the plane, distant $2\sqrt{3}$ units from it.

ILLUSTRATION (THE CASE $n = 0$). *To find the lengths of the perpendiculars from the points* $A\,(1, 2, 3)$, $B\,(5, 0, 0)$ *to the plane* $3x-4y = 5$.

The equation is written with positive coefficient for y in the form

$$-3x+4y+5 = 0,$$

or, after division by $\sqrt{\{(-3)^2+4^2\}}$,

$$(-\tfrac{3}{5})x+\tfrac{4}{5}y+1 = 0.$$

Hence the length of the perpendicular from A is

$$(-\tfrac{3}{5}).1+\tfrac{4}{5}.2+1 = +2.$$

The point A is therefore on the 'Y' side of the plane, distant 2 units from it.

The length of the perpendicular from B is

$$(-\tfrac{3}{5}).5+\tfrac{4}{5}.0+1 = -2.$$

The point B is therefore on the 'Y'' side of the plane, distant 2 units from it.

7. The plane through three given points

To find the equation of the plane determined by the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

The equation is of the form

$$ax+by+cz+d=0,$$

where

$$ax_1+by_1+cz_1+d=0,$$
$$ax_2+by_2+cz_2+d=0,$$
$$ax_3+by_3+cz_3+d=0.$$

If desired, the last three equations can be solved for the ratios $a:b:c:d$ and the equation of the plane is then determined. It is better, however, to eliminate the ratios $a:b:c:d$ from the four equations so as to obtain *the equation of the plane in the* DETERMINANTAL *form*

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

This is a convenient point at which to introduce some notation that will be wanted later. The equation, when expanded in terms of the first row, is

$$\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} x - \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} z - \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0,$$

or

$$Ax+By+Cz=D,$$

where

$$A \equiv \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \quad B \equiv \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}, \quad C \equiv \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

$$D \equiv \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

If Δ is defined by the identity

$$\Delta^2 \equiv A^2+B^2+C^2,$$

then the direction cosines of the normal to the plane are (numerically)

$$A/\Delta, \quad B/\Delta, \quad C/\Delta.$$

COROLLARY. *The length of the perpendicular from an arbitrary point* (x_4, y_4, z_4) *to the plane is the numerical value of*

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \div \Delta.$$

In fact, the equation of the plane is, with 'direction cosine' coefficients,

$$(A/\Delta)x + (B/\Delta)y + (C/\Delta)z - (D/\Delta) = 0,$$

being precisely the same, by definition, as

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} \div \Delta = 0.$$

The length of the perpendicular is found by writing (x_4, y_4, z_4) for (x, y, z) in the left-hand side of the first equation, and, consequently, in the left-hand side of the second equation also. Rearrangement of the order of the rows, which does not affect the numerical value, then gives the quoted formula.

8. The points of an arbitrary plane

Suppose that $A\,(x_1, y_1, z_1)$, $B\,(x_2, y_2, z_2)$, $C\,(x_3, y_3, z_3)$ are three given non-collinear points. The position of a point $P\,(x, y, z)$ of the plane ABC may be specified as follows:

(i) Let $L\,(\xi, \eta, \zeta)$ be the point of BC such that

$$\overrightarrow{BL}/\overrightarrow{LC}$$

has the given value u.

(ii) Let P be the point of AL such that

$$\overrightarrow{AP}/\overrightarrow{PL}$$

has the given value v.

Then (p. 18)
$$\xi = \frac{x_2 + ux_3}{1+u},$$

and
$$x = \frac{x_1 + v\xi}{1+v}.$$

Hence

$$x = \frac{x_1 + \dfrac{v(x_2+ux_3)}{1+u}}{1+v}$$

$$= \lambda x_1 + \mu x_2 + \nu x_3,$$

say, where

$$\lambda = \frac{1}{1+v}, \qquad \mu = \frac{v}{(1+u)(1+v)}, \qquad \nu = \frac{uv}{(1+u)(1+v)}.$$

Moreover, λ, μ, ν satisfy the relation

$$\lambda+\mu+\nu = 1.$$

Hence *the coordinates of the points of the plane ABC may be expressed in the form*

$$x = \lambda x_1+\mu x_2+\nu x_3, \qquad y = \lambda y_1+\mu y_2+\nu y_3,$$
$$z = \lambda z_1+\mu z_2+\nu z_3,$$

where the parameters λ, μ, ν *are connected by the relation*

$$\lambda+\mu+\nu = 1.$$

EXAMPLES

1. Find the coordinates of the three points in which the plane through the points $(2, 5, 1)$, $(3, -2, 6)$, $(1, 4, -3)$ meets the axes of coordinates.

2. Prove that the coordinates (x, y, z) of any point in the plane through the three points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ satisfy the equation

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1.$$

3. Prove that the plane through the points $(2, 7, 5)$, $(4, 2, 1)$, $(2, 3, 2)$ passes through the origin.

4. Find the coordinates of the point in which the line joining the origin to the point $(-1, -2, 3)$ meets the plane through the points $(4, 1, 2)$, $(3, 5, 7)$, $(1, 1, 1)$.

5. Find the coordinates of the point in which the line joining the points $(0, 1, 2)$, $(2, 1, 0)$ meets the plane through the points $(0, 0, 0)$, $(5, 3, 7)$, $(-2, 6, 1)$.

6. Find the lengths of the perpendiculars from the points $(0, 0, 0)$, $(1, 2, 3)$, $(3, -2, 1)$ to the planes

(i) $2x-y+2z = 9$,

(ii) $3x+4y-12z = 26$,

(iii) $12x-3y-4z = -52$.

7. Find the equations of the faces of the tetrahedron whose vertices are the points $(0, 3, 5)$, $(2, 0, -2)$, $(3, 7, 0)$, $(1, 2, 3)$.

Find also the coordinates of the points in which the line joining the origin to the point $(3, -4, 12)$ meets these planes.

8. Find the altitudes of the triangle whose vertices are the points $(0, 0, 1)$, $(3, 4, 5)$, $(-2, 3, 1)$, and verify that these altitudes are concurrent.

9. Find the point of intersection of the altitudes of the triangle whose vertices are the points $(0, 0, 0)$, $(1, 1, 1)$, (a, b, c).

9. The planes through a straight line

Two planes meet in a straight line, and their equations consequently determine the points of that line. Thus the TWO EQUATIONS TO DETERMINE A STRAIGHT LINE may be taken in the general form

$$\begin{cases} a_1 x+b_1 y+c_1 z+d_1 = 0, \\ a_2 x+b_2 y+c_2 z+d_2 = 0. \end{cases}$$

This choice of two equations is not unique. The equation

$$a_1 x+b_1 y+c_1 z+d_1+\lambda(a_2 x+b_2 y+c_2 z+d_2) = 0$$

also represents, for any value of λ, a plane through the line; for (i) it is linear in x, y, z and therefore represents some plane, and (ii) it is satisfied by the coordinates of all points for which

$$a_1 x+b_1 y+c_1 z+d_1 = 0, \qquad a_2 x+b_2 y+c_2 z+d_2 = 0$$

simultaneously.

By varying the value of λ, the equations of all planes through the straight line may be found.

NOTATION. It is often convenient to use a single symbol, say L, to denote the expression $ax+by+cz+d$; thus

$$L \equiv ax+by+cz+d.$$

Then the equation $$L = 0$$

represents a plane. If, further,

$$L_1 \equiv a_1 x+b_1 y+c_1 z+d_1, \qquad L_2 \equiv a_2 x+b_2 y+c_2 z+d_2,$$

then what has just been proved is that *the equation*

$$L_1+\lambda L_2 = 0$$

represents a plane passing through the line of intersection of the planes $L_1 = 0$, $L_2 = 0$.

The system of planes through a given line is often called a PENCIL.

NOTE. The single parameter λ is sometimes replaced by its 'homogeneous' equivalent μ/λ, and the equation of the pencil is then taken in the form

$$\lambda L_1 + \mu L_2 = 0.$$

In strict accuracy, this has the advantage of including the plane $L_2 = 0$, whereas the form $L_1 + \lambda L_2 = 0$ does not. But this advantage is often lost when the extra symbol complicates the working.

10. The intersection of three planes

(Extensions of the work of this paragraph are basically important, and the results are worthy of close attention. The detailed analysis is, perhaps, less necessary yet.)

Consider the points common to three given planes

$$L_1 \equiv a_1 x + b_1 y + c_1 z + d_1 = 0,$$
$$L_2 \equiv a_2 x + b_2 y + c_2 z + d_2 = 0,$$
$$L_3 \equiv a_3 x + b_3 y + c_3 z + d_3 = 0.$$

The planes are normally expected to intersect in one point, whose coordinates are found by solving the three equations; but exceptions occur. Before the general discussion, three particular examples will indicate where the trouble arises.

(i) THE PLANES

$$L_1 \equiv 2x + 3y + 4z - 9 = 0,$$
$$L_2 \equiv x + y - 8z + 6 = 0,$$
$$L_3 \equiv 5x + 6y - 12z + 1 = 0.$$

Eliminate† z:

$$M_1 \equiv 2L_1 + L_2 \equiv 5x + 7y - 12 = 0,$$
$$M_2 \equiv 3L_1 + L_3 \equiv 11x + 15y - 26 = 0.$$

Eliminate y:

$$15M_1 - 7M_2 \equiv (75 - 77)x - (180 - 182) = 0,$$

or

$$-2x + 2 = 0.$$

† The new names M_1, M_2 are inserted for convenience of reference.

Hence $$x = 1,$$

so that $$y = 1, \quad z = 1.$$

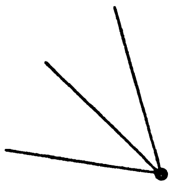

Fig. 20

The three planes thus meet in the unique point (1, 1, 1) (Fig. 20).

(ii) THE PLANES

$$L_1 \equiv 2x+3y+4z-9 = 0,$$
$$L_2 \equiv x+y-8z+6 = 0,$$
$$L_3 \equiv 5x+6y-20z+12 = 0.$$

Eliminate z:

$$M_1 \equiv 2L_1+L_2 \equiv 5x+7y-12 = 0,$$
$$M_2 \equiv 5L_1+L_3 \equiv 15x+21y-33 = 0.$$

The two equations $M_1 = 0$, $M_2 = 0$ are, however, incompatible, for they require

$$5x+7y = 12,$$
$$5x+7y = 11,$$

which is impossible.

Hence the equations are insoluble, and so *the three planes have no common point.*

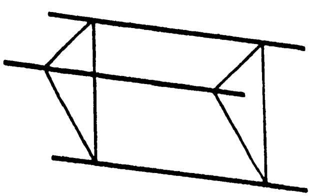

Fig. 21

Geometrically, the three planes are so related that the line of intersection of any two of them is parallel to the third; thus the three lines of intersection are all parallel (Fig. 21).

(iii) THE PLANES

$$L_1 \equiv 2x+3y+4z-9 = 0,$$
$$L_2 \equiv x+y-8z+6 = 0,$$
$$L_3 \equiv 5x+6y-20z+9 = 0.$$

Eliminate z:

$$M_1 \equiv 2L_1+L_2 \equiv 5x+7y-12 = 0,$$
$$M_2 \equiv 5L_1+L_3 \equiv 15x+21y-36 = 0.$$

The two equations $M_1 = 0$, $M_2 = 0$ are, however, identical, and so the process of elimination cannot be carried further.

There is, in fact, an identical relation

$$3M_1 - M_2 \equiv 0,$$

or

$$L_1 + 3L_2 - L_3 \equiv 0,$$

which shows that every point of the line $L_1 = 0$, $L_2 = 0$ lies (p. 28) in the plane $L_3 = 0$. Thus the three equations have an infinite number of solutions.

The expression of L_1, L_2 in the forms

$$L_1 \equiv 2(x-1)+3(y-1)+4(z-1) = 0,$$
$$L_2 \equiv (x-1)+(y-1)-8(z-1) = 0$$

gives, on solution, the equalities

$$\frac{x-1}{-28} = \frac{y-1}{20} = \frac{z-1}{-1};$$

thus, setting each of these ratios equal to λ, *the three planes have in common the line* (Fig. 22) *whose points are expressed parametrically in the form*

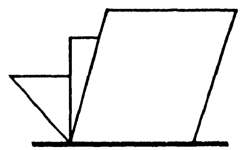

FIG. 22

$$x = 1-28\lambda, \quad y = 1+20\lambda, \quad z = 1-\lambda.$$

Returning to the general equations, denote by Δ the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and by $A_1, B_1, \ldots, C_3$ the cofactors of $a_1, b_1, \ldots, c_3$. It is assumed that the nine cofactors are not all zero, otherwise the planes would be parallel (or coincident) and the solution trivial; in particular, it is convenient to make the assumption

$$A_1 \equiv b_2 c_3 - b_3 c_2 \neq 0.$$

Multiply the given equations by A_1, A_2, A_3 and add, using the standard formulae in the theory of determinants,

$$A_1 a_1 + A_2 a_2 + A_3 a_3 = \Delta,$$
$$A_1 b_1 + A_2 b_2 + A_3 b_3 = 0,$$
$$A_1 c_1 + A_2 c_2 + A_3 c_3 = 0.$$

Thus $$\Delta x+(A_1 d_1+A_2 d_2+A_3 d_3) = 0,$$
or
$$\Delta x = -\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$
There are now two possibilities:

(i) THE DETERMINANT Δ IS NOT ZERO.

On dividing by Δ, the solution is
$$x = -\Delta^{-1}\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$
Similarly,
$$y = -\Delta^{-1}\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \qquad z = -\Delta^{-1}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$
Hence *when* $\Delta \neq 0$, *the three planes meet in a unique point.*

(ii) THE DETERMINANT Δ IS ZERO.

There are, again, two possibilities:

(*a*) *The determinant*
$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$
is not zero.

There is then no solution for the equation for x, and so *the three planes have no common point.*

(*b*) *The determinant*
$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$
is zero.

The equation is then satisfied by any value of x, say by $x = \lambda$.

To follow the solution further, return to the original equations, taking (to conform with the agreement $A_1 \neq 0$) the second and third:
$$b_2 y+c_2 z = -(a_2\lambda+d_2),$$
$$b_3 y+c_3 z = -(a_3\lambda+d_3).$$

Eliminate z:

$$(b_2c_3-b_3c_2)y = c_2(a_3\lambda+d_3)-c_3(a_2\lambda+d_2).$$

Eliminate y:

$$(b_2c_3-b_3c_2)z = b_3(a_2\lambda+d_2)-b_2(a_3\lambda+d_3).$$

Since $A_1 \equiv b_2c_3-b_3c_2 \neq 0$, the values of y, z are determined in terms of the parameter λ.

Hence *the three planes have an infinity of common points; that is, they have a line in common.*

NOTE. The restriction $A_1 \neq 0$ merely ensures that the common line is not parallel to the x-axis, in which case the analysis becomes a little more complicated. As the line cannot be parallel to all three axes at once, the restriction is not of an essential nature.

11. The planes through the line

$$(x-a)/l = (y-b)/m = (z-c)/n$$

To prove that *the equation of any plane through the line*

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

may be expressed in the form

$$A(x-a)+B(y-b)+C(z-c) = 0,$$

where the constants A, B, C are subject to the condition

$$Al+Bm+Cn = 0.$$

The plane is required to satisfy the two conditions:

(i) it contains the point (a, b, c),

(ii) its normals are perpendicular to the line of direction (l, m, n) lying in it.

From the first condition, the equation can be expressed in the form

$$A(x-a)+B(y-b)+C(z-c) = 0,$$

where A, B, C are constants; from the second condition, the two directions defined by (A, B, C) and (l, m, n) are perpendicular, so that

$$Al+Bm+Cn = 0.$$

The ratios $A:B:C$ in any particular example must be determined, where necessary, by some further condition on the plane.

ILLUSTRATION. *To find the equation of the plane through the line*

$$\frac{x-2}{3} = \frac{y-3}{5} = \frac{z}{7}$$

and passing through the point $(1, -2, 3)$.

Any plane through the line is

$$A(x-2)+B(y-3)+Cz = 0,$$

where $$3A+5B+7C = 0.$$

If the plane passes through $(1, -2, 3)$, then

$$-A-5B+3C = 0.$$

Solving the last two equations for the ratios $A:B:C$,

$$\frac{A}{25} = \frac{B}{-8} = \frac{C}{-5},$$

so that the plane is

$$25(x-2)-8(y-3)-5z = 0,$$

or $$25x-8y-5z = 26.$$

Alternatively, determinantal elimination of the ratios $A:B:C$ gives the equation in the form

$$\begin{vmatrix} x-2 & y-3 & z \\ 3 & 5 & 7 \\ -1 & -5 & 3 \end{vmatrix} = 0.$$

EXAMPLES

1. Find the equation of the plane through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

2. Find the equation of each plane which cuts the axis OX at a (positive or negative) distance 2 units from O, which cuts OY at a distance 3 units from O, and which cuts OZ at a distance 4 units from O.

3. Determine which of the points $(0, -2, 2)$, $(3, -2, 4)$, $(1, 3, 5)$, $(2, 1, -1)$, $(0, 0, 3)$, $(4, 2, 2)$, $(-1, -2, -3)$, $(1, 3, 1)$, $(2, 0, 4)$ are on the same side of the plane $2x+3y+4z = 5$ as the point $(0, 1, 2)$.

4. Find the distance of each of the points $(1, 2, 3)$, $(-1, 2, 3)$, $(1, -2, 3)$, $(1, 2, -3)$ from the plane $x+y+z = 1$, from the plane $x+y+z = 0$, and from the plane $x+y = 0$.

5. Find the point of intersection of the planes

$$x+2y+z = 4,$$
$$3x-y-z = 1,$$
$$x+7y-2z = 6.$$

6. Find the equation of the plane through the point $(3, 7, 1)$ parallel to the plane $x+2y-z = 3$.

7. Find the equation of the plane through the point $(1, -1, 2)$ perpendicular to a line whose direction ratios are $(1, -1, 2)$.

8. Find the equation of the two planes at a distance of 3 units from the origin and which pass through the two points $(2, 0, 5)$, $(0, 2, 5)$.

9. Find the three points in the plane $z = 0$ which lie on a line of intersection of two of the three planes

$$x+y-z = 6, \quad 2x+y+4z = 4, \quad x-3y+5z = 0.$$

10. Find the locus of a point which moves so that its distances from the planes $2x+y+2z = 3$, $2x-2y-z = 5$ are (numerically) equal.

11. Prove that the lines

$$A(1, 2, -1),\ B(3, 5, 3) \quad \text{and} \quad A'(3, 7, 2),\ B'(-1, 1, -6)$$

are parallel.

12. Prove that the points $(2, 1, 0)$, $(3, 3, 3)$, $(8, 6, 9)$, $(7, 4, 6)$ are the vertices of a parallelogram.

13. Find equations for the sides of the triangle whose vertices are $(1, -7, 2)$, $(5, 3, 0)$, $(2, 1, 6)$.

14. The vertices of a triangle are $A(1, 2, -3)$ $B(5, 0, 2)$, $C(3, 4, 1)$. Find the equations of the medians of the triangle, and the coordinates of the centroid.

15. The vertices of a tetrahedron are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4). Find the equations of each of the three lines joining the middle points of opposite sides, and prove that those lines are concurrent. Also prove the latter result more simply.

16. Prove that the line common to the planes $x+2y-3z+4 = 0$, $x+y+z-6 = 0$ passes through the point $(1, 2, 3)$, and obtain the equations of the line in the form $(x-1)/l = (y-2)/m = (z-3)/n$.

17. Obtain the equations of the line common to the planes

$$x+3y-z-1 = 0, \quad x+4y+z-6 = 0$$

in the form $\quad (x-x_1)/l = (y-y_1)/m = (z-z_1)/n.$

18. Obtain the equations of the line common to the planes

$$2x-3y+z = 0, \quad 5x+4y+3z-12 = 0$$

in the form $\quad (x-x_1)/l = (y-y_1)/m = (z-z_1)/n.$

19. Find the direction cosines of the line common to the planes

$$x-y+2z+3 = 0, \quad 5x+y-2z+4 = 0.$$

20. Find the direction cosines of each side of the triangle whose vertices are the points $(0, 5, 1)$, $(2, 3, 7)$, $(-1, 4, -3)$.

21. Find the equation of the plane through the points $(2, 1, 5)$, $(3, -2, 4)$, $(1, -3, 3)$.

22. Find the coordinates of the point common to the three planes $y+2z = 3$, $z+2x = 3$, $x+2y = 12$.

23. Prove that every point common to the two planes

$$3x-4y+7z+2 = 0, \qquad x+y-2z+3 = 0$$

also belongs to the plane $x-6y+11z-4 = 0$.

24. Find the coordinates of the point in which the line joining the points $(1, 1, 1)$, $(3, 2, 1)$ meets the plane $x-3y = 0$.

25. Prove that, for all values of k, the equation $x+y+2+k(z-3) = 0$ represents a plane through the line common to the planes $x+y+2 = 0$, $z-3 = 0$. Prove also that, conversely, the equation of *any* plane which passes through the line common to these two planes can be written in that form.

26. Find the equation of the plane, through the line of intersection of the planes $2x+4y+z-1 = 0$ and $x-y-z+6 = 0$, which passes through the origin.

27. Prove that the line common to the two planes $x+y+k(z-3) = 0$, $2x-3y+z+k'(x-4) = 0$ meets the line common to the planes $x+y = 0$, $z-3 = 0$ and also the line common to the planes $2x-3y+z = 0$, $x-4 = 0$.

28. Find the three points in which the line, common to the plane through the points $(0, 1, 2)$, $(2, 1, 0)$, $(1, 0, 1)$ and the plane through the points $(1, 1, 1)$, $(1, 2, 3)$, $(2, 3, -4)$, meets the coordinate planes.

12. The common perpendicular of two skew lines

Let the equations of two given skew lines be

$$\frac{x-a_1}{l_1} = \frac{y-b_1}{m_1} = \frac{z-c_1}{n_1},$$

$$\frac{x-a_2}{l_2} = \frac{y-b_2}{m_2} = \frac{z-c_2}{n_2};$$

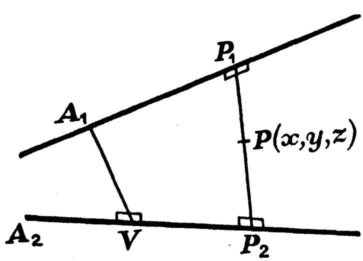

FIG. 23

for the moment, adopt the 'n-positive' convention. The point $A_1(a_1, b_1, c_1)$ lies on the first line and the point $A_2(a_2, b_2, c_2)$ on the second. Suppose that the common perpendicular of the two lines meets them in P_1, P_2 (Fig. 23).

(i) THE ANGLE BETWEEN THE LINES. The angle between the lines is θ, where (p. 15)

$$\cos\theta = l_1 l_2+m_1 m_2+n_1 n_2.$$

It is useful to note that

$$\sin^2\theta = 1-\cos^2\theta$$
$$= (l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1 l_2+m_1 m_2+n_1 n_2)^2$$
$$= (m_1 n_2-m_2 n_1)^2+(n_1 l_2-n_2 l_1)^2+(l_1 m_2-l_2 m_1)^2,$$

on reduction. The value of $\sin\theta$ is (since θ lies in the interval $0, \pi$) the *positive* square root of this expression.

(ii) THE DIRECTION COSINES OF $P_1 P_2$. Suppose that the direction cosines of $P_1 P_2$ are (λ, μ, ν). Then, by perpendicularity,

$$l_1\lambda+m_1\mu+n_1\nu = 0,$$
$$l_2\lambda+m_2\mu+n_2\nu = 0.$$

Hence $$\frac{\lambda}{m_1 n_2-m_2 n_1} = \frac{\mu}{n_1 l_2-n_2 l_1} = \frac{\nu}{l_1 m_2-l_2 m_1},$$

so that the direction *ratios* are

$$(m_1 n_2-m_2 n_1,\ n_1 l_2-n_2 l_1,\ l_1 m_2-l_2 m_1).$$

It follows from (i) that the direction *cosines* are found by dividing the ratios by $\sin\theta$. [If the 'positive n' convention is being used, the division may be by $-\sin\theta$.]

(iii) THE LENGTH OF $P_1 P_2$. Since $P_1 P_2$ is the projection of $A_1 A_2$ on the direction (λ, μ, ν), its length is (p. 14) the numerical value of $$\lambda(a_1-a_2)+\mu(b_1-b_2)+\nu(c_1-c_2),$$
or
$$\frac{(a_1-a_2)(m_1 n_2-m_2 n_1)+(b_1-b_2)(n_1 l_2-n_2 l_1)+ +(c_1-c_2)(l_1 m_2-l_2 m_1)}{\sin\theta}.$$

This expression may be exhibited more compactly in the determinantal form

$$\frac{\begin{vmatrix} a_1-a_2 & b_1-b_2 & c_1-c_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\{(m_1 n_2-m_2 n_1)^2+(n_1 l_2-n_2 l_1)^2+(l_1 m_2-l_2 m_1)^2\}}}.$$

(iv) THE EQUATIONS FOR $P_1 P_2$. Let $P(x, y, z)$ be any point of $P_1 P_2$. Draw $A_1 V$ perpendicular to $A_2 P_2$. Then (p. 14)

$$\overrightarrow{A_1 P_1} = \text{projection of } \overrightarrow{A_1 P} \text{ on the direction } (l_1, m_1, n_1)$$
$$= l_1(x-a_1)+m_1(y-b_1)+n_1(z-c_1),$$

and $\overrightarrow{VP_2}$ = projection of $\overrightarrow{A_1 P}$ on the direction (l_2, m_2, n_2)

$$= l_2(x-a_1)+m_2(y-b_1)+n_2(z-c_1).$$

But $\overrightarrow{VP_2}$ = projection of $\overrightarrow{A_1 P_1}$ on the line $A_2 P_2$

$$= \overrightarrow{A_1 P_1}\cos\theta \quad \text{(p. 13)}.$$

Hence

$$l_2(x-a_1)+m_2(y-b_1)+n_2(z-c_1)$$
$$= \{l_1(x-a_1)+m_1(y-b_1)+n_1(z-c_1)\}(l_1 l_2+m_1 m_2+n_1 n_2).$$

Similarly

$$l_1(x-a_2)+m_1(y-b_2)+n_1(z-c_2)$$
$$= \{l_2(x-a_2)+m_2(y-b_2)+n_2(z-c_2)\}(l_1 l_2+m_1 m_2+n_1 n_2).$$

These two equations determine the line $P_1 P_2$.

An ALTERNATIVE FORM FOR THE EQUATIONS may be obtained as follows:

Any plane through the line $A_1 P_1$ is (p. 33)

$$p(x-a_1)+q(y-b_1)+r(z-c_1) = 0,$$

where $$pl_1+qm_1+rn_1 = 0.$$

If p, q, r are chosen so that

$$p\lambda+q\mu+r\nu = 0,$$

then the normal (p,q,r) to the plane is perpendicular to the line $P_1 P_2$ of direction cosines (λ, μ, ν), so that the plane also contains $P_1 P_2$. Eliminating $p:q:r$, the equation of one plane through the line $P_1 P_2$ is found in the form

$$\begin{vmatrix} x-a_1 & y-b_1 & z-c_1 \\ l_1 & m_1 & n_1 \\ \lambda & \mu & \nu \end{vmatrix} = 0.$$

Similarly a second plane through the line is

$$\begin{vmatrix} x-a_2 & y-b_2 & z-c_2 \\ l_2 & m_2 & n_2 \\ \lambda & \mu & \nu \end{vmatrix} = 0,$$

and these two equations taken together form equations for the line.

COROLLARY. It follows from (iii), or is easily proved independently, that *the condition for the two lines*

$$\frac{x-a_1}{l_1} = \frac{y-b_1}{m_1} = \frac{z-c_1}{n_1},$$

$$\frac{x-a_2}{l_2} = \frac{y-b_2}{m_2} = \frac{z-c_2}{n_2}$$

to be concurrent is

$$\begin{vmatrix} a_1-a_2 & b_1-b_2 & c_1-c_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

They then lie in the plane

$$\lambda(x-a_1)+\mu(y-b_1)+\nu(z-c_1) = 0,$$

or

$$\begin{vmatrix} x-a_1 & y-b_1 & z-c_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

13. Convenient equations for two skew lines

To prove that *the equations of two given skew lines can be taken in the form*

$$y = mx, \qquad z = c$$

and

$$y = -mx, \qquad z = -c.$$

We begin with a somewhat more general treatment, which is sometimes useful.

Let AB, CD be the two given lines. Draw their common perpendicular PQ. Take any point O on PQ as origin, and the line PQ as the axis OZ; suppose that $\vec{OP} = p$, $\vec{OQ} = q$. The axes OX, OY are then (in the first instance) any two perpendicular lines through O, each perpendicular to OZ.

The line AB lies entirely in the plane $z = p$; moreover, the plane OZA through AB is given by an equation of the form $\alpha x+\beta y = 0$, since it passes through OZ. Hence *AB is given by the equations*

$$\alpha x+\beta y = 0, \qquad z = p.$$

Similarly, *CD is given by the equations*

$$\gamma x+\delta y = 0, \qquad z = q.$$

To simplify these equations to the form quoted initially, consider two modifications:

(i) Take O to be the middle point of PQ; then we may write $\overrightarrow{OP} = c$, $\overrightarrow{OQ} = -c$.

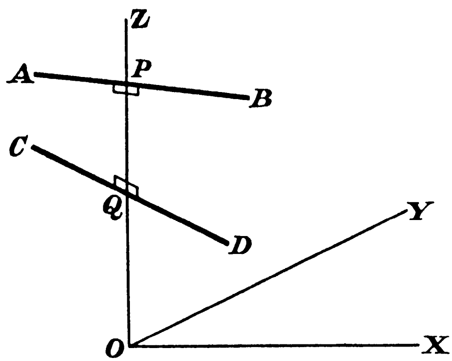

FIG. 24

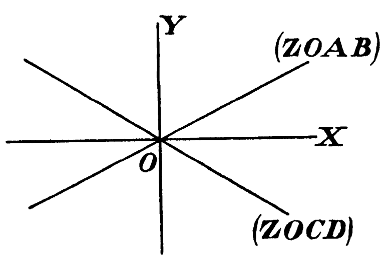

FIG. 25

(ii) Take the planes ZOX, ZOY as those which bisect the angles between the planes $ZOAB$, $ZOCD$; the diagram (Fig. 25) then shows the section in the plane XOY. If the angle between the planes is 2θ, then one of the planes, say $ZOAB$ has equation

$$y = x \tan\theta,$$

and the other has equation

$$y = -x \tan\theta.$$

Writing m for $\tan\theta$, the equations are

$$y = mx, \qquad z = c \qquad \text{for } AB,$$

and

$$y = -mx, \qquad z = -c \qquad \text{for } CD.$$

COROLLARY. If the given lines AB, CD are perpendicular, their equations may be taken in the still simpler form

$$x = 0, \qquad z = c$$

and

$$y = 0, \qquad z = -c.$$

14. Triads of mutually perpendicular lines

The axes OX, OY, OZ form an example of three straight lines each of which is perpendicular to the two others. Suppose, more generally, that p_1, p_2, p_3 are three mutually perpendicular lines meeting in the origin O, and let their direction cosines be (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3), where

$$l_1^2+m_1^2+n_1^2 = 1,$$
$$l_2^2+m_2^2+n_2^2 = 1,$$
$$l_3^2+m_3^2+n_3^2 = 1.$$

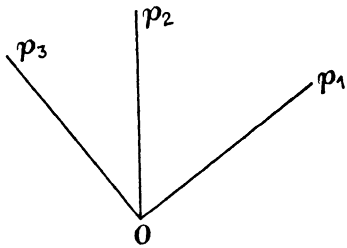

FIG. 26

Denote by Δ the determinant

$$\Delta \equiv \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix},$$

and recall the formula (expansion of Δ by the first row)

$$\Delta = l_1(m_2 n_3 - m_3 n_2) + m_1(n_2 l_3 - n_3 l_2) + n_1(l_2 m_3 - l_3 m_2).$$

Since the lines are perpendicular in pairs,

$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0,$$
$$l_3 l_1 + m_3 m_1 + n_3 n_1 = 0,$$
$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0.$$

(i) We prove first that *the value of* Δ *is* ± 1:

By direct multiplication of determinants,

$$\Delta^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} =$$

$$\begin{vmatrix} l_1^2+m_1^2+n_1^2, & l_1 l_2+m_1 m_2+n_1 n_2, & l_1 l_3+m_1 m_3+n_1 n_3 \\ l_2 l_1+m_2 m_1+n_2 n_1, & l_2^2+m_2^2+n_2^2, & l_2 l_3+m_2 m_3+n_2 n_3 \\ l_3 l_1+m_3 m_1+n_3 n_1, & l_3 l_2+m_3 m_2+n_3 n_2, & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

so that $$\Delta = \pm 1.$$

To resolve the ambiguity in the sign of Δ, observe first that, in the particular case when $\vec{p_1}, \vec{p_2}, \vec{p_3}$ are the axes $\vec{OX}, \vec{OY}, \vec{OZ}$ respectively, with direction cosines $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, the value of Δ is $+1$. If one pair, say $\vec{OY}$ and $\vec{OZ}$, is interchanged, the value of Δ is -1. In the first case, the lines form a *right-handed set* (p. 3), in the second, a *left-handed set.*

Suppose, then, that $\vec{p_1}, \vec{p_2}, \vec{p_3}$ are in general position, forming a right-handed set in that order. Imagine their direction cosines to vary continuously, by very small steps, until $\vec{p_1}$ falls along $\vec{OX}$ and $\vec{p_2}$ along $\vec{OY}$. Since the set is right-handed, $\vec{p_3}$ falls automatically along $\vec{OZ}$. The value of Δ changes during this motion by small steps only, if at all; but its value is restricted to be $+1$ or -1, so that small steps are excluded. Hence $\Delta = +1$ for a right-handed set.

Similarly $\Delta = -1$ for a left-handed set.

(ii) We obtain next NINE RELATIONS, *of which a typical one is*

$$m_2 n_3 - m_3 n_2 = \Delta l_1,$$

where $\Delta = \pm 1$, *as above.*

Solve the two equations

$$l_1 l_2+m_1 m_2+n_1 n_2 = 0,$$

$$l_1 l_3+m_1 m_3+n_1 n_3 = 0$$

for the ratios $l_1 : m_1 : n_1$. Thus

$$\frac{l_1}{m_2 n_3 - m_3 n_2} = \frac{m_1}{n_2 l_3 - n_3 l_2} = \frac{n_1}{l_2 m_3 - l_3 m_2}.$$

(The denominators are not all zero, otherwise the lines p_2, p_3 would coincide.)

If each of these ratios is denoted temporarily by $1/k$, so that

$$m_2 n_3 - m_3 n_2 = k l_1,$$

and so on, the formula of expansion for Δ (p. 41) gives the relation

$$\begin{aligned} \Delta &= l_1 . k l_1 + m_1 . k m_1 + n_1 . k n_1 \\ &= k(l_1^2 + m_1^2 + n_1^2), \end{aligned}$$

so that

$$k = \Delta.$$

There are thus NINE RELATIONS,

$$\begin{aligned} m_2 n_3 - m_3 n_2 &= \Delta l_1, & n_2 l_3 - n_3 l_2 &= \Delta m_1, & l_2 m_3 - l_3 m_2 &= \Delta n_1, \\ m_3 n_1 - m_1 n_3 &= \Delta l_2, & n_3 l_1 - n_1 l_3 &= \Delta m_2, & l_3 m_1 - l_1 m_3 &= \Delta n_2, \\ m_1 n_2 - m_2 n_1 &= \Delta l_3, & n_1 l_2 - n_2 l_1 &= \Delta m_3, & l_1 m_2 - l_2 m_1 &= \Delta n_3. \end{aligned}$$

(iii) We obtain finally *six alternative orthogonality relations*:

Suppose that the lines p_1, p_2, p_3 are regarded as a system of coordinate axes OU, OV, OW. The cosines of the angles between OX and OU, OV, OW are l_1, l_2, l_3, so that, referred to the new axes, the direction cosines of OX are (l_1, l_2, l_3); similarly the direction cosines of OY, OZ are (m_1, m_2, m_3), (n_1, n_2, n_3). Since OX, OY, OZ are mutually orthogonal, there are SIX RELATIONS

$$\begin{aligned} l_1^2 + l_2^2 + l_3^2 &= 1, \\ m_1^2 + m_2^2 + m_3^2 &= 1, \\ n_1^2 + n_2^2 + n_3^2 &= 1, \\ m_1 n_1 + m_2 n_2 + m_3 n_3 &= 0, \\ n_1 l_1 + n_2 l_2 + n_3 l_3 &= 0, \\ l_1 m_1 + l_2 m_2 + l_3 m_3 &= 0. \end{aligned}$$

15. Rotation of axes

To find the formulae for a transformation of axes, without change of origin, from OX, OY, OZ to a right-handed triad OU, OV, OW

whose direction cosines, referred to OX, OY, OZ are (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3).

Let a point P have coordinates (x, y, z) referred to OX, OY, OZ and (u, v, w) referred to OU, OV, OW. Since u, v, w are the projections of OP on OU, OV, OW, we have (p. 14) the relations

$$u = l_1 x + m_1 y + n_1 z,$$
$$v = l_2 x + m_2 y + n_2 z,$$
$$w = l_3 x + m_3 y + n_3 z.$$

Further, the direction cosines of OX, OY, OZ with respect to OU, OV, OW are (l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3). Hence, by similar argument,

$$x = l_1 u + l_2 v + l_3 w,$$
$$y = m_1 u + m_2 v + m_3 w,$$
$$z = n_1 u + n_2 v + n_3 w.$$

These equations serve to express u, v, w in terms of x, y, z; and x, y, z in terms of u, v, w.

The two sets of formulae may be recollected with the help of the scheme:

	x	y	z
u	l_1	m_1	n_1
v	l_2	m_2	n_2
w	l_3	m_3	n_3

16. The area of a triangle

We begin with a lemma:

To prove that, *if a triangle ABC is projected orthogonally into a triangle A′B′C′ in a plane inclined at an angle θ to the plane ABC, then the areas of the triangles are connected by the relation*

$$\Delta A'B'C' = \Delta ABC \cos\theta.$$

We may, without loss of generality, take the plane of projection to pass through A, so that A, A' coincide. Let AX be the common line of the two planes. Since BB', CC' are both perpendicular to the plane $AB'C'$, they are parallel and therefore coplanar. Hence BC, $B'C'$ meet in a point U of the common line AX.

If BP is drawn perpendicular to AX, then, since BB' (perpendicular to the plane $B'AX$) is also perpendicular to AX, it follows that AX is perpendicular to the plane containing BP and BB', so that $B'P$ is perpendicular to AX. Thus $\angle B'PB = \theta$, the angle between the planes. Hence

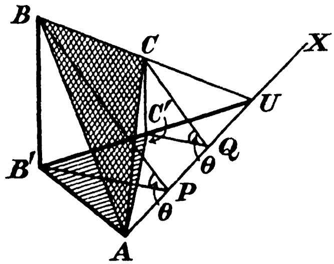

FIG. 27

$$\begin{aligned}\Delta B'AU &= \tfrac{1}{2}B'P.AU \\ &= \tfrac{1}{2}(BP\cos\theta).AU \\ &= \tfrac{1}{2}BP.AU\cos\theta \\ &= \Delta BAU\cos\theta.\end{aligned}$$

Similarly, $$\Delta C'AU = \Delta CAU\cos\theta.$$

Thus $$\begin{aligned}\Delta AB'C' &= \Delta B'AU - \Delta C'AU \\ &= (\Delta BAU - \Delta CAU)\cos\theta \\ &= \Delta ABC\cos\theta.\end{aligned}$$

If U were between B and C, then it would also be between B' and C', so that the proof would hold as before, save that subtraction would be replaced by addition.

The primary problem can now be undertaken:

To find an expression for the area of the triangle whose vertices are the points $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$.

The projections on the plane $z = 0$ of the vertices P, Q, R have in that plane coordinates $P'(x_1, y_1)$, $Q'(x_2, y_2)$, $R'(x_3, y_3)$, and it is a familiar theorem of plane geometry that (apart, possibly, from sign)

$$\Delta P'Q'R' = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

In the notation used before (p. 25)

$$\Delta P'Q'R' \quad \tfrac{1}{2}C.$$

Now the direction cosines of the normal to the plane PQR are (p. 25)

$$A/\sqrt{(A^2+B^2+C^2)}, \quad B/\sqrt{(A^2+B^2+C^2)}, \quad C/\sqrt{(A^2+B^2+C^2)},$$

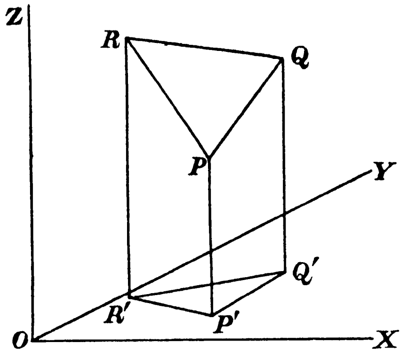

FIG. 28

so that the cosine of the angle between the planes PQR, $P'Q'R'$ is $C/\sqrt{(A^2+B^2+C^2)}$. Hence, by the Lemma,

$$\tfrac{1}{2}C = \{C/\sqrt{(A^2+B^2+C^2)}\}\Delta PQR,$$

and so† *the area of the triangle PQR is*

$$\tfrac{1}{2}\sqrt{(A^2+B^2+C^2)},$$

or

$$\frac{1}{2}\left\{\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2\right\}^{\frac{1}{2}}.$$

COROLLARY. Let R coincide with the origin O. Then *the area of the triangle OPQ is*

$$\tfrac{1}{2}\sqrt{\{(y_1 z_2 - y_2 z_1)^2 + (z_1 x_2 - z_2 x_1)^2 + (x_1 y_2 - x_2 y_1)^2\}}.$$

17. The volume of a tetrahedron

To find the volume of the tetrahedron whose vertices are

$$P(x_1, y_1, z_1), \quad Q(x_2, y_2, z_2), \quad R(x_3, y_3, z_3), \quad S(x_4, y_4, z_4).$$

† If $C = 0$, use projection on $x = 0$ or $y = 0$ instead.

It has been proved (p. 26) that the length of the perpendicular from the point S to the plane PQR is

$$p \equiv \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \div \sqrt{(A^2+B^2+C^2)},$$

and also (p. 46) that the area of the triangle PQR is

$$\Delta PQR \equiv \tfrac{1}{2}\sqrt{(A^2+B^2+C^2)}.$$

Hence the volume of the tetrahedron is

$$\begin{aligned} &\tfrac{1}{3}\text{ base} \times \text{altitude} \\ &= \tfrac{1}{3} p \Delta PQR \\ &= \tfrac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}. \end{aligned}$$

18. Oblique axes

It is not always necessary to choose coordinate axes which are mutually perpendicular; for many problems they are better

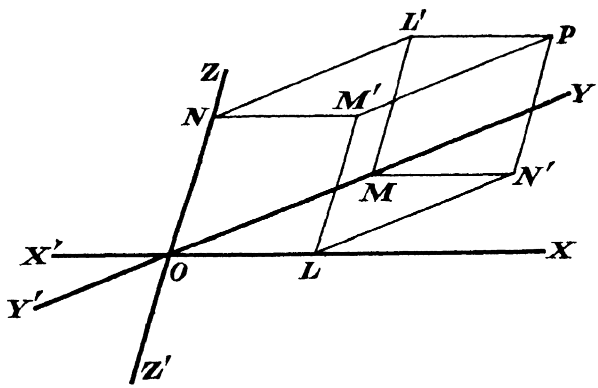

FIG. 29

OBLIQUE. If OX, OY, OZ are three concurrent lines, the coordinates of a point P may then be defined from the 'box' $OLMNPL'M'N'$, obtained (Fig. 29) by drawing the planes through P parallel to the coordinate planes. Thus

$$x \equiv \overrightarrow{OL}, \qquad y \equiv \overrightarrow{OM}, \qquad z \equiv \overrightarrow{ON}.$$

The distinction between left-handed and right-handed axes loses its point, and there is no advantage to be gained by any 'n-positive' convention.

Many of the formulae (but not those involving the use of the theorem of Pythagoras) remain true for oblique axes; for example, the equation

$$ax+by+cz+d = 0$$

represents a plane, and the equations

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

represent a straight line—though the interpretation of l, m, n as cosines is no longer valid.

The illustration which follows shows how oblique axes may be used to simplify equations.

ILLUSTRATION. (i) *A convenient form for the equations of three straight lines.*

Let AA', BB', CC' be three given skew lines. Through AA' draw the plane parallel to BB' and the plane parallel to CC'; by constructing planes similarly through BB' and through CC', complete the (non-rectangular) 'box' shown in the diagram (Fig. 30). Take the centre of the 'box' as origin O, and the axes OX, OY, OZ parallel to $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$ respectively. The equations of the planes may then be exhibited according to the scheme:

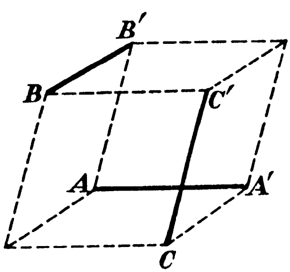

FIG. 30

$$A'CC',\ x-a = 0; \qquad AB'B,\ x+a = 0;$$
$$B'AA',\ y-b = 0; \qquad BC'C,\ y+b = 0;$$
$$C'BB',\ z-c = 0; \qquad CA'A,\ z+c = 0.$$

The equations of the three lines are therefore given by the scheme:

$$AA': \quad y-b = 0, \quad z+c = 0;$$
$$BB': \quad z-c = 0, \quad x+a = 0;$$
$$CC': \quad x-a = 0, \quad y+b = 0.$$

NOTE. This result includes by implication a possible form for the equations of two given skew lines. An alternative, and more usual, form will be found on p. 39.

(ii) *To determine whether there is a line meeting each of three given skew lines so that the portion intercepted between two of the given lines is bisected by the third.*

Let the equations of the three given lines be

$$AA': \quad y-b = 0, \quad z+c = 0;$$
$$BB': \quad z-c = 0, \quad x+a = 0;$$
$$CC': \quad x-a = 0, \quad y+b = 0.$$

The coordinates of a point of the line AA' can be taken in the form $P(\xi, b, -c)$; in like manner points $Q(-a, \eta, c)$, $R(a, -b, \zeta)$ can be taken on BB', CC' respectively. If P is the middle point of QR, then

$$\xi = \tfrac{1}{2}(-a+a), \quad b = \tfrac{1}{2}(\eta-b), \quad -c = \tfrac{1}{2}(c+\zeta),$$

so that $$\xi = 0, \quad \eta = 3b, \quad \zeta = -3c.$$

There is therefore precisely *one* line meeting each of the three given lines so that the portion intercepted between BB' and CC' is bisected by AA'. The line joins the points $(-a, 3b, c)$, $(a, -b, -3c)$, and is bisected at the point $(0, b, -c)$.

MISCELLANEOUS EXAMPLES

1. Find the equation of the plane which is perpendicular to the plane $3x-4y+9 = 0$ and passes through its line of intersection with the plane $7x-y-12z+16 = 0$, and prove that the perpendicular distance of the point $(3, 2, 1)$ from this line of intersection is $\sqrt{5}$.

2. Find the equation of the plane which contains the point $(4, 1, 1)$ and passes through the straight line common to the two planes

$$x+2y+z = 1 \quad \text{and} \quad 3x+y+2z = 3.$$

Show that the plane passing through $(4, 1, 1)$ and perpendicular to the line common to the two planes given above has the equation

$$3x+y-5z = 8.$$

Hence, or otherwise, show that the straight line drawn from $(4, 1, 1)$ to be perpendicular to the common straight line above can be expressed in the form

$$\frac{x-4}{3} = \frac{y-1}{1} = \frac{z-1}{2}.$$

3. Find the volume of the tetrahedron formed by the four planes

$$x = 0, \quad y = 0, \quad z = 0, \quad x+3y+2z = 6.$$

4. If the coordinate plane $z = 0$ is horizontal, find the direction cosines of a line of greatest slope on the plane $lx+my+nz = 0$.

5. Find the equation of the plane containing the point (3, 9, 14) and the line

$$y = 11x-3, \qquad z = 15x+4.$$

By dropping the perpendicular from the point $Q(10, -2, 17)$, or otherwise, find the point of the plane nearest to Q, and find the distance between the plane and Q.

6. Find the point of intersection of the planes

$$3x - y - 5z = 6,$$
$$-x+5y-3z = 12,$$
$$-x-9y+25z = -16.$$

The planes intersect in pairs to give three lines through the common point; find the equations of the line through this point which is equally inclined to the three lines of intersection.

7. Find the equations of all the lines through the origin which make equal angles with the three lines

$$x-y = x-z = 0,$$
$$x+y = x-z = 0,$$
$$x+y = x+z = 0.$$

8. Points A, B, C are chosen, one on each of the coordinate axes of a rectangular cartesian system, in such a way that the lines joining them to a given point $P(a, b, c)$, not on any of the coordinate planes, are mutually perpendicular. Find the coordinates of A, B, C and prove that the plane ABC bisects at right angles the line joining P to the origin.

9. Show that the line

$$\frac{x+1}{3} = \frac{y-2}{6} = \frac{z-3}{4}$$

is parallel to the plane

$$2x+3y-6z+7 = 0,$$

and find the distance of the line from the plane.

10. Three vertices A, B, C of a cube have coordinates (2, 9, 12), (1, 8, 8), (−2, 11, 8) respectively. Find (i) the other vertex of the cube lying in the plane ABC, (ii) the equation of the other face of the cube passing through AB, (iii) the two possible positions of the centre of the cube.

11. Find the equation of the surface traced out by lines which intersect the two lines

$$y = 1,\ z = -1 \quad \text{and} \quad x = -1,\ z = 1$$

and are perpendicular to the line

$$x = y = z.$$

12. Find the equations of the straight line through the origin meeting both of the straight lines

$$x+y+z = x-2y+3z = 6,$$

and

$$2x-y+z = 3x+3y-2z = 6,$$

and find the angles between that line and each of the two given lines.

13. Find the equations of the image of the line

$$\frac{x-1}{-1} = \frac{y}{2} = \frac{z-1}{1}$$

in the plane $x+y+z = 4$, and show that the cosine of the acute angle between the line and its image is $\frac{5}{9}$.

14. Show that the following two lines intersect:

$$\frac{x-9}{3} = \frac{y-1}{5} = \frac{z-18}{-7},$$

$$\frac{x-5}{13} = \frac{y-11}{5} = \frac{z+6}{3}.$$

Find the equation of the plane containing these lines, and obtain also the equations of the normal to the plane through the point of intersection of the two lines.

15. Find the equation of the plane parallel to and equidistant from the two lines

$$\frac{x+27}{4} = \frac{y-32}{3} = \frac{z-2}{2},$$

$$\frac{x-31}{2} = \frac{y+25}{2} = \frac{z+14}{-7}.$$

16. Prove that, if the lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \quad \text{and} \quad \frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'}$$

are coplanar, then so are

$$\frac{x-a}{l'} = \frac{y-b}{m'} = \frac{z-c}{n'} \quad \text{and} \quad \frac{x-a'}{l} = \frac{y-b'}{m} = \frac{z-c'}{n}.$$

17. Find the equation of the plane through the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

parallel to the line $$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}.$$

18. Prove that, if the direction cosines of two lines through the origin are (l, m, n), (l', m', n'), then the cosine of the angle between them is $\pm(ll'+mm'+nn')$.

A right circular cone has its vertex at the origin, and its axis is the line $x = y = z$. The generators all make an angle of $\frac{1}{6}\pi$ with this line. Prove that the coordinates of any point (x, y, z) of the cone satisfy the equation $$4(x+y+z)^2 = 3(x^2+y^2+z^2).$$

19. Show that, in general, one and only one straight line can be drawn through a given point $P_1(x_1, y_1, z_1)$ to meet each of the lines

$$z = a, \quad y = 0 \quad \text{and} \quad x = 0, \quad z = -a.$$

Find the direction ratios of this line, and obtain the locus of points P_1 for which the line is parallel to a given plane

$$Ax+By+Cz = 0.$$

20. Prove that the lines

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$$

and
$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$$

are coplanar, and find the equation of the plane in which they lie.

21. Find the direction ratios of the orthogonal projection of the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

on the plane
$$Ax+By+Cz = 0.$$

Show that the orthogonal projections on that plane of two lines with direction ratios (l, m, n), (l', m', n') are perpendicular if

$$(A^2+B^2+C^2)(ll'+mm'+nn') = (Al+Bm+Cn)(Al'+Bm'+Cn').$$

22. The plane
$$x+2y-z = 8$$
is rotated about its intersection with the plane

$$5x-2y+7z = 17$$

through an angle of 60° in both directions. Find the equations of the plane in its new positions.

23. Find the length and equations of the common perpendicular to the two lines

$$\frac{x+3}{-4} = \frac{y-6}{3} = \frac{z}{2}$$

and
$$\frac{x+2}{-4} = \frac{y}{1} = \frac{z-7}{1}.$$

24. Find the coordinates of the mirror image of the point (p, q, r) in the plane
$$ax+by+cz+d = 0.$$

A ray from the origin is reflected successively in the planes

$$x+y-z+1 = 0,$$
$$x-y+2z-1 = 0,$$

and then passes again through the origin. Find the points at which it meets the two planes.

25. Two perpendicular lines OA, OB, of lengths a, b respectively, lie in a horizontal plane and vertical posts of heights h, k are erected at A, B respectively. Prove that the plane through O and the tops of these posts makes with the horizontal plane an angle θ, where

$$a^2b^2\tan^2\theta = a^2k^2+b^2h^2.$$

26. The feet of the perpendiculars from a point P to the mutually perpendicular coordinate planes OYZ, OZX, OXY are L, M, N. Show that the line OP makes equal angles with the three planes OMN, ONL, OLM, and that the plane OPL is equally inclined to the planes OLM, OLN.

27. A, B, C are the points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$, and M is the foot of the perpendicular from the origin O to the plane ABC. Find the coordinates of M and the length OM. By considering the volume of the tetrahedron $OABC$, deduce that the area of the triangle ABC is

$$\tfrac{1}{2}\sqrt{(b^2c^2+c^2a^2+a^2b^2)}.$$

28. A point P lies in a plane which is parallel to each of two given skew lines, and the perpendiculars from P to the lines are equal in length. Prove that the locus of P is a rectangular hyperbola.

29. A straight line meets each of two fixed perpendicular non-intersecting straight lines so that the intercept on it is of constant length. Find the locus of the middle point of the intercept.

30. Prove that in general just one line can be drawn meeting two lines in space and parallel to a third.

31. A variable point P lies in a fixed plane and λ, μ are two skew lines parallel to this plane. Prove that, if the perpendicular distances of P from λ, μ are equal, the locus of P is a rectangular hyperbola whose centre lies on the common normal of λ, μ.

32. Points P, P' are taken, one on each of the lines

$$y = mx,\ z = c \quad \text{and} \quad y = -mx,\ z = -c,$$

such that PP' subtends a right angle at the mid-point of their shortest distance. Prove that the locus of the mid-point of PP' is a hyperbola, and that the line PP' describes the quadric

$$(m^2-1)(y^2-m^2x^2)+m^2(z^2-c^2) = 0.$$

33. Prove that by a suitable choice of (non-rectangular) coordinate axes the equations of a skew triad of lines can be written in the forms

$$\begin{cases} y-b = 0, \\ z+c = 0, \end{cases} \qquad \begin{cases} z-c = 0, \\ x+a = 0, \end{cases} \qquad \begin{cases} x-a = 0, \\ y+b = 0, \end{cases}$$

and that the lines lie on the surface

$$ayz+bzx+cxy+abc = 0.$$

Show that any line which meets each of these three lines is given by any two of the equations

$$c\beta y-b\gamma z+bc\alpha = 0, \quad a\gamma z-c\alpha x+ca\beta = 0, \quad b\alpha x-a\beta y+ab\gamma = 0,$$

where $\alpha:\beta:\gamma$ are parameters and

$$\alpha+\beta+\gamma = 0.$$

34. Show that the transversals of the three skew lines

$$\begin{cases} y-mc = 0, \\ z+nb = 0, \end{cases} \qquad \begin{cases} z-na = 0, \\ x+lc = 0, \end{cases} \qquad \begin{cases} x-lb = 0, \\ y+ma = 0 \end{cases}$$

generate the surface

$$l(b+c)yz+m(c+a)zx+n(a+b)xy+$$
$$+mna(b-c)x+nlb(c-a)y+lmc(a-b)z+2lmnabc = 0.$$

35. Prove that there is a transversal λ to three skew lines l, m, n that intersects l in the mid-point of its intersections with m, n.

If μ and ν are similarly defined (the mid-points being the intersections with m and n respectively), prove that l intersects λ in the mid-point of its intersections with μ, ν.

36. The triangle with vertices $(5, -4, 3)$, $(4, -1, -2)$, $(10, -5, 2)$ is projected orthogonally on to the plane $x-y = 3$. Find the vertices and the area of the new triangle.

37. Find the coordinates of the vertices of the tetrahedron formed by the planes

$$z = 2, \quad 3x+4y = 13, \quad 2x+y-2z = 8, \quad 6x+2y+3z = 14,$$

and find the centre of the *inscribed* sphere.

38. Prove that rectangular cartesian coordinates can be chosen so that three given points have coordinates

$$A(a, 0, 0), \quad B(0, b, 0), \quad C(0, 0, c)$$

provided that ABC is not an obtuse-angled triangle.

Find then the coordinates of the orthocentre of the triangle ABC.

39. Three mutually perpendicular planes meet in O. The projections on the three planes of a point P in general position are L, M, N. Prove that the line OP makes the same angle with each of the planes OMN, ONL, OLM.

III

VECTORS

1. Suffix notation

IN this chapter we indicate coordinates by suffixes 1, 2, 3 instead of letters x, y, z. The axes are assumed to be orthogonal and right-handed, and the coordinates of a point X are denoted by (x_1, x_2, x_3); the coordinates of points A, B,... are denoted by (a_1, a_2, a_3), (b_1, b_2, b_3),..., the letters a, b identifying the points and the suffixes 1, 2, 3 the separate coordinates.

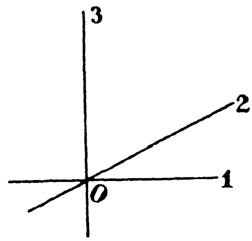

FIG. 31

Similarly (l_1, l_2, l_3) are used for the direction cosines of a line l, and the equation

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = b$$

denotes a typical plane.

The axes will be named $O1$, $O2$, $O3$ (Fig. 31).

2. The summation convention

The formula (p. 15)

$$\cos\theta = l_1 m_1 + l_2 m_2 + l_3 m_3$$

for the angle between two lines with direction cosines (l_1, l_2, l_3), (m_1, m_2, m_3) is often abbreviated to the form

$$\cos\theta = \sum_{\lambda=1}^{3} l_\lambda m_\lambda.$$

Further simplicity is gained by the use of the SUMMATION CONVENTION, under which ***repeated Greek suffixes in a product imply summation over the values*** 1, 2, 3. Thus the formula for $\cos\theta$ is

$$\cos\theta = l_\lambda m_\lambda.$$

Again, the plane $\quad a_1 x_1 + a_2 x_2 + a_3 x_3 = b$

is

$$a_\lambda x_\lambda = b.$$

Not all writers insist on the repeated suffix of summation being *Greek*; but, if it is not, some notation is required for a repeated suffix which is not summed. (In the present notation, $l_\lambda m_\lambda$ means $l_1 m_1 + l_2 m_2 + l_3 m_3$, whereas $l_i m_i$ is just a single term.)

The repeated Greek suffix is called a DUMMY SUFFIX, and may be given any other Greek name: for example,

$$l_\lambda m_\lambda = l_\mu m_\mu = l_\alpha m_\alpha,$$

each being the sum $l_1 m_1 + l_2 m_2 + l_3 m_3$ in which λ, μ, α do not appear.

Care must be taken during manipulation not to use as a dummy any suffix already appearing in an algebraic expression. For example, if

$$y = a_\lambda x_\lambda,$$

meaning

$$y = a_1 x_1 + a_2 x_2 + a_3 x_3,$$

and if

$$x_i = b_{i\lambda} z_\lambda,$$

meaning

$$x_i = b_{i1} z_1 + b_{i2} z_2 + b_{i3} z_3,$$

the expression for y in terms of z_1, z_2, z_3 is

$$y = a_\lambda b_{\lambda\mu} z_\mu,$$

the dummy suffixes λ, μ being summed *independently* over the values 1, 2, 3. It is necessary to use the suffix μ in the substitution $x_\lambda = b_{\lambda\mu} z_\mu$, as λ is already in use.

3. Vectors

By a VECTOR we mean a given magnitude in a given direction, which can be *represented* by a displacement in space and which, when so represented, is subject to the same mathematical laws as if it were in fact a displacement. Many physical quantities are vectors: for example, velocity, acceleration, force.

The word SCALAR is used for a number, such as volume, not linked with direction.

In order to set up an abstract theory of vectors, let X be a typical point whose coordinates, referred to one particular set of orthogonal axes $O(1,2,3)$, are (x_1, x_2, x_3). The NUMBER-TRIPLET (x_1, x_2, x_3), conceived as a single entity, is denoted by the single symbol $\mathbf{x}$. When the individual COMPONENTS have to be emphasized, the more extended notation $\mathbf{x}(x_1, x_2, x_3)$ is used.

The symbol $\mathbf{x}$ is called the COORDINATE, or POSITION, VECTOR of the point X. It may be regarded in two complementary ways:

(i) it gives the *position* of X;

(ii) it gives the *displacement* $\overrightarrow{OX}$ of X from the origin O.

4. Transformation according to the vector rule

A vector is given by its magnitude and direction and is subject to the same mathematical laws as displacement in space. Suppose, then, that $\mathbf{x}$ is a displacement whose components for a given set of right-handed orthogonal axes $O1$, $O2$, $O3$ are (x_1, x_2, x_3), and for a second such set $O1'$, $O2'$, $O3'$ are (x_1', x_2', x_3'). Let the direction cosines of $O1'$, $O2'$, $O3'$ referred to $O1$, $O2$, $O3$ be (l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3). The two sets of components are connected by the relations (p. 44) of which typical ones are

$$x_1' = l_1 x_1 + l_2 x_2 + l_3 x_3, \qquad x_1 = l_1 x_1' + m_1 x_2' + n_1 x_3'.$$

Since these formulae govern displacements, they must also govern vectors. Hence, if $\mathbf{u}$ is a vector with components (u_1, u_2, u_3) referred to $O1$, $O2$, $O3$ and components (u_1', u_2', u_3') referred to $O1'$, $O2'$, $O3'$, then these components obey the transformation rule

$$\begin{aligned} u_1' &= l_1 u_1 + l_2 u_2 + l_3 u_3, & u_1 &= l_1 u_1' + m_1 u_2' + n_1 u_3', \\ u_2' &= m_1 u_1 + m_2 u_2 + m_3 u_3, & u_2 &= l_2 u_1' + m_2 u_2' + n_2 u_3', \\ u_3' &= n_1 u_1 + n_2 u_2 + n_3 u_3, & u_3 &= l_3 u_1' + m_3 u_2' + n_3 u_3'. \end{aligned}$$

We say that $\mathbf{u}(u_1, u_2, u_3)$ TRANSFORMS ACCORDING TO THE VECTOR RULE.

These formulae suffer from an awkwardness that we overcome by the use of a DOUBLE SUFFIX NOTATION for direction cosines. Denote by

$$l_{ij} \quad (i,j = 1, 2, 3)$$

the cosine of the angle between the axis $\mathbf{O}i$ of the first system and the axis $\mathbf{O}j'$ of the second.†

† We occasionally use the notations (l_1, l_2, l_3) and l_{ij} close together, but no confusion need arise.

For example,
$$l_{13} = \cos(O1, O3') = n_1,$$
$$l_{21} = \cos(O2, O1') = l_2,$$
$$l_{22} = \cos(O2, O2') = m_2.$$

The law of transformation is then given by six equations, of which

$$u_1' = l_{11}u_1+l_{21}u_2+l_{31}u_3, \quad u_1 = l_{11}u_1'+l_{12}u_2'+l_{13}u_3'$$

are typical. More briefly, with the summation convention, the RULE OF VECTOR TRANSFORMATION is

$$u_i' = l_{\lambda i}u_\lambda, \quad u_i = l_{i\lambda}u_\lambda \quad (i = 1, 2, 3),$$

the right-hand sides being summed, in each case, over the values 1, 2, 3 of λ.

Note, incidentally, the identities

$$\left.\begin{aligned} l_{i\lambda}l_{j\lambda} &= 1 \quad (i=j) \\ &= 0 \quad (i \neq j) \end{aligned}\right\}$$

and

$$\left.\begin{aligned} l_{\lambda i}l_{\lambda j} &= 1 \quad (i=j) \\ &= 0 \quad (i \neq j) \end{aligned}\right\}.$$

5. The direction cosine vector

Suppose that a line L has direction cosines (l_1, l_2, l_3) referred to axes $O1$, $O2$, $O3$ and (l_1', l_2', l_3') referred to $O1'$, $O2'$, $O3'$. Then l_i' is the cosine of the angle between L and the axis Oi'. But, referred to $O1$, $O2$, $O3$, the direction cosines of L are (l_1, l_2, l_3) and the direction cosines of Oi' are (l_{1i}, l_{2i}, l_{3i}). Hence, by the ordinary formula (p. 15) for the cosine of the angle between two lines,

$$\begin{aligned} l_i' &= l_1 l_{1i}+l_2 l_{2i}+l_3 l_{3i} \\ &= l_{\lambda i}l_\lambda. \end{aligned}$$

Similarly, l_i is the cosine of the angle between L and the axis Oi. But, referred to $O1'$, $O2'$, $O3'$, the direction cosines of L are (l_1', l_2', l_3') and the direction cosines of Oi are (l_{i1}, l_{i2}, l_{i3}). Hence

$$\begin{aligned} l_i &= l_1' l_{i1}+l_2' l_{i2}+l_3' l_{i3} \\ &= l_{i\lambda}l_\lambda'. \end{aligned}$$

Thus *direction cosines transform according to the vector rule. The three magnitudes l_i therefore define a* DIRECTION COSINE VECTOR **l**.

ILLUSTRATION. *To express the nine orthogonality relations* (p. 42) *in double suffix notation.*

A typical relation is, say,

$$l_1 m_2 - m_1 l_2 = \Delta n_3,$$

where, for right-handed axes, $\Delta = +1$; thus

$$l_1 m_2 - m_1 l_2 = n_3.$$

In double suffix notation, this is

$$l_{11} l_{22} - l_{12} l_{21} = l_{33}.$$

Let us write i, j, k for 1, 2, 3 in the first suffix of each direction cosine, and p, q, r for 1, 2, 3 in the second: thus,

$$l_{ip} l_{jq} - l_{iq} l_{jp} = l_{kr}.$$

If now i, j, k is any permutation in CYCLIC ORDER of 1, 2, 3 (that is, 1, 2, 3 or 2, 3, 1 or 3, 1, 2) and p, q, r is, independently, any permutation in cyclic order of 1, 2, 3, then the resulting equation is one of the nine.

For example, with $i, j, k = 2, 3, 1$ and $p, q, r = 3, 1, 2$, the equation is

$$l_{23} l_{31} - l_{21} l_{33} = l_{12},$$

or

$$n_2 l_3 - l_2 n_3 = m_1.$$

Thus *the orthogonality relation is*

$$l_{ip} l_{jq} - l_{iq} l_{jp} = l_{kr}.$$

6. The algebra of vectors

In order to manipulate vectors, some definitions must now be given.

Let $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ be vectors and $a, b, c, \ldots$ scalars. Then *the expression*

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{z} + \ldots$$

is defined to mean the vector $\mathbf{w}$ *whose components* (w_1, w_2, w_3) *are formed according to the rule*

$$w_i = ax_i + by_i + cz_i + \ldots .$$

[Note that, in the notation of § 4 (p. 58), with obvious adaptations,

$$\begin{aligned} l_{\lambda i} w_\lambda &= a(l_{\lambda i} x_\lambda) + b(l_{\lambda i} y_\lambda) + \ldots \\ &= ax'_i + by'_i + \ldots \\ &= w'_i, \end{aligned}$$

and
$$l_{i\lambda}w'_\lambda = a(l_{i\lambda}x'_\lambda)+b(l_{i\lambda}y'_\lambda)+\dots$$
$$= ax_i+by_i+\dots$$
$$= w_i.$$
Hence **w** transforms according to the vector rule.]

Three particular cases deserve special mention:

(i) The SUM **x**+**y** of two vectors **x**, **y** is the vector **w** for which
$$w_i = x_i+y_i.$$

(ii) The DIFFERENCE **x**−**y** of two vectors **x**, **y** is the vector **w** for which
$$w_i = x_i-y_i.$$

(iii) The SCALAR MULTIPLE a**x** of the vector **x** by the scalar a is the vector **w** for which
$$w_i = ax_i.$$

7. The parallelogram rule

Let **x**, **y** be two given vectors whose sum is
$$\mathbf{z} \equiv \mathbf{x}+\mathbf{y}.$$
If X, Y, Z are the points whose coordinate vectors are **x**, **y**, **z**,

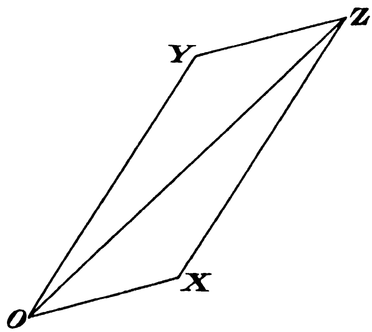

FIG. 32

and if O is the origin, then the two segments XY, OZ have the same middle point, of coordinate vector†
$$\tfrac{1}{2}(\mathbf{x}+\mathbf{y}).$$
Hence *Z is the fourth vertex of the parallelogram of which OX, OY are adjacent sides.*

† Its components are $\frac{1}{2}(x_1+y_1)$, $\frac{1}{2}(x_2+y_2)$, $\frac{1}{2}(x_3+y_3)$.

In terms of the 'displacement' interpretation of vectors (p. 56), *the sum* $\mathbf{z} \equiv \mathbf{x}+\mathbf{y}$ *of two vectors* $\mathbf{x}$, $\mathbf{y}$ *is represented by the diagonal* $\overrightarrow{OZ}$ *of the parallelogram of which* $\overrightarrow{OX}$, $\overrightarrow{OY}$ *are adjacent sides.*

Briefly, *vectors are added according to the* PARALLELOGRAM RULE.

COROLLARY. *The vector* $\mathbf{y}-\mathbf{x}$ *is interpreted by the line through the origin equal and parallel to* $\overrightarrow{XY}$.

ILLUSTRATION: THE CENTROID OF A TETRAHEDRON.

Let $ABCD$ be a tetrahedron, and denote by L, M, N, P, Q, R

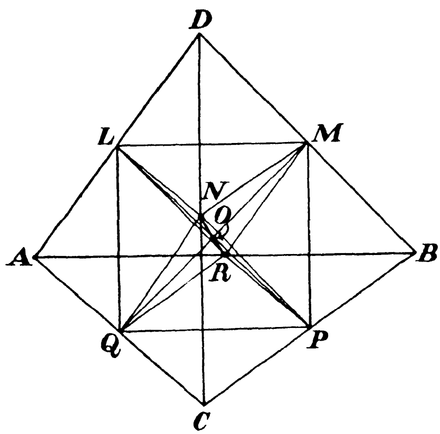

FIG. 33

the middle points of the edges DA, DB, DC, BC, CA, AB. Then, in terms of vectors referred to any arbitrary origin,

$$\mathbf{l} = \tfrac{1}{2}(\mathbf{a}+\mathbf{d}), \qquad \mathbf{p} = \tfrac{1}{2}(\mathbf{b}+\mathbf{c}),$$
$$\mathbf{m} = \tfrac{1}{2}(\mathbf{b}+\mathbf{d}), \qquad \mathbf{q} = \tfrac{1}{2}(\mathbf{c}+\mathbf{a}),$$
$$\mathbf{n} = \tfrac{1}{2}(\mathbf{c}+\mathbf{d}), \qquad \mathbf{r} = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}).$$

Thus $$\mathbf{n}-\mathbf{m} = \mathbf{q}-\mathbf{r} = \tfrac{1}{2}(\mathbf{c}-\mathbf{b}),$$

so that, as above, $$\overrightarrow{MN} = \overrightarrow{RQ} = \tfrac{1}{2}\overrightarrow{BC},$$

the three lines being parallel. Thus

$$MNQR$$

is a parallelogram. Similarly

$$NLRP, \quad LMPQ$$

are parallelograms. Hence *the three lines LP, MQ, NR pass through a point O at which each is bisected.*

The last result may also be verified directly: the middle point of LP is

$$\tfrac{1}{2}(\mathbf{l}+\mathbf{p}),$$

or

$$\tfrac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}),$$

and this, by symmetry, is the same for the lines MQ, NR also.

8. The scalar product of two vectors

Given two vectors $\mathbf{x}$, $\mathbf{y}$, the scalar number

$$x_1 y_1 + x_2 y_2 + x_3 y_3$$

or, with the summation convention,

$$x_\lambda y_\lambda$$

is called their SCALAR PRODUCT and is denoted by the symbol $\mathbf{x}.\mathbf{y}$. Since $x_\lambda y_\lambda \equiv y_\lambda x_\lambda$, the scalar product is also $\mathbf{y}.\mathbf{x}$. Thus

$$\mathbf{x}.\mathbf{y} \equiv \mathbf{y}.\mathbf{x} \equiv x_\lambda y_\lambda.$$

The scalar product $\mathbf{x}.\mathbf{x}$ is written $\mathbf{x}^2$, so that

$$\mathbf{x}^2 \equiv x_1^2 + x_2^2 + x_3^2.$$

The positive square root

$$+\sqrt{(x_1^2+x_2^2+x_3^2)}$$

is called the MAGNITUDE of $\mathbf{x}$, and is written $|\mathbf{x}|$ or x. Note this use of the symbol x for the magnitude of $\mathbf{x}$; it will be used without explicit remark.

Since
$$\mathbf{x}.(\mathbf{y}+\mathbf{z}) \equiv x_\lambda(y_\lambda+z_\lambda) = x_\lambda y_\lambda + x_\lambda z_\lambda$$
$$= \mathbf{x}.\mathbf{y}+\mathbf{x}.\mathbf{z},$$

it follows that *scalar products obey the* DISTRIBUTIVE LAW

$$\mathbf{x}.(\mathbf{y}+\mathbf{z}) = \mathbf{x}.\mathbf{y}+\mathbf{x}.\mathbf{z}.$$

Since
$$(a\mathbf{x}).(b\mathbf{y}) \equiv (ax_\lambda)(by_\lambda) = abx_\lambda y_\lambda$$
$$= ab\mathbf{x}.\mathbf{y},$$

it follows that *scalar factors of either vector may be taken 'outside' a scalar product.*

9. Distance and angle

In this and the following paragraphs there are gathered together in vector form several results already established in terms of cartesian coordinates.

(i) The relation $\qquad \mathbf{x}^2 \equiv x_1^2+x_2^2+x_3^2$

shows that *the distance OX of the point X from the origin O is given by the formula*

$$OX^2 = \mathbf{x}^2, \quad \text{or} \quad OX = |\mathbf{x}|.$$

COROLLARY. *The* DIRECTION COSINE VECTOR *of the direction* $\overrightarrow{OX}$ *is* **l**, *where* **l** *is given by the relation*

$$\mathbf{x} = x\mathbf{l},$$

or

$$\mathbf{l} = x^{-1}\mathbf{x}.$$

(ii) The relation

$$\mathbf{l}.\mathbf{m} \equiv \mathbf{m}.\mathbf{l} \equiv l_1 m_1+l_2 m_2+l_3 m_3$$

shows that *the angle θ between the two directions* **l**, **m** *is given by the formula* (p. 15)

$$\cos\theta = \mathbf{l}.\mathbf{m} = \mathbf{m}.\mathbf{l}.$$

[Note that **l**, **m** are vectors of *unit* magnitude.]

COROLLARY. *The two directions* **l**, **m** *are perpendicular if, and only if,*

$$\mathbf{l}.\mathbf{m} = 0 \quad (\textit{or}\ \mathbf{m}.\mathbf{l} = 0).$$

(iii) A geometrical interpretation for the scalar product $\mathbf{x}.\mathbf{y}$ may be founded on a combination of (i), (ii). If **l**, **m** are the direction cosines of OX, OY, then

$$\mathbf{x} = x\mathbf{l}, \qquad \mathbf{y} = y\mathbf{m},$$

where x, y are the magnitudes of **x**, **y**; also

$$\mathbf{l}.\mathbf{m} = \cos\theta,$$

where θ is the angle between the lines. Thus

$$\begin{aligned} \mathbf{x}.\mathbf{y} &= (x\mathbf{l}).(y\mathbf{m}) \\ &= xy\mathbf{l}.\mathbf{m} \quad \text{(p. 62)} \\ &= xy\cos\theta. \end{aligned}$$

Hence $\mathbf{x}.\mathbf{y}$ *is equal to* $OX.OY\cos XOY$.

COROLLARY. When $y = 1$, so that $\mathbf{y} \equiv \mathbf{m}$, the above formula becomes

$$\mathbf{x}.\mathbf{m} = x\cos\theta,$$

so that (see p. 13) $\mathbf{x}.\mathbf{m}$ *is equal to the projection of* OX *on the direction* $\mathbf{m}$.

(iv) THE DISTANCE BETWEEN TWO POINTS. The results of this paragraph serve to verify what was perhaps already obvious, that *the length of the segment* XY *is given by the formula*

$$XY^2 = (\mathbf{x}-\mathbf{y})^2.$$

In fact,

$$\begin{aligned}(\mathbf{x}-\mathbf{y})^2 &= \mathbf{x}.(\mathbf{x}-\mathbf{y})-\mathbf{y}.(\mathbf{x}-\mathbf{y}) \\ &= \mathbf{x}^2-2\mathbf{x}.\mathbf{y}+\mathbf{y}^2 \\ &= x^2-2xy\cos\theta+y^2 \\ &= XY^2.\end{aligned}$$

Thus (p. 61)

$$XY = |\mathbf{x}-\mathbf{y}|.$$

ILLUSTRATION: THE PROJECTION OF A POINT ON A LINE. Let P, with position vector $\mathbf{p}$, be a given point, and l a given line through the origin with direction cosine vector $\mathbf{l}$. Let Q, with position vector $\mathbf{q}$, be the projection of P on l.

Since l passes through the origin, there is a scalar k such that

$$\mathbf{q} = k\mathbf{l}.$$

Now PQ is perpendicular to l, so that

$$(\mathbf{p}-\mathbf{q}).\mathbf{l} = 0,$$

or

$$\mathbf{p}.\mathbf{l} = \mathbf{q}.\mathbf{l} = k\mathbf{l}.\mathbf{l} = k.$$

Hence

$$\mathbf{q} = (\mathbf{p}.\mathbf{l})\mathbf{l}.$$

The components of these vectors thus satisfy the relation

$$\begin{aligned}q_i &= (p_\lambda l_\lambda)l_i \\ &= (l_i l_\lambda)p_\lambda.\end{aligned}$$

If we now define nine numbers a_{ij} (for i, $j = 1, 2, 3$ independently) by the relation

$$a_{ij} = l_i l_j,$$

then *the projection of* $\mathbf{p}$ *on the line* $\mathbf{l}$ *is the point* $\mathbf{q}$, *where*

$$q_i = a_{i\lambda}p_\lambda.$$

Note the relation

$$a_{i\lambda}a_{j\lambda} = (l_i l_\lambda)(l_j l_\lambda)$$
$$= (l_i l_j)(l_\lambda l_\lambda)$$
$$= l_i l_j,$$

since $l_\lambda l_\lambda = 1$. Thus
$$a_{i\lambda}a_{j\lambda} = a_{ij}.$$

10. The unit axes-vectors

The three vectors

$$\mathbf{i} \equiv (1,0,0), \qquad \mathbf{j} \equiv (0,1,0), \qquad \mathbf{k} \equiv (0,0,1)$$

are called the UNIT AXES-VECTORS. They satisfy the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1,$$
$$\mathbf{jk} = \mathbf{ki} = \mathbf{ij} = 0.$$

Their importance lies in the fact that an arbitrary vector $\mathbf{x}(x_1, x_2, x_3)$ may be expressed in terms of them in the form

$$\mathbf{x} \equiv x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}.$$

11. The straight line and the plane

The formulae obtained earlier in this book may be restated concisely in vector form. It is understood that a point like A has corresponding coordinate vector $\mathbf{a}$.

(i) THE DISTANCE FORMULA (p. 17).

The coordinate vector of a point X on the line through the point A with direction cosine vector $\mathbf{l}$ is given by the formula

$$\mathbf{x} = \mathbf{a}+r\mathbf{l},$$

where $r \equiv \overrightarrow{AX}$.

(ii) THE RATIO FORMULA (p. 17).

If X is the point on the straight line AB such that

$$\overrightarrow{AX}/\overrightarrow{XB} = k,$$

then
$$(1+k)\mathbf{x} = \mathbf{a}+k\mathbf{b}.$$

(iii) THE EQUATION OF A PLANE (p. 20).

Let P be the foot of the perpendicular from the origin O to the plane, so that $\overrightarrow{OP}$ is of (positive or negative) length p and

direction $\mathbf{l}$. Then the coordinates of a point X of the plane satisfy the equation
$$\mathbf{l}.\mathbf{x} = p.$$

(iv) PARAMETRIC FORM FOR A PLANE (p. 26).

The coordinate vector of an arbitrary point X of the plane ABC may be taken in the form
$$\mathbf{x} = \lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{c},$$
where the parameters λ, μ, ν satisfy the relation
$$\lambda+\mu+\nu = 1.$$

(v) THE PERPENDICULAR FROM A POINT TO A PLANE (p. 23).

If N is the foot of the perpendicular from the point Q to the plane $\mathbf{l}.\mathbf{x}-p = 0$, then
$$\overrightarrow{NQ} = \mathbf{l}.\mathbf{q}-p.$$

ILLUSTRATION. THE ORTHOGONAL TETRAHEDRON. To prove that, *if in a tetrahedron $ABCD$ the edges BD, CA are perpendicular and the edges CD, AB are perpendicular, then the edges AD, BC are also perpendicular.*

Since BD, CA are perpendicular, it follows (p. 63) that
$$(\mathbf{d}-\mathbf{b}).(\mathbf{a}-\mathbf{c}) = 0,$$
or
$$\mathbf{d}.\mathbf{a}-\mathbf{d}.\mathbf{c}-\mathbf{a}.\mathbf{b}+\mathbf{b}.\mathbf{c} = 0.$$
Similarly
$$\mathbf{d}.\mathbf{b}-\mathbf{d}.\mathbf{a}-\mathbf{b}.\mathbf{c}+\mathbf{c}.\mathbf{a} = 0.$$
Adding,
$$\mathbf{d}.\mathbf{b}-\mathbf{d}.\mathbf{c}-\mathbf{a}.\mathbf{b}+\mathbf{c}.\mathbf{a} = 0,$$
or
$$(\mathbf{d}-\mathbf{a}).(\mathbf{b}-\mathbf{c}) = 0.$$
Hence AD, BC are perpendicular.

12. The vector product

In many applications of vectors, and especially in the physical applications, there is associated with two given vectors $\mathbf{x}$ and $\mathbf{y}$, inclined at an angle θ $(0 < \theta < \pi)$, a third vector $\mathbf{z}$ determined by the properties that it is

(i) of magnitude $xy\sin\theta$,

(ii) perpendicular to $\mathbf{x}$ and $\mathbf{y}$,

(iii) in the sense such that a right-handed cork-screw, turning from $\mathbf{x}$ towards $\mathbf{y}$, drives along $\mathbf{z}$.

This is the vector which we now define and whose properties we proceed to investigate. We approach it from a somewhat different point of view, but recover these three basic properties almost immediately, in § 13.

DEFINITION. Given two vectors $\mathbf{x}$, $\mathbf{y}$, their VECTOR PRODUCT, written

$$\mathbf{x} \wedge \mathbf{y},$$

is defined to be the vector with components

$$(x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1).$$

Thus, interchanging the roles of x and y,

$$\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}.$$

Vector products obey the DISTRIBUTIVE LAW

$$\mathbf{x} \wedge (\mathbf{y} + \mathbf{z}) = \mathbf{x} \wedge \mathbf{y} + \mathbf{x} \wedge \mathbf{z},$$

since, for example, the first component on the left is

$$x_2(y_3 + z_3) - x_3(y_2 + z_2),$$

or

$$(x_2 y_3 - x_3 y_2) + (x_2 z_3 - x_3 z_2).$$

Significant form may be given to the vector product by writing it in terms of the unit axes-vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ (p. 65). Then, from the definition,

$$\mathbf{x} \wedge \mathbf{y} \equiv (x_2 y_3 - x_3 y_2)\mathbf{i} + (x_3 y_1 - x_1 y_3)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k},$$

or, symbolically,

$$\mathbf{x} \wedge \mathbf{y} \equiv \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix}.$$

It is worthy of remark that $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *themselves satisfy the relations*

$$\mathbf{j} \wedge \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \wedge \mathbf{i} = \mathbf{j}, \quad \mathbf{i} \wedge \mathbf{j} = \mathbf{k}.$$

The definition shows that *the vector product of any vector with itself is zero; that is,*

$$\mathbf{x} \wedge \mathbf{x} = 0.$$

The Corollary follows that, *if the vectors* $\mathbf{x}$, $\mathbf{y}$ *are* ***parallel***, *so that* $\mathbf{y} = k\mathbf{x}$ *for some scalar* k, *then*

$$\mathbf{x} \wedge \mathbf{y} = 0.$$

Before deriving the geometrical interpretation of the vector product, we first define also the SCALAR TRIPLE PRODUCT, written $(\mathbf{xyz})$, of three vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. This is *the scalar number whose value is the determinant*

$$(\mathbf{xyz}) \equiv \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Expansion in terms of the first row gives the identity

$$(\mathbf{xyz}) = x_1(y_2 z_3 - y_3 z_2) + x_2(y_3 z_1 - y_1 z_3) + x_3(y_1 z_2 - y_2 z_1),$$

or
$$(\mathbf{xyz}) = \mathbf{x}.(\mathbf{y} \wedge \mathbf{z});$$

similarly
$$(\mathbf{xyz}) = \mathbf{y}.(\mathbf{z} \wedge \mathbf{x}) = \mathbf{z}.(\mathbf{x} \wedge \mathbf{y}).$$

Note that *the scalar triple product is unaffected by cyclic interchange of the vectors* (this is what we have just proved), *but that it is changed in sign if two of the vectors are interchanged.* For example,

$$\begin{aligned} (\mathbf{xzy}) &= \mathbf{x}.(\mathbf{z} \wedge \mathbf{y}) \\ &= \mathbf{x}.(-\mathbf{y} \wedge \mathbf{z}) \\ &= -\mathbf{x}.(\mathbf{y} \wedge \mathbf{z}) \\ &= -(\mathbf{xyz}). \end{aligned}$$

A PHYSICAL INTERPRETATION for the scalar triple product is afforded by the following result:

If V is the volume of the tetrahedron $OXYZ$, then (p. 46)

$$\begin{aligned} 6V &= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \\ &= (\mathbf{xyz}) \end{aligned}$$

numerically. (See also p. 70.)

NOTE. When the constituent vectors are complicated, commas are sometimes inserted for clarity. Thus the scalar triple product of $\mathbf{x}-\mathbf{a}$, $\mathbf{y}-\mathbf{b}$, $\mathbf{z}-\mathbf{c}$ is

$$(\mathbf{x}-\mathbf{a}, \mathbf{y}-\mathbf{b}, \mathbf{z}-\mathbf{c}).$$

13. Geometrical interpretation of the vector product $\mathbf{x} \wedge \mathbf{y}$

(i) THE MAGNITUDE. By definition, the magnitude is (p. 62)

$$+\sqrt{\{(x_2 y_3 - x_3 y_2)^2 + (x_3 y_1 - x_1 y_3)^2 + (x_1 y_2 - x_2 y_1)^2\}},$$

or (p. 46)
$$2\Delta OXY,$$

so that *the magnitude of* $\mathbf{x} \wedge \mathbf{y}$ *is* $xy \sin\theta$, *where* x, y *are the magnitudes of* $\mathbf{x}$, $\mathbf{y}$ *and* θ *is the angle between them.* Note that x, y are positive, and that $\sin\theta$ is also positive, since (p. 8) θ lies between 0 and π.

(ii) THE DIRECTION. Since

$$\mathbf{x}.(\mathbf{x} \wedge \mathbf{y}) \equiv (\mathbf{xxy}) \qquad \text{(p. 68)}$$
$$\equiv \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$
$$= 0,$$

it follows (p. 63) that $\mathbf{x} \wedge \mathbf{y}$ is perpendicular to $\mathbf{x}$, and, similarly, to $\mathbf{y}$. Thus $\mathbf{x} \wedge \mathbf{y}$ *is in the direction perpendicular to both* $\mathbf{x}$ *and* $\mathbf{y}$.

(iii) THE SENSE. The actual sense of $\mathbf{x} \wedge \mathbf{y}$ along the perpendicular is still not determined, and a rule must be provided to

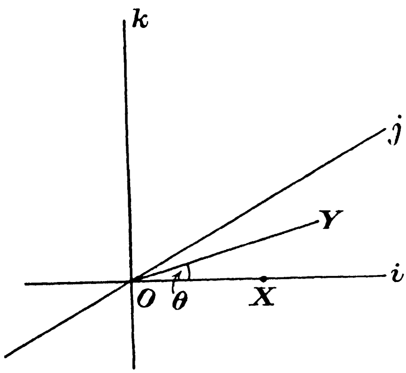

FIG. 34

distinguish between the two possible cases. Imagine the points X, Y to remain fixed, but the *axes* to move continuously about O until $\mathbf{i}$ lies along $\overrightarrow{OX}$; then rotate the coordinate system about OX until the plane $O\mathbf{ij}$ passes through OY in that sense for which Y is on the 'positive' side of $O\mathbf{i}$. The position of the coordinate system is then determined.

Now the position of Y, referred to these axes, is given by the coordinate vector $y\mathbf{m}$, where

$$\mathbf{m} \equiv (\cos\theta, \sin\theta, 0),$$

and, in this position, the value of the vector product $\mathbf{x} \wedge \mathbf{y}$, where $\mathbf{x} \equiv (x, 0, 0)$, $\mathbf{y} \equiv (y \cos\theta, y \sin\theta, 0)$, is $(0, 0, xy \sin\theta)$, so that

$$\mathbf{x} \wedge \mathbf{y} = xy \sin\theta\, \mathbf{k}.$$

Finally, observe that the vector product has remained constant throughout the rotation of the axes, a sudden change of sense at any point being excluded by the continuity of the motion. The sense may therefore be described by this final position; and, since θ is between 0 and π in the sense from $O\mathbf{i}$ to $O\mathbf{j}$ (and on towards $-O\mathbf{i}$ if θ is obtuse), the rule is:

A right-handed cork-screw, turning from $\mathbf{x}$ *towards* $\mathbf{y}$ *drives forward in the sense* $\mathbf{x} \wedge \mathbf{y}$.

We have therefore established the three basic properties with which we introduced the vector product on p. 66.

ILLUSTRATION. *Geometrical derivation of the formula* $\frac{1}{6}(\mathbf{xyz})$ *for the volume of the tetrahedron* $OXYZ$. (Compare p. 68.)

The magnitude of the vector product $\mathbf{y} \wedge \mathbf{z}$ is $OY \,.\, OZ \sin YOZ$, which is twice the area of the triangle YOZ. If, then, $\mathbf{u}$ is the unit vector perpendicular to the plane YOZ and in the same sense as $\mathbf{y} \wedge \mathbf{z}$, it follows that

$$\mathbf{y} \wedge \mathbf{z} = 2\mathbf{u}\Delta YOZ.$$

Hence

$$\begin{aligned}(\mathbf{xyz}) &= \mathbf{x} \,.\, (\mathbf{y} \wedge \mathbf{z}) \\ &= 2(\mathbf{x} \,.\, \mathbf{u})\Delta YOZ.\end{aligned}$$

But the numerical value of $\mathbf{x} \,.\, \mathbf{u}$ is the length of the projection of OX in the direction of $\mathbf{u}$; that is, $\pm\mathbf{x} \,.\, \mathbf{u}$ is the length of the perpendicular from X on to the plane YOZ, say $\pm\mathbf{x} \,.\, \mathbf{u} = p$. Thus, numerically,

$$\begin{aligned}(\mathbf{xyz}) &= 2p\Delta YOZ \\ &= 2(3 \text{ volume } OXYZ),\end{aligned}$$

so that, numerically,

$$\text{volume } OXYZ = \tfrac{1}{6}(\mathbf{xyz}).$$

ILLUSTRATION FROM MECHANICS. THE MOMENT OF A FORCE. Suppose that a force $\mathbf{F}$, with components F_1, F_2, F_3, acts in a straight line passing through the point with position vector

$\mathbf{x}(x_1, x_2, x_3)$. Then the MOMENT *about the origin O* of the force is defined to be the vector

$$\mathbf{m} \equiv \mathbf{x} \wedge \mathbf{F}.$$

Suppose that $\mathbf{l}(l_1, l_2, l_3)$ is the direction cosine vector of a line L through O. Then the MOMENT *about the line L* of the force is defined to be the component of $\mathbf{m}$ in the direction of L; that is,

$$m_L \equiv (\mathbf{l}.\mathbf{m})$$
$$\equiv (\mathbf{lxF}).$$

If θ is the angle between $\mathbf{x}$ and $\mathbf{F}$, then the magnitude of $\mathbf{m}$ is $xF \sin\theta$. But $x \sin\theta$ is the length p of the perpendicular from O on to the line of action of $\mathbf{F}$. Hence

$$|\mathbf{m}| = p|\mathbf{F}|.$$

ILLUSTRATION. THE VECTOR PRODUCT AND THE VECTOR RULE FOR TRANSFORMATION. It is instructive to verify that *the vector*

$$\mathbf{w} \equiv \mathbf{x} \wedge \mathbf{y}$$

transforms according to the vector rule (p. 58).

Let i, j, k be a cyclic permutation of the numbers 1, 2, 3. Then, by definition of $\mathbf{w}$,

$$w_i = x_j y_k - x_k y_j.$$

Under the transformation

$$x_i = l_{i\lambda} x'_\lambda,$$

the expression for w_i is

$$w_i = l_{j\lambda} x'_\lambda l_{k\mu} x'_\mu - l_{k\lambda} x'_\lambda l_{j\mu} x'_\mu.$$

Consider a term $x'_p y'_q$ for *given* p, q; the coefficient is

$$l_{jp} l_{kq} - l_{jq} l_{kp}.$$

Consider also a term $x'_q y'_p$; the coefficient is

$$l_{jq} l_{kp} - l_{jp} l_{kq}.$$

If p, q are equal, these coefficients are zero; if not, they are equal and opposite, so that together they contribute to w_i the amount

$$(l_{jp} l_{kq} - l_{jq} l_{kp})(x'_p y'_q - x'_q y'_p).$$

Suppose now that q follows p in the cyclic order 1, 2, 3; this does not affect the argument, since otherwise p, q could be interchanged, thereby changing the signs of *both* brackets in the last expression. Let r be the third member of the cyclic sequence p, q, r. Then (p. 59)

$$l_{jp}\,l_{kq}-l_{jq}\,l_{kp} = l_{ir}.$$

Also, by definition,

$$x'_p\,y'_q-x'_q\,y'_p = w'_r.$$

Giving p, q in turn the pairs of values 2, 3; 3, 1; 1, 2, the total expression for w_i appears in the form

$$w_i = l_{i1}\,w'_1+l_{i2}\,w'_2+l_{i3}\,w'_3$$
$$= l_{i\lambda}\,w'_\lambda.$$

Similarly, $$w'_i = l_{\lambda i}\,w_i.$$

Hence $\mathbf{w} \equiv \mathbf{x}\wedge\mathbf{y}$ transforms according to the vector rule.

14. Linearly dependent vectors

A number of vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$,... are said to be LINEARLY DEPENDENT if there exists a relation

$$a\mathbf{x}+b\mathbf{y}+c\mathbf{z}+d\mathbf{w}+... = 0$$

(or more than one such relation) with the scalars a, b, c, d,... not all zero.

If the relation holds only when $a = b = c = d = ... = 0$, then the vectors are said to be LINEARLY INDEPENDENT.

We shall see that two, or three, vectors *may* be linearly dependent, but that four *must* be.

(i) TWO VECTORS. If there exists a relation

$$a\mathbf{x}+b\mathbf{y} = 0,$$

with a, b not zero, then

$$\mathbf{y} = (-a/b)\mathbf{x},$$

so that *the points X, Y lie in a line through O.*

(ii) THREE VECTORS. If there exists a relation

$$a\mathbf{x}+b\mathbf{y}+c\mathbf{z} = 0,$$

with a, b, c not zero, then†

$$\frac{-c}{a+b}\mathbf{z} = \frac{a}{a+b}\mathbf{x}+\frac{b}{a+b}\mathbf{y},$$

† If $a+b = 0$, take $b+c$ with $\mathbf{x}$ or $c+a$ with $\mathbf{y}$ instead. The three expressions $b+c$, $c+a$, $a+b$ cannot all be zero.

so that (p. 65) the point whose coordinate vector is

$$-\frac{c}{a+b}\mathbf{z}$$

lies on the line XY; that is, there is a point of the line OZ which lies on XY, so that *the points X, Y, Z lie in a plane through O.*

(This assumes that there is not a second linear relation connecting the vectors; if there were, then the points X, Y, Z would be collinear with O.)

The converse of this result is of interest as a preparation for the consideration of four vectors. Suppose that X, Y, Z are three points lying in a plane through O. Then OZ meets XY, so that there exists a point whose coordinate vector can be expressed in *each* of the forms

$$p\mathbf{z}, \qquad \frac{q\mathbf{x}+r\mathbf{y}}{q+r}.$$

Hence there exists an identity

$$p(q+r)\mathbf{z} = q\mathbf{x}+r\mathbf{y},$$

which is of the linear form

$$a\mathbf{x}+b\mathbf{y}+c\mathbf{z} = 0.$$

COROLLARY. If, exceptionally,

$$a+b+c = 0,$$

then the points X, Y, Z are collinear.

(iii) FOUR VECTORS. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$ be four vectors no three of which are linearly dependent; then, by (ii), no three of the points X, Y, Z, W lie in a plane through O, so that, in particular, OW meets the plane XYZ in a point. There exists therefore a point whose coordinate vector can be expressed in each of the forms

$$p\mathbf{w}, \qquad \frac{q\mathbf{x}+r\mathbf{y}+s\mathbf{z}}{q+r+s},$$

and so there exists an identity

$$p(q+r+s)\mathbf{w} = q\mathbf{x}+r\mathbf{y}+s\mathbf{z}$$

which is of the linear form

$$a\mathbf{x}+b\mathbf{y}+c\mathbf{z}+d\mathbf{w} = 0.$$

Hence *four vectors are necessarily connected by (at least) one linear relation.*

An alternative important statement is:

If $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ *are linearly independent vectors, so that* OX, OY, OZ *are not coplanar, then any other vector* $\mathbf{w}$ *can be expressed in terms of them in the form*

$$\mathbf{w} = a\mathbf{x}+b\mathbf{y}+c\mathbf{z}.$$

15. The vector triple product

Given three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, form the vector product

$$\mathbf{w} = \mathbf{b}\wedge\mathbf{c}$$

and then the further product

$$\mathbf{a}\wedge\mathbf{w}$$

or

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}).$$

This is called the VECTOR TRIPLE PRODUCT OF $\mathbf{a}$ WITH $\mathbf{b}\wedge\mathbf{c}$.

To establish the formula

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{a}.\mathbf{b})\mathbf{c}.$$

By definition,

$$\mathbf{b}\wedge\mathbf{c} \equiv (b_2c_3-b_3c_2,\ b_3c_1-b_1c_3,\ b_1c_2-b_2c_1),$$

so that, similarly, the first component of $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$ is

$$a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3).$$

Grouped in terms of b_1 and c_1 (the first components of $\mathbf{b}$ and $\mathbf{c}$), this is

$$(a_2c_2+a_3c_3)b_1-(a_2b_2+a_3b_3)c_1,$$

or, inserting and cancelling a term $a_1b_1c_1$,

$$(a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1,$$

or

$$(\mathbf{a}.\mathbf{c})b_1-(\mathbf{a}.\mathbf{b})c_1.$$

A similar formula is obtained for the second and third components, so that

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{a}.\mathbf{b})\mathbf{c}.$$

NOTE. The position of the bracket is important; for

$$\begin{aligned}(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c} &= -\mathbf{c}\wedge(\mathbf{a}\wedge\mathbf{b})\\ &= -\{(\mathbf{c}.\mathbf{b})\mathbf{a}-(\mathbf{c}.\mathbf{a})\mathbf{b}\}\\ &= (\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{b}.\mathbf{c})\mathbf{a}.\end{aligned}$$

EXAMPLES

1. Prove that

(i) $$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})+\mathbf{b}\wedge(\mathbf{c}\wedge\mathbf{a})+\mathbf{c}\wedge(\mathbf{a}\wedge\mathbf{b}) = \mathbf{0},$$

(ii) $$(\mathbf{u}\wedge\mathbf{v})\wedge(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{uvc})\mathbf{b}-(\mathbf{uvb})\mathbf{c}.$$

2. Prove that the line $\mathbf{x} = \mathbf{a}+r\mathbf{l}$ meets the plane $(\mathbf{m}.\mathbf{x}) = p$ in the point

$$\{\mathbf{m}\wedge(\mathbf{a}\wedge\mathbf{l})+p\mathbf{l}\}/(\mathbf{l}.\mathbf{m}).$$

16. Expression of a vector in terms of three vector products

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be three linearly independent vectors, so that OX, OY, OZ are not coplanar. Then the common perpendiculars through O of the pairs (OY, OZ), (OZ, OX), (OX, OY) are not coplanar either, and so *an arbitrary vector can be expressed in terms of the* **linearly independent** *vector products* $\mathbf{y}\wedge\mathbf{z}$, $\mathbf{z}\wedge\mathbf{x}$, $\mathbf{x}\wedge\mathbf{y}$ *in the form*

$$\mathbf{w} = a\mathbf{y}\wedge\mathbf{z}+b\mathbf{z}\wedge\mathbf{x}+c\mathbf{x}\wedge\mathbf{y}.$$

The scalars a, b, c can be expressed conveniently as follows:

Form the scalar product of each side with $\mathbf{x}$. Then

$$\mathbf{x}.\mathbf{w} = a\mathbf{x}.(\mathbf{y}\wedge\mathbf{z})+b\mathbf{x}.(\mathbf{z}\wedge\mathbf{x})+c\mathbf{x}.(\mathbf{x}\wedge\mathbf{y})$$

$$= a(\mathbf{xyz}) \qquad \text{(p. 68)}$$

Hence $$a = \frac{\mathbf{x}.\mathbf{w}}{(\mathbf{xyz})},$$

with similar results for b, c.

The vector $\mathbf{w}$ *is therefore given by the relation*

$$(\mathbf{xyz})\mathbf{w} = (\mathbf{x}.\mathbf{w})\mathbf{y}\wedge\mathbf{z}+(\mathbf{y}.\mathbf{w})\mathbf{z}\wedge\mathbf{x}+(\mathbf{z}.\mathbf{w})\mathbf{x}\wedge\mathbf{y},$$

where, it should be remembered, the four expressions in brackets are all scalars.

DEFINITION. Given a set of three vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, the set of vectors

$$\mathbf{x}' \equiv \frac{\mathbf{y}\wedge\mathbf{z}}{(\mathbf{xyz})}, \qquad \mathbf{y}' \equiv \frac{\mathbf{z}\wedge\mathbf{x}}{(\mathbf{xyz})}, \qquad \mathbf{z}' \equiv \frac{\mathbf{x}\wedge\mathbf{y}}{(\mathbf{xyz})}$$

is said to be RECIPROCAL to the set $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$.

The equation just proved is

$$\mathbf{w} = (\mathbf{x}.\mathbf{w})\mathbf{x}'+(\mathbf{y}.\mathbf{w})\mathbf{y}'+(\mathbf{z}.\mathbf{w})\mathbf{z}'.$$

17. The plane through three given points

We prove first that *the direction of the normal to the plane containing three points A, B, C is that of the vector*

$$\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a}+\mathbf{a}\wedge\mathbf{b}.$$

The plane contains the line AB, of direction $\mathbf{b}-\mathbf{a}$, and the line AC, of direction $\mathbf{c}-\mathbf{a}$. The normal is thus in the direction

$$(\mathbf{b}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{a}),$$

or
$$\mathbf{b}\wedge(\mathbf{c}-\mathbf{a})-\mathbf{a}\wedge(\mathbf{c}-\mathbf{a}),$$

or
$$\mathbf{b}\wedge\mathbf{c}-\mathbf{b}\wedge\mathbf{a}-\mathbf{a}\wedge\mathbf{c}+\mathbf{a}\wedge\mathbf{a},$$

or
$$\mathbf{b}\wedge\mathbf{c}+\mathbf{a}\wedge\mathbf{b}+\mathbf{c}\wedge\mathbf{a}+\mathbf{0},$$

or
$$\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a}+\mathbf{a}\wedge\mathbf{b}.$$

Further, *the magnitude of this vector is twice the area of the triangle ABC*. It was proved (p. 68) that the magnitude of the product $\mathbf{x}\wedge\mathbf{y}$ is $2\Delta OXY$; and $(\mathbf{b}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{a})$ is the similar expression referred to an initial point A instead of the origin O.

Finally, *the equation of the plane ABC is*

$$\mathbf{x}.(\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a}+\mathbf{a}\wedge\mathbf{b}) = (\mathbf{abc}):$$

Since the normal is in the direction

$$\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a}+\mathbf{a}\wedge\mathbf{b},$$

the equation of the plane is in the form (p. 66)

$$\mathbf{x}.(\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a}+\mathbf{a}\wedge\mathbf{b}) = k$$

for some value of k. But the equation is to be satisfied when $\mathbf{x} = \mathbf{a}$, and so

$$\mathbf{a}.\mathbf{b}\wedge\mathbf{c}+\mathbf{a}.\mathbf{c}\wedge\mathbf{a}+\mathbf{a}.\mathbf{a}\wedge\mathbf{b} = k,$$

or
$$(\mathbf{abc})+0+0 = k,$$

so that
$$k = (\mathbf{abc}).$$

Hence the equation is

$$\mathbf{x}.(\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a}+\mathbf{a}\wedge\mathbf{b}) = (\mathbf{abc}).$$

18. Other forms for the equation of a plane

The following results are easily established and the proofs are left as EXAMPLES.

(i) *The equation of the plane through a point of position vector* **a** *and parallel to two directions* **l**, **m** *is*

$$(\mathbf{x}-\mathbf{a}, \mathbf{l}, \mathbf{m}) = 0.$$

(ii) *The equation of the plane through two points of position vectors* $\mathbf{a}, \mathbf{b}$ *and parallel to the direction* $\mathbf{l}$ *is*

$$(\mathbf{x}, \mathbf{b}-\mathbf{a}, \mathbf{l}) = (\mathbf{abl}).$$

19. The common perpendicular of two lines

Let l be a line through a point $\mathbf{a}$ in the direction $\mathbf{l}$ and m a line through a point $\mathbf{b}$ in the direction $\mathbf{m}$.

The common perpendicular of the lines is in the direction of the vector product $\mathbf{l} \wedge \mathbf{m}$. If $\mathbf{l}$, $\mathbf{m}$ are (as we assume) unit vectors and θ the angle between them, then the *unit* vector in the direction of the common perpendicular is

$$\mathbf{d} \equiv \mathbf{l} \wedge \mathbf{m} \operatorname{cosec} \theta.$$

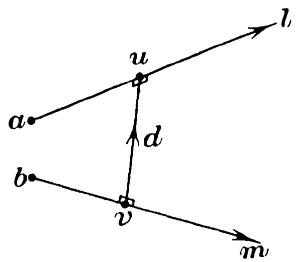

FIG. 35

The length of the common perpendicular, being the projection of the vector $\mathbf{a}-\mathbf{b}$, has magnitude d, where

$$d = (\mathbf{a}-\mathbf{b}).(\mathbf{l} \wedge \mathbf{m}) \operatorname{cosec} \theta$$

$$= (\mathbf{a}-\mathbf{b}, \mathbf{l}, \mathbf{m}) \operatorname{cosec} \theta.$$

COROLLARY. If the two lines are *coplanar*, then $d = 0$, so that (compare p. 76)

$$(\mathbf{a}-\mathbf{b}, \mathbf{l}, \mathbf{m}) = 0.$$

An adaptation of this Corollary leads to equations for the common perpendicular. Suppose that a typical point on this line is $\mathbf{x}$. Then the lines through $\mathbf{a}$ in the direction $\mathbf{l}$, and through $\mathbf{x}$ in the direction $\mathbf{d}$, are coplanar. Hence

$$(\mathbf{x}-\mathbf{a}, \mathbf{l}, \mathbf{d}) = 0,$$

so that
$$(\mathbf{x}-\mathbf{a}, \mathbf{l}, \mathbf{l} \wedge \mathbf{m}) = 0.$$

Thus *two equations for the line are* (p. 38)

$$(\mathbf{x}-\mathbf{a}, \mathbf{l}, \mathbf{l} \wedge \mathbf{m}) = 0,$$

$$(\mathbf{x}-\mathbf{b}, \mathbf{m}, \mathbf{l} \wedge \mathbf{m}) = 0.$$

MISCELLANEOUS EXAMPLES

1. Prove that $|\mathbf{u}|^2|\mathbf{v}|^2 = (\mathbf{u}.\mathbf{v})^2+|\mathbf{u}\wedge\mathbf{v}|^2$.

2. If $\mathbf{b}$ is not parallel to $\mathbf{c}$, show that any vector $\mathbf{a}$ can be expressed uniquely in the form

$$\mathbf{a} = l\mathbf{b}+m\mathbf{c}+n\mathbf{b}\wedge\mathbf{c},$$

where l, m, n are numbers, and determine l, m.

Show that $\mathbf{b}\wedge(\mathbf{b}\wedge\mathbf{c})$ is coplanar with $\mathbf{b}$ and $\mathbf{c}$, and show with the aid of a diagram that $\quad \mathbf{b}\wedge(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{b}.\mathbf{c})\mathbf{b}-(\mathbf{b}^2)\mathbf{c}$.

3. The set of vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ *reciprocal* to the set $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ is defined by

$$\mathbf{b}_1 = \frac{\mathbf{a}_2\wedge\mathbf{a}_3}{(\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3)}, \qquad \mathbf{b}_2 = \frac{\mathbf{a}_3\wedge\mathbf{a}_1}{(\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3)}, \qquad \mathbf{b}_3 = \frac{\mathbf{a}_1\wedge\mathbf{a}_2}{(\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3)}.$$

Show that $\quad (\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3) = (\mathbf{b}_1\,\mathbf{b}_2\,\mathbf{b}_3)$.

Axes parallel to $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are taken at a point O, and a plane cuts off intercepts proportional to $\mathbf{a}_1/h_1$, $\mathbf{a}_2/h_2$, $\mathbf{a}_3/h_3$ on the axes. Show that this plane is perpendicular to the direction

$$h_1\mathbf{b}_1+h_2\mathbf{b}_2+h_3\mathbf{b}_3.$$

4. The position vectors of two points A, B are $\mathbf{a}$, $\mathbf{b}$ respectively. Write down the position vector of the point C on AB such that

$$AC/CB = \lambda/1.$$

Four points A_1, B_1, A_2, B_2 are given in space; C_1, C_2 are points on A_1B_1, A_2B_2 respectively such that

$$\frac{A_1C_1}{C_1B_1} = \frac{A_2C_2}{C_2B_2} = \frac{\lambda}{1}.$$

Show that there exists a direction to which A_1A_2, B_1B_2, C_1C_2 are all orthogonal.

If A_3, B_3, C_3 are points on A_1A_2, B_1B_2, C_1C_2 respectively such that

$$\frac{A_1A_3}{A_3A_2} = \frac{B_1B_3}{B_3B_2} = \frac{C_1C_3}{C_3C_2} = \frac{\mu}{1},$$

show that A_3, B_3, C_3 are collinear.

5. Expand the expressions

$$(\mathbf{l}\wedge\mathbf{m}).(\mathbf{l}\wedge\mathbf{n}), \qquad (\mathbf{l}\wedge\mathbf{m})\wedge(\mathbf{l}\wedge\mathbf{n}).$$

Hence prove that the angle between two faces of a regular tetrahedron is $\cos^{-1}(\frac{1}{3})$.

6. Prove the formula

$$(\mathbf{u}\wedge\mathbf{v})\wedge\mathbf{w} = (\mathbf{u}.\mathbf{w})\mathbf{v}-(\mathbf{v}.\mathbf{w})\mathbf{u}.$$

Solve for $\mathbf{x}$ the vector equation

$$\mathbf{x}\wedge\mathbf{a} = \mathbf{b}-\mathbf{x}.$$

7. Given three non-coplanar vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$, show that there is a unique vector $\mathbf{l}'$ perpendicular to $\mathbf{m}$ and to $\mathbf{n}$ and such that $\mathbf{l}.\mathbf{l}' = 1$, and express it in terms of $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$.

Vectors $\mathbf{m}'$, $\mathbf{n}'$ are defined similarly. Show that any vector $\mathbf{a}$ can be expressed uniquely in the form

$$\mathbf{a} = a_1\mathbf{l}+a_2\mathbf{m}+a_3\mathbf{n},$$

and determine the coefficients a_1, a_2, a_3 in terms of $\mathbf{a}$, $\mathbf{l}'$, $\mathbf{m}'$, $\mathbf{n}'$. Show also that $\mathbf{a}$ can be expressed uniquely in the form

$$\mathbf{a} = a_1'\mathbf{l}'+a_2'\mathbf{m}'+a_3'\mathbf{n}',$$

and determine a_1', a_2', a_3' in terms of $\mathbf{a}$, $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$. Hence show that, for any vectors $\mathbf{a}$, $\mathbf{b}$,

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (\mathbf{a}.\mathbf{l}')(\mathbf{b}.\mathbf{l})+(\mathbf{a}.\mathbf{m}')(\mathbf{b}.\mathbf{m})+(\mathbf{c}.\mathbf{n}')(\mathbf{c}.\mathbf{n}) \\ &= (\mathbf{a}.\mathbf{l})(\mathbf{b}.\mathbf{l}')+(\mathbf{a}.\mathbf{m})(\mathbf{b}.\mathbf{m}')+(\mathbf{c}.\mathbf{n})(\mathbf{c}.\mathbf{n}').\end{aligned}$$

8. Four points A, B, C, D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. Prove that the vector $\tfrac{1}{2}\{(\mathbf{b}\wedge\mathbf{c})+(\mathbf{c}\wedge\mathbf{a})+(\mathbf{a}\wedge\mathbf{b})\}$

has magnitude equal to the area of the triangle ABC and direction perpendicular to its plane.

Prove also that the volume of the tetrahedron $ABCD$ is (numerically)

$$\tfrac{1}{6}\{(\mathbf{abc})-(\mathbf{bcd})-(\mathbf{cad})-(\mathbf{abd})\}.$$

9. If $\mathbf{a}$, $\mathbf{b}$ are given vectors, $\mathbf{b}$ being perpendicular to $\mathbf{a}$, and k is a given scalar, show that the solution of the equations

$$\mathbf{a}.\mathbf{x} = k, \qquad \mathbf{a}\wedge\mathbf{x} = \mathbf{b}$$

for an unknown vector $\mathbf{x}$ is unique, and find it.

Examine whether the solution is valid if $\mathbf{b}$ is not perpendicular to $\mathbf{a}$.

10. Solve the following equation for λ, μ, ν in terms of the three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$:

$$\lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{c} = \mathbf{d}.$$

11. P is the foot of the perpendicular from a point B, with position vector $\mathbf{b}$, to the line $\mathbf{r} = \mathbf{a}+\lambda\mathbf{t}$. Show that the equation of the line BP is

$$\mathbf{r} = \mathbf{b}+\mu\mathbf{t}\wedge\{(\mathbf{a}-\mathbf{b})\wedge\mathbf{t}\},$$

and find the position vector of P.

12. The position vectors of three points A, B, C are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Find the condition for the point D with position vector $\mathbf{d} = \lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{c}$ to lie in the plane ABC.

If AB, CD meet in E, and AC, BD in F, find the position vectors of E, F. Show that the point of intersection of AD, EF has position vector $(\lambda\mathbf{a}+\mathbf{d})/(1+\lambda)$.

13. Points P, Q have position vectors $\mathbf{p}$, $\mathbf{q}$ respectively, and a plane Ω passes through the origin with normal along the vector $\mathbf{n}$. Find the position vectors $\mathbf{u}$, $\mathbf{v}$ of the feet of the perpendiculars from P, Q to Ω. Express the angle subtended at the origin by the feet in terms of $\mathbf{p}$, $\mathbf{q}$, $\mathbf{n}$.

Apply your formulae to the case

$$\mathbf{p} = (1,2,-2), \quad \mathbf{q} = (5,2,2), \quad \mathbf{n} = (2,1,-1),$$

and show that the angle is $\cos^{-1}(-\tfrac{4}{7})$.

14. Show that the most general solution of the equation

$$\mathbf{x} \wedge \mathbf{a} = \mathbf{b}$$

is

$$\mathbf{x} = t\mathbf{a} + (\mathbf{a} \wedge \mathbf{b})/a^2,$$

where t is an arbitrary scalar.

15. Find the condition for the line whose vector equation is

$$\mathbf{r} = \mathbf{a} + t\mathbf{m}$$

to be parallel to the plane $\mathbf{r}.\mathbf{n} = p,$

where $\mathbf{r} \equiv (x, y, z)$, $\mathbf{a}$ is a constant vector, $\mathbf{m}$, $\mathbf{n}$ are unit vectors, and t, p are scalar constants.

Show that the condition is satisfied by the line

$$\frac{x+1}{3} = \frac{y-2}{6} = \frac{z-3}{4}$$

and the plane $2x+3y-6z+7 = 0,$

and find the distance of the line from the plane.

16. The unit vectors $\mathbf{a}$, $\mathbf{b}$ are perpendicular and the unit vector $\mathbf{c}$ is inclined at an angle θ to both $\mathbf{a}$ and $\mathbf{b}$. Show that

$$\mathbf{c} = \alpha(\mathbf{a}+\mathbf{b}) + \beta\mathbf{a} \wedge \mathbf{b},$$

where $\alpha = \cos\theta$, $\beta^2 = -\cos 2\theta$.

The vector $\mathbf{x}$ satisfies the equation

$$\mathbf{x} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge \mathbf{x}.$$

Show that $2\mathbf{x} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge (\mathbf{a} \wedge \mathbf{b}) + (\mathbf{c}.\mathbf{x})\mathbf{c}$

and hence obtain the solution

$$\mathbf{x} = \tfrac{1}{2}\alpha(1-\beta^{-1})\mathbf{a} - \tfrac{1}{2}\alpha(1+\beta^{-1})\mathbf{b} + (1-\alpha^2)\beta^{-1}\mathbf{c}.$$

17. Vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, of magnitudes a, b, c, are such that $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and to $\mathbf{c}$. Show that the equation of the plane through the three points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is

$$\left\{\frac{\mathbf{a}}{a} + \frac{(c^2 - \mathbf{b}.\mathbf{c})\mathbf{b} + (b^2 - \mathbf{b}.\mathbf{c})\mathbf{c}}{b^2c^2 - (\mathbf{b}.\mathbf{c})^2}\right\}.\mathbf{r} = 1.$$

Hence, or otherwise, find the equation of the plane through the points $(1, 1, 1)$, $(-1, 2, -1)$, $(-1, -1, 2)$.

18. Prove that $(\mathbf{b} \wedge \mathbf{c}, \mathbf{c} \wedge \mathbf{a}, \mathbf{a} \wedge \mathbf{b}) = (\mathbf{abc})^2.$

19. Lines l, m, n are drawn in the faces DBC, DCA, DAB of a tetrahedron $ABCD$ so that l is perpendicular to DA, m is perpendicular to DB and n is perpendicular to DC. Prove that l, m, n are coplanar.

20. Prove that the six planes, each through one edge of a tetrahedron and bisecting the opposite edge, are concurrent.

21. Points A, B, C have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and O is the origin. Prove that the volume of the tetrahedron $OABC$ is

$$\tfrac{1}{6}abc \sqrt{\left\{\begin{vmatrix} 1 & \cos AOB & \cos AOC \\ \cos BOA & 1 & \cos BOC \\ \cos COA & \cos COB & 1 \end{vmatrix}\right\}}.$$

IV

TENSORS; δ_{ij} AND ϵ_{ijk}

1. Introduction

THE idea of a vector was explained in Chapter III. The basic requirement is conformity to the rule of transformation

$$u'_i = l_{\lambda i} u_\lambda, \qquad u_i = l_{i\lambda} u'_\lambda$$

(either of which implies the other); the right-hand sides are summed over the values 1, 2, 3 of λ, and l_{ij} denotes the cosine of the angle between the axis Oi of the first coordinate system and the axis Oj' of the second.

The aim of this chapter is to enlarge the concept. But first we develop a calculus somewhat wider than is really necessary, limiting it later by an extension of the rule of transformation. The *arrays* of which we speak are essentially matrices, but the word *matrix* itself is not used, so that implications may be avoided. In particular, we do not need a formula for the multiplication of two arrays; the place occupied by products in matrix theory is, to a large extent, taken over by the summation convention.

2. Rectangular arrays

By a RECTANGULAR ARRAY we mean a set of mn numbers arranged as a rectangle in the form

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{matrix}$$

A typical ELEMENT a_{ij} appears in the ith row and the jth column, and the array itself consists of m rows and n columns. For brevity, the array is denoted by the notation

$$(a_{ij}).$$

A LINEAR COMBINATION of the arrays (a_{ij}), (b_{ij}), (c_{ij}),... is an array (h_{ij}) whose typical element h_{ij} is connected with the corresponding elements a_{ij}, b_{ij}, c_{ij},... by a relation of the form

$$h_{ij} = \alpha a_{ij}+\beta b_{ij}+\gamma c_{ij}+...,$$

where α, β, γ,... are constants independent of i, j. In particular, the SUM $(a_{ij}+b_{ij})$ of the two arrays (a_{ij}), (b_{ij}) has typical element $a_{ij}+b_{ij}$.

The existence of a linear combination implies that the arrays all have the same number of rows and the same number of columns.

ILLUSTRATION. If

$$(a_{ij}) \equiv \begin{pmatrix} x & y & z \\ a & b & c \end{pmatrix}, \qquad (b_{ij}) \equiv \begin{pmatrix} p & q & r \\ u & v & w \end{pmatrix},$$

then
$$2(a_{ij})-(b_{ij}) \equiv \begin{pmatrix} 2x-p & 2y-q & 2z-r \\ 2a-u & 2b-v & 2c-w \end{pmatrix}.$$

3. Some definitions

When $m = n$, the array (a_{ij}), of m rows and columns, is said to be SQUARE.

We now assume, for the rest of this work, that

$$m = n = 3,$$

so that a typical array has three rows and three columns. The only exceptions will be connected with 'vectors' (where one of m, n is 1 and the other is 3) and 'scalars' (where $m = n = 1$), and the context will make clear what is happening.

An array (a_{ij}) is said to be SYMMETRIC if

$$a_{ij} = a_{ji}$$

for all pairs of values of i, j; and SKEW-SYMMETRIC, or ANTI-SYMMETRIC, if

$$a_{ij} = -a_{ji}$$

for all pairs of values of i, j. In the latter case, $a_{ii} = 0$ for all values of i.

An arbitrary array can always be expressed as the sum of a symmetric part and an antisymmetric part. For two numbers p, q can always be written

$$p = \tfrac{1}{2}(u+v), \qquad q = \tfrac{1}{2}(u-v)$$

by taking $u = p+q, \qquad v = p-q.$

Thus

$$a_{ij} = \tfrac{1}{2}(a_{ij}+a_{ji})+\tfrac{1}{2}(a_{ij}-a_{ji})$$
$$= c_{ij}+d_{ij},$$

say, where

$$c_{ij} = \tfrac{1}{2}(a_{ij}+a_{ji}), \qquad d_{ij} = \tfrac{1}{2}(a_{ij}-a_{ji}).$$

Hence, by definition of addition,

$$(a_{ij}) = (c_{ij})+(d_{ij}).$$

But $$c_{ji} = \tfrac{1}{2}(a_{ji}+a_{ij}) = c_{ij},$$

and $$d_{ji} = \tfrac{1}{2}(a_{ji}-a_{ij}) = -d_{ij},$$

so that (c_{ij}) is *symmetric* and (d_{ij}) is *antisymmetric*.

4. The symbol δ_{ij}

The symbol δ_{ij} is defined by the rule:

$$\delta_{ij} = 1 \quad (i = j)$$
$$= 0 \quad (i \neq j).$$

Many formulae may be written concisely with its help.

Suppose, for example, that

$$(l_{11}, l_{21}, l_{31}), \quad (l_{12}, l_{22}, l_{32}), \quad (l_{13}, l_{23}, l_{33})$$

are the direction cosines of three mutually perpendicular lines OU, OV, OW; the first suffix names the *coordinate* and the second the *line*. Then, for each line,

$$l_{1i}^2+l_{2i}^2+l_{3i}^2 = 1 \quad (i = 1,2,3),$$

or $$l_{\lambda i}l_{\lambda i} = 1.$$

Also for each pair of lines, by orthogonality,

$$l_{1i}l_{1j}+l_{2i}l_{2j}+l_{3i}l_{3j} = 0 \quad (i \neq j),$$

or $$l_{\lambda i}l_{\lambda j} = 0.$$

These two results are comprised in the single formula

$$l_{\lambda i}l_{\lambda j} = \delta_{ij}.$$

The alternative equations (p. 43) reversing the roles of OU, OV, OW and the given axes are

$$l_{i\lambda}l_{j\lambda} = \delta_{ij}.$$

IMPORTANT. *The value of* $\delta_{\lambda\lambda}$ *is* **3**. For

$$\delta_{\lambda\lambda} = \delta_{11}+\delta_{22}+\delta_{33}$$
$$= 3.$$

Note the RULE OF SUBSTITUTION:

If (p_{ij}) *is an array of three rows and columns, then*

$$p_{i\lambda}\delta_{j\lambda} = p_{ij},$$

for the left-hand side consists of three terms, of which two are zero ($\lambda \neq j$) and one is $p_{ij}\delta_{jj}$ (not summed), or p_{ij}.

In particular $$\delta_{i\lambda}\delta_{j\lambda} = \delta_{ij}.$$

[The effect of multiplication by $\delta_{j\lambda}$ is to substitute the suffix j.]

5. The symbol ϵ_{ijk}

The symbol ϵ_{ijk} is defined by the rule:

$\epsilon_{ijk} = 0$ if any two of i, j, k are equal;

$\epsilon_{ijk} = +1$ if i, j, k are 1, 2, 3, or 2, 3, 1, or 3, 1, 2, so that the numbers i, j, k occur in the CYCLIC ORDER 1, 2, 3, 1,...;

$\epsilon_{ijk} = -1$ if i, j, k are 3, 2, 1, or 2, 1, 3, or 1, 3, 2, so that the numbers i, j, k occur in the ANTICYCLIC order 3, 2, 1, 3, 2,... .

For example,

$$\epsilon_{121} = 0, \quad \epsilon_{321} = -1, \quad \epsilon_{231} = +1.$$

Thus $$\epsilon_{ijk}\delta_{ij} = 0$$

since the first factor is zero if i, j are equal and the second factor is zero if i, j are unequal. Hence, also,

$$\epsilon_{\lambda\mu k}\delta_{\lambda\mu} = 0$$

when summed over λ and μ.

6. Determinants

REMARK. The elementary properties of determinants are assumed known, and have been used freely in earlier chapters. The purpose of this brief introduction is to show how the symbol ϵ_{ijk} can be incorporated into the general theory.

Consider the expression

$$\epsilon_{\lambda\mu\nu}a_{p\lambda}a_{q\mu}a_{r\nu},$$

where (a_{ij}) is a given array, and where p, q, r are given numbers whose values are 1, 2, or 3 independently.

When p, q, r have the particular values 1, 2, 3 respectively, the expression is

$$\epsilon_{\lambda\mu\nu}\, a_{1\lambda}\, a_{2\mu}\, a_{3\nu},$$

or, writing non-zero terms in full,

$$a_{11}\,a_{22}\,a_{33}+a_{12}\,a_{23}\,a_{31}+a_{13}\,a_{21}\,a_{22}-$$
$$-a_{13}\,a_{22}\,a_{31}-a_{12}\,a_{21}\,a_{33}-a_{11}\,a_{23}\,a_{32}.$$

This is called the DETERMINANT OF THE ARRAY (a_{ij}), and is written

$$|a_{ij}|,$$

or, in full,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Suppose, more generally, that p, q, r are *distinct*, and occur in the *cyclic* order 1, 2, 3, 1,.... Then

$$\epsilon_{\lambda\mu\nu}\, a_{p\lambda}\, a_{q\mu}\, a_{r\nu} \equiv \epsilon_{\lambda\mu\nu}\, a_{1\lambda}\, a_{2\mu}\, a_{3\nu},$$

the names of the *dummy* suffixes λ, μ, ν on the left being moved in cyclic order so that λ comes with p, μ comes with q, and ν comes with r.

If, however, p, q, r are *distinct* and occur in *anticyclic* order 3, 2, 1, 3,..., then (since $\epsilon_{\lambda\mu\nu} = -\epsilon_{\lambda\nu\mu}$)

$$\epsilon_{\lambda\mu\nu}\, a_{p\lambda}\, a_{q\mu}\, a_{r\nu} = -\epsilon_{\lambda\nu\mu}\, a_{p\lambda}\, a_{r\nu}\, a_{q\mu}$$
$$= -\epsilon_{\lambda\mu\nu}\, a_{p\lambda}\, a_{r\mu}\, a_{q\nu}$$

on interchanging the names λ, μ. But the numbers p, r, q are in *cyclic* order, so that, applying the preceding paragraph to the right-hand side,

$$\epsilon_{\lambda\mu\nu}\, a_{p\lambda}\, a_{q\mu}\, a_{r\nu} = -\epsilon_{\lambda\mu\nu}\, a_{1\lambda}\, a_{2\mu}\, a_{3\nu}.$$

Finally, if p, q, r are *not distinct*, then

$$\epsilon_{\lambda\mu\nu}\, a_{p\lambda}\, a_{q\mu}\, a_{r\nu} = 0,$$

since interchange of the equal numbers (say q and r) reverses the sign of the expression by the preceding paragraph, while also leaving it unaffected.

To summarize,

$$\begin{aligned}\epsilon_{\lambda\mu\nu}\, a_{p\lambda}\, a_{q\mu}\, a_{r\nu} &= +|a_{ij}| \quad (p,\ q,\ r \text{ cyclic order})\\ &= -|a_{ij}| \quad (p,\ q,\ r \text{ anticyclic order})\\ &= 0 \qquad\quad (p,\ q,\ r \text{ not all different}).\end{aligned}$$

Hence
$$\epsilon_{\lambda\mu\nu}\, a_{p\lambda}\, a_{q\mu}\, a_{r\nu} = \epsilon_{pqr}|a_{ij}|.$$

NOTES. (i) If we write, as a temporary notation,

$$a_{p\lambda} \equiv b_{1\lambda}, \quad a_{q\mu} \equiv b_{2\mu}, \quad a_{r\nu} \equiv b_{3\nu},$$

then

$$\begin{aligned}\epsilon_{\lambda\mu\nu}\, a_{p\lambda}\, a_{q\mu}\, a_{r\nu} &\equiv \epsilon_{\lambda\mu\nu}\, b_{1\lambda}\, b_{2\mu}\, b_{3\nu}\\ &\equiv \begin{vmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{vmatrix}\\ &\equiv \begin{vmatrix} a_{p1} & a_{p2} & a_{p3}\\ a_{q1} & a_{q2} & a_{q3}\\ a_{r1} & a_{r2} & a_{r3}\end{vmatrix}.\end{aligned}$$

(ii) If (c_{ij}) is the array defined by the relation

$$c_{ij} = a_{ji},$$

then

$$\begin{aligned}|c_{ij}| &= \epsilon_{\lambda\mu\nu}\, c_{1\lambda}\, c_{2\mu}\, c_{3\nu}\\ &= \epsilon_{\lambda\mu\nu}\, a_{\lambda 1}\, a_{\mu 2}\, a_{\nu 3}\\ &= a_{11}\, a_{22}\, a_{33} + a_{21}\, a_{32}\, a_{13} + a_{31}\, a_{12}\, a_{23} -\\ &\qquad - a_{31}\, a_{22}\, a_{13} - a_{21}\, a_{12}\, a_{33} - a_{11}\, a_{32}\, a_{23}\\ &= |a_{ij}|.\end{aligned}$$

Thus *the value of a determinant is unaltered if its rows and columns are interchanged.*

ILLUSTRATION. *To prove that*

$$\begin{vmatrix} \delta_{p1} & \delta_{p2} & \delta_{p3}\\ \delta_{q1} & \delta_{q2} & \delta_{q3}\\ \delta_{r1} & \delta_{r2} & \delta_{r3}\end{vmatrix} = \epsilon_{pqr}.$$

The left-hand side is

$$\begin{aligned}&\epsilon_{\lambda\mu\nu}\, \delta_{p\lambda}\, \delta_{q\mu}\, \delta_{r\nu}\\ &= \epsilon_{pqr}.\end{aligned}$$

7. Expansion by cofactors

The expression

$$|a_{ij}| \equiv \epsilon_{\lambda\mu\nu} a_{1\lambda} a_{2\mu} a_{3\nu}$$

may be evaluated by stages, in the form

$$\epsilon_{1\mu\nu} a_{11} a_{2\mu} a_{3\nu} + \epsilon_{2\mu\nu} a_{12} a_{2\mu} a_{3\nu} + \epsilon_{3\mu\nu} a_{13} a_{2\mu} a_{3\nu},$$

or $$a_{11}(\epsilon_{1\mu\nu} a_{2\mu} a_{3\nu}) + a_{12}(\epsilon_{2\mu\nu} a_{2\mu} a_{3\nu}) + a_{13}(\epsilon_{3\mu\nu} a_{2\mu} a_{3\nu}).$$

The coefficient of a_{1i}, namely

$$\epsilon_{i\mu\nu} a_{2\mu} a_{3\nu},$$

is called the COFACTOR of a_{1i} in $|a_{ij}|$, and the above expression for the determinant is called its EXPANSION IN TERMS OF THE FIRST ROW. Similar expansions are used for the other rows, or for columns.

For example, $\epsilon_{1\mu\nu} a_{2\mu} a_{3\nu} = a_{22} a_{33} - a_{23} a_{32}$.

8. The product of two determinants

There are several alternative forms for the product of two given determinants. A convenient one is given by the following theorem:

If

$$a \equiv \epsilon_{\lambda\mu\nu} a_{1\lambda} a_{2\mu} a_{3\nu},$$
$$b \equiv \epsilon_{\lambda\mu\nu} b_{1\lambda} b_{2\mu} b_{3\nu}$$

are two given determinants, then their product may be written in the form

$$c \equiv \epsilon_{\lambda\mu\nu} c_{1\lambda} c_{2\mu} c_{3\nu},$$

where the array (c_{ij}) *is defined by the relation*

$$c_{ij} = a_{i\lambda} b_{\lambda j}.$$

Consider the expression

$$\begin{aligned} \epsilon_{\lambda\mu\nu} c_{1\lambda} c_{2\mu} c_{3\nu} &\equiv \epsilon_{\lambda\mu\nu} a_{1\alpha} b_{\alpha\lambda} a_{2\beta} b_{\beta\mu} a_{3\gamma} b_{\gamma\nu} \\ &= (\epsilon_{\lambda\mu\nu} b_{\alpha\lambda} b_{\beta\mu} b_{\gamma\nu}) a_{1\alpha} a_{2\beta} a_{3\gamma} \\ &= (\epsilon_{\alpha\beta\gamma} b) a_{1\alpha} a_{2\beta} a_{3\gamma} \end{aligned}$$

by § 6. Hence

$$\begin{aligned} c &= b\epsilon_{\alpha\beta\gamma} a_{1\alpha} a_{2\beta} a_{3\gamma} \\ &= ba. \end{aligned}$$

This method of multiplying is called MULTIPLICATION ACCORDING TO THE MATRIX RULE.

Two alternative formulae may be derived quickly from the fact (p. 86) that the value of a determinant is unaltered when rows and columns are interchanged:

(a) $$ab = c,$$

where $$c = \epsilon_{\lambda\mu\nu} c_{1\lambda} c_{2\mu} c_{3\nu}$$

and $$c_{ij} = a_{i\lambda} b_{j\lambda};$$

(b) $$ab = c,$$

where $$c = \epsilon_{\lambda\mu\nu} c_{1\lambda} c_{2\mu} c_{3\nu}$$

and $$c_{ij} = a_{\lambda i} b_{\lambda j}.$$

ILLUSTRATION. *To establish the formula*

$$\epsilon_{ij\lambda} \epsilon_{pq\lambda} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}.$$

By the Illustration in § 6 (p. 86),

$$\epsilon_{ijk} \epsilon_{pqr} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \begin{vmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{vmatrix},$$

on interchanging rows and columns in the second determinant. Now multiplication on the right gives (p. 87)

$$\epsilon_{ijk} \epsilon_{pqr} = \begin{vmatrix} \delta_{i\lambda}\delta_{\lambda p} & \delta_{i\lambda}\delta_{\lambda q} & \delta_{i\lambda}\delta_{\lambda r} \\ \delta_{j\lambda}\delta_{\lambda p} & \delta_{j\lambda}\delta_{\lambda q} & \delta_{j\lambda}\delta_{\lambda r} \\ \delta_{k\lambda}\delta_{\lambda p} & \delta_{k\lambda}\delta_{\lambda q} & \delta_{k\lambda}\delta_{\lambda r} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}.$$

Write $$\Delta_{qr} \equiv \begin{vmatrix} \delta_{iq} & \delta_{ir} \\ \delta_{jq} & \delta_{jr} \end{vmatrix} \equiv -\Delta_{rq},$$

with similar notation for $\Delta_{rp} \equiv -\Delta_{pr}$, $\Delta_{pq} \equiv -\Delta_{qp}$. Then, expanding the determinant by the bottom row (p. 87),

$$\epsilon_{ijk} \epsilon_{pqr} = \delta_{kp} \Delta_{qr} + \delta_{kq} \Delta_{rp} + \delta_{kr} \Delta_{pq}.$$

Hence

$$\begin{aligned} \epsilon_{ij\lambda} \epsilon_{pq\lambda} &= \delta_{\lambda p} \Delta_{q\lambda} + \delta_{\lambda q} \Delta_{\lambda p} + \delta_{\lambda\lambda} \Delta_{pq} \\ &= \Delta_{qp} + \Delta_{qp} + 3\Delta_{pq} \\ &= \Delta_{pq} \\ &= \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}. \end{aligned}$$

ILLUSTRATION. *Given a vector* $\mathbf{u}$, *to associate with it a skew-symmetric array* (u_{ij}).

Consider the expression

$$u_{ij} \equiv \epsilon_{ij\lambda} u_\lambda.$$

When $i = j$, its value is zero. Otherwise, the only non-zero term is $\epsilon_{ijk} u_k$, where k is different from both i and j.

If j follows i in the cyclic order 1, 2, 3 then $\epsilon_{ijk} = +1$; if j comes before i, then $\epsilon_{ijk} = -1$. Hence (u_{ij}) is the skew-symmetric array

$$\begin{pmatrix} 0 & +u_3 & -u_2 \\ -u_3 & 0 & +u_1 \\ +u_2 & -u_1 & 0 \end{pmatrix}.$$

ILLUSTRATION. *To express the orthogonality relation* (p. 59)

$$l_{ip} l_{jq} - l_{iq} l_{jp} = l_{kr}$$

(*where* i, j, k *and* p, q, r *both occur in cyclic order* 1, 2, 3, 1,...) *in the form*

$$\epsilon_{\lambda\mu n} l_{i\lambda} l_{j\mu} = \epsilon_{ij\lambda} l_{\lambda n},$$

where i, j, n *take independently any of the values* 1, 2, 3.

Let l, m be the two numbers different from n, so named (as we may) that l, m, n is the cyclic order. The non-zero terms in $\epsilon_{\lambda\mu n} l_{i\lambda} l_{j\mu}$ are

$$\epsilon_{lmn} l_{il} l_{jm} + \epsilon_{mln} l_{im} l_{jl} \equiv l_{il} l_{jm} - l_{im} l_{jl}.$$

The non-zero term in $\epsilon_{ij\lambda} l_{\lambda n}$ is

$$\epsilon_{ijk} l_{kn}.$$

If $i = j$, each side of the proposed relation is zero.

If $i \neq j$, let k be the number different from each. When i, j, k is the cyclic order, the orthogonality relation

$$l_{il} l_{jm} - l_{im} l_{jl} = l_{kn}$$

is obtained; when i, j, k are in anti-cyclic order, the orthogonality relation is again obtained, save that the sign of each side is changed.

The result therefore holds generally.

9. Scalar and vector products

The symbol ϵ_{ijk} may be used to give a concise expression for the vector product (p. 66) of two given vectors

$$\mathbf{u} \equiv (u_1, u_2, u_3), \qquad \mathbf{v} \equiv (v_1, v_2, v_3).$$

For comparison, note that the scalar product $\mathbf{u}.\mathbf{v}$ (p. 62) can be written

$$\mathbf{u}.\mathbf{v} \equiv \delta_{\lambda\mu} u_\lambda v_\mu.$$

The corresponding expression for the vector product $\mathbf{u} \wedge \mathbf{v}$ is

$$\mathbf{u} \wedge \mathbf{v})_i \equiv \epsilon_{i\lambda\mu} u_\lambda v_\mu,$$

where $\mathbf{u} \wedge \mathbf{v})_i$ denotes the ith component. If i, j, k are in cyclic order, then the ith component on the right is

$$\epsilon_{ijk} u_j v_k + \epsilon_{ikj} u_k v_j \equiv u_j v_k - u_k v_j,$$

agreeing with the definition on p. 67.

ILLUSTRATION. *To prove the formula for the scalar triple product* (p. 68)

$$\mathbf{u}.(\mathbf{v} \wedge \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

The left-hand side is

$$u_\lambda(\mathbf{v} \wedge \mathbf{w})_\lambda = u_\lambda \epsilon_{\lambda\mu\nu} v_\mu w_\nu$$
$$= \epsilon_{\lambda\mu\nu} u_\lambda v_\mu w_\nu,$$

which (p. 85) is equal to the determinant on the right.

ILLUSTRATION. *To prove the formula for the vector triple product* (p. 74)

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}.$$

The ith component of the left-hand side is

$$\epsilon_{i\lambda\rho} a_\lambda (\mathbf{b} \wedge \mathbf{c})_\rho = \epsilon_{i\lambda\rho} a_\lambda \epsilon_{\mu\nu\rho} b_\mu c_\nu$$
$$= \epsilon_{i\lambda\rho} \epsilon_{\mu\nu\rho} a_\lambda b_\mu c_\nu.$$

For the summation with respect to ρ, we use the formula (p. 88)

$$\epsilon_{i\lambda\rho} \epsilon_{\mu\nu\rho} = \delta_{i\mu} \delta_{\lambda\nu} - \delta_{i\nu} \delta_{\lambda\mu},$$

so that

$$\epsilon_{i\lambda\rho}\epsilon_{\mu\nu\rho}a_\lambda b_\mu c_\nu = \delta_{i\mu}\delta_{\lambda\nu}a_\lambda b_\mu c_\nu - \delta_{i\nu}\delta_{\lambda\mu}a_\lambda b_\mu c_\nu$$
$$= (\delta_{i\mu}b_\mu)(\delta_{\lambda\nu}a_\lambda c_\nu) - (\delta_{i\nu}c_\nu)(\delta_{\lambda\mu}a_\lambda b_\mu)$$
$$= b_i(a_\lambda c_\lambda) - c_i(a_\lambda b_\lambda)$$
$$= (\mathbf{a}.\mathbf{c})b_i - (\mathbf{a}.\mathbf{b})c_i,$$

which is the ith component of the right-hand side.

10. Tensors

An array (t_{ij}) is called a TENSOR when it has a certain law of transformation, analogous to that given earlier (p. 58) for vectors. Let $O(1, 2, 3)$ be a given set of right-handed orthogonal axes, and $O(1', 2', 3')$ a similar set referred to the same origin O, and let l_{ij} be the cosine of the angle between the axis Oi of the first set and the axis Oj' of the second.

The law of transformation for vectors is

$$u_i = l_{i\lambda}u'_\lambda, \qquad u'_i = l_{\lambda i}u_\lambda.$$

The analogous LAW OF TENSOR TRANSFORMATION is defined to be

$$t_{ij} = l_{i\lambda}l_{j\mu}t'_{\lambda\mu}, \qquad t'_{ij} = l_{\lambda i}l_{\mu j}t_{\lambda\mu},$$

the repeated suffixes λ, μ being summed independently over the values 1, 2, 3.

The direction cosines are subject to the orthogonality condition (p. 89)

$$\epsilon_{\lambda\mu n}l_{i\lambda}l_{j\mu} = \epsilon_{ij\lambda}l_{\lambda n}.$$

ILLUSTRATION. Before examining the significance of the restriction imposed by the law of tensor transformation, consider as a particular example the angular momentum of a particle of unit mass at the point (x_1, x_2, x_3), moving with velocity (v_1, v_2, v_3). By definition, the angular momentum is (h_1, h_2, h_3), where

$$h_1 = x_2 v_3 - x_3 v_2,$$
$$h_2 = x_3 v_1 - x_1 v_3,$$
$$h_3 = x_1 v_2 - x_2 v_1.$$

Let an array (t_{ij}) be defined by the formula (compare p. 89)

$$(t_{ij}) \equiv \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$

then

$$h_i = t_{i\lambda} v_\lambda.$$

[For example,

$$h_1 = t_{11} v_1 + t_{12} v_2 + t_{13} v_3$$
$$= -x_3 v_2 + x_2 v_3.]$$

The importance of (t_{ij}) lies in the fact that its elements are *functions of position* for the particle. They are *independent* of the velocity.

Consider next an alternative set of coordinate axes, also right-handed and orthogonal. Referred to them, denote the position of the particle by (x'_i) and its velocity by (v'_i). Using *precisely the same definitions as before*, the angular momentum is (h'_i), where

$$h'_i = t'_{i\lambda} v'_\lambda,$$

and where

$$(t'_{ij}) \equiv \begin{pmatrix} 0 & -x'_3 & x'_2 \\ x'_3 & 0 & -x'_1 \\ -x'_2 & x'_1 & 0 \end{pmatrix}.$$

The quantity (x_i) is a vector, and so, by the law of vector transformation,

$$x_i = l_{i\lambda} x'_\lambda.$$

To prove that (t_{ij}) is a tensor, we must verify that it obeys the law of tensor transformation

$$t_{ij} = l_{i\lambda} l_{j\mu} t'_{\lambda\mu}.$$

Since (p. 89)

$$t_{ij} = -\epsilon_{ij\lambda} x_\lambda,$$

it follows that, in terms of (x'_i),

$$t_{ij} = -\epsilon_{ij\lambda} l_{\lambda\mu} x'_\mu,$$

so that, by the orthogonality relation in the form (p. 89)

$$\epsilon_{ij\lambda} l_{\lambda\mu} = \epsilon_{\alpha\beta\mu} l_{i\alpha} l_{j\beta},$$

the value of t_{ij} is given by

$$t_{ij} = -\epsilon_{\alpha\beta\mu} l_{i\alpha} l_{j\beta} x'_\mu$$
$$= l_{i\alpha} l_{j\beta}(-\epsilon_{\alpha\beta\mu} x'_\mu)$$
$$= l_{i\alpha} l_{j\beta} t'_{\alpha\beta}.$$

Hence (t_{ij}) transforms according to the tensor law.

What has happened should be clearly understood. The array

$$(t_{ij}) \equiv \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

can be evaluated in any set of rectangular cartesian coordinates, but it does not automatically follow that the two sets of numbers

$$\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -x'_3 & x'_2 \\ x'_3 & 0 & -x'_1 \\ -x'_2 & x'_1 & 0 \end{pmatrix}$$

will transform according to the rule

$$t_{ij} = l_{i\lambda} l_{j\mu} t'_{\lambda\mu}$$

even when (x_i), (x'_i) satisfy the relation

$$x_i = l_{i\lambda} x'_\lambda.$$

That is, transformation of the components of an array u_{ij} (assumed functions of x_i) into corresponding components of an array u^*_{ij} according to the vector rule $x_i = l_{i\lambda} x'_\lambda$ does *not* ensure that the arrays necessarily satisfy the relation

$$u_{ij} = l_{i\lambda} l_{j\mu} u^*_{\lambda\mu}.$$

Only for a TENSOR is this true, and the law of transformation must be established for any array before it can be treated as a tensor.

ILLUSTRATION. *The array (l_{ij}) is* **not** *a tensor.*

The idea of a tensor may be emphasized by citing an array which is *not*. Let three mutually perpendicular lines OU, OV, OW have direction cosines (referred to axes $O1$, $O2$, $O3$)

$$(b_{11}, b_{21}, b_{31}), \quad (b_{12}, b_{22}, b_{32}), \quad (b_{13}, b_{23}, b_{33}),$$

formed into an array (b_{ij}). *This array is not a tensor*; that is to say, when the direction cosines are calculated for another set of coordinate axes and grouped similarly into an array (b'_{ij}), the relation

$$b_{ij} = l_{i\lambda} l_{j\mu} b'_{\lambda\mu}$$

does not necessarily hold. To prove this, it is sufficient to consider the case when OU, OV, OW are taken as the new axes of coordinates, so that

$$l_{ij} \equiv b_{ij}.$$

The right-hand side of the proposed relation is then

$$b_{i\lambda} b_{j\mu} b'_{\lambda\mu}.$$

But, referred to the new axes, the direction cosines of OU, OV, OW are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, so that

$$b'_{ij} \equiv \delta_{ij}.$$

Hence

$$\begin{aligned} b_{i\lambda} b_{j\mu} b'_{\lambda\mu} &= b_{i\lambda} b_{j\lambda} \\ &= \delta_{ij} \\ &\neq b_{ij}. \end{aligned}$$

The law of tensor transformation therefore does not hold.

11. δ_{ij} as a tensor

To prove that δ_{ij} *is a tensor, transforming according to the law*

$$\delta_{ij} = l_{i\lambda} l_{j\mu} \delta'_{\lambda\mu}$$

(where $\delta'_{\lambda\mu} = 1$ when $\lambda = \mu$, and $\delta'_{\lambda\mu} = 0$ when $\lambda \neq \mu$).

We have remarked (p. 83) the relation

$$\delta_{ij} = l_{i\lambda} l_{j\lambda}$$

whose right-hand side is, by definition of $\delta'_{\lambda\mu}$, equal to

$$l_{i\lambda} l_{j\mu} \delta'_{\lambda\mu}.$$

Hence

$$\delta_{ij} = \delta'_{\lambda\mu} l_{i\lambda} l_{j\mu}.$$

A tensor whose respective components are the same for all sets of coordinate systems is said to be ISOTROPIC.

12. The inertia tensor

The genesis of a tensor is well illustrated by the inertia tensor, which we now derive.

Let a typical particle P of a rigid body have mass m and position (x_i). If the body rotates about the origin as a fixed point with angular velocity (ω_i), it is known that P has velocity (v_i), where

$$v_1 = \omega_2 x_3 - \omega_3 x_2, \quad v_2 = \omega_3 x_1 - \omega_1 x_3, \quad v_3 = \omega_1 x_2 - \omega_2 x_1.$$

Thus $$v_i = \epsilon_{i\lambda\mu}\,\omega_\lambda\,x_\mu.$$

The angular momentum about O for the system of particles is (h_i), where (p. 92)

$$h_i = \sum m\epsilon_{i\lambda\mu}\,x_\lambda\,v_\mu,$$

summed over all the particles of the body.

We seek to express (h_i) in terms of (ω_i) in the form

$$h_i = I_{i\lambda}\,\omega_\lambda,$$

where (I_{ij}) is a tensor whose components depend only on the characteristics of the body, and not on ω_i.

Eliminate v_i between the relations for h_i and v_i given above, renaming some of the dummy suffixes for convenience:

$$h_i = \sum m\epsilon_{i\lambda\rho}\,x_\lambda\,v_\rho,$$

where $$v_\rho = \epsilon_{\nu\mu\rho}\,x_\mu\,\omega_\nu,$$

so that $$\begin{aligned} h_i &= \sum m\epsilon_{i\lambda\rho}\,\epsilon_{\nu\mu\rho}\,x_\lambda\,x_\mu\,\omega_\nu \\ &= I_{i\nu}\,\omega_\nu, \end{aligned}$$

where the array (I_{ij}) is given by the formula

$$I_{ij} = \sum m\epsilon_{i\lambda\rho}\,\epsilon_{j\mu\rho}\,x_\lambda\,x_\mu.$$

Summing first with respect to ρ, this is (p. 88)

$$\begin{aligned} I_{ij} &= \sum m(\delta_{ij}\,\delta_{\lambda\mu} - \delta_{i\mu}\,\delta_{\lambda j})x_\lambda\,x_\mu \\ &= \sum m(\delta_{ij}\,x_\lambda\,x_\lambda - x_i\,x_j). \end{aligned}$$

The expression (I_{ij}) is called the INERTIA TENSOR WITH RESPECT TO THE GIVEN ORIGIN for the rigid body. To summarize,

$$\begin{aligned} I_{ij} &\equiv \sum m(\delta_{ij}\,x_\lambda\,x_\lambda - x_i\,x_j) \\ &\equiv \sum m\epsilon_{i\lambda\rho}\,\epsilon_{j\mu\rho}\,x_\lambda\,x_\mu \\ &\equiv \begin{pmatrix} \sum m(x_2^2+x_3^2) & -\sum mx_1\,x_2 & -\sum mx_1\,x_3 \\ -\sum mx_2\,x_1 & \sum m(x_3^2+x_1^2) & -\sum mx_2\,x_3 \\ -\sum mx_3\,x_1 & -\sum mx_3\,x_2 & \sum m(x_1^2+x_2^2) \end{pmatrix}. \end{aligned}$$

The expressions

$$A \equiv \sum m(x_2^2+x_3^2), \quad B \equiv \sum m(x_3^2+x_1^2), \quad C \equiv \sum m(x_1^2+x_2^2)$$

are called the MOMENTS OF INERTIA of the body about the axes $O1$, $O2$, $O3$, and the expressions

$$F \equiv \sum mx_2\,x_3, \quad G \equiv \sum mx_3\,x_1, \quad H \equiv \sum mx_1\,x_2$$

are called the PRODUCTS OF INERTIA of the body with respect to the axes $(O2, O3)$, $(O3, O1)$, $(O1, O2)$.

We have still to prove that (I_{ij}) is a tensor, transforming according to the rule

$$I_{ij} = l_{i\lambda} l_{j\mu} I'_{\lambda\mu}.$$

Consider first the term $\quad \delta_{ij} x_\lambda x_\lambda.$

We have proved (p. 94) that δ_{ij} is itself a tensor, so that

$$\delta_{ij} = l_{i\lambda} l_{j\mu} \delta'_{\lambda\mu}.$$

Also $x_\lambda x_\lambda$, being the square of the distance of P from O, is equal to $x'_\nu x'_\nu$. Hence

$$\delta_{ij} x_\lambda x_\lambda = l_{i\lambda} l_{j\mu} \delta'_{\lambda\mu} x'_\nu x'_\nu.$$

Further, by direct substitution,

$$\begin{aligned} x_i x_j &= l_{i\lambda} x'_\lambda l_{j\mu} x'_\mu \\ &= l_{i\lambda} l_{j\mu} x'_\lambda x'_\mu. \end{aligned}$$

Hence
$$\begin{aligned} I_{ij} &\equiv \sum m(\delta_{ij} x_\lambda x_\lambda - x_i x_j) \\ &= \sum m(l_{i\lambda} l_{j\mu} \delta'_{\lambda\mu} x'_\nu x'_\nu - l_{i\lambda} l_{j\mu} x'_\lambda x'_\mu) \\ &= l_{i\lambda} l_{j\mu} \sum m(\delta'_{\lambda\mu} x'_\nu x'_\nu - x'_\lambda x'_\mu) \\ &= l_{i\lambda} l_{j\mu} I'_{\lambda\mu}. \end{aligned}$$

The array (I_{ij}) is therefore a tensor.

ILLUSTRATION. To prove that *the kinetic energy of the rigid body is*

$$\tfrac{1}{2} I_{\lambda\mu} \omega_\lambda \omega_\mu.$$

The kinetic energy is

$$\tfrac{1}{2} \sum m v_\rho v_\rho$$

summed over all the particles of the body. Also (p. 95)

$$v_\rho = \epsilon_{\rho\lambda\alpha} \omega_\lambda x_\alpha = \epsilon_{\lambda\alpha\rho} \omega_\lambda x_\alpha,$$

so that the kinetic energy is

$$\tfrac{1}{2} \sum m\epsilon_{\lambda\alpha\rho} \omega_\lambda x_\alpha \epsilon_{\mu\beta\rho} \omega_\mu x_\beta = \tfrac{1}{2} \sum m\epsilon_{\lambda\alpha\rho} \epsilon_{\mu\beta\rho} x_\alpha x_\beta \omega_\lambda \omega_\mu.$$

But (p. 95) $\qquad I_{\lambda\mu} = \sum m\epsilon_{\lambda\alpha\rho} \epsilon_{\mu\beta\rho} x_\alpha x_\beta.$

Hence the kinetic energy is

$$\tfrac{1}{2} I_{\lambda\mu} \omega_\lambda \omega_\mu.$$

13. ϵ_{ijk} as a tensor

In this brief survey, we have confined attention to tensors (t_{ij}) with two suffixes i, j. Further developments are possible, but we restrict ourselves to proving, without further comment, the theorem:

The array ϵ_{ijk} satisfies the equation

$$\epsilon_{ijk} = l_{i\lambda}\, l_{j\mu}\, l_{k\nu}\, \epsilon_{\lambda\mu\nu}.$$

From the orthogonality relation (p. 89) in the form

$$l_{i\lambda}\, l_{j\mu}\, \epsilon_{\lambda\mu\nu} = \epsilon_{ij\rho}\, l_{\rho\nu},$$

the right-hand side is $\quad \epsilon_{ij\rho}\, l_{\rho\nu}\, l_{k\nu},$

or (p. 83) $\quad \epsilon_{ij\rho}\, \delta_{\rho k} = \epsilon_{ijk}.$

ILLUSTRATION. *Particles of mass m, $2m$, $3m$ are placed respectively at the points* $(3, 0, 0)$, $(1, 0, 1)$, $(5, 3, 1)$. *To find the inertia tensor referred to these axes.*

By the formula $a_{ii} = \sum m(x_j^2 + x_k^2)$,

$$a_{11} = m.0+2m.1+3m.10 = 32m,$$
$$a_{22} = m.9+2m.2+3m.26 = 91m,$$
$$a_{33} = m.9+2m.1+3m.34 = 113m.$$

By the formula $a_{ij} = -\sum m x_i x_j$,

$$-a_{23} = m.0+2m.0+3m.3 = 9m,$$
$$-a_{31} = m.0+2m.1+3m.5 = 17m,$$
$$-a_{12} = m.0+2m.0+3m.15 = 45m.$$

The tensor is thus

$$\begin{pmatrix} 32m & -45m & -17m \\ -45m & 91m & -9m \\ -17m & -9m & 113m \end{pmatrix}.$$

MISCELLANEOUS EXAMPLES

1. Prove the identities

(i) $\delta_{\mu\nu}\delta_{\nu\lambda}\delta_{\lambda\mu} = 3$, (ii) $\delta_{i\lambda}\delta_{\lambda\mu}\delta_{\mu j} = \delta_{ij}$, (iii) $\lambda\mu\delta_{\lambda\mu} = 14$.

2. Prove the identities

(i) $\epsilon_{i\lambda\mu}\epsilon_{j\lambda\mu} = 2\delta_{ij}$, (ii) $\epsilon_{i\mu\nu}\epsilon_{j\nu\lambda}\epsilon_{k\lambda\mu} = \epsilon_{ijk}$,

(iii) $\lambda\mu\nu\epsilon_{\lambda\mu\nu} = 0$.

3. Prove that
$$\epsilon_{ijk} = -\tfrac{1}{2}\begin{vmatrix} i^2 & i & 1 \\ j^2 & j & 1 \\ k^2 & k & 1 \end{vmatrix},$$
and deduce that
$$\epsilon_{ijk}\epsilon_{pqr} = \begin{vmatrix} (i-p)^2 & (i-q)^2 & (i-r)^2 \\ (j-p)^2 & (j-q)^2 & (j-r)^2 \\ (k-p)^2 & (k-q)^2 & (k-r)^2 \end{vmatrix}.$$

4. Prove that, if
$$\Delta = \begin{vmatrix} \delta_{ij} & \delta_{ik} \\ \delta_{ik} & \delta_{jk} \end{vmatrix},$$
then $\Delta = 0$ except when $i = k \neq j$, and that its value then is -1.

5. Evaluate
$$\text{(i) } \delta_{i\lambda}\epsilon_{ij\lambda}, \quad \text{(ii) } \delta_{\lambda\mu}\delta_{\lambda\mu}, \quad \text{(iii) } \epsilon_{\lambda\mu\nu}\epsilon_{\lambda\mu\nu}.$$

6. Prove that, if (x_i) is a vector, then $(x_i x_j)$ is a tensor.

7. An array is given by
$$(a_{ij}) \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
for all systems of coordinate axes. Prove that it is not a tensor.

8. Particles of unit mass are situated at the eight vertices of a cube of side two units. Find the components I_{ij} of the inertia tensor referred to a system of coordinates with origin at the centre and with axes parallel to the edges.

9. A uniform circular disk of density ρ lies in the plane $z = 0$ and occupies the circle $x^2+y^2-2x = 0$ in that plane. Find the inertia tensor referred to those axes.

10. Particles of mass m, $2m$, $3m$ are placed respectively at the points $(3,0,0)$, $(0,5,0)$, $(0,0,7)$. Prove that the inertia tensor referred to these axes is
$$\begin{pmatrix} 197m & 0 & 0 \\ 0 & 156m & 0 \\ 0 & 0 & 59m \end{pmatrix}.$$

11. Given three functions u_1, u_2, u_3 of the variables x_1, x_2, x_3, prove that (a_{ij}) is a tensor, where
$$a_{ij} = \frac{\partial u_i}{\partial x_j}.$$

12. Prove the formulae

(i) $(\mathbf{a}\wedge\mathbf{b}).(\mathbf{c}\wedge\mathbf{d}) = (\mathbf{a}.\mathbf{c})(\mathbf{b}.\mathbf{d})-(\mathbf{a}.\mathbf{d})(\mathbf{b}.\mathbf{c})$,

(ii) $(\mathbf{a}\wedge\mathbf{b})\wedge(\mathbf{c}\wedge\mathbf{d}) = (\mathbf{cda})\mathbf{b}-(\mathbf{bcd})\mathbf{a}$
$= (\mathbf{dab})\mathbf{c}-(\mathbf{abc})\mathbf{d}$,

where, for example, $(\mathbf{abc}) = \mathbf{a}.(\mathbf{b}\wedge\mathbf{c})$.

V

THE SPHERE AND THE CIRCLE

1. The sphere

A SPHERE is a surface traced by a point whose distance from a fixed point, the CENTRE, has a constant magnitude, the RADIUS. For centre (α, β, γ) and radius r, the equation is

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2 = r^2.$$

In vector notation, if $\mathbf{x}$ is the position vector of a point on a sphere of centre $\boldsymbol{\alpha}$ and radius r, then

$$(\mathbf{x}-\boldsymbol{\alpha})^2 = r^2 \quad \text{or} \quad |\mathbf{x}-\boldsymbol{\alpha}| = r.$$

NOTE. The GENERAL QUADRATIC FORM in the variables x, y, z contains three parts:

(i) a quadratic expression

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$

(ii) a linear expression

$$2ux+2vy+2wz,$$

(iii) a constant d.

It is thus

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d.$$

The equation of the sphere, on expansion, is

$$x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z+(\alpha^2+\beta^2+\gamma^2-r^2) = 0.$$

Hence ***necessary*** *conditions for the general quadratic equation to represent a sphere are*

$$a = b = c,$$

$$f = g = h = 0.$$

Conversely, *these conditions are also* ***sufficient****, provided also that* $a \neq 0$ *and that*

$$u^2+v^2+w^2-da > 0.$$

For the equation may then be written

$$ax^2+ay^2+az^2+2ux+2vy+2wy+d = 0,$$

or, after division by (non-zero) a,

$$\left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)^2+\left(z+\frac{w}{a}\right)^2 = \frac{u^2+v^2+w^2}{a^2}-\frac{d}{a}.$$

If the right-hand side is positive, this equation ensures that the variable point (x, y, z) is at constant distance

$$\sqrt{\{(u^2+v^2+w^2-da)/a^2\}}$$

from the fixed point $(-u/a, -v/a, -w/a)$.

In practice the value of a is usually taken to be unity. We have then shown that the equation

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0$$

represents a sphere of centre $(-u, -v, -w)$ and radius

$$+\sqrt{(u^2+v^2+w^2-d)}.$$

The equation of the sphere of centre the origin and radius a is

$$x^2+y^2+z^2 = a^2,$$

or, in vector notation, $\mathbf{x}^2 = a^2$.

It may be remarked that *the point* $P(x_1, y_1, z_1)$ *lies inside the sphere*

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0$$

if

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d < 0$$

and outside if

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d > 0.$$

The point lies inside if its distance from the centre is less than the radius; that is, if (squaring)

$$(x_1+u)^2+(y_1+v)^2+(z_1+w)^2 < u^2+v^2+w^2-d,$$

or

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d < 0.$$

The 'outside' test is proved similarly.

2. The sphere with a given diameter

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ be two given points and $P(x, y, z)$ a variable point of the sphere on AB as diameter. The angle APB is a right angle, so that the direction ratios

$$(x-x_1, y-y_1, z-z_1), \qquad (x-x_2, y-y_2, z-z_2)$$

represent perpendicular lines. Hence *the equation of the sphere on* (x_1, y_1, z_1), (x_2, y_2, z_2) *as diameter is*

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2) = 0.$$

In vector notation, this is

$$(\mathbf{x}-\mathbf{x}_1).(\mathbf{x}-\mathbf{x}_2) = 0.$$

3. Joachimstal's ratio equation

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ be two given points. The coordinates of the point dividing the segment $\overrightarrow{PQ}$ in the (positive or negative) ratio $\lambda:1$ are (p. 17)

$$\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}, \frac{z_1+\lambda z_2}{1+\lambda}\right).$$

To find a quadratic equation for the two values of λ *for which this point lies on the sphere*

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0.$$

The following notation, which is typical of much that will occur later, helps to make the statements more concise. Write

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d,$$
$$S_1 \equiv x_1 x+y_1 y+z_1 z+u(x+x_1)+v(y+y_1)+w(z+z_1)+d,$$
$$S_{11} \equiv x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d,$$
$$S_{12} \equiv x_1 x_2+y_1 y_2+z_1 z_2+u(x_1+x_2)+v(y_1+y_2)+w(z_1+z_2)+d.$$

Then $$S_{12} \equiv S_{21}.$$

The equation of the sphere is thus

$$S = 0,$$

and the conditions for P, Q to lie on it are

$$S_{11} = 0, \qquad S_{22} = 0$$

respectively.

The point

$$\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}, \frac{z_1+\lambda z_2}{1+\lambda}\right)$$

lies on the sphere if, on substituting its coordinates and then multiplying by $(1+\lambda)^2$,

$$(x_1+\lambda x_2)^2+(y_1+\lambda y_2)^2+(z_1+\lambda z_2)^2+$$
$$+2(1+\lambda)\{u(x_1+\lambda x_2)+v(y_1+\lambda y_2)+w(z_1+\lambda z_2)\}+d(1+\lambda)^2 = 0.$$

Arrange in powers of λ and use the notation just defined:

$$S_{22}\lambda^2+2S_{12}\lambda+S_{11} = 0.$$

This is *the quadratic equation whose roots serve to determine the two points where the line meets the sphere.*

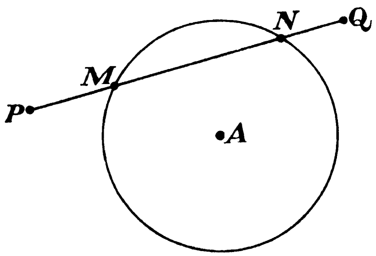

FIG. 36

The two points do not have real existence unless the roots of the quadratic are real; that is, unless

$$S_{12}^2-S_{11}S_{22} \geqslant 0.$$

When real, they are denoted by the letters M, N (Fig. 36).

In vector notation, the point

$$\frac{\mathbf{x}_1+\lambda\mathbf{x}_2}{1+\lambda}$$

lies on the sphere $\qquad (\mathbf{x}-\boldsymbol{\alpha})^2 = \mathbf{r}^2$

if

$$\{(\mathbf{x}_2-\boldsymbol{\alpha})^2-r^2\}\lambda^2+2\{(\mathbf{x}_1-\boldsymbol{\alpha}).(\mathbf{x}_2-\boldsymbol{\alpha})-r^2\}\lambda+\{(\mathbf{x}_1-\boldsymbol{\alpha})^2-r^2\} = 0.$$

It is, however, very doubtful whether the language of vectors is of much use here.

4. Tangency

A straight line is said to be a TANGENT to a sphere at a point L (or to TOUCH it at L) if it meets the sphere at L and at no other point.

(i) THE TANGENT PLANE. In the notation of § 3, let the point $P(x_1, y_1, z_1)$ be chosen to lie on the sphere, so that

$$S_{11} = 0.$$

The equation for λ has thus one root zero; that is, one of the two points of intersection of the line and the sphere is at P. If, in addition, the line is chosen to be a tangent at P, there can be (by definition) no root other than zero, so that the second root of the quadratic equation is also zero. Hence $S_{12} = 0$. Thus *the condition for $Q(x_2, y_2, z_2)$ to lie on a tangent line at $P(x_1, y_1, z_1)$ is* $\qquad S_{12} = 0.$

As Q varies subject to this condition, its coordinates satisfy the equation found by replacing x_2, y_2, z_2 by current coordinates x, y, z, namely $S_1 = 0$. Thus *the coordinates of any point on any line touching the sphere at $P(x_1, y_1, z_1)$ satisfy the equation*

$$S_1 \equiv x_1 x + y_1 y + z_1 z + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0,$$

or

$$(x_1+u)x + (y_1+v)y + (z_1+w)z + (ux_1+vy_1+wz_1+d) = 0.$$

This is the equation of a plane, called the TANGENT PLANE at P to the sphere. It contains all the tangent lines through P.

COROLLARY. The direction ratios of the normal to the tangent plane at P are (p. 21)

$$(x_1+u,\ y_1+v,\ z_1+w),$$

and these (p. 19) are also the direction ratios of the radius joining P to the centre $(-u, -v, -w)$. Hence *the tangent plane at P is perpendicular to the radius through P.*

(ii) THE TANGENT CONE. Suppose next that, in the work of § 3, the points P, Q do not lie on the sphere, but that the line PQ is a tangent. The two points M, N (Fig. 36) then coincide, so that Joachimstal's equation has *equal roots*. The condition for this is

$$S_{11} S_{22} = S_{12}^2.$$

If, then, P is regarded as given, while Q moves in such a way that the line PQ always touches the sphere, the coordinates of Q satisfy the relation found by replacing x_2, y_2, z_2 by the current variables x, y, z, namely

$$S_{11} S = S_1^2.$$

The locus of Q is called the TANGENT CONE from P to the sphere, so that *the coordinates (x, y, z) of any point Q on the tangent cone from $P(x_1, y_1, z_1)$ satisfy the equation*

$$S_{11} S = S_1^2.$$

For example, the tangent cone to the sphere

$$x^2+y^2+z^2 = a^2$$

is $$(x_1^2+y_1^2+z_1^2-a^2)(x^2+y^2+z^2-a^2) = (x_1 x+y_1 y+z_1 z-a^2)^2;$$

and the tangent cone from the origin to the general sphere is

$$d(x^2+y^2+z^2+2ux+2vy+2wz+d) = (ux+vy+wz+d)^2,$$

or $$(u^2-d)x^2+(v^2-d)y^2+(w^2-d)z^2+2vwyz+2wuzx+2uvxy = 0.$$

5. Pole and polar; harmonic separation

The interpretation of the relation $S_{12} = 0$ in terms of tangency at $P(x_1, y_1, z_1)$ when P is on the sphere ($S_{11} = 0$) suggests consideration of the relation $S_{12} = 0$ under the more general condition $S_{11} \neq 0$.

Joachimstal's equation

$$S_{22}\lambda^2 + 2S_{12}\lambda + S_{11} = 0$$

becomes, under the conditions $S_{12} = 0$, $S_{11} \neq 0$,

$$S_{22}\lambda^2 + S_{11} = 0,$$

and then the two values† of λ are equal in magnitude but opposite in sign. By definition of λ, the two values are (p. 17)

$$\overrightarrow{PM}/\overrightarrow{MQ}, \qquad \overrightarrow{PN}/\overrightarrow{NQ},$$

so that M and N divide $\overrightarrow{PQ}$ internally (λ positive) and externally (λ negative) in the same ratio.

DEFINITION. *Four points P, Q, M, N such that*

$$\overrightarrow{PM}/\overrightarrow{MQ} = -\overrightarrow{PN}/\overrightarrow{NQ},$$

(so that M and N divide $\overrightarrow{PQ}$ internally and externally in the same ratio) *are said to form a* HARMONIC RANGE. The points M, N are called HARMONIC CONJUGATES with respect to P, Q.

COROLLARY. *If M, N are harmonic conjugates with respect to P, Q, then P, Q are also harmonic conjugates with respect to M, N:*

For the relation

$$\overrightarrow{PM}/\overrightarrow{MQ} = -\overrightarrow{PN}/\overrightarrow{NQ}$$

is also

$$\overrightarrow{PM}/\overrightarrow{PN} = -\overrightarrow{MQ}/\overrightarrow{NQ},$$

or

$$\overrightarrow{MP}/\overrightarrow{PN} = -\overrightarrow{MQ}/\overrightarrow{QN}.$$

We return to the main problem. Since the relation $S_{12} = 0$ gives $\overrightarrow{PM}/\overrightarrow{MQ} = -\overrightarrow{PN}/\overrightarrow{NQ}$, it follows that *the points P, Q are*

† The (real) values of λ exist only if S_{11}, S_{22} have opposite signs; that is (p. 100), if one of the points P, Q is inside the sphere and the other outside. But we do not wish to emphasize this aspect unduly.

separated harmonically by the two points in which the line PQ meets the sphere. Two points, such as P, Q, related to the sphere in this way are said to be CONJUGATE with respect to it. Thus *the condition for the two points* $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ *to be conjugate with respect to the sphere* $S = 0$ *is*

$$S_{12} = 0.$$

Suppose now that the point P is regarded as given. Then *those points* Q *such that* P, Q *are conjugate with respect to the sphere lie in the plane given by the equation*

$$S_1 = 0.$$

This plane is called the POLAR PLANE of P with respect to the sphere; also P is the POLE of its polar plane.

Note that, if P lies on the sphere, then the polar plane of P, given by the equation $S_1 = 0$, is (p. 103) the tangent plane at P.

The relation (p. 101)

$$S_{12} = S_{21}$$

shows that, *if the polar plane of* P *passes through* Q (so that $S_{12} = 0$), *then the polar plane of* Q *passes through* P (since $S_{21} = 0$).

6. The segment theorem; diameters

The work of this section is very similar to that of § 3 (p. 101), but it deals with distances and directions instead of ratios.

Let $P(x_1, y_1, z_1)$ be a given point and (l, m, n) the direction cosines of a line through P. The point $Q(x, y, z)$ on this line, such that $\overrightarrow{PQ} = r$, satisfies the relations

$$x = x_1 + lr, \quad y = y_1 + mr,$$
$$z = z_1 + nr.$$

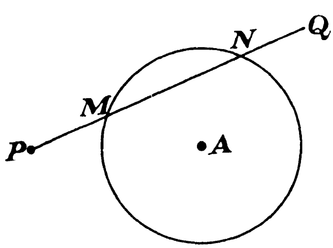

FIG. 37

The line cuts the sphere in two points M, N. To prove that *the two values of r corresponding to* M, N *satisfy the equation*

$$r^2 + 2r\{(x_1+u)l + (y_1+v)m + (z_1+w)n\} + S_{11} = 0.$$

Substitute the values of x, y, z into the equation of the sphere; thus

$$(lr+x_1)^2+...+2u(lr+x_1)+...+d = 0.$$

Arrange in powers of r, remembering that $l^2+m^2+n^2 = 1$; thus

$$r^2+2r\{(x_1+u)l+(y_1+v)m+(z_1+w)n\}+S_{11} = 0.$$

This equation is called the r-EQUATION of the point $P(x_1, y_1, z_1)$ and the direction (l, m, n) for the sphere S.

In vector notation, the point $\mathbf{x}_1+r\mathbf{l}$ lies on the sphere

$$(\mathbf{x}-\boldsymbol{\alpha})^2 = k^2, \quad \text{or} \quad |\mathbf{x}-\boldsymbol{\alpha}| = k,$$

if
$$r^2+2\mathbf{l}.(\mathbf{x}-\boldsymbol{\alpha})r+\{(\mathbf{x}_1-\boldsymbol{\alpha})^2-k^2\} = 0.$$

THE RECTANGLE THEOREMS. *It is important to remember that the formulae now to be given pre-suppose that S is expressed in a form such that the coefficients of x^2, y^2, z^2 are* **unity**.

(i) To prove that, *if a variable line through a fixed point $P(x_1, y_1, z_1)$ meets a given sphere in points M, N, then $\overrightarrow{PM}.\overrightarrow{PN}$ is constant.*

The r-equation for the fixed point (x_1, y_1, z_1) and (variable) direction (l, m, n) is

$$r^2+\{...\}r+S_{11} = 0,$$

so that, if $\overrightarrow{PM} \equiv r_1$, $\overrightarrow{PN} \equiv r_2$, the product of the roots is given by the formula

$$r_1 r_2 = S_{11};$$

thus
$$\overrightarrow{PM}.\overrightarrow{PN} = S_{11}.$$

But the right-hand side is independent of l, m, n and is therefore constant.

Note that, if P is outside the sphere, $\overrightarrow{PM}$, $\overrightarrow{PN}$ have the same signs, so that their product is positive; if P is inside the sphere, $\overrightarrow{PM}$, $\overrightarrow{PN}$ have opposite signs so that their product is negative. Hence *the point P lies outside or inside the sphere according as S_{11} is positive or negative.* (Compare p. 100.)

(ii) To prove that, *if $P(x_1, y_1, z_1)$ lies outside the sphere $S = 0$, then the length t of a tangent from P to the sphere is given by the formula*

$$t^2 = S_{11}.$$

This is merely the formula $r_1 r_2 = S_{11}$ when $r_1 = r_2 = t$.

DIAMETERS. Suppose next that P is the middle point of a chord MN whose direction is (l, m, n). Then $\vec{PM} = -\vec{PN}$, and so the r-equation of P for that direction has roots which are equal and opposite. The coefficient of r thus vanishes, so that

$$l(x_1+u)+m(y_1+v)+n(z_1+w) = 0.$$

Hence, replacing x_1, y_1, z_1 by current variables x, y, z, *the middle points of chords in the given direction* (l, m, n) *all lie in the plane*

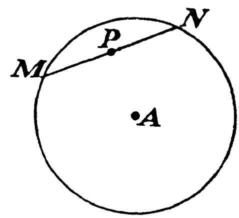

FIG. 38

$$lx+my+nz+(lu+mv+nw) = 0.$$

This plane passes through the centre $(-u, -v, -w)$ of the sphere and is perpendicular to the given direction (l, m, n).

The condition

$$l(x_1+u)+m(y_1+v)+n(z_1+w) = 0$$

is satisfied for *all* values of l, m, n when $P(x_1, y_1, z_1)$ is at the centre $(-u, -v, -w)$ of the sphere; that is, *all chords through the centre of the sphere are bisected there.*

There are many problems in which it is convenient to take as a starting-point a circle of given centre drawn on the sphere. The following theorem is useful:

To prove that *the equation of the plane cutting the sphere* $S = 0$ *in a circle of centre* $P(x_1, y_1, z_1)$ *is*

$$S_1 = S_{11}.$$

The centre of the sphere is $A(-u, -v, -w)$, so that the direction ratios of AP are

$$(x_1+u,\ y_1+v,\ z_1+w).$$

Hence the plane, being perpendicular to AP and passing through P, is

$$(x_1+u)(x-x_1)+(y_1+v)(y-y_1)+(z_1+w)(z-z_1) = 0,$$

or

$$(x_1+u)x+(y_1+v)y+(z_1+w)z$$
$$= (x_1+u)x_1+(y_1+v)y_1+(z_1+w)z_1,$$

or, adding $ux_1+vy_1+wz_1+d$ to each side,

$$S_1 = S_{11}.$$

ILLUSTRATION. To prove that *the centre of a circle cut on the sphere S by a plane through the given point $P(x_1, y_1, z_1)$ lies on the sphere whose equation is*

$$S = S_1.$$

Suppose that the centre of a typical circle is $Q(x_2, y_2, z_2)$. Then the equation of the plane of the circle is

$$S_2 = S_{22}.$$

This passes through P if

$$S_{12} = S_{22}.$$

The locus of Q, found by replacing x_2, y_2, z_2 by current coordinates x, y, z, is therefore

$$S_1 = S.$$

7. Orthogonal spheres

DEFINITION. Two intersecting spheres

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wy+d = 0,$$
$$S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d' = 0$$

are called ORTHOGONAL (cutting AT RIGHT ANGLES) if the tangent planes at a common point $P(x_1, y_1, z_1)$ are perpendicular.

To prove that *the condition for S, S' to be orthogonal is*

$$2uu'+2vv'+2ww' = d+d'.$$

The tangent planes

$$(x_1+u)x+(y_1+v)y+(z_1+w)z+... = 0,$$
$$(x_1+u')x+(y_1+v')y+(z_1+w')z+... = 0$$

are perpendicular if and only if

$$(x_1+u)(x_1+u')+(y_1+v)(y_1+v')+(z_1+w)(z_1+w') = 0,$$

or

$$x_1^2+y_1^2+z_1^2+(u+u')x_1+(v+v')y_1+(w+w')z_1+$$
$$+uu'+vv'+ww' = 0.$$

Since P lies on each sphere,

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d = 0,$$
$$x_1^2+y_1^2+z_1^2+2u'x_1+2v'y_1+2w'z_1+d' = 0.$$

Add and divide by 2:

$$x_1^2+y_1^2+z_1^2+(u+u')x_1+(v+v')y_1+(w+w')z_1+\tfrac{1}{2}(d+d') = 0.$$

The preceding equation of condition thus gives

$$uu'+vv'+ww' = \tfrac{1}{2}(d+d').$$

COROLLARIES. (i) *The tangent planes to two orthogonal spheres are perpendicular at* **every** *common point.* The condition

$$2uu'+2vv'+2ww' = d+d'$$

is, in fact, independent of the point $P(x_1, y_1, z_1)$ from which the argument started.

(ii) *If two orthogonal spheres, of radii a, b, have their centres distant k apart, then*

$$k^2 = a^2+b^2.$$

For

$$a^2 = u^2+v^2+w^2-d, \qquad b^2 = u'^2+v'^2+w'^2-d',$$

$$k^2 = (u-u')^2+(v-v')^2+(w-w')^2,$$

so that

$$a^2+b^2-k^2 = 2uu'+2vv'+2ww'-d-d'$$
$$= 0.$$

(iii) *If two spheres are orthogonal, then the centre of either lies in the tangent plane to the other at any common point.*

The tangent plane to S at $P(x_1, y_1, z_1)$ contains the centre $(-u', -v', -w')$ of S' if

$$-(x_1+u)u'-(y_1+v)v'-(z_1+w)w'+ux_1+vy_1+wz_1+d = 0,$$

where

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d = 0,$$
$$x_1^2+y_1^2+z_1^2+2u'x_1+2v'y_1+2w'z_1+d' = 0.$$

Subtract the last two equations:

$$2(u-u')x_1+2(v-v')y_1+2(w-w')z_1+d-d' = 0.$$

Subtract from this the orthogonality relation

$$2uu'+2vv'+2ww'-d-d' = 0$$

and divide by 2:

$$(u-u')x_1+(v-v')y_1+(w-w')z_1-uu'-vv'-ww'+d = 0.$$

Rearranging, this is the required condition

$$-(x_1+u)u'-(y_1+v)v'-(z_1+w)w'+ux_1+vy_1+wz_1+d = 0.$$

NOTE. The results of these Corollaries are otherwise obvious, and could have been used as a basis for the discussion. The treatment actually used does, however, lay greater emphasis on the root conception of orthogonality.

8. Pairs of spheres; circles

Let the equations of two given spheres be

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0,$$

$$S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d' = 0,$$

of centres $A\,(-u, -v, -w)$, $A'\,(-u', -v', -w')$.

Consider the equation

$$S-kS' = 0.$$

When $k = 1$, this represents the plane

$$2(u-u')x+2(v-v')y+2(w-w')z+d-d' = 0,$$

and we regard this case temporarily as excluded. When $k \neq 1$, the equation, after division by $1-k$, is

$$S_k \equiv x^2+y^2+z^2+2u_k x+2v_k y+2w_k z+d_k = 0,$$

where

$$u_k = \frac{u-ku'}{1-k}, \quad v_k = \frac{v-kv'}{1-k}, \quad w_k = \frac{w-kw'}{1-k}, \quad d_k = \frac{d-kd'}{1-k}.$$

This equation represents a *sphere* of centre $B\,(-u_k, -v_k, -w_k)$.

Now the equation $S-kS' = 0$ is satisfied whenever,

$$S = 0, \qquad S' = 0$$

simultaneously. Hence *the sphere S_k (and the plane when $k = 1$) passes through all the points, if any, common to the two given spheres S, S'.*

In particular, *the curve common to two intersecting spheres lies entirely in a plane.* The intersection is therefore a CIRCLE.

Observe carefully that ***two*** *equations are necessary to specify a circle in space.* A natural choice would be the equations of the plane containing the circle and of a sphere through it. Alternatively, the equations of two spheres might be selected, the equation of the plane, if required, being obtained by the process just described. If the equations $U = 0$, $V = 0$ represent *either* a plane and a sphere *or* two spheres, then all spheres through their circle of intersection are given by the equation $U-kV = 0$ for appropriate values of k.

Illustration. *To find the equation of the sphere through the origin and the circle*

$$x^2+y^2+z^2+6x+8y+10 = 0, \qquad 4x+3y+2z+5 = 0.$$

The equation of any sphere through the circle is

$$x^2+y^2+z^2+6x+8y+10-k(4x+3y+2z+5) = 0,$$

and it passes through the origin if

$$10-5k = 0,$$

or

$$k = 2.$$

Hence the equation is

$$x^2+y^2+z^2-2x+2y-4z = 0.$$

Note: The effective existence of the circle depends on whether the plane cuts the sphere or not. A simple test to settle this point is that *the sphere cuts the plane provided that the distance of its centre from the plane is less than its radius.*

Consider, for example, the sphere

$$x^2+y^2+z^2+6x+8y+10 = 0$$

and the plane

$$4x+3y+2z+5 = 0$$

of the preceding Illustration. The centre of the sphere is $(-3, -4, 0)$, which is at a distance

$$\frac{-12-12+5}{\pm\sqrt{29}} = \frac{19}{\sqrt{29}}$$

from the plane. Also the radius of the sphere is

$$\sqrt{(9+16-10)} = \sqrt{15}.$$

But

$$19/\sqrt{29} < \sqrt{15},$$

and so the plane cuts the sphere.

Finally, *to find the radius of the circle in which the plane*

$$lx+my+nz+p = 0$$

cuts the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0:$$

If a is the radius of the circle, r the radius of the sphere, and b the distance of the centre $(-u, -v, -w)$ from the plane, then

$$\begin{aligned} a^2 &= r^2-b^2 \\ &= (u^2+v^2+w^2-d)-\left\{\frac{-lu-mv-nw+p}{\sqrt{(l^2+m^2+n^2)}}\right\}^2 \\ &= \frac{(l^2+m^2+n^2)(u^2+v^2+w^2-d)-(lu+mv+nw-p)^2}{(l^2+m^2+n^2)}. \end{aligned}$$

COROLLARY. *The condition for the plane to* **touch** *the sphere is*

$$(l^2+m^2+n^2)(u^2+v^2+w^2-d) = (lu+mv+nw-p)^2,$$

and the condition for the plane to **intersect** *the sphere in a circle is*

$$(l^2+m^2+n^2)(u^2+v^2+w^2-d) > (lu+mv+nw-p)^2.$$

9. The radical plane

There is another way of interpreting the equation

$$S-kS' = 0$$

considered in § 8. Denote by t, t' the lengths of the tangents from the point $P(x, y, z)$ to the spheres

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0,$$

$$S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d' = 0,$$

so that (p. 106) $\quad t^2 = S, \quad t'^2 = S'.$

Then *the locus of a point which moves so that*

$$t = mt'$$

is the sphere (plane if $m = 1$)

$$S = m^2S'.$$

The plane is called the RADICAL PLANE of the two given spheres, and the system of spheres defined by the equation

$$S = m^2S'$$

for varying m is called a COAXAL SYSTEM. Each two spheres selected from the coaxal system have the plane $S = S'$ as their radical plane.

The above statement gives the definition of the radical plane in its most graphic form, but it needs modifying if, for example, there are values of x, y, z for which S is negative (compare

p. 100). To meet this difficulty, define the POWER of a point $P(x_1, y_1, z_1)$ with respect to a sphere

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0$$

to be the function

$$S_{11} \equiv x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d.$$

Then the RADICAL PLANE of two spheres S, S' is *the locus of a point whose powers with respect to the spheres are equal.*

The result may be extended. The RADICAL LINE of three spheres S, S', S'' is the locus of a point whose powers with respect to the three spheres are equal. The locus is given by the *two* equations

$$S = S' = S'',$$

and is a straight line. [For example, it is the line of intersection of the two planes $S-S' = 0$, $S-S'' = 0$, which, in the general case, are not parallel.]

Similarly the RADICAL CENTRE of four spheres S, S', S'', S''' is that point (unique for general positions of the spheres, with which alone we concern ourselves) whose powers with respect to the four spheres are equal. The point is given by the *three* equations

$$S = S' = S'' = S'''.$$

It follows easily (compare p. 109) that *the sphere with centre any point* (i) *on the radical plane of two spheres,* (ii) *on the radical line of three, or* (iii) *at the radical centre of four, and having its radius equal to the tangents from the point to the spheres, cuts orthogonally each of the two, three, or four spheres.*

10. Coaxal system; simplified equation

The radical plane of the two spheres

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0,$$

$$S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d' = 0$$

is

$$2(u-u')x+2(v-v')y+2(w-w')z+(d-d') = 0,$$

and the direction cosines of its normals are

$$(u-u', v-v', w-w').$$

Hence *the radical plane of two spheres is perpendicular to their line of centres.*

If, then, the line of centres is taken to be the axis $y = z = 0$, the equations of the spheres appear in the simpler form

$$S \equiv x^2+y^2+z^2+2ux+d = 0,$$

$$S' \equiv x^2+y^2+z^2+2u'x+d' = 0,$$

and the radical plane is

$$2(u-u')x+(d-d') = 0.$$

Suppose, further, that the origin is chosen to be that point where the line of centres meets the radical plane; then

$$d-d' = 0.$$

Hence *the equations of the two given spheres may be reduced to the simplified form*

$$S \equiv x^2+y^2+z^2+2ux+d = 0,$$

$$S' \equiv x^2+y^2+z^2+2u'x+d = 0.$$

The spheres of the coaxal system are then

$$x^2+y^2+z^2+\frac{2(u-\lambda u')}{1-\lambda}x+d = 0$$

for varying λ. Thus, writing

$$\frac{u-\lambda u'}{1-\lambda} \equiv \mu,$$

the equation for the spheres of a coaxal system may be expressed in the form

$$x^2+y^2+z^2+2\mu x+d = 0$$

for varying μ.

The radical plane is $\quad x = 0.$

This plane meets the spheres of the system, if at all, in the circle

$$x = 0, \qquad y^2+z^2+d = 0.$$

The circle exists if d is negative, but not if it is positive.

We confine our attention now to non-intersecting spheres, for which d is positive; say $d = a^2$. Then the spheres are

$$x^2+y^2+z^2+2\mu x+a^2 = 0.$$

In particular, the two 'spheres' given by $\mu = -a$ and $\mu = +a$ are

$$(x\mp a)^2+y^2+z^2 = 0,$$

and so reduce to the two points $(a, 0, 0)$, $(-a, 0, 0)$. They are called the LIMITING POINTS of the coaxal system.

Consider next any sphere (if existing) which cuts *each* sphere of the coaxal system orthogonally. Such a sphere has equation

$$x^2+y^2+z^2+2px+2qy+2rz+w = 0,$$

where, for orthogonality (p. 108),

$$2p\mu-a^2-w = 0.$$

For this to be true for *all* values of μ, we need the relations

$$p = 0, \qquad w = -a^2,$$

so that the equation of a typical sphere is

$$x^2+y^2+z^2+2qy+2rz-a^2 = 0.$$

But this sphere passes through the two limiting points $(\pm a, 0, 0)$. Hence *every sphere cutting the spheres of a coaxal system orthogonally passes through the limiting points.*

MISCELLANEOUS EXAMPLES

1. Find the equation of the sphere whose centre is the point $(2, 2, 1)$ and which touches the plane $3x+4y+12z = 0$.

The plane $z = h$ cuts the sphere in a circle. Prove that the radius of the circle is $\sqrt{\{(3-h)(1+h)\}}$, and deduce the equations of those tangent planes to the sphere which are parallel to the plane $z = 0$.

2. Prove that the two circles

$$x^2+y^2+z^2-2x+3y+4z-5 = 0, \qquad 5y+6z+1 = 0$$

and

$$x^2+y^2+z^2-3x-4y+5z-6 = 0, \qquad x+2y-7z = 0$$

lie on the same sphere, and find its equation.

3. Find the equation of the sphere whose centre is the point $(1, 2, 3)$ and which touches the plane given by the equation $3x+2y+z+4 = 0$.

Find also the radius of the circle in which the sphere is cut by the plane $x+y+z = 0$.

4. The line

$$\frac{x+2}{3} = \frac{y+1}{4} = \frac{z-8}{-5}$$

intersects the sphere

$$x^2+y^2+z^2-2x-6y+4z-11 = 0$$

in the points P_1, P_2. Find the coordinates of P_1, P_2, and obtain also the equations of the line that passes through the centre of the sphere and through the mid-point of P_1P_2.

5. Find the equation of the sphere with centre $(3, 0, 8)$ which cuts off a chord of length 16 units on the line

$$2x+y-z = 7, \qquad 4x-4y-5z = 29.$$

6. Find the equations of the straight line through the point $P(18, 23, -3)$ and the centre C of the sphere

$$S \equiv x^2+y^2+z^2+8x-6y+14z+38 = 0.$$

Hence, or otherwise, find the equations of the spheres which have their centres on the line PC, pass through P, and touch S.

7. A point P moves on the surface of a sphere

$$x^2+y^2+z^2+2x-4y+1 = 0$$

in such a way that its distance from the point $U(2, 1, -3)$ is always 3. Find the equation of the plane in which P always lies.

The line UP cuts the sphere again in Q. Find the equation of the plane in which Q always lies, and the distance between these two planes.

8. Find the equation of the sphere whose centre is the origin and whose radius is 5 units.

Find the range of values of λ for which the plane

$$3x+4y+12z = \lambda$$

cuts the sphere, and find the radius of the circle of intersection when $\lambda = 39$.

9. Find the centre and radius of the circle in which the spheres

$$x^2+y^2+z^2-8x-10y-4z-15 = 0,$$
$$x^2+y^2+z^2+2x+10y+6z+5 = 0$$

intersect, and obtain the equation of the sphere on which this circle is a great circle.

10. A sphere passes through the points $(4, 3, -2)$, $(-1, -1, 1)$, $(3, 0, -2)$, $(2, 3, 2)$. Find its equation.

Find the centre and radius of the section of the sphere by the plane $x-y = 0$ and the equations of the projection of this section on to the plane $x = 0$.

11. A sphere has its centre at the point $(0, -2, 1)$, and it touches the plane which passes through the point $(1, 1, 0)$ and the line

$$\frac{x-1}{1} = \frac{y+1}{4} = \frac{z+1}{1}.$$

Find the radius of the sphere and its point of contact with the plane.

12. Find the equation of the sphere through the points $(1, -2, 0)$, $(0, -2, -1)$, $(1, -1, -1)$, $(1, -3, -1)$, and show that the plane

$$x+y-z = 0$$

passes through the centre of the sphere.

13. Find the equation of the sphere with centre $(1, 2, 3)$ and radius 5.

Show that the plane $3x+4y+12z = 86$ cuts the sphere in a circle of radius 4, and find the equation of the parallel plane at the same distance from the centre but on the opposite side.

14. Points A, B, C, D have coordinates $(3, 5, 2)$, $(1, 3, 0)$, $(3, 4, 1)$, $(-1, 6, -1)$ respectively. Find the points in which the straight line CD meets the sphere of which AB is a diameter.

15. Show that the spheres

$$x^2+y^2+z^2-2x-2z-2 = 0,$$
$$x^2+y^2+z^2-8x-4y-2z+20 = 0$$

do not intersect.

Obtain conditions for the plane

$$lx+my+nz = d,$$

where $$l^2+m^2+n^2 = 1,$$

to touch both spheres. Deduce that all such planes pass through one or other of two fixed points collinear with the centres of the spheres, and find the coordinates of these points.

16. Find the equation of the sphere through the origin O and the points $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$.

If U is the centre of this sphere, show that the sphere on OU as diameter passes through the mid-points of the six edges of the tetrahedron $OABC$.

17. A is the point $(0,0,1)$, P is a point of the sphere of unit radius and centre the origin O, and Q is a point of the plane $z = a-1$, where $-1 < a < 1$. If AP and OP are perpendicular to PQ and OQ respectively, show that the positions of P are confined to a certain circle on the sphere and those of Q to the region exterior to the circle

$$x^2+y^2 = a^2(1-a)/(1+a)$$

in the plane.

18. Find the values of d for which the plane

$$3x-2y+z = d$$

touches the sphere

$$x^2+y^2+z^2-2x-4y+2z-8 = 0,$$

and obtain the coordinates of the points of contact.

19. The points A, B, C, D have coordinates $(5,-3,2)$, $(6,-2,2)$, $(5,-2,3)$, $(6,-3,3)$ respectively. Show that spheres may be centred on these points so that each sphere touches the three others externally.

A plane (not intersecting the sphere about A) is laid in contact with the spheres about B, C, D. Find its distance from A, and its equation.

Find the equation of the sphere through all four points.

20. Find the condition for the plane

$$lx+my+nz = p$$

to cut the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz-c = 0$$

in a (real) circle.

Prove that the plane $$x+2y-z = 4$$

cuts the sphere $$x^2+y^2+z^2-x+z-2 = 0$$

in a circle of unit radius, and find the equation of the sphere which has this circle as one of its great circles.

21. Prove that the tangent lines from the origin of coordinates to the sphere
$$(x-a)^2+(y-b)^2+(z-c)^2 = k^2$$
are the generators of the cone given by the equation
$$(a^2+b^2+c^2-k^2)(x^2+y^2+z^2) = (ax+by+cz)^2.$$

22. Find the plane, the centre, and the radius of the circle common to the two spheres
$$x^2+y^2+z^2-4z+1 = 0,$$
$$x^2+y^2+z^2-4x-2y-1 = 0.$$

23. Find the length of the chord cut on the line
$$x-2y+3 = 0, \qquad 2x-2y-z+5 = 0$$
by the sphere
$$x^2+y^2+z^2-2x+3y-16 = 0.$$

24. Find the centre and radius of the circle common to the two spheres
$$x^2+y^2+z^2-3y-5z-2 = 0,$$
$$x^2+y^2+z^2-4x-5y-7z+12 = 0.$$

25. If $\mathbf{n}$ is a unit vector, show that the condition for the plane $\mathbf{n}.\mathbf{r} = p$ to touch the sphere $(\mathbf{r}-\mathbf{c})^2 = a^2$ is
$$(p-\mathbf{n}.\mathbf{c})^2 = a^2.$$

A cone has its vertex at the origin and consists of tangents to a sphere of radius a and centre $\mathbf{b}$. Show that the position vectors $\mathbf{r}$ of points on the cone satisfy the equation
$$(\mathbf{r}.\mathbf{b})^2 = (b^2-a^2)r^2.$$

26. Three planes have equations
$$\mathbf{r}.\mathbf{l} = 0, \quad \mathbf{r}.\mathbf{m} = 0, \quad \mathbf{r}.\mathbf{n} = 0,$$
where $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ are unit vectors. Give the conditions for a vector $\mathbf{p}$ to be equally inclined to $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$.

Find $\mathbf{p}$ when $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ point in the directions $(1, 2, 2)$, $(2, 3, 6)$, $(0, 3, 4)$. Deduce that there is a cone of semi-vertical angle $\cos^{-1}(1/\sqrt{26})$ touching all three planes, and give the vector equation of this cone.

27. The position vector $\mathbf{x}$ of a point P at time t satisfies the differential equation
$$\frac{d\mathbf{x}}{dt} = \mathbf{w} \wedge \mathbf{x},$$
where $\mathbf{w}$ is a fixed vector. Show that P lies on a fixed sphere and also in a fixed plane.

Deduce that P moves on a circle, and show that it describes the circle with constant speed.

VI

THE CENTRAL QUADRICS

THE central conics $$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

of plane geometry extend naturally to the surfaces

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

in space of three dimensions. As in the plane, the character of the surface depends significantly on the selection of the alternative signs; but, equally, there are many properties common to all the different types, and these may be explored conveniently by grouping the surfaces under the comprehensive equation $$Ax^2 + By^2 + Cz^2 = 1.$$

The surfaces given by this equation are all called CENTRAL QUADRICS.

1. The cone and the cylinder

Before studying the central quadrics, brief reference may be made to two other types of surface:

(i) THE CONE. A surface traced out (*generated*) by straight lines all passing through a fixed point is called a CONE. In particular, a cone whose equation is of the second degree in x, y, z is called a QUADRIC CONE. The fixed point is called the VERTEX of the cone.

The equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

where the left-hand side is homogeneous of degree 2 in x, y, z, represents a quadric cone with vertex at the origin. It contains the whole of each line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

through the origin for which the direction cosines (l, m, n) satisfy the equation

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm = 0.$$

(ii) THE CYLINDER. A surface generated by straight lines all parallel to a fixed direction is called a CYLINDER. In particular, a cylinder whose equation is of degree 2 in x, y, z is called a QUADRIC CYLINDER.

The equation

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

represents a quadric cylinder whose generating lines are parallel to the z-axis. It contains the whole of each line parallel to the z-axis and passing through a point of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

in the plane $$z = 0.$$

ILLUSTRATION. THE ORTHOGONAL CONE. To prove that *the cone*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$$

possesses triads of mutually orthogonal generators if

$$a+b+c = 0.$$

Let (λ, μ, ν) be the direction cosines of an arbitrary generator, so that

$$a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu = 0.$$

The plane perpendicular to this generator cuts the cone in two further generators each perpendicular to it; the problem is to find a condition for the two to be perpendicular to each other.

A line through the origin in direction (l, m, n) is perpendicular to (λ, μ, ν) if

$$\lambda l+\mu m+\nu n = 0$$

and is a generator if

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm = 0.$$

These two equations have two sets of solutions (l_1, m_1, n_1) and (l_2, m_2, n_2), giving the required lines. The ratios for m/n are found by eliminating l; so multiply the second equation by λ^2 and substitute $-(\mu m+\nu n)$ for λl:

$$a(\mu m+\nu n)^2-2\lambda(gn+hm)(\mu m+\nu n)+\lambda^2(bm^2+2fmn+cn^2) = 0.$$

Hence

$$(a\mu^2-2h\lambda\mu+b\lambda^2)m^2+(...)mn+(a\nu^2-2g\nu\lambda+c\lambda^2)n^2 = 0,$$

the coefficient of mn being irrelevant.

The product $(m_1 m_2/n_1 n_2)$ of the two roots m_1/n_1 and m_2/n_2 of this quadratic equation satisfies the relation

$$\frac{m_1 m_2}{n_1 n_2} = \frac{a\nu^2-2g\nu\lambda+c\lambda^2}{a\mu^2-2h\lambda\mu+b\lambda^2}.$$

Similarly,
$$\frac{l_1 l_2}{n_1 n_2} = \frac{b\nu^2-2f\mu\nu+c\mu^2}{a\mu^2-2h\lambda\mu+b\lambda^2}.$$

Hence

$$\begin{aligned}\frac{l_1 l_2+m_1 m_2+n_1 n_2}{n_1 n_2} &= \frac{l_1 l_2}{n_1 n_2}+\frac{m_1 m_2}{n_1 n_2}+1 \\ &= \frac{(b\nu^2-2f\mu\nu+c\mu^2)+(a\nu^2-2g\nu\lambda+c\lambda^2)+(a\mu^2-2h\lambda\mu+b\lambda^2)}{a\mu^2-2h\lambda\mu+b\lambda^2}.\end{aligned}$$

But, since (λ,μ,ν) is a generator,

$$-2f\mu\nu-2g\nu\lambda-2h\lambda\mu = a\lambda^2+b\mu^2+c\nu^2,$$

and so, after reduction,

$$\frac{l_1 l_2+m_1 m_2+n_1 n_2}{n_1 n_2} = \frac{(a+b+c)(\lambda^2+\mu^2+\nu^2)}{a\mu^2-2h\lambda\mu+b\lambda^2}.$$

The denominators will, in general, be non-zero, so that the condition $l_1 l_1+m_1 m_2+n_1 n_2 = 0$ is necessary and sufficient for the condition $a+b+c = 0$. That is, *the three generators* (λ,μ,ν), (l_1,m_1,n_1), (l_2,m_2,n_2) *are mutually orthogonal if and only if*

$$a+b+c = 0.$$

Since (λ,μ,ν) is any generator of the cone, the condition $a+b+c = 0$ implies the existence of an infinite number of such triads.

ILLUSTRATION. THE EQUATION OF A CYLINDER. *To find the equation of a right circular cylinder of radius a whose axis is the straight line through the origin in the direction* (l,m,n).

Let $P(x,y,z)$ be an arbitrary point of the cylinder, and draw

PM perpendicular to the axis. Then, by the formula on p. 14, if O is the origin,

$$OM^2 = (lx+my+nz)^2.$$

Also $$OP^2 = x^2+y^2+z^2.$$

But, by the theorem of Pythagoras,

$$OP^2-OM^2 = PM^2 = a^2,$$

so that *the equation of the cylinder is*

$$x^2+y^2+z^2-(lx+my+nz)^2 = a^2.$$

2. Notes on the particular central quadrics (p. 119)

(i) THE ELLIPSOID. When A, B, C are all positive, they may be written in the form

$$A = 1/a^2, \quad B = 1/b^2, \quad C = 1/c^2,$$

so that the equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

The surface is called an ELLIPSOID.

When a, b, c are equal, the surface is a SPHERE; when two of them are equal, say $b = c$, the surface is an ELLIPSOID OF REVOLUTION about the x-axis, being PROLATE if $a > b$ and OBLATE if $a < b$.

The surface may be visualized as a somewhat distorted sphere, or as a distortion of the surface obtained by rotating an ellipse about one or other of its axes.

(ii) THE HYPERBOLOID OF ONE SHEET. When two of A, B, C are positive and one negative, they may be written in, say, the form

$$A = 1/a^2, \quad B = 1/b^2, \quad C = -1/c^2,$$

so that the equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1.$$

The surface is called a HYPERBOLOID OF ONE SHEET.

To visualize it, imagine the hyperbola

$$\frac{x^2}{a^2}-\frac{z^2}{c^2} = 1$$

rotated about the z-axis, forming a single 'sheet', infinite in extent. This corresponds to the particular case

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=1,$$

and the more general surface may be conceived as a distortion.

(iii) THE HYPERBOLOID OF TWO SHEETS. When one of A, B, C is positive and two negative, they may be written in, say, the form $$A = 1/a^2, \quad B = -1/b^2, \quad C = -1/c^2,$$
so that the equation is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

The surface is called a HYPERBOLOID OF TWO SHEETS.

To visualize it, imagine the hyperbola

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$$

rotated about the x-axis, forming two 'sheets', both infinite in extent. This corresponds to the particular case

$$\frac{x^2}{a^2}-\frac{y^2}{c^2}-\frac{z^2}{c^2}=1,$$

and the more general surface may be conceived as a distortion.

3. Joachimstal's equation; tangency

The method used earlier for a sphere (pp. 101–5) can be adapted to quadric surfaces in general and to the central quadrics in particular. The statement of the argument may now be more brief. We use the notation:

$$S \equiv Ax^2+By^2+Cz^2-1,$$
$$S_1 \equiv Ax_1x+By_1y+Cz_1z-1,$$
$$S_{12} \equiv Ax_1x_2+By_1y_2+Cz_1z_2-1,$$
$$S_{11} \equiv Ax_1^2+By_1^2+Cz_1^2-1.$$

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ be two given points. The coordinates of the point dividing the line $\overrightarrow{PQ}$ in the ratio $\lambda/1$ are

$$\left(\frac{x_1+\lambda x_2}{1+\lambda}, \quad \frac{y_1+\lambda y_2}{1+\lambda}, \quad \frac{z_1+\lambda z_2}{1+\lambda}\right),$$

and this point lies on the quadric $S = 0$ if

$$S_{22}\lambda^2+2S_{12}\lambda+S_{11} = 0.$$

This is a quadratic in λ, and so *an arbitrary line meets the surface in two points*. The two points do not have real existence unless $S_{12}^2-S_{11}S_{22} \geqslant 0$. We denote them, when existing, by the letters M, N. (Compare Fig. 36, p. 102.)

(i) TANGENT LINES. A straight line is called a TANGENT to a quadric, which it is said to TOUCH, at a point L if it does not meet it at any point other than L.

(ii) THE TANGENT PLANE. If the point $P(x_1, y_1, z_1)$ lies on the quadric, then

$$S_{11} = 0,$$

and one root of the equation in λ is zero. If the line PQ is a tangent, the second root must be zero too, so that

$$S_{12} = 0.$$

As Q varies, subject to this condition, its coordinates satisfy the equation

$$S_1 = 0$$

found by replacing x_2, y_2, z_2 by the current variables x, y, z. This is the equation of a plane, and so *the tangent lines to the quadric at the point* $P(x_1, y_1, z_1)$ *all lie in the plane*

$$S_1 \equiv Ax_1x+By_1y+Cz_1z-1 = 0.$$

This plane is called the TANGENT PLANE at P to the surface.

(iii) THE TANGENT CONE. If the given point $P(x_1, y_1, z_1)$ does not lie on the quadric, but if the line joining it to a point $Q(x_2, y_2, z_2)$, also not (in general) on the quadric, touches the quadric elsewhere, then the equation in λ has equal roots, so that

$$S_{11}S_{22} = S_{12}^2.$$

As Q varies, subject to this condition, its coordinates satisfy the equation

$$S_{11}S = S_1^2.$$

This, from the nature of its derivation, represents a cone, and it is, indeed, the *quadric* cone (p. 119)

$$(Ax_1^2+By_1^2+Cz_1^2-1)(Ax^2+By^2+Cz^2-1)$$
$$= (Ax_1x+By_1y+Cz_1z-1)^2.$$

The cone is called the TANGENT CONE from P to the surface. It meets the surface

$$S = 0$$

where

$$S_1 = 0,$$

that is, at the points where the quadric is cut by the plane

$$Ax_1 x + By_1 y + Cz_1 z - 1 = 0.$$

(iv) THE POLAR PLANE. Two points $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ are said to be CONJUGATE with respect to the quadric $S = 0$ when the line joining them meets the quadric in two points M, N such that P, Q are separated harmonically by M, N (p. 104).

The condition for this is that the roots of the equation in λ should be equal in magnitude and opposite in sign; that is, that

$$S_{12} \equiv Ax_1 x_2 + By_1 y_2 + Cz_1 z_2 - 1 = 0.$$

When P is given, the locus of a point Q conjugate to it is called the POLAR PLANE of P, given by the equation

$$S_1 \equiv Ax_1 x + By_1 y + Cz_1 z - 1 = 0.$$

If P is on the quadric, then its polar plane is the tangent plane at P. A point P is called the POLE of its polar plane.

4. Tangent planes and tangential equations

(i) To prove that *the condition for the plane*

$$lx + my + nz = p$$

to touch the quadric

$$Ax^2 + By^2 + Cz^2 = 1$$

is

$$\frac{l^2}{A} + \frac{m^2}{B} + \frac{n^2}{C} = p^2.$$

Suppose that the plane touches the quadric, the point of contact being (x_1, y_1, z_1); then the tangent plane is (p. 124)

$$Ax_1 x + By_1 y + Cz_1 z = 1.$$

Comparing the two forms of equation,

$$Ax_1 p = l, \quad By_1 p = m, \quad Cz_1 p = n.$$

But the point (x_1, y_1, z_1) lies on the quadric, so that

$$Ax_1^2 + By_1^2 + Cz_1^2 = 1,$$

or $$A(l/Ap)^2+B(m/Bp)^2+C(n/Cp)^2 = 1,$$

or $$\frac{l^2}{A}+\frac{m^2}{B}+\frac{n^2}{C}=p^2.$$

Note, conversely, that (as is easy to prove) *the planes* $lx+my+nz = p$ *for which a relation*

$$A'l^2+B'm^2+C'n^2 = p^2$$

is satisfied all touch the quadric

$$\frac{x^2}{A'}+\frac{y^2}{B'}+\frac{z^2}{C'} = 1.$$

The relation $$\frac{l^2}{A}+\frac{m^2}{B}+\frac{n^2}{C}=p^2$$

is called the TANGENTIAL EQUATION of the quadric

$$Ax^2+By^2+Cz^2 = 1.$$

(ii) To prove that *the pole of the plane*

$$lx+my+nz = p$$

with respect to the quadric

$$Ax^2+By^2+Cz^2 = 1$$

is $(l/Ap, m/Bp, n/Cp)$.

Suppose that the pole is (x_1, y_1, z_1). Then the polar plane is

$$Ax_1x+By_1y+Cz_1z = 1,$$

and comparison of this with the given equation for the plane gives the formulae

$$Ax_1p = l, \quad By_1p = m, \quad Cz_1p = n,$$

so that $$x_1 = \frac{l}{Ap}, \quad y_1 = \frac{m}{Bp}, \quad z_1 = \frac{n}{Cp}.$$

(iii) To prove that *two tangent planes can be drawn through an arbitrary line.*

If the line is given by the two equations

$$a_1x+b_1y+c_1z+d_1 = 0,$$
$$a_2x+b_2y+c_2z+d_2 = 0,$$

then any plane through it is

$$(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+(d_1+\lambda d_2) = 0.$$

The plane touches the quadric if

$$\frac{(a_1+\lambda a_2)^2}{A}+\frac{(b_1+\lambda b_2)^2}{B}+\frac{(c_1+\lambda c_2)^2}{C} = (d_1+\lambda d_2)^2.$$

This is a quadratic equation in λ, whose two roots determine the two tangent planes through the line.

5. The normals to a central quadric

We begin with a lemma: to prove that *the length of the perpendicular from the origin to the tangent plane at* (x_1, y_1, z_1) *to the quadric* $Ax^2+By^2+Cz^2 = 1$ *is*

$$(A^2x_1^2+B^2y_1^2+C^2z_1^2)^{-\frac{1}{2}}.$$

The tangent plane is

$$Ax_1x+By_1y+Cz_1z-1 = 0,$$

so that (p. 23) the length of the perpendicular from the origin is

$$\frac{1}{\sqrt{\{(Ax_1)^2+(By_1)^2+(Cz_1)^2\}}}$$

or

$$(A^2x_1^2+B^2y_1^2+C^2z_1^2)^{-\frac{1}{2}}.$$

Denote this length by the symbol π_1. The length of the perpendicular from the origin to the tangent at a general point (x, y, z) is denoted correspondingly by π.

DEFINITION. The NORMAL to a surface at a point P is defined to be the straight line through P perpendicular to the tangent plane there.

(For example, the normal to a sphere at a point P is the radius through P.)

For the quadric

$$Ax^2+By^2+Cz^2 = 1,$$

the tangent plane is

$$Ax_1x+By_1y+Cz_1z = 1,$$

with direction ratios (Ax_1, By_1, Cz_1). Hence *the equations of the normal are*

$$\frac{x-x_1}{Ax_1} = \frac{y-y_1}{By_1} = \frac{z-z_1}{Cz_1}.$$

Note that, if π_1 is the length of the perpendicular from the origin on the tangent plane at (x_1, y_1, z_1) then *the direction cosines of the normal are*

$$(Ax_1\pi_1,\ By_1\pi_1,\ Cz_1\pi_1).$$

One or two elementary properties of normals may be added:

To prove that *six normals pass through a given point, and that these six normals all lie on a quadric cone.*

The normal at the point P_1 (x_1, y_1, z_1) is given by the equations

$$\frac{x-x_1}{Ax_1} = \frac{y-y_1}{By_1} = \frac{z-z_1}{Cz_1},$$

and it passes through the given point $Q\,(\alpha, \beta, \gamma)$ provided that

$$\frac{\alpha-x_1}{Ax_1} = \frac{\beta-y_1}{By_1} = \frac{\gamma-z_1}{Cz_1}$$
$$= \lambda,$$

say. Hence

$$x_1 = \frac{\alpha}{1+A\lambda},\quad y_1 = \frac{\beta}{1+B\lambda},\quad z_1 = \frac{\gamma}{1+C\lambda}.$$

Now P_1 lies on the quadric, so that

$$Ax_1^2+By_1^2+Cz_1^2 = 1,$$

or, after multiplying by $(1+A\lambda)^2(1+B\lambda)^2(1+C\lambda)^2$,

$$A\alpha^2(1+B\lambda)^2(1+C\lambda)^2+B\beta^2(1+C\lambda)^2(1+A\lambda)^2+$$
$$+C\gamma^2(1+A\lambda)^2(1+B\lambda)^2$$
$$= (1+A\lambda)^2(1+B\lambda)^2(1+C\lambda)^2.$$

This equation is sextic in λ, and each of its six roots gives one set of values for x_1, y_1, z_1, so that six normals can be drawn through Q.

To find the cone, write the equation of condition

$$\frac{\beta-y_1}{By_1} = \frac{\gamma-z_1}{Cz_1}$$

in the form

$$B(y_1-\beta+\beta)(\gamma-z_1) = C(z_1-\gamma+\gamma)(\beta-y_1),$$

or $$(B-C)(y_1-\beta)(z_1-\gamma) = B\beta(\gamma-z_1)-C\gamma(\beta-y_1).$$

Similarly

$$(C-A)(z_1-\gamma)(x_1-\alpha) = C\gamma(\alpha-x_1)-A\alpha(\gamma-z_1),$$
$$(A-B)(x_1-\alpha)(y_1-\beta) = A\alpha(\beta-y_1)-B\beta(\alpha-x_1).$$

Multiply by $A\alpha$, $B\beta$, $C\gamma$ and add:

$$A(B-C)\alpha(y_1-\beta)(z_1-\gamma)+ \\ +B(C-A)\beta(z_1-\gamma)(x_1-\alpha)+ \\ +C(A-B)\gamma(x_1-\alpha)(y_1-\beta) = 0.$$

Hence the point (x_1, y_1, z_1) satisfies the equation

$$A(B-C)\alpha(y-\beta)(z-\gamma)+ \\ +B(C-A)\beta(z-\gamma)(x-\alpha)+ \\ +C(A-B)\gamma(x-\alpha)(y-\beta) = 0.$$

This equation represents a quadric cone with vertex at the point $Q(\alpha, \beta, \gamma)$. In fact, the point $(\alpha+lr, \beta+mr, \gamma+nr)$ lies on the surface for *all* values of r if (l, m, n) satisfies the equation

$$A(B-C)\alpha mn+B(C-A)\beta nl+C(A-B)\gamma lm = 0;$$

that is, the surface consists of a system of straight lines passing through Q. Hence *the equation of the cone is*

$$\sum A(B-C)\alpha(y-\beta)(z-\gamma) = 0.$$

6. The r-equation

Let $P(x_1, y_1, z_1)$ be a given point and (l, m, n) the direction cosines of a line through P. If $Q(x, y, z)$ is the point on this line such that

$$\overrightarrow{PQ} = r,$$

then (p. 11)

$$x = x_1+lr, \quad y = y_1+mr, \quad z = z_1+nr.$$

To find the two values of r for which Q lies on the quadric

$$S \equiv Ax^2+By^2+Cz^2-1 = 0.$$

The point Q lies on the quadric if

$$A(x_1+lr)^2+B(y_1+mr)^2+C(z_1+nr)^2-1 = 0,$$

or

$$(Al^2+Bm^2+Cn^2)r^2+2(Ax_1l+By_1m+Cz_1n)r+ \\ +(Ax_1^2+By_1^2+Cz_1^2-1) = 0,$$

or

$$(Al^2+Bm^2+Cn^2)r^2+2(Ax_1l+By_1m+Cz_1n)r+S_{11} = 0.$$

This may be called the r-EQUATION of the point (x_1, y_1, z_1) and the direction (l, m, n) for the quadric S.

7. The plane section with given centre

To prove that, *given a point* $P(x_1, y_1, z_1)$, *there is* (*in general*) *a unique plane cutting the quadric in a conic with its centre at* P, *the equation of the plane being*

$$S_1 = S_{11}$$

or $$Ax_1(x-x_1)+By_1(y-y_1)+Cz_1(z-z_1) = 0.$$

Given $P(x_1, y_1, z_1)$, draw an arbitrary line through it in the direction (l, m, n). This line meets the quadric in two points whose distances from P are the roots of the equation (§ 6)

$$(Al^2+Bm^2+Cn^2)r^2+2(Ax_1 l+By_1 m+Cz_1 n)r+S_{11} = 0.$$

Choose l, m, n so that P is the middle point of the chord joining these two points; the values of r in the quadratic equation are then equal and opposite, so that

$$Ax_1 l+By_1 m+Cz_1 n = 0.$$

This is the condition to which l, m, n are subjected.

If, now, $U(x, y, z)$ is an arbitrary point of this chord, then the direction ratios of UP are $(x-x_1, y-y_1, z-z_1)$, so that

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}.$$

Substituting in the equation of condition,

$$Ax_1(x-x_1)+By_1(y-y_1)+Cz_1(z-z_1) = 0.$$

This is the equation of a plane containing all chords having P as middle point; that is, P is the centre of the section of the given quadric by this plane.

THE CENTRE OF THE QUADRIC. The equation of condition is satisfied automatically for all values of l, m, n in the particular case when

$$x_1 = 0, \quad y_1 = 0, \quad z_1 = 0,$$

so that P is the origin of coordinates. Thus *every chord through the point* $O(0, 0, 0)$ *is bisected there.*

The point, if any, at which all chords of a quadric are bisected is called the CENTRE of the quadric. Any chord through the centre is called a DIAMETER. Any plane through the centre is called a DIAMETRAL PLANE.

8. Conjugate diameters

(i) To prove that *the middle points of chords of the quadric*

$$S \equiv Ax^2+By^2+Cz^2-1 = 0$$

in the direction (l, m, n) *lie in the plane*

$$Alx+Bmy+Cnz = 0.$$

Let (x_1, y_1, z_1) be the middle point of a chord in the direction (l, m, n). Then (p. 130)

$$Alx_1+Bmy_1+Cnz_1 = 0.$$

The locus of the point is thus the plane

$$Alx+Bmy+Cnz = 0.$$

This plane is called the diametral plane CONJUGATE to the direction (l, m, n).

(ii) To prove that *the centres of the sections of the quadric* S *by planes parallel to the given diametral plane*

$$ux+vy+wz = 0$$

all lie on the straight line

$$\frac{Ax}{u} = \frac{By}{v} = \frac{Cz}{w}.$$

If (x_1, y_1, z_1) is a centre, the corresponding plane is (p. 130)

$$Ax_1(x-x_1)+By_1(y-y_1)+Cz_1(z-z_1) = 0,$$

and so, since this is parallel to the given plane,

$$\frac{Ax_1}{u} = \frac{By_1}{v} = \frac{Cz_1}{w}.$$

The locus of the point is thus the straight line

$$\frac{Ax}{u} = \frac{By}{v} = \frac{Cz}{w}.$$

This line is called the diameter CONJUGATE to the given plane.

NOTE. Given a direction (l, m, n), the conjugate diametral plane is, by (i),
$$Alx+Bmy+Cnz = 0,$$

and the diameter conjugate to this plane is, by (ii),

$$\frac{Ax}{Al} = \frac{By}{Bm} = \frac{Cz}{Cn},$$

or

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

This is the diameter in the given direction.

DEFINITION. A diameter is called a PRINCIPAL DIAMETER when it is perpendicular to the plane conjugate to it.

If such a diameter has direction cosines (l, m, n), the conjugate plane is

$$Alx + Bmy + Cnz = 0,$$

and the direction cosines of the normals to it are

$$(Al, Bm, Cn).$$

For a principal diameter, this direction is the same as (l, m, n), so that there exists a number k, not zero, such that

$$Al = kl, \qquad Bm = km, \qquad Cn = kn.$$

To assess the significance of these equations, *suppose first that A, B, C are unequal.*

Then two of l, m, n must be zero; otherwise, if, say, m, n were not zero, we should have the contradiction

$$B = k = C.$$

Hence *when A, B, C are unequal, there are three principal directions, namely*

$$(1, 0, 0), \; (0, 1, 0), \; (0, 0, 1)$$

and the principal diameters are the axes of coordinates.

Suppose next that $B = C$, but that A is different.

(i) If neither of m, n is zero, then

$$B = C = k,$$

so that, since $A \neq k$, $\qquad l = 0.$

(ii) If l is not zero, then

$$A = k,$$

so that, since B, $C \neq k$,

$$m = n = 0.$$

In the first case, the principal diameter may be *any* line in the plane $x = 0$; in the second case, the principal diameter is the x-axis. The surface is, in fact, formed by the revolution of an ellipse about the x-axis; the axis of revolution is one principal diameter, and there is, further, an infinite set of principal diameters all lying in the diametral plane perpendicular to the axis.

Suppose next that $A = B = C$.

Then l, m, n may have arbitrary values. The surface is a sphere, and all its diameters are principal diameters.

We now give a few PROPERTIES OF CONJUGATE DIAMETERS:

Let $P_1(x_1, y_1, z_1)$ be a point on the quadric. The line OP_1 has direction ratios (x_1, y_1, z_1) and so the direction of the conjugate diametral plane is

$$L_1 \equiv Ax_1 x + By_1 y + Cz_1 z = 0.$$

Thus *the diametral plane of OP_1 is parallel to the tangent plane at P_1.*

If $P_2(x_2, y_2, z_2)$ is another point of the quadric, the corresponding conjugate plane is

$$L_2 \equiv Ax_2 x + By_2 y + Cz_2 z = 0.$$

Suppose now that P_2 is chosen to lie in the plane conjugate to OP_1. Then

$$L_{12} \equiv Ax_1 x_2 + By_1 y_2 + Cz_1 z_2 = 0,$$

so that P_1 lies in the plane conjugate to OP_2.

Take the point $P_3(x_3, y_3, z_3)$ to be one of the intersections of the quadric with the line of intersection of the planes $L_1 = 0$, $L_2 = 0$. There are then the three relations

$$Ax_2 x_3 + By_2 y_3 + Cz_2 z_3 = 0,$$

$$Ax_3 x_1 + By_3 y_1 + Cz_3 z_1 = 0,$$

$$Ax_1 x_2 + By_1 y_2 + Cz_1 z_2 = 0.$$

These relations are completely symmetrical. The lines OP_1, OP_2, OP_3 are so related that each lies in the diametral planes conjugate to the other two, and each is the diameter conjugate to the plane containing the other two. They are called

MUTUALLY CONJUGATE DIAMETERS. The fact that P_1, P_2, P_3 lie on the quadric is expressed by the three equations

$$Ax_1^2+By_1^2+Cz_1^2 = 1,$$
$$Ax_2^2+By_2^2+Cz_2^2 = 1,$$
$$Ax_3^2+By_3^2+Cz_3^2 = 1.$$

Consider, in particular, the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

The six relations are

$$\frac{x_2 x_3}{a^2}+\frac{y_2 y_3}{b^2}+\frac{z_2 z_3}{c^2} = 0,$$
$$\frac{x_3 x_1}{a^2}+\frac{y_3 y_1}{b^2}+\frac{z_3 z_1}{c^2} = 0,$$
$$\frac{x_1 x_2}{a^2}+\frac{y_1 y_2}{b^2}+\frac{z_1 z_2}{c^2} = 0,$$

where

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2} = 1,$$
$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2} = 1,$$
$$\frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2} = 1.$$

The three triplets

$$\left(\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}\right), \quad \left(\frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}\right), \quad \left(\frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}\right)$$

are thus the direction cosines of three mutually perpendicular lines. They therefore satisfy the six further relations (p. 43)

$$y_1 z_1+y_2 z_2+y_3 z_3 = 0,$$
$$z_1 x_1+z_2 x_2+z_3 x_3 = 0,$$
$$x_1 y_1+x_2 y_2+x_3 y_3 = 0,$$

where

$$x_1^2+x_2^2+x_3^2 = a^2,$$
$$y_1^2+y_2^2+y_3^2 = b^2,$$
$$z_1^2+z_2^2+z_3^2 = c^2.$$

For example,

$$\begin{aligned} OP_1^2+OP_2^2+OP_3^2 &= (x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)+(x_3^2+y_3^2+z_3^2) \\ &= (x_1^2+x_2^2+x_3^2)+(y_1^2+y_2^2+y_3^2)+(z_1^2+z_2^2+z_3^2) \\ &= a^2+b^2+c^2. \end{aligned}$$

Hence *the value of* $OP_1^2+OP_2^2+OP_3^2$ *is constant for all sets of mutually conjugate diameters.*

9. Generators; first method

A surface containing an infinite system of straight lines is said to be GENERATED by them; the lines themselves are called GENERATORS. Elementary examples are the cone and the cylinder. It is also easy to verify that the surface

$$yz = x$$

contains the straight line

$$y = \lambda x, \qquad \lambda z = 1$$

for all values of λ, so that, as λ varies, the line generates the surface.

We proceed to examine the central quadric

$$Ax^2+By^2+Cz^2 = 1.$$

Let (x_1, y_1, z_1) be an arbitrary point of the quadric, so that

$$Ax_1^2+By_1^2+Cz_1^2 = 1,$$

and let

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$$

be an arbitrary line through it. The line meets the quadric where

$$A(x_1+lr)^2+B(y_1+mr)^2+C(z_1+nr)^2 = 1,$$

so that $r = 0$ or else

$$2(Alx_1+Bmy_1+Cnz_1)+(Al^2+Bm^2+Cn^2)r = 0.$$

If the line is a GENERATOR, this equation must be satisfied for all values of r, so that

$$Alx_1+Bmy_1+Cnz_1 = 0,$$

$$Al^2+Bm^2+Cn^2 = 0.$$

These two equations, solved simultaneously for the ratios $l:m:n$, have two solutions, corresponding to *two* generators that can be drawn through (x_1, y_1, z_1); but we have no assurance yet that the quadratic on which the solution will depend has real roots. This we now investigate.

The ratio m/n is found by eliminating l from the two equations. Multiplying the second by Ax_1^2, we have

$$(-Bmy_1-Cnz_1)^2+ABx_1^2m^2+ACx_1^2n^2 = 0,$$

or $$B(Ax_1^2+By_1^2)m^2+2BCy_1z_1mn+C(Ax_1^2+Cz_1^2)n^2 = 0.$$

The two values so obtained for m/n are real and distinct provided that

$$B^2C^2y_1^2z_1^2-BC(Ax_1^2+By_1^2)(Ax_1^2+Cz_1^2) > 0,$$

or $$ABCx_1^2(Ax_1^2+By_1^2+Cz_1^2) < 0,$$

or, since $Ax_1^2+By_1^2+Cz_1^2 = 1$,

$$ABC < 0.$$

Hence either one or three of A, B, C must be negative. We must, however, exclude the possibility of three negative, since $Ax^2+By^2+Cz^2$ could never then be $+1$; hence one of A, B, C is negative and two positive. That is, *the only ruled central quadric* (excluding cone or cylinder) *is the hyperboloid of one sheet.*

10. Generators; second method

We take advantage of the result just proved, that the only central quadric with real generators is the hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1.$$

The form $$1-\frac{x^2}{a^2} = \frac{y^2}{b^2}-\frac{z^2}{c^2},$$

or $$\left(1+\frac{x}{a}\right)\left(1-\frac{x}{a}\right) = \left(\frac{y}{b}+\frac{z}{c}\right)\left(\frac{y}{b}-\frac{z}{c}\right),$$

then shows that the two systems of lines

$$\begin{cases} 1+\dfrac{x}{a} = \lambda\left(\dfrac{y}{b}+\dfrac{z}{c}\right), \\ \lambda\left(1-\dfrac{x}{a}\right) = \dfrac{y}{b}-\dfrac{z}{c} \end{cases}$$

and
$$\begin{cases} 1+\dfrac{x}{a}=\mu\left(\dfrac{y}{b}-\dfrac{z}{c}\right), \\ \mu\left(1-\dfrac{x}{a}\right)=\dfrac{y}{b}+\dfrac{z}{c} \end{cases}$$

are generators of the quadrics for all values of the parameters λ, μ. They are called generators of the λ-SYSTEM and the μ-SYSTEM respectively.

It is easy to verify, by attempts at actual solution, that *two generators of the same system have no point of intersection.*

On the other hand, we prove that *every λ-generator has one point in common with every μ-generator*:

At a common point we have the equations

$$\frac{1+(x/a)}{\lambda\mu}=\frac{1-(x/a)}{1}=\frac{(y/b)+(z/c)}{\mu}=\frac{(y/b)-(z/c)}{\lambda},$$

so that
$$\frac{x/a}{\lambda\mu-1}=\frac{y/b}{\mu+\lambda}=\frac{z/c}{\mu-\lambda}=\frac{1}{\lambda\mu+1}.$$

These relations therefore give the coordinates of the common point, incidentally expressing the points of the hyperboloid of one sheet in the PARAMETRIC FORM

$$x=\frac{(\lambda\mu-1)a}{\lambda\mu+1},\quad y=\frac{(\mu+\lambda)b}{\lambda\mu+1},\quad z=\frac{(\mu-\lambda)c}{\lambda\mu+1}.$$

This point is sometimes called THE POINT (λ,μ).

11. Some elementary properties of generators

The basic properties with which we are concerned arise as immediate consequences of results which the reader may regard as obvious from definition but which we shall also prove by independent algebra:

(i) *The tangent plane to a quadric at a point P contains the two generators through P.*

The surface
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

may be expressed in parametric form

$$\frac{x}{a}=\frac{\lambda\mu-1}{\lambda\mu+1},\quad \frac{y}{b}=\frac{\mu+\lambda}{\lambda\mu+1},\quad \frac{z}{c}=\frac{\mu-\lambda}{\lambda\mu+1}.$$

The two generating lines through the point (λ, μ) are given by λ = constant and μ = constant respectively, where

(*a*) λ constant, is the line

$$1+\frac{x}{a} = \lambda\left(\frac{y}{b}+\frac{z}{c}\right),$$

$$\lambda\left(1-\frac{x}{a}\right) = \frac{y}{b}-\frac{z}{c};$$

(*b*) μ constant, is the line

$$1+\frac{x}{a} = \mu\left(\frac{y}{b}-\frac{z}{c}\right),$$

$$\mu\left(1-\frac{x}{a}\right) = \frac{y}{b}+\frac{z}{c}.$$

If, then, P is the point (λ_1, μ_1), the two generating lines through it are defined by the relations

$$\lambda = \lambda_1, \qquad \mu = \mu_1$$

respectively.

Now the tangent plane at P is

$$\frac{x}{a}\frac{\lambda_1\mu_1-1}{\lambda_1\mu_1+1}+\frac{y}{b}\frac{\mu_1+\lambda_1}{\lambda_1\mu_1+1}-\frac{z}{c}\frac{\mu_1-\lambda_1}{\lambda_1\mu_1+1} = 1,$$

and it contains those points (λ, μ) of the quadric whose parameters satisfy the equation

$$\frac{(\lambda\mu-1)(\lambda_1\mu_1-1)}{(\lambda\mu+1)(\lambda_1\mu_1+1)}+\frac{(\mu+\lambda)(\mu_1+\lambda_1)}{(\lambda\mu+1)(\lambda_1\mu_1+1)}-\frac{(\mu-\lambda)(\mu_1-\lambda_1)}{(\lambda\mu+1)(\lambda_1\mu_1+1)} = 1.$$

Hence, multiplying and rearranging,

$$(\mu+\lambda)(\mu_1+\lambda_1)-(\mu-\lambda)(\mu_1-\lambda_1)$$
$$= (\lambda\mu+1)(\lambda_1\mu_1+1)-(\lambda\mu-1)(\lambda_1\mu_1-1),$$

or
$$2\lambda\mu_1+2\lambda_1\mu = 2\lambda\mu+2\lambda_1\mu_1,$$

or
$$\lambda\mu-\lambda\mu_1-\lambda_1\mu+\lambda_1\mu_1 = 0,$$

or
$$(\lambda-\lambda_1)(\mu-\mu_1) = 0.$$

Hence *either* λ has the constant value λ_1 *or* μ has the constant value μ_1. That is, the section of the quadric by the tangent plane at (λ_1, μ_1) consists of the two generators through that point.

(ii) *Every plane through a generator is a tangent plane to the quadric.*

A plane through, say, the λ-generator

$$1+\frac{x}{a} = \lambda\left(\frac{y}{b}+\frac{z}{c}\right),$$

$$\lambda\left(1-\frac{x}{a}\right) = \frac{y}{b}-\frac{z}{c}$$

is $$\left\{\left(1+\frac{x}{a}\right)-\lambda\left(\frac{y}{b}+\frac{z}{c}\right)\right\}+k\left\{\lambda\left(1-\frac{x}{a}\right)-\left(\frac{y}{b}-\frac{z}{c}\right)\right\} = 0,$$

or, on rearranging,

$$\left\{\left(1+\frac{x}{a}\right)-k\left(\frac{y}{b}-\frac{z}{c}\right)\right\}+\lambda\left\{k\left(1-\frac{x}{a}\right)-\left(\frac{y}{b}+\frac{z}{c}\right)\right\}.$$

The plane therefore contains the generator

$$1+\frac{x}{a} = k\left(\frac{y}{b}-\frac{z}{c}\right),$$

$$k\left(1-\frac{x}{a}\right) = \frac{y}{b}+\frac{z}{c}$$

of the μ-system. Since it contains a λ-generator and a μ-generator, it is the tangent plane at the point where these generators meet.

(iii) *Two tangent planes can be drawn through an arbitrary line.*

The arbitrary line meets the quadric in two points A, B; through A pass two generators λ_1, μ_1, and through B pass two generators λ_2, μ_2. Also λ_1, μ_2, being generators of opposite systems, meet in a point P, and λ_2, μ_1 meet in a point Q.

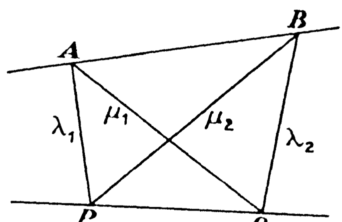

Fig. 39

Now the plane PAB, containing λ_1, μ_2, is the tangent plane at P; and the plane QAB, containing λ_2, μ_1, is the tangent plane at Q. There therefore exist certainly these two tangent planes through the line AB.

Further, there cannot be any other tangent planes through AB; for the tangent plane at a point R would meet the quadric in two generators which, meeting AB, could do so only at the points A or B common to that line and the quadric. Hence the point R must be P or Q.

12. Polar lines

Given two points $A\,(x_1, y_1, z_1)$, $B\,(x_2, y_2, z_2)$, the coordinates of any point P of the line AB may be expressed in the form

$$\left(\frac{x_1+\lambda x_2}{1+\lambda}, \quad \frac{y_1+\lambda y_2}{1+\lambda}, \quad \frac{z_1+\lambda z_2}{1+\lambda}\right).$$

The polar plane of this point with respect to the quadric

$$S \equiv Ax^2+By^2+Cz^2-1 = 0$$

is, after multiplication by $1+\lambda$,

$$A(x_1+\lambda x_2)x+B(y_1+\lambda y_2)y+C(z_1+\lambda z_2)z-(1+\lambda) = 0,$$

or $$(Ax_1+By_1+Cz_1-1)+\lambda(Ax_2+By_2+Cz_2-1) = 0,$$

or $$S_1+\lambda S_2 = 0.$$

For all values of λ this plane passes through the line given by the equations

$$S_1 = 0, \qquad S_2 = 0.$$

Hence *the polar planes of all points of a line l pass through a line l'.*

The line l' is called the POLAR LINE of l with respect to the quadric S.

The line l' is defined by *any two* points of l. In particular, since the polar plane of a point on a quadric is the tangent plane there, the two lines AB, PQ (Fig. 39) defined in § 11 (iii) are so related that PQ is the polar line of AB.

Moreover, it is easy to prove that *the polar line of l' is the line l itself.* Referring again to Fig. 39, we observe that the polar plane of P is PAB and that the polar plane of Q is QAB. These planes meet in the line AB, which is thus the polar line of PQ.

Finally, a basic theorem should be recorded:

If U is an arbitrary point of a line l and V an arbitrary point *of its polar line l', and if the line UV meets the quadric in M, N, then the points M, N separate the points U, V harmonically.*

The polar plane of U passes through l' and so contains V. Hence, by definition, U, V are conjugate with respect to the quadric. They therefore (p. 125) separate M, N harmonically.

MISCELLANEOUS EXAMPLES

1. The feet of the perpendiculars from a variable point P to a skew pair of lines are L, M. Prove that (i) if PL^2+PM^2 is constant, the locus of P is an ellipsoid, (ii) if LM is constant, the locus of P is a circular cylinder.

2. P is a variable point on a fixed straight line and L, M are the feet of the perpendiculars from P to two fixed perpendicular planes. Prove that the locus of a point, which divides the segment LM in a given ratio, is a straight line.

3. A point P moves so that the line joining the feet of the perpendiculars drawn from P to two skew lines l_1, l_2 subtends a right angle at the mid-point of the common perpendicular transversal t of l_1 and l_2. Show that the locus of P is a hyperbolic cylinder whose axis is t.

4. Find the coordinates of the foot of the perpendicular from the point (ξ, η, ζ) to the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}.$$

Find the equation of the locus of the point P such that the perpendicular distances of P from the lines

$$y = x\tan\theta, \quad z = c$$

and

$$y = -x\tan\theta, \quad z = -c$$

are in the ratio $\lambda:1$.

5. A point P moves in space so that its distances from the two lines

$$x-a = 0, \quad y = 0$$

and

$$x = 0, \quad z-a = 0$$

are equal. Show that the locus of P is the surface whose equation is

$$y^2-z^2-2ax+2az = 0.$$

6. Show that the locus of mid-points of chords of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

parallel to the diameter

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

is a plane ϖ, and find its equation.

Show that the locus of the centres of sections of the ellipsoid by planes through the line
$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$
is an ellipse lying in ϖ.

7. Prove that the line
$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\sin\theta} = \frac{z}{c}$$
is a generator of the surface
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1.$$

Prove that the normals to this surface at all points of the generators through $(a\cos\theta, b\sin\theta, 0)$ meet the plane $z = 0$ at points which lie in in the plane
$$\frac{ax\cos\theta}{a^2+c^2}+\frac{by\sin\theta}{b^2+c^2} = 1.$$

8. Prove that one generator of each system on the hyperboloid
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1$$
passes through any point of the surface, and show that the locus of intersections of perpendicular generators is the intersection of the hyperboloid with a sphere.

9. A regular octahedron has its vertices at the eight points $(\pm k, 0, 0)$, $(0, \pm k, 0)$, $(0, 0, \pm k)$. Prove that each of the planes
$$x+y+z = \pm k$$
contains three edges of the octahedron, and that the other six edges lie on the quadric
$$x^2+y^2+z^2-2yz-2zx-2xy = k^2.$$

Prove that, if $\alpha+\beta+\gamma = 0$, the plane
$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma} = 0$$
meets the quadric in a pair of parallel straight lines.

10. Find the equation of the surface generated by lines l which meet the three given lines
$$x = y+z+a = 0,$$
$$y = z+x+b = 0,$$
$$z = x+y+c = 0.$$

Show that equations of the lines l are
$$ax+by+cz+\lambda = 0,$$
$$a(b+c)x+b(c+a)y+c(a+b)z+abc = \lambda(x+y+z)$$
for different values of λ.

11. Find the equations of the two generators of the hyperboloid of revolution

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2} = 1$$

which pass through the point $(a\cos\theta, a\sin\theta, 0)$.

Show that the angle between these generators is independent of θ, and determine the relation between c and a if this angle is a right angle.

12. Prove that the coordinates of a point of the quadric

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

may be expressed in terms of two variable parameters λ, μ in the form

$$\frac{x}{a} = \frac{1+\lambda\mu}{\lambda+\mu}, \qquad \frac{y}{b} = \frac{1-\lambda\mu}{\lambda+\mu}, \qquad \frac{z}{c} = \frac{\lambda-\mu}{\lambda+\mu}.$$

(i) Prove that the curve on the quadric for which

$$A\lambda\mu+B\lambda+C\mu+D = 0$$

is a conic, which is a parabola if

$$(B-C)^2+4AD = 0.$$

(ii) Find the parameters of the points of the quadric

$$x^2-\frac{y^2}{4}+\frac{z^2}{9} = 1$$

which lie on the line

$$\frac{x+3}{4} = \frac{y+8}{14} = \frac{z-12}{-21}.$$

Hence, or otherwise, find any points of the quadric whose tangent planes contain this line.

13. Find a common tangent plane of the three ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1,$$

$$\frac{x^2}{b^2}+\frac{y^2}{c^2}+\frac{z^2}{a^2} = 1,$$

$$\frac{x^2}{c^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2} = 1.$$

14. Prove that the common tangent planes of the three ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1,$$

$$\frac{x^2}{b^2}+\frac{y^2}{c^2}+\frac{z^2}{a^2} = 1,$$

$$\frac{x^2}{c^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2} = 1$$

touch a sphere of radius

$$\sqrt{\{\tfrac{1}{3}(a^2+b^2+c^2)\}}$$

and that the points of contact of the planes lie on a sphere of radius

$$\sqrt{\left(\frac{a^4+b^4+c^4}{a^2+b^2+c^2}\right)}.$$

15. Find the equations of the normal to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

at the point (x_1, y_1, z_1).

The normal at P cuts the plane $z = 0$ at A, and the point Q is taken such that AQ is parallel to the z-axis and $AQ = AP$. Prove that Q lies on the surface

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2} = 1.$$

16. Show that the locus of the centres of the conics in which the quadric

$$ax^2+by^2+cz^2+d = 0$$

is cut by a variable plane through the line

$$\frac{x-f}{l} = \frac{y-g}{m} = \frac{z-h}{n}$$

is the conic

$$ax(x-f)+by(y-g)+cz(z-h) = 0,$$

$$alx+bmy+cnz = 0.$$

Interpret this result geometrically when $f = g = h = 0$.

17. Find the coordinates of the centre of the conic in which the plane

$$lx+my+nz = p$$

cuts the quadric

$$ax^2+by^2+cz^2+d = 0.$$

If this conic is a parabola, prove that

$$bcl^2+cam^2+abn^2 = 0$$

and find the direction of its axis.

18. Prove that the locus of the foot of the perpendicular from any point of the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

to the polar plane of the point with respect to the quadric

$$ax^2+by^2+cz^2+d = 0$$

is a hyperbola having the line

$$\frac{x}{al} = \frac{y}{bm} = \frac{z}{cn}$$

as one asymptote.

By finding the other asymptote, or otherwise, verify that the hyperbola is rectangular.

19. Prove that, if

$$\frac{ll'}{a^2(b^2-c^2)} = \frac{mm'}{b^2(c^2-a^2)} = \frac{nn'}{c^2(a^2-b^2)},$$

the lines $$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}, \qquad \frac{x}{l'} = \frac{y}{m'} = \frac{z}{n'}$$

are principal axes of a section of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

20. Prove that the direction cosines of the outward normal to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

at the point $P(\xi,\eta,\zeta)$ are

$$\left(\frac{p\xi}{a^2}, \frac{p\eta}{b^2}, \frac{p\zeta}{c^2}\right),$$

where p is the length of the perpendicular from the centre to the tangent plane at P.

Prove also that, if q is the length of the normal chord at p,

$$\frac{\xi^2}{a^4}+\frac{\eta^2}{b^4}+\frac{\zeta^2}{c^4} = \frac{1}{p^2},$$

$$\frac{\xi^2}{a^6}+\frac{\eta^2}{b^6}+\frac{\zeta^2}{c^6} = \frac{2}{p^3q}.$$

21. Prove that, in general, six normals to the quadric

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = 1$$

pass through a given point.

If this point lies on the line

$$\frac{a(b-c)x}{l} = \frac{b(c-a)y}{m} = \frac{c(a-b)z}{n},$$

prove that the feet of the normals are on the curve of intersection of the quadric and the cone

$$lyz+mzx+nxy = 0.$$

22. Find the equations of the normal to the quadric

$$ax^2+by^2+cz^2 = 1$$

at the point $P(\alpha,\beta,\gamma)$ on the quadric.

Assuming a, b, c all different and non-zero and P not to lie on a coordinate plane, prove that, if the normal at P cuts the coordinate planes at L, M, N, then $LM:MN$ is independent of the position of P on the quadric.

23. Prove that the normal to the quadric

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = 1$$

at the point (x_1, y_1, z_1) meets the plane $z = 0$ in the point

$$\left\{\frac{(a-c)x_1}{a}, \frac{(b-c)y_1}{b}, 0\right\}.$$

Prove that, for all values of λ, the normals to

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda} = 1$$

that pass through a fixed point (α, β, γ) meet the plane $z = 0$ in points of the conic

$$(b-c)\beta x+(c-a)\alpha y+(a-b)xy = 0.$$

24. Show that the normals to an ellipsoid at all points of a section parallel to one of the axes intersect the principal planes in a straight line and two ellipses.

25. Prove that two normals to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

lie in the plane $$lx+my+nz = 0$$

and that the line joining their feet has direction ratios

$$a^2(b^2-c^2)mn, \quad b^2(c^2-a^2)nl, \quad c^2(a^2-b^2)lm.$$

26. Show that the tangent planes to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

which are perpendicular to the plane

$$lx+my+nz = 0$$

touch the ellipsoid at points in the plane

$$\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2} = 0.$$

Show that in general two normals to the ellipsoid lie in a given plane. Find the coordinates of the two points on the ellipsoid the normals at which lie in the plane

$$by-cz = \tfrac{1}{2}(b^2-c^2).$$

27. Prove that, if ω is the angle between the central radius to the point $P(\alpha, \beta, \gamma)$ of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

and the normal at P, then

$$\tan^2\omega = \beta^2\gamma^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)^2+\gamma^2\alpha^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)^2+\alpha^2\beta^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)^2.$$

28. Normals are drawn to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

at the points in which it is met by the plane

$$lx+my+nz = p.$$

Show that the points in which they meet the plane $x = 0$ lie on a conic.

29. Find necessary and sufficient conditions for the points

$$P_1(x_1, y_1, z_1), \quad P_2(x_2, y_2, z_2), \quad P_3(x_3, y_3, z_3)$$

to be the ends of mutually conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

If the three diameters vary so that OP_1, OP_2 lie respectively in the fixed planes

$$\frac{\alpha_1 x}{a^2}+\frac{\beta_1 y}{b^2}+\frac{\gamma_1 z}{c^2} = 0,$$

$$\frac{\alpha_2 x}{a^2}+\frac{\beta_2 y}{b^2}+\frac{\gamma_2 z}{c^2} = 0,$$

show that the locus of OP_3 is the cone

$$a^2(\beta_1 z-\gamma_1 y)(\beta_2 z-\gamma_2 y)+b^2(\gamma_1 x-\alpha_1 z)(\gamma_2 x-\alpha_2 z)+$$
$$+c^2(\alpha_1 y-\beta_1 x)(\alpha_2 y-\beta_2 x) = 0.$$

30. The extremities of conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$. Show that the area of the projection of the triangle OBC on any coordinate plane is proportional to the projection of OA on the corresponding axis.

Show that the pole P of the plane ABC is

$$(x_1+x_2+x_3,\ y_1+y_2+y_3,\ z_1+z_2+z_3)$$

and that the equations of the polar line of AP are

$$\frac{x-x_1}{x_2-x_3} = \frac{y-y_1}{y_2-y_3} = \frac{z-z_1}{z_2-z_3}.$$

31. Show that any set of three equal conjugate diameters of the spheroid

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2} = 1$$

lie on a right circular cone, and that the cosine of the angle between any two is $(a^2-b^2)/(a^2+2b^2)$.

32. The equation of a central quadric S is

$$ax^2+by^2+cz^2 = 1,$$

where a, b, c are real and $a > b > c$. Show that there are two systems of real circles on S lying in planes parallel to

$$x\sqrt{(a-b)}\pm z\sqrt{(b-c)} = 0.$$

The centres of the two circles through the point $P(\xi, \eta, \zeta)$ of S are Q and R. Show that the coordinates of the middle point of QR are

$$\left\{\frac{c(a-b)\xi}{b(a-c)},\ 0,\ \frac{a(b-c)\zeta}{b(a-c)}\right\}.$$

33. Prove that the sum of the squares of three conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \quad (a > b > c)$$

is constant, and that the volume of the parallelepiped formed by the tangent planes at their extremities is also constant.

Hence, or otherwise, prove that the radius of a circular section at a distance p from the centre is

$$b\sqrt{\left(1-\frac{p^2b^2}{a^2c^2}\right)}.$$

34. Show that the circular sections of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \quad (a > b > c)$$

passing through the point $(a, 0, 0)$ are both of radius r, where

$$\frac{r^2}{b^2} = \frac{b^2-c^2}{a^2-c^2}.$$

35. Normals are drawn at all points of the circular sections of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \quad (a > b > c).$$

Prove that they meet the plane $x = 0$ in the ellipse

$$\frac{b^2y^2}{a^2-b^2}+\frac{c^2z^2}{a^2-c^2} = a^2-b^2.$$

36. Find the equation of the circular cylinder of radius a with its axis along the line $x/l = y/m = z/n$.

Prove that the common points of two equal circular cylinders, whose axes intersect at an angle 2α, lie on two ellipses of eccentricities $\cos\alpha$ and $\sin\alpha$.

37. Find the equation of the surface generated by the lines joining the pairs of points

$$\{a\cos(\theta+\alpha),\, b\sin(\theta+\alpha),\, c\}, \qquad \{a\cos(\theta-\alpha),\, b\sin(\theta-\alpha),\, -c\},$$

where θ is a parameter and α a constant.

Prove that the planes $z = \pm h$ cut the surface in two equal ellipses and that a variable generator of either system meets these ellipses in points whose eccentric angles differ by $2\tan^{-1}\{(h\tan\alpha)/c\}$.

38. Prove that the point

$$x = \alpha(\beta+1), \quad y = \beta(\alpha+1), \quad z+\alpha\beta = 0,$$

where α, β are parameters, lies on a quadric and that passing through every point of the quadric there are two generators, of which one is parallel to the plane $x+z = 0$ and the other is parallel to the plane $y+z = 0$.

Prove also that when α is constant the locus of the point is a generator.

39. Prove that the equation of the surface generated by the family of lines which are parallel to the plane

$$2x+y = 0$$

and which intersect the two lines

$$x = y+z = 1,$$

$$y = z+x = 2$$

is

$$2x^2+y^2+yz+2zx+3xy-12x-6y-4z+12 = 0.$$

Show that the surface contains another family of straight lines, parallel to the plane

$$x+y+z = 0.$$

40. The section of a fixed sphere by a variable plane p is projected from a fixed point O of the sphere by a quadric cone. Prove that, if this cone has sets of three mutually perpendicular generators, then the plane p passes through a fixed point P.

If the cone has sets of three mutually perpendicular tangent planes, prove that the plane p touches a fixed spheroid whose centre is at P and whose axis of revolution lies along OP.

41. Chords MPN of a quadric

$$ax^2+by^2+cz^2 = 1$$

are drawn through the fixed point $P(\xi,\eta,\zeta)$ so that $PM = \lambda PN$. Show that they lie on the cone

$$4\lambda\{a\xi(\xi-x)+b\eta(\eta-y)+c\zeta(\zeta-z)\}^2+$$
$$+(\lambda-1)^2(a\xi^2+b\eta^2+c\zeta^2-1)\{a(\xi-x)^2+b(\eta-y)^2+c(\zeta-z)^2\} = 0.$$

Discuss the special case $\lambda = 1$.

42. Prove that the equations of the right circular cones which pass through the rectangular axes Ox, Oy, Oz are

$$yz\pm zx\pm xy = 0.$$

Prove that the lines through a point P which are normal to the quadric

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = 1$$

lie on a quadric cone, and show that this cone is right circular if P lies on one of the lines

$$(b-c)x = \pm(c-a)y = \pm(a-b)z.$$

43. Find the equation of the cone which projects the conic

$$ax^2+by^2+c = 0, \qquad z = 0$$

from the point $P(f,g,h)$.

If the conic is fixed and the sections of the cones by planes parallel to $y = 0$ are circles, prove that the locus of P is a conic in the plane $x = 0$.

44. Find the equation of the cone generated by lines passing through the point $(a, 0, c)$ and meeting the circle

$$x^2+y^2-r^2 = 0, \qquad z = 0.$$

Prove that any plane parallel to the plane

$$2cax+(r^2+c^2-a^2)z = 0$$

cuts the cone in a circle.

45. Find the equation of the cone generated by straight lines drawn from the origin to cut the circle through the three points $(1, 0, 0)$, $(0, 2, 0)$, $(2, 1, 1)$, and prove that the acute angle between the two straight lines in which the plane

$$x = 2y$$

cuts the cone is $\qquad \cos^{-1}\sqrt{(5/14)}$.

46. Find the equation of the cone which has its vertex at the origin of coordinates and passes through the circle in which the plane $x = a$ cuts the sphere $x^2+y^2+(z-c)^2 = r^2$.

Prove that the circle in which this cone cuts the sphere again lies in the plane

$$(c^2-r^2)(x+a) = 2caz.$$

47. Find the equation of the tangent cone from the point $(0, 0, c)$ to the sphere $x^2+y^2+z^2 = a^2$, in the form

$$(c^2-a^2)(x^2+y^2) = a^2(z-c)^2.$$

Show that the common points of this tangent cone and the tangent cone from the point $(0, 0, d)$ to the same sphere lie on two circles, whose planes are given by

$$(c+d)(z^2+a^2)-2(a^2+cd)z = 0.$$

Interpret this result geometrically when $c+d = 0$.

48. Find the condition that the plane $x = 1$ should cut the cone $ax^2+by^2+cz^2 = 0$ in a circle.

If this condition is satisfied, find the equation of the sphere through this circle and the origin.

VII

THE PARABOLOIDS

THE surfaces now to be considered are natural extensions of the parabola

$$y^2 = 4ax$$

in which the left-hand side is quadratic and the right-hand side linear. They are given by the equations

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \frac{2z}{c},$$

or, in composite form,

$$Ax^2 + By^2 = 2z.$$

We may assume that A is positive; otherwise we could multiply the equation by -1 and change the sense of z.

The surfaces $$Ax^2 + By^2 = 2z$$

are called PARABOLOIDS.

1. Notes on the particular paraboloids

The surface $$Ax^2 + By^2 = 2z$$

is cut by the planes $x =$ constant and by the planes $y =$ constant in two systems of parabolas. The sections by the plane $z =$ constant require more attention.

(i) THE ELLIPTIC PARABOLOID. When A, B are positive, they may be written

$$A = c/a^2, \quad B = c/b^2 \quad (c \text{ positive})$$

so that the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}.$$

The surface is called an ELLIPTIC PARABOLOID.

When $a = b$, the surface is a figure of revolution formed by rotating the parabola

$$\frac{x^2}{a^2} = \frac{2z}{c}$$

about the z-axis. The general elliptic paraboloid may be visualized as a distortion of this figure.

If h is any *positive* number, the plane

$$z = -h$$

does not meet the surface, since it is not possible for the positive number

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

to be equal to the negative number

$$-\frac{2h}{c}.$$

On the other hand, the plane

$$z = h$$

cuts the surface in an ellipse for all positive values of h. The ellipse projects orthogonally on the plane $z = 0$ to give the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{2h}{c}.$$

As h varies, this equation defines a system of similar and similarly situated ellipses.

(ii) THE HYPERBOLIC PARABOLOID. When A is positive and B negative, they may be written

$$A = c/a^2, \quad B = -c/b^2 \quad (c \text{ positive})$$

so that the equation is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = \frac{2z}{c}.$$

The surface is called a HYPERBOLIC PARABOLOID.

If h is any *positive* number, the plane

$$z = h$$

cuts the surface in the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = \frac{2h}{c}, \qquad z = h,$$

and the plane $$z = -h$$
cuts it in the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = -\frac{2h}{c}, \qquad z = -h.$$

These hyperbolas project orthogonally on the plane $z = 0$ to give the two systems (for varying h)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = \pm\frac{2h}{c}.$$

These hyperbolas all have the asymptotes

$$\frac{x}{a}+\frac{y}{b} = 0, \qquad \frac{x}{a}-\frac{y}{b} = 0,$$

but belong to opposite ('conjugate') systems according as the positive or negative sign is taken; for positive sign they meet

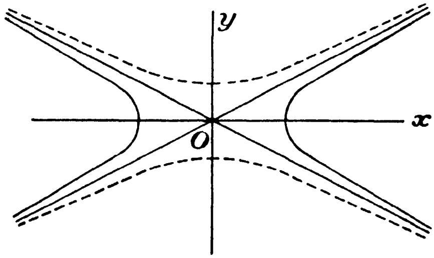

FIG. 40

the x-axis, and for negative sign they meet the y-axis. (Compare the diagram, Fig. 40.)

The surface is usually described as 'saddle-shaped' and may be visualized by considering the section, two straight lines, in the plane $z = 0$ and then imagining the changes as the plane of section moves, firstly upwards and secondly downwards.

2. Joachimstal's equation; tangency

Write

$$\begin{aligned} S &\equiv Ax^2+By^2-2z, \\ S_1 &\equiv Ax_1x+By_1y-(z+z_1), \\ S_{12} &\equiv Ax_1x_2+By_1y_2-(z_1+z_2), \\ S_{11} &\equiv Ax_1^2+By_1^2-2z_1. \end{aligned}$$

The point dividing the line $\overrightarrow{PQ}$ in the ratio $\lambda/1$, where $P \equiv (x_1, y_1, z_1)$, $Q \equiv (x_2, y_2, z_2)$, is

$$\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}, \frac{z_1+\lambda z_2}{1+\lambda}\right),$$

and lies on the quadric if, on substituting, multiplying by $(1+\lambda)^2$, and rearranging,

$$S_{22}\lambda^2+2S_{12}\lambda+S_{11} = 0.$$

The argument now follows that given (p. 124) for the central quadrics, and the results may be summarized concisely:

(i) The line meets the quadric in two points (having real existence only if $S_{12}^2-S_{11}S_{22} \geqslant 0$).

(ii) There is at an arbitrary point $P(x_1, y_1, z_1)$ on the quadric a TANGENT PLANE

$$S_1 \equiv Ax_1 x+By_1 y-(z+z_1) = 0.$$

(iii) From an arbitrary point $P(x_1, y_1, z_1)$ not on the quadric a TANGENT CONE $\quad S_{11}S = S_1^2$
may be drawn.

(iv) Two points $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ are CONJUGATE with respect to the quadric if

$$S_{12} \equiv Ax_1 x_2+By_1 y_2-(z_1+z_2) = 0,$$

and the locus of points conjugate to P is the POLAR PLANE

$$S_1 \equiv Ax_1 x+By_1 y-(z+z_1) = 0.$$

(v) The NORMAL at (x_1, y_1, z_1), being the line through it perpendicular to the tangent plane

$$Ax_1 x+By_1 y-(z+z_1) = 0,$$

is

$$\frac{x-x_1}{Ax_1} = \frac{y-y_1}{By_1} = \frac{z-z_1}{-1}.$$

3. Tangent plane and tangential equations

(i) To prove that the *condition for the plane*

$$lx+my+nz = p$$

to touch the quadric $\quad Ax^2+By^2 = 2z$

is

$$\frac{l^2}{A}+\frac{m^2}{B}+2np = 0.$$

Suppose that the plane touches the quadric, the point of contact being (x_1, y_1, z_1); then the tangent plane is

$$Ax_1 x+By_1 y = z+z_1.$$

Comparing the two forms of equation,

$$Ax_1 n = -l, \quad By_1 n = -m, \quad z_1 n = -p.$$

But the point (x_1, y_1, z_1) lies on the quadric, so that

$$Ax_1^2 + By_1^2 = 2z_1,$$

or

$$A(l/An)^2 + B(m/Bn)^2 = 2(-p/n),$$

or

$$\frac{l^2}{A} + \frac{m^2}{B} + 2np = 0.$$

Conversely, *the planes* $lx+my+nz = p$ *for which the relation*

$$\frac{l^2}{A} + \frac{m^2}{B} + 2np = 0$$

is satisfied all touch the quadric

$$Ax^2 + By^2 = 2z.$$

The relation

$$\frac{l^2}{A} + \frac{m^2}{B} + 2np = 0$$

is called the TANGENTIAL EQUATION of the quadric

$$Ax^2 + By^2 = 2z.$$

(ii) To prove that *the pole of the plane*

$$lx + my + nz = p$$

with respect to the quadric

$$Ax^2 + By^2 = 2z$$

is $(-l/An, \ -m/Bn, \ -p/n)$.

Suppose that the pole is (x_1, y_1, z_1). Then the polar plane is

$$Ax_1 x + By_1 y = z + z_1,$$

and comparison of this with the given equation for the plane gives the formulae

$$Ax_1 n = -l, \quad By_1 n = -m, \quad z_1 n = -p,$$

so that

$$x_1 = \frac{-l}{An}, \quad y_1 = \frac{-m}{Bn}, \quad z_1 = \frac{-p}{n}.$$

4. The r-equation

Let $P(x_1, y_1, z_1)$ be a given point and (l, m, n) the direction cosines of a line through P. If $Q(x, y, z)$ is the point on this line such that

$$\overrightarrow{PQ} = r,$$

then (p. 11)

$$x = x_1+lr, \quad y = y_1+mr, \quad z = z_1+nr.$$

To find the two values of r for which Q lies on the quadric

$$S \equiv Ax^2+By^2-2z = 0.$$

The point Q lies on the quadric if

$$A(x_1+lr)^2+B(y_1+mr)^2-2(z_1+nr) = 0,$$

or $$(Al^2+Bm^2)r^2+2(Ax_1 l+By_1 m-n)r+S_{11} = 0.$$

This is the r-EQUATION of the point (x_1, y_1, z_1) and the direction (l, m, n) for the quadric S.

5. The plane section with given centre

To prove that, *given a point $P(x_1, y_1, z_1)$, there is (in general) a unique plane cutting the quadric in a conic with its centre at P, the equation of the plane being*

$$S_1 = S_{11},$$

or $$Ax_1(x-x_1)+By_1(y-y_1)-(z-z_1) = 0.$$

The condition for an arbitrary line through P having direction (l, m, n) to be bisected at P is (compare p. 130 and the preceding paragraph)

$$Ax_1 l+By_1 m-n = 0.$$

If $U(x, y, z)$ is an arbitrary point of that line, then

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n},$$

so that U lies in the plane whose equation is

$$Ax_1(x-x_1)+By_1(y-y_1)-(z-z_1) = 0,$$

or, on reduction, $$S_1 = S_{11}.$$

It should be observed that there is no point $P(x_1, y_1, z_1)$ for which this equation vanishes identically; the term $-z$, with constant coefficient, precludes that possibility. In other words, the *paraboloids are quadrics which do not have centres*. They are called NON-CENTRAL quadrics.

6. Conjugate diameters

(i) To prove that *the middle points of chords of the quadric*

$$S \equiv Ax^2+By^2-2z = 0$$

in the direction (l, m, n) *lie in the plane*

$$Alx+Bmy-n = 0.$$

Let (x_1, y_1, z_1) be the middle point of a chord in the direction (l, m, n). Then (p. 156)

$$Ax_1 l+By_1 m-n = 0.$$

The locus of this point is the plane

$$Alx+Bmy-n = 0.$$

This is called the diametral plane CONJUGATE to the direction (l, m, n).

Note that *the diametral plane corresponding to any direction* (l, m, n) *is parallel to the z-axis.*

(ii) To prove that *the centres of the sections of the quadric* S *by planes parallel to the given plane*

$$ux+vy+wz = 0$$

all lie on the straight line

$$x = \frac{-u}{Aw}, \quad y = \frac{-v}{Bw}.$$

If (x_1, y_1, z_1) is the centre, the corresponding plane is (p. 156)

$$Ax_1(x-x_1)+By_1(y-y_1)-(z-z_1) = 0,$$

and so, since this is parallel to the given plane,

$$\frac{Ax_1}{u} = \frac{By_1}{v} = \frac{-1}{w}.$$

The locus of the point is thus the straight line

$$x = -u/Aw, \qquad y = -v/Bw.$$

This line is called the diameter CONJUGATE to the given plane. It is *parallel to the z-axis.*

Note that the point (x_1, y_1, z_1) does not exist if $w = 0$.

7. Generators; first method

Let (x_1, y_1, z_1) be an arbitrary point of the paraboloid

$$Ax^2+By^2-2z = 0,$$

so that
$$Ax_1^2+By_1^2-2z_1 = 0,$$

and let
$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$$

be an arbitrary line through it. The line meets the quadric where
$$A(x_1+lr)^2+B(y_1+mr)^2-2(z_1+nr) = 0,$$

so that $r = 0$ or

$$2(Alx_1+Bmy_1-n)+(Al^2+Bm^2) = 0.$$

If the line is a GENERATOR, this equation must be satisfied for all values of r, so that

$$Alx_1+Bmy_1-n = 0,$$
$$Al^2+Bm^2 = 0.$$

The second of these equations shows that *the two coefficients A, B must have opposite signs*, so that, for (real) generators the surface must be a HYPERBOLIC PARABOLOID. The equation of the surface is thus (p. 152)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{2z}{c} = 0,$$

and the conditions are

$$\frac{lx_1}{a^2}-\frac{my_1}{b^2} = \frac{n}{c},$$

$$\frac{l^2}{a^2}-\frac{m^2}{b^2} = 0.$$

Hence, easily,
$$\frac{l}{a} = \frac{m}{\pm b} = \frac{n}{c(x_1/a \mp y_1/b)},$$

so that the equations of the two generators through the point (x_1, y_1, z_1) are

$$\frac{x-x_1}{a} = \frac{y-y_1}{\pm b} = \frac{z-z_1}{c(x_1/a \mp y_1/b)}.$$

These generators are parallel to the planes

$$\frac{x}{a} \mp \frac{y}{b} = 0$$

respectively.

8. Generators; second method

The hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$$

may be written in the form

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = \frac{2z}{c},$$

which shows that the two systems of lines

$$\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 2\lambda, \\ \lambda\left(\dfrac{x}{a} - \dfrac{y}{b}\right) = \dfrac{z}{c} \end{cases}$$

and

$$\begin{cases} \dfrac{x}{a} - \dfrac{y}{b} = 2\mu, \\ \mu\left(\dfrac{x}{a} + \dfrac{y}{b}\right) = \dfrac{z}{c} \end{cases}$$

are generators for all values of the parameters λ, μ. They are called generators of the λ-SYSTEM and μ-SYSTEM respectively.

Two generators of the same system have no point of intersection, but every λ-generator has one point in common with every μ-generator, given, on direct solution of the equations, by the formulae

$$x = a(\lambda + \mu), \quad y = b(\lambda - \mu), \quad z = 2c\lambda\mu.$$

This gives a PARAMETRIC FORM for the points of the hyperbolic paraboloid.

The λ-generators are parallel to the plane

$$\frac{x}{a} + \frac{y}{b} = 0$$

and the μ-generators are parallel to the plane

$$\frac{x}{a}-\frac{y}{b}=0.$$

MISCELLANEOUS EXAMPLES

1. Two generators of the paraboloid

$$\frac{x^2}{a}-\frac{y^2}{b}=4z$$

pass through the point $(\xi, 0, \zeta)$. Prove that the cosine of the angle between them is $(a-b+\zeta)/(a+b+\zeta)$.

2. Prove that the perpendiculars from the origin $(0,0,0)$ to the generators of the paraboloid

$$\frac{x^2}{a}-\frac{y^2}{b}=2z$$

lie on two quadric cones.

3. The plane $y=\lambda z$ cuts the quadric

$$\frac{y^2}{b}+\frac{z^2}{c}=x$$

in a parabola. Prove that the ends of the latus rectum lie, for all values of λ, on the cone

$$y^2+z^2=4x^2.$$

4. A straight line meets each of the parabolas

$$y^2=ax,\ z=0 \quad\text{and}\quad z^2=-bx,\ y=0.$$

It is also parallel to one or other of the planes

$$\frac{y^2}{a}=\frac{z^2}{b}.$$

Show that it lies on the paraboloid

$$\frac{y^2}{a}-\frac{z^2}{b}=x.$$

5. A, B are two arbitrary points on a paraboloid, and the tangent planes at A, B meet in a line l. Show that the plane through l and the middle point of AB is parallel to the axis of the paraboloid.

6. Prove that the line drawn perpendicular to the line $x=0$, $y+z=0$ from a point of the parabola $y^2=ax$, $z=0$ lies on the hyperbolic paraboloid $y^2-z^2=ax$.

7. Prove that a line parallel to the plane $y-z=0$ and meeting each of the parabolas $y^2=x$, $z=0$ and $z^2=x$, $y=0$ lies on the quadric $(y-z)^2=x$.

VIII

THE GENERAL QUADRIC

THE most general equation of the second degree in the variables x, y, z may be written in the form

$$S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d = 0.$$

We shall require an abridged notation, of which the examples which follow are typical:

$$S_{11} \equiv ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+$$
$$+2ux_1+2vy_1+2wz_1+d,$$

$$S_{12} \equiv ax_1x_2+by_1y_2+cz_1z_2+f(y_1z_2+y_2z_1)+g(z_1x_2+z_2x_1)+$$
$$+h(x_1y_2+x_2y_1)+u(x_1+x_2)+v(y_1+y_2)+w(z_1+z_2)+d$$
$$\equiv x_1(ax_2+hy_2+gz_2+u)+y_1(hx_2+by_2+fz_2+v)+$$
$$+z_1(gx_2+fy_2+cz_2+w)+(ux_2+vy_2+wz_2+d)$$
$$\equiv x_2(ax_1+hy_1+gz_1+u)+y_2(hx_1+by_1+fz_1+v)+$$
$$+z_2(gx_1+fy_1+cz_1+w)+(ux_1+vy_1+wz_1+d)$$
$$\equiv S_{21},$$

$$S_1 \equiv (ax_1+hy_1+gz_1+u)x+(hx_1+by_1+fz_1+v)y+$$
$$+(gx_1+fy_1+cz_1+w)z+(ux_1+vy_1+wz_1+d).$$

The triplet (x_1, y_1, z_1) used in these definitions will often be replaced by (l_1, m_1, n_1). The context will make clear what is meant.

We shall also have to use the determinants

$$\Delta \equiv \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}, \qquad D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

It is assumed throughout that the coefficients $a, b, c, d, f, g, h, u, v, w$ are all REAL.

Before dealing with the general quadric, we devote a section to the homogeneous quadratic form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy.$$

The direct manipulation is sometimes awkward, but considerable lightening can be obtained by use of the double suffix notation described earlier for tensors. This we give as an Appendix to Part I, and the reader who feels sufficiently familiar with the use of the notation may proceed straight to it if he prefers. (An alternative treatment, nearly equivalent, is provided by the use of MATRICES.) One or two of the results contained in the 'ordinary' treatment are, however, sufficiently graphic to have an interest of their own.

PART I

THE QUADRATIC FORM

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$

1. Preliminary remark

We write

$$F \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$
$$\Omega \equiv x^2+y^2+z^2,$$

and use notation F_1 F_{11}, F_{12}, Ω_1, Ω_{11}, Ω_{12} in the sense analogous to that described for S.

The ultimate aim is to express F in the form

$$F \equiv \lambda_1 \xi^2+\lambda_2 \eta^2+\lambda_3 \zeta^2$$

by means of an orthogonal transformation (p. 43) which maintains the identity

$$\Omega \equiv \xi^2+\eta^2+\zeta^2.$$

2. The characteristic equation

The equation

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

is called the CHARACTERISTIC EQUATION of the determinant D. It is a cubic, with three roots, not necessarily distinct, known variously as the CHARACTERISTIC ROOTS, LATENT ROOTS, or EIGENVALUES.

There is a familiar theorem of algebra that, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

then a triad (α, β, γ) of numbers† *not all zero* exists such that

$$a_1\alpha+b_1\beta+c_1\gamma = 0,$$
$$a_2\alpha+b_2\beta+c_2\gamma = 0,$$
$$a_3\alpha+b_3\beta+c_3\gamma = 0.$$

If the co-factors A_1, B_1,..., C_3 are all zero, then

$$a_1:b_1:c_1 = a_2:b_2:c_2 = a_3:b_3:c_3,$$

and there is an infinite set of these triads, consisting of all triads such that

$$a_i\alpha+b_i\beta+c_i\gamma = 0 \quad (i = 1,2,3).$$

Otherwise the ratios $\alpha:\beta:\gamma$ are unique.

In particular, if λ_i is any root of the characteristic equation, then there exists at least one triad (l_i, m_i, n_i) satisfying the relations

$$(a-\lambda_i)l_i+hm_i+gn_i = 0,$$
$$hl_i+(b-\lambda_i)m_i+fn_i = 0,$$
$$gl_i+fm_i+(c-\lambda_i)n_i = 0,$$

or, in more convenient form,

$$al_i+hm_i+gn_i = \lambda_i l_i,$$
$$hl_i+bm_i+fn_i = \lambda_i m_i,$$
$$gl_i+fm_i+cn_i = \lambda_i n_i.$$

These three equations lead to a fundamental identity. Let them be multiplied in order by any three numbers l_j, m_j, n_j. Then

$$F_{ij} = \lambda_i\Omega_{ij}.$$

That is, *if λ_i is any root of the characteristic equation and (l_i, m_i, n_i) a corresponding triad, and if (l_j, m_j, n_j) is* ***any*** *other arbitrary triad, then*

$$F_{ij} = \lambda_i\Omega_{ij}.$$

† Correctly speaking, it is the *two ratios* $\alpha:\beta:\gamma$ whose existence we assert. Each of the numbers α, β, γ can be multiplied by any non-zero factor.

The equation is also (p. 161) expressible in the form

$$F_{ji} = \lambda_i \Omega_{ji}.$$

Two important results follow quickly:

(i) *If* λ_1, λ_2 *are* **unequal** *characteristic roots, and* (l_1, m_1, n_1), (l_2, m_2, n_2) *corresponding triads, then*

$$F_{12} = 0, \qquad \Omega_{12} = 0.$$

(ii) *The characteristic roots are all real.*

From the characteristic root λ_1, we have

$$F_{12} = \lambda_1 \Omega_{12};$$

from the characteristic root λ_2, we have

$$F_{12} = \lambda_2 \Omega_{12}.$$

(i) If $\lambda_1 \neq \lambda_2$, then the two relations are incompatible, unless

$$F_{12} = 0, \qquad \Omega_{12} = 0.$$

(ii) If λ_1 is a complex root, take λ_2 to be its conjugate complex $\bar{\lambda}_1$. When λ_1 is not real, $\lambda_2 \neq \lambda_1$, so that, as we have just proved,

$$\Omega_{12} = 0.$$

Also, if (l_1, m_1, n_1) corresponds to λ_1, then $(\bar{l}_1, \bar{m}_1, \bar{n}_1)$† is a triad corresponding to $\bar{\lambda}_1$. Hence the relation is

$$\Omega_{12} \equiv l_1 \bar{l}_1 + m_1 \bar{m}_1 + n_1 \bar{n}_1 = 0.$$

This is not possible (since the products $l_1 \bar{l}_1$, $m_1 \bar{m}_1$, $n_1 \bar{n}_1$ are real and positive) unless $l_1 = m_1 = n_1 = 0$, which contradicts the condition (p. 163) that these numbers can be chosen not all zero. Hence λ_1 must be real.

Note, in passing, that *a cubic equation with real coefficients cannot have two* **equal** *complex roots*; for if λ_1 were a repeated complex root, so also would be $\bar{\lambda}_1$, and the cubic equation would have four roots.

For the work which follows, we shall usually regard l_i, m_i, n_i as defining a certain direction. It will then be convenient to take values of the ratios $l_i : m_i : n_i$ which make them actual *direction cosines*. When this is done, there is an identity $l_i^2 + m_i^2 + n_i^2 = 1$, or

$$\Omega_{ii} = 1.$$

† The sign of i is changed in each of λ_1, l_1, m_1, n_1.

The relation $l_i l_j + m_i m_j + n_i n_j = 0$, or

$$\Omega_{ij} = 0$$

then means that the two directions (l_i, m_i, n_i) and (l_j, m_j, n_j) are *orthogonal.*

For actual direction cosines, the fundamental identity (p. 163)

$$F_{ii} = \lambda_i \Omega_{ii}$$

becomes

$$F_{ii} = \lambda_i.$$

3. Orthogonal transformation of the characteristic equation

(The work in this section is more easily comprehended in the briefer notation with double suffixes. These details are given for the benefit of those who have not yet reached that stage.)

Let (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) be three mutually orthogonal directions. Then, by direct multiplication of determinants (p. 87), using α, β, γ to denote $a-\lambda$, $b-\lambda$, $c-\lambda$, we have

$$\begin{vmatrix} \alpha & h & g \\ h & \beta & f \\ g & f & \gamma \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha l_1 + hm_1 + gn_1, & \alpha l_2 + hm_2 + gn_2, & \alpha l_3 + hm_3 + gn_3 \\ hl_1 + \beta m_1 + fn_1, & hl_2 + \beta m_2 + fn_2, & hl_3 + \beta m_3 + fn_3 \\ gl_1 + fm_1 + \gamma n_1, & gl_2 + fm_2 + \gamma n_2, & gl_3 + fm_3 + \gamma n_3 \end{vmatrix}.$$

Further multiplication according to the same rule gives a relation which may be expressed concisely in the form

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \begin{vmatrix} \alpha & h & g \\ h & \beta & f \\ g & f & \gamma \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_2 \end{vmatrix} = \begin{vmatrix} F_{11}^* & F_{12}^* & F_{13}^* \\ F_{21}^* & F_{22}^* & F_{23}^* \\ F_{31}^* & F_{32}^* & F_{33}^* \end{vmatrix},$$

where $\quad F^* \equiv \alpha x^2 + \beta y^2 + \gamma z^2 + 2fyz + 2gzx + 2hxy.$

In particular, if (l_i, m_i, n_i) are actual direction cosines, then (p. 42)

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1.$$

Also, with $a-\lambda$, $b-\lambda$, $c-\lambda$ for α, β, γ, definition of F^*_{ij} gives

$$F^*_{ij} \equiv (a-\lambda)l_i l_j+(b-\lambda)m_i m_j+(c-\lambda)n_i n_j+$$
$$+f(m_i n_j+m_j n_i)+g(n_i l_j+n_j l_i)+h(l_i m_j+l_j m_i)$$
$$\equiv F_{ij}-\lambda\Omega_{ij}$$
$$\equiv \begin{cases} F_{ii}-\lambda & (i=j) \\ F_{ij} & (i \neq j). \end{cases}$$

Hence

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = \begin{vmatrix} F_{11}-\lambda & F_{12} & F_{13} \\ F_{21} & F_{22}-\lambda & F_{23} \\ F_{31} & F_{32} & F_{33}-\lambda \end{vmatrix}.$$

4. The orthogonal transformation of F

Let (l_i, m_i, n_i), $i = 1, 2, 3$, be the direction cosines of three mutually orthogonal directions, and consider the transformation (p. 44)

$$x = l_1\xi+l_2\eta+l_3\zeta,$$
$$y = m_1\xi+m_2\eta+m_3\zeta,$$
$$z = n_1\xi+n_2\eta+n_3\zeta.$$

Then, under it,

$$F \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$\equiv a(l_1\xi+l_2\eta+l_3\zeta)^2+...+$$
$$+2f(m_1\xi+m_2\eta+m_3\zeta)(n_1\xi+n_2\eta+n_3\zeta)+...$$
$$\equiv F_{11}\xi^2+F_{22}\eta^2+F_{33}\zeta^2+2F_{23}\eta\zeta+2F_{31}\zeta\xi+2F_{12}\xi\eta.$$

Similarly, $$\Omega \equiv \xi^2+\eta^2+\zeta^2.$$

(i) *Suppose that the characteristic roots $\lambda_1, \lambda_2, \lambda_3$ are all different.* Then (p. 165) the corresponding directions are mutually orthogonal. Take them for the three directions (l_i, m_i, n_i) just used. Also (p. 165) we then have

$$F_{ii} = \lambda_i, \quad F_{ij} = 0 \quad (i \neq j).$$

Hence $$F \equiv \lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2,$$

so that F is reduced to the sum of three squares (or, possibly, less than three if one of λ_1, λ_2, λ_3 is zero). The coefficients λ_1, λ_2, λ_3 are the characteristic roots of the determinant D.

(ii) *Suppose that the characteristic roots $\lambda_1, \lambda_2, \lambda_3$ are not all* *different*. Let λ_1 be any one of the characteristic roots, and (l_1, m_1, n_1) the corresponding direction cosines. The perpendicular directions (l_2, m_2, n_2), (l_3, m_3, n_3) are undetermined, save that each is perpendicular to (l_1, m_1, n_1)—they are not otherwise linked in any way to the characteristic roots. Then (p. 165)

$$F_{11} = \lambda_1,$$

and, for *any* triads (l_2, m_2, n_2), (l_3, m_3, n_3),

$$F_{12} = \lambda_1 \Omega_{12}, \qquad F_{13} = \lambda_1 \Omega_{13}.$$

Since (l_2, m_2, n_2), (l_3, m_3, n_3) are chosen perpendicular to (l_1, m_1, n_1), it follows that $\Omega_{12} = \Omega_{13} = 0$, so that, by the preceding equations,

$$F_{12} = 0, \qquad F_{13} = 0.$$

Hence, by direct substitution,

$$F \equiv \lambda_1 \xi^2 + F_{22}\eta^2 + F_{33}\zeta^2 + 2F_{23}\eta\zeta.$$

Now (p. 166) the characteristic equation can be taken in the form

$$\begin{vmatrix} F_{11}-\lambda & F_{12} & F_{13} \\ F_{21} & F_{22}-\lambda & F_{23} \\ F_{31} & F_{32} & F_{33}-\lambda \end{vmatrix} = 0,$$

or, here,

$$\begin{vmatrix} F_{11}-\lambda & 0 & 0 \\ 0 & F_{22}-\lambda & F_{23} \\ 0 & F_{32} & F_{33}-\lambda \end{vmatrix} = 0.$$

The two roots other than $\lambda_1 \equiv F_{11}$ are thus given by the quadratic

$$(F_{22}-\lambda)(F_{33}-\lambda) - F_{23}^2 = 0$$

or

$$\lambda^2 - (F_{22}+F_{33})\lambda + (F_{22}F_{33} - F_{23}^2) = 0.$$

If, as is allowable, we choose λ_1 so that these are the two equal roots (not excluding the possibility that they may also be equal to λ_1), then

$$(F_{22}+F_{33})^2 - 4(F_{22}F_{33} - F_{23}^2) = 0,$$

or

$$(F_{22}-F_{33})^2 + 4F_{23}^2 = 0.$$

Since all numbers are real,† it follows that

$$F_{22} = F_{33},$$

$$F_{23} = 0.$$

† See the more detailed treatment on p. 174.

The equation for λ is thus

$$\lambda^2-2F_{22}\lambda+F_{22}^2 = 0,$$

so that
$$\lambda_2 = \lambda_3 = F_{22} = F_{33}.$$

Hence, once again, the expression for F is

$$F \equiv \lambda_1 \xi^2+\lambda_2 \eta^2+\lambda_3 \zeta^2,$$

where, now, $\lambda_2 = \lambda_3$ (possibly also $= \lambda_1$).

Appendix to Part I

1. The use of a double-suffix notation

The geometry of the quadratic form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

is closely linked with the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

This linkage can be emphasized by the use of a double suffix notation, in which we write

$$a_{11} = a, \quad a_{22} = b, \quad a_{33} = c,$$
$$a_{23} = a_{32} = f, \quad a_{31} = a_{13} = g, \quad a_{12} = a_{21} = h.$$

The quadratic form is then (with x_1, x_2, x_3 for x, y, z)

$$a_{11} x_1^2+a_{22} x_2^2+a_{33} x_3^2+(a_{23}+a_{32})x_2 x_3+$$
$$+(a_{31}+a_{13})x_3 x_1+(a_{12}+a_{21})x_1 x_2,$$

with determinant
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

We therefore make a fresh start, and consider the array

$$(a_{ij}) \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The determinant
$$| a_{ij} |$$

is called the DETERMINANT OF THE ARRAY, and the expression

$$a_{\lambda\mu} x_\lambda x_\mu$$

(whose detailed expansion appears above) is called the QUADRATIC FORM ASSOCIATED WITH THE ARRAY.

The array is assumed SYMMETRICAL, so that

$$a_{ij} = a_{ji},$$

and it is also assumed that all its elements are REAL.

2. The characteristic equation

The equation

$$\begin{vmatrix} a_{11}-t & a_{12} & a_{13} \\ a_{21} & a_{22}-t & a_{23} \\ a_{31} & a_{32} & a_{33}-t \end{vmatrix} = 0,$$

or, briefly, $$| a_{ij}-t\delta_{ij} | = 0,$$

is called the CHARACTERISTIC EQUATION of the array (a_{ij}). The left-hand side is the determinant of the array

$$(a_{ij}-t\delta_{ij}).$$

The three roots, not necessarily different, are called the CHARACTERISTIC ROOTS, or LATENT ROOTS, or EIGENVALUES of (a_{ij}).

By a familiar theorem of algebra, a necessary and sufficient condition for the vanishing of a determinant $|c_{ij}|$ is the existence of an array $(u_i) \equiv (u_1, u_2, u_3)$ such that the three equations

$$c_{i\lambda} u_\lambda = 0 \quad (i = 1, 2, 3)$$

are satisfied simultaneously, *where the numbers* u_1, u_2, u_3 *are* ***not all zero***. The numbers u_1, u_2, u_3 may be multiplied by any non-zero factor, as only their *ratios* are significant.

Applied to the characteristic equation, this theorem shows that, *for every value of t such that*

$$|a_{ij}-t\delta_{ij}| = 0,$$

a set of ratios exists such that

$$(a_{i\lambda}-t\delta_{i\lambda})u_\lambda = 0,$$

or

$$a_{i\lambda} u_\lambda = t\delta_{i\lambda} u_\lambda = tu_i.$$

COROLLARY. *If* $(v_i) \equiv (v_1, v_2, v_3)$ *is any* **arbitrary** *array, then*

$$a_{\lambda\mu} u_\lambda v_\mu = tu_\lambda v_\mu.$$

In particular, $$a_{\lambda\mu} u_\lambda u_\mu = tu_\lambda u_\mu.$$

When necessary, the characteristic roots will be distinguished by suffixes, with the notation t_1, t_2, t_3. The letter for a general suffix will usually be selected from p, q, r. Thus a characteristic array corresponding to t_p will appear in the form (u_{1p}, u_{2p}, u_{3p}), or

$$(u_{ip}).$$

In practice, a characteristic array will often be regarded as a direction cosine vector (p. 58). The notation

$$(l_{ip})$$

will then be used (corresponding to t_p), where

$$a_{i\lambda}\, l_{\lambda p} = t_p\, l_{ip},$$

not summed for p. As usual (p. 58) the direction cosines are subject to the relation

$$l_{\lambda p}\, l_{\lambda p} = 1.$$

We can now deduce easily the FUNDAMENTAL IDENTITY FOR CHARACTERISTIC ROOTS, *that*

$$a_{\lambda\mu}\, l_{\lambda p}\, l_{\mu p} = t_p.$$

The left-hand side is

$$\begin{aligned}(a_{\lambda\mu}\, l_{\mu p}) l_{\lambda p} &= (t_p\, l_{\lambda p}) l_{\lambda p} \\ &= t_p (l_{\lambda p}\, l_{\lambda p}) \\ &= t_p.\end{aligned}$$

3. Some properties associated with two distinct characteristic roots

Let t_p, t_q be two *distinct* characteristic roots, and (l_{ip}), (l_{iq}) the corresponding direction cosines. We prove that:

(i) *the directions* (l_{ip}), (l_{iq}) *are orthogonal,*

(ii) $a_{\lambda\mu}\, l_{\lambda p}\, l_{\mu q} = 0$,

(iii) t_p, t_q *are necessarily real.*

By § 2,
$$a_{i\lambda}\, l_{\lambda p} = t_p\, l_{ip},$$
so that
$$a_{\mu\lambda}\, l_{\lambda p}\, l_{\mu q} = t_p\, l_{\mu p}\, l_{\mu q},$$
or, since $a_{\lambda\mu} = a_{\mu\lambda}$,
$$a_{\lambda\mu}\, l_{\lambda p}\, l_{\mu q} = t_p\, l_{\lambda p}\, l_{\lambda q}.$$
Similarly,
$$a_{\lambda\mu}\, l_{\lambda p}\, l_{\mu q} = t_q\, l_{\lambda p}\, l_{\lambda q}.$$

But these two equations are incompatible when $t_p \neq t_q$ unless each side is zero. Hence

(i) $$l_{\lambda p}\, l_{\lambda q} = 0,$$

so that the direction cosines (l_{ip}), (l_{iq}) are orthogonal;

(ii) $$a_{\lambda\mu}\, l_{\lambda p}\, l_{\mu q} = 0.$$

To prove that t_p, t_q are real, assume that, on the contrary, t_p is complex. Its conjugate complex $\bar{t}_p$ is then different from t_p, and may be taken as t_q in the preceding work. If (l_{ip}) is the array associated with t_p, then its conjugate complex $(\bar{l}_{ip})$ is the array associated with $\bar{t}_p$, since the calculations in the two cases are identical except for the sign to be given to $\sqrt{(-1)}$. The relation $l_{\lambda p}\, l_{\lambda q} = 0$ just proved thus becomes

$$l_{\lambda p}\, \bar{l}_{\lambda p} = 0,$$

which is impossible unless

$$l_{ip} \equiv 0,$$

contradicting the condition (p. 169) that l_{1p}, l_{2p}, l_{3p} are not all zero.

[If $$l_{ip} = u_i + v_i\sqrt{(-1)},$$

then $$\bar{l}_{ip} = u_i - v_i\sqrt{(-1)},$$

so that $$l_{\lambda p}\bar{l}_{\lambda p} = (u_1^2+v_1^2)+(u_2^2+v_2^2)+(u_3^2+v_3^2),$$

which cannot vanish unless $u_i = 0$, $v_i = 0$ for $i = 1, 2, 3$.]

4. The orthogonal transformation of a quadratic form

Let $$F \equiv a_{\lambda\mu}\, x_\lambda\, x_\mu$$

be a given quadratic form, and consider the transformation (p. 58)

$$x_i = l_{i\lambda} x'_\lambda,$$

giving

$$F \equiv a_{\lambda\mu}\, l_{\lambda\alpha}\, x'_\alpha\, l_{\mu\beta}\, x'_\beta$$

$$\equiv (a_{\lambda\mu}\, l_{\lambda\alpha}\, l_{\mu\beta}) x'_\alpha\, x'_\beta.$$

This is the corresponding quadratic form in x'_i, and is associated with the array (b_{pq}), where

$$(b_{pq}) \equiv (a_{\lambda\mu}\, l_{\lambda p}\, l_{\mu q}).$$

5. The orthogonal transformation associated with the characteristic roots; roots all different

When the characteristic roots t_1, t_2, t_3 of (a_{ij}) are all different, they define (§ 3, (i)) three mutually orthogonal directions

$$(l_{i1}),\ (l_{i2}),\ (l_{i3}).$$

If these are taken as the direction cosine vectors (l_{ij}) in the transformation of § 4, then $a_{\lambda\mu} x_\lambda x_\mu$ becomes

$$b_{\lambda\mu} x'_\lambda x'_\mu,$$

where $$b_{pq} \equiv a_{\lambda\mu} l_{\lambda p} l_{\mu q}.$$

Now we have proved (p. 170) that, *when p, q are different,*

$$a_{\lambda\mu} l_{\lambda p} l_{\mu q} = 0,$$

so that $$b_{pq} = 0 \quad (p \neq q).$$

We have also proved (p. 170) that

$$a_{\lambda\mu} l_{\lambda p} l_{\mu p} = t_p,$$

so that $$b_{pp} = t_p.$$

Hence *the quadratic form is expressed as a sum of squares in x'_1, x'_2, x'_3 in the form*

$$t_1 x_1'^2 + t_2 x_2'^2 + t_3 x_3'^2,$$

where t_1, t_2, t_3 are the distinct roots of the characteristic equation.

6. The orthogonal transformation; roots not all different

Let t_1 be any one of the characteristic roots, and

$$(l_{i1}) \equiv (l_{11}, l_{21}, l_{31})$$

the corresponding direction cosines. Select any two perpendicular directions (l_{i2}), (l_{i3}) subject only to the condition (so far) that each is perpendicular to (l_{i1}). Then $l_{\lambda p} l_{\lambda q} = \delta_{pq}$; in particular,

$$l_{\lambda 1} l_{\lambda 2} = 0, \qquad l_{\lambda 1} l_{\lambda 3} = 0.$$

Also (p. 170) $$a_{\lambda\mu} l_{\lambda 1} l_{\mu 1} = t_1.$$

Moreover (p. 170), $$a_{i\mu} l_{\mu 1} = t_1 l_{i1},$$

so that
$$\begin{aligned} a_{\lambda\mu} l_{\mu 1} l_{\lambda 2} &= t_1 l_{\lambda 1} l_{\lambda 2} \\ &= 0 \qquad \text{(as above)}, \end{aligned}$$

and
$$a_{\lambda\mu} l_{\mu 1} l_{\lambda 3} = t_1 l_{\lambda 1} l_{\lambda 3} = 0.$$

Now, with the substitution $x_i = l_{i\lambda} x'_\lambda$,
$$F \equiv a_{\lambda\mu} x_\lambda x_\mu = b_{\alpha\beta} x'_\alpha x'_\beta,$$
where
$$b_{pq} = a_{\lambda\mu} l_{\lambda p} l_{\mu q}.$$
The three relations just proved are thus
$$b_{11} = t_1; \quad b_{12} = b_{21} = 0; \quad b_{13} = b_{31} = 0.$$

We prove next the basic theorem, that *the two arrays* (a_{ij}), (b_{ij}) *have the same characteristic equation.*

Since, as above, $\delta_{ij} = l_{\lambda i} l_{\lambda j}$, it follows that
$$\begin{aligned} b_{ij} - t\delta_{ij} &= a_{\lambda\mu} l_{\lambda i} l_{\mu j} - t l_{\lambda i} l_{\lambda j} \\ &= a_{\lambda\mu} l_{\lambda i} l_{\mu j} - t\delta_{\lambda\mu} l_{\lambda i} l_{\mu j} \\ &= l_{\lambda i}(a_{\lambda\mu} - t\delta_{\lambda\mu}) l_{\mu j}. \end{aligned}$$
Hence, by the rule for the multiplication of determinants (p. 87),
$$\begin{aligned} |b_{ij} - t\delta_{ij}| &= |l_{ij}||a_{ij} - t\delta_{ij}||l_{ij}| \\ &= |a_{ij} - t\delta_{ij}|, \end{aligned}$$
since $|l_{ij}|^2 = 1$. The two equations
$$|b_{ij} - t\delta_{ij}| = 0,$$
$$|a_{ij} - t\delta_{ij}| = 0$$
are therefore identical.

The characteristic equation is thus
$$\begin{vmatrix} b_{11}-t & b_{12} & b_{13} \\ b_{21} & b_{22}-t & b_{23} \\ b_{31} & b_{32} & b_{33}-t \end{vmatrix} = 0,$$
or
$$\begin{vmatrix} t_1-t & 0 & 0 \\ 0 & b_{22}-t & b_{23} \\ 0 & b_{32} & b_{33}-t \end{vmatrix} = 0.$$

If we now decide to choose t_1 so that the other two are the equal roots (though they may also be equal to t_1 in a special case), the equation
$$\begin{vmatrix} b_{22}-t & b_{23} \\ b_{32} & b_{33}-t \end{vmatrix} = 0,$$

or
$$t^2-(b_{22}+b_{33})t+(b_{22}b_{33}-b_{23}^2) = 0,$$
has equal roots. The condition for this is
$$(b_{22}+b_{33})^2-4(b_{22}b_{33}-b_{23}^2) = 0,$$
or
$$(b_{22}-b_{33})^2+4b_{23}^2 = 0.$$

Now t_1 is real (otherwise, as in the remark on p. 164, there would be a *repeated* root $\bar{t}_1$, which is impossible since t_1 would have to be repeated too), so that (l_{i1}) is real; and the two direction cosines (l_{i2}), (l_{i3}) perpendicular to it are real also. Hence, from the condition for equal roots, we have
$$b_{22}-b_{33} = 0, \qquad b_{23} = 0,$$
and so

(i) the characteristic roots are t_1, b_{22}, b_{33},

(ii) the quadratic form is
$$t_1 x_1'^2+b_{22}x_2'^2+b_{33}x_3'^2,$$
where $b_{22} = b_{33}$, each being $t_2 = t_3$. Hence the quadratic form, in terms of t_1, t_2, t_3 as coefficients, is
$$t_1 x_1'^2+t_2 x_2'^2+t_3 x_3'^2 \quad (t_2 = t_3),$$
or
$$t_1 x_1'^2+t_2(x_2'^2+x_3'^2).$$

It is possible for t_1 also to be equal to t_2 and t_3. The form is then
$$t_1(x_1'^2+x_2'^2+x_3'^2).$$

ILLUSTRATION. *The inertia quadric*

The MOMENT OF INERTIA of a body about an axis is defined as
$$\sum mp^2,$$
where p is the distance from the axis of a typical particle P of mass m and where the summation is over all the particles of the body.

Suppose that the axis passes through the origin, with direction cosines $(a_i) \equiv (a_1, a_2, a_3)$, and that the position of P is $(x_i) \equiv (x_1, x_2, x_3)$. Then (p. 64) the projection OP' of OP on the axis is given by
$$OP' = a_\lambda x_\lambda.$$
Also
$$OP^2 = x_\lambda x_\lambda.$$

Hence
$$p^2 = OP^2 - OP'^2$$
$$= x_\lambda x_\lambda - a_\lambda x_\lambda a_\mu x_\mu$$
$$= (\delta_{\lambda\mu} x_\nu x_\nu - x_\lambda x_\mu) a_\lambda a_\mu$$

since $a_\lambda a_\lambda = 1$ in the summation $\delta_{\lambda\mu} a_\lambda a_\mu$.

The moment of inertia $I_{(a)}$ about the axis is thus given by

$$I_{(a)} = \sum m(\delta_{\lambda\mu} x_\nu x_\nu - x_\lambda x_\mu) a_\lambda a_\mu$$
$$= I_{\lambda\mu} a_\lambda a_\mu,$$

where (I_{ij}) is the inertia tensor of the body for the origin O (p. 94).

Hence *the moment of inertia about the axis* (a_i) *is given by the formula*

$$I_{(a)} = I_{\lambda\mu} a_\lambda a_\mu.$$

This result can be given an alternative form of expression:

The equation
$$I_{\lambda\mu} x_\lambda x_\mu = 1$$

represents a quadric, known as the INERTIA QUADRIC at O for the body. The point distant r from O in the direction (a_i) lies on the quadric if its coordinates (ra_i) satisfy the equation: that is, if

$$r^2(I_{\lambda\mu} a_\lambda a_\mu) = 1,$$

so that
$$I_{(a)} = 1/r^2.$$

Thus *the moment of inertia about any axis is equal to* $1/r^2$, *where* r *is the radius of the inertia quadric in the direction of the axis.*

For a genuinely three-dimensional body the value of $I_{(a)}$ is always positive, so that *the inertia quadric is an* ELLIPSOID.

By the general theory, the ellipsoid has three principal axes, and these are known as the PRINCIPAL AXES OF INERTIA for the body. The corresponding moments of inertia are called the PRINCIPAL MOMENTS. They are the moments of inertia of the body about the principal axes.

When two of the principal moments are equal, the ellipsoid is one of revolution.

When three of the principal moments are equal, the ellipsoid is a sphere. In this case, the moments of inertia of the body about *all* lines through O are equal.

MISCELLANEOUS EXAMPLES

1. Find the characteristic roots of

$$\begin{pmatrix} 6 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

2. Prove that the equations

$$\begin{aligned} 4x_1-x_2-x_3 &= tx_1, \\ -x_1+2x_2+x_3 &= tx_2, \\ -x_1+x_2+2x_3 &= tx_3 \end{aligned}$$

are soluble for values of x_1, x_2, x_3, not all zero, for just three values of t. Determine these values and the corresponding ratios for $x_1 : x_2 : x_3$.

Show that, if (x_{1i}, x_{2i}, x_{3i}) is a solution corresponding to the value t_i, then

$$x_{\lambda i} x_{\lambda j} = 0 \quad (i \neq j).$$

3. P is the point (x_1, x_2, x_3), and P' is the point (x_1', x_2', x_3'), where

$$x_1' = 3x_1+x_2-2x_3, \quad x_2' = -4x_1-x_2+7x_3, \quad x_3' = 4x_1+2x_2-4x_3.$$

Show that if P' lies on OP (where O is the origin), then P must lie on one or other of three lines through O, and find these lines.

4. Show that the characteristic values of

$$\tfrac{1}{3}\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}$$

are -1, 1, 3, and obtain the characteristic vectors.

5. A given transformation is $x_i' = a_{i\lambda} x_\lambda$, where

$$(a_{ij}) = \tfrac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}.$$

Find the three characteristic values (of which two form a complex conjugate pair) and construct characteristic vectors.

By considering the transformation law for the real and imaginary parts of two conjugate characteristic vectors, or otherwise, show that the transformation is a rotation about the real characteristic vector as axis.

6. Given that $$l_{\lambda i} l_{\lambda j} = \delta_{ij}$$

(from which it follows that $l_{i\lambda} l_{j\lambda} = \delta_{ij}$), show that

$$|l_{ij}| = \pm 1.$$

Show that, in cases where the upper sign is valid,

$$2l_{ij} = \epsilon_{\lambda\mu i} \epsilon_{\alpha\beta j} l_{\lambda\alpha} l_{\mu\beta}.$$

Hence, or otherwise, prove that

$$|l_{ij} - \delta_{ij}| = 0,$$

and deduce that, in any orthogonal transformation of axes, points lying on a certain line are transformed into themselves.

7. Prove that, if (l_{ij}) is the array formed by the direction cosines of three mutually orthogonal lines, then the characteristic equation

$$|l_{ij}-t\delta_{ij}| = 0$$

has the property that, if k is a root, so also is $1/k$.

8. An array (a_{ij}) has the property

$$a_{i\lambda}a_{\lambda j} = \delta_{ij}.$$

A non-zero vector $(x_i) \equiv (x_1, x_2, x_3)$ is not a characteristic vector of (a_{ij}). Prove that $(x_i+a_{i\lambda}x_\lambda)$ and $(x_i-a_{i\lambda}x_\lambda)$ are independent characteristic vectors of (a_{ij}).

9. A homogeneous solid has the form of a right circular cylinder of radius a and height $a\sqrt{3}$. A point P lies on one of the circular edges. Show that the inertia quadric at P is a spheroid, the axis of which passes through the centre of the cylinder.

10. A rigid body consists of four particles of masses m, $2m$, $3m$, $4m$ situated respectively at the points (a,a,a), $(a,-a,-a)$, $(-a,a,-a)$, $(-a,-a,a)$. The particles are rigidly connected by a light framework. Find the inertia tensor at the origin and hence show that the principal moments of inertia are $(20+2\sqrt{5})ma^2$, $20ma^2$, $(20-2\sqrt{5})ma^2$.

11. The edges at the corner O of a uniform cube are taken as rectangular axes. Prove that the equation of the inertia quadric at this corner is of the form

$$4(x_1^2+x_2^2+x_3^2)-3(x_2x_3+x_3x_1+x_1x_2) = \text{constant}.$$

12. Find the principal moments of inertia of a solid hemisphere of mass M and radius a at its centre of mass.

With a point O on the circular rim as origin and with right-handed rectangular axes Ox, a diameter, Oy, the tangent to the circular rim, and Oz, perpendicular to the plane face and on the same side as the centre of mass, calculate the moments and products of inertia at O. Hence find the principal axes at O, and show that the moments of inertia about them are $\frac{11}{40}Ma^2$, $\frac{7}{5}Ma^2$, $\frac{61}{40}Ma^2$.

13. For a chain of three uniform mutually perpendicular rods each of mass m and length $2a$, show that the equation of the inertia quadric at the mass centre referred to axes parallel to the rods may be written in the form

$$\tfrac{1}{3}ma^2(5x^2+5y^2+3z^2-3yz+3zx+xy) = 1.$$

Deduce that one principal moment of inertia at the mass centre is ma^2, and find the others.

14. Find the lengths and direction cosines of the principal semi-axes of the quadric surface

$$3x^2+3y^2+5z^2-2yz-2zx+2xy = 1.$$

15. Show that one of the characteristic roots of

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & -2 \\ 2 & -2 & 3 \end{pmatrix}$$

is zero, and determine the others.

Find the axes, and state the form, of the surface

$$2x^2+4y^2+3z^2-4yz+4zx = 12.$$

16. Prove that the equation

$$x^2+y^2+z^2+yz+zx+xy = a^2$$

represents a quadric of revolution with axis $x = y = z$, and verify that the 'polar' axis is half the 'equatorial'.

17. Prove that the equation

$$x^2+y^2-3z^2-4xy = 1$$

represents a hyperboloid of one sheet.

18. Prove that the equation

$$yz+zx+xy = 1$$

represents a surface formed by the revolution of a hyperbola of eccentricity $\sqrt{(3/2)}$.

19. Prove that the equation

$$(cy-bz)^2+(az-cx)^2+(bx-ay)^2 = 1$$

represents a right circular cylinder.

20. Prove that the lines meeting the three lines

$$x = 0,\ y = 1; \quad y = 0,\ z = 1; \quad z = 0,\ x = 1$$

generate a quadric of revolution.

21. Prove that the equation

$$(y-z)^2+2(z-x)^2-3(x-y)^2 = 1$$

represents a hyperbolic cylinder whose principal sections are rectangular hyperbolas.

22. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar and $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$ is a characteristic vector of (h_{ij}) with characteristic value t, prove that $\mathbf{a}'-t\mathbf{a}$, $\mathbf{b}'-t\mathbf{b}$, $\mathbf{c}'-t\mathbf{c}$ are coplanar, where, for example,

$$a'_i = h_{i\lambda}\, a_\lambda.$$

Given that

$$\mathbf{a}'.\mathbf{b}\wedge\mathbf{c}+\mathbf{a}.\mathbf{b}'\wedge\mathbf{c}+\mathbf{a}.\mathbf{b}\wedge\mathbf{c}' = 6\mathbf{a}.\mathbf{b}\wedge\mathbf{c},$$

$$\mathbf{a}.\mathbf{b}'\wedge\mathbf{c}'+\mathbf{a}'.\mathbf{b}\wedge\mathbf{c}'+\mathbf{a}'.\mathbf{b}'\wedge\mathbf{c} = 11\mathbf{a}.\mathbf{b}\wedge\mathbf{c},$$

$$\mathbf{a}'.\mathbf{b}'\wedge\mathbf{c}' = 6\mathbf{a}.\mathbf{b}\wedge\mathbf{c},$$

prove that one of the characteristic values is unity, and find the others.

PART II

DIAMETERS, CENTRES; CLASSIFICATION

1. The segment equation

Consider now the general equation

$$S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+ \\ +2ux+2vy+2wz+d = 0,$$

using the notation of § 1 (p. 162).

Let $P(x_1, y_1, z_1)$ be a given point and (l, m, n) the direction cosines of a line through P. If $Q(x, y, z)$ is a point on this line such that $\overrightarrow{PQ} = r$, then (p. 11)

$$x = x_1+lr, \quad y = y_1+mr, \quad z = z_1+nr.$$

The point Q lies on the quadric if

$$a(x_1+lr)^2+...+d = 0,$$

or

$$r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)+ \\ +2r\{(ax_1+hy_1+gz_1+u)l+(hx_1+by_1+fz_1+v)m+ \\ +(gx_1+fy_1+cz_1+w)n\}+S_{11} = 0.$$

This is a quadratic in r whose roots, when real, correspond to the two values of $\overrightarrow{PQ}$ for which Q lies on the surface. The equation may be called the r-EQUATION *of the point* (x_1, y_1, z_1) *and the direction* (l, m, n) *for the quadric* S.

2. The diametral plane; centres

The two roots of the r-equation are equal and opposite when

$$(ax_1+hy_1+gz_1+u)l+(hx_1+by_1+fz_1+v)m+ \\ +(gx_1+fy_1+cz_1+w)n = 0,$$

and the point $P(x_1, y_1, z_1)$ is then the middle point of the chord through it in the direction (l, m, n). If, in particular, (l, m, n) is regarded as given, then P lies in the plane

$$(al+hm+gn)x+(hl+bm+fn)y+(gl+fm+cn)z+ \\ +(ul+vm+wn) = 0.$$

This is called the DIAMETRAL PLANE CONJUGATE TO THE DIRECTION (l, m, n).

Special interest attaches to the position, if any, of the point P when its coordinates satisfy simultaneously the three equations

$$\begin{aligned} ax_1+hy_1+gz_1+u &= 0, \\ hx_1+by_1+fz_1+v &= 0, \\ gx_1+fy_1+cz_1+w &= 0. \end{aligned}$$

The r-equation then has equal and opposite roots for all values of (l, m, n); that is, every chord through P is bisected there.

Several cases may arise (compare p. 29):

(i) The solution of the three equations may be unique. The point P is then called the CENTRE of the quadric; the quadric is of CENTRAL type.

(ii) The three equations may be insoluble. The quadric is then NON-CENTRAL.

(iii) The three equations may have an infinite number of solutions. These may define points which lie on a straight line, as when the quadric is a cylinder or two intersecting planes; or in a plane, as when the quadric consists of two parallel (or coincident) planes.

We examine these possibilities in more detail.

3. The general analysis

Suppose that the axes are rotated, without change of origin, in the way described in Part I of this chapter. The equation of the quadric then assumes the form

$$\lambda_1 \xi^2+\lambda_2 \eta^2+\lambda_3 \zeta^2+2\alpha\xi+2\beta\eta+2\gamma\zeta+d = 0,$$

where λ_1, λ_2, λ_3 are the characteristic roots obtained before, and α, β, γ are constants whose values can be calculated as required.†

The equations for the centre are

$$\begin{aligned} \lambda_1 \xi+\alpha &= 0, \\ \lambda_2 \eta+\beta &= 0, \\ \lambda_3 \zeta+\gamma &= 0. \end{aligned}$$

† In fact,

$$\alpha = ul_1+vm_1+wn_1, \quad \beta = ul_2+vm_2+wn_2, \quad \gamma = ul_3+vm_3+wn_3.$$

The argument may be divided into sections according as the number of zeros in $\lambda_1, \lambda_2, \lambda_3$ is 0, 1, or 2. (When $\lambda_1 = \lambda_2 = \lambda_3 = 0$, the quadric is not 'genuine'.)

4. The cases when $\lambda_1, \lambda_2, \lambda_3$ are not zero

When $\lambda_1, \lambda_2, \lambda_3$ are not zero, the three equations for the centre can all be solved:

$$\xi = -\alpha/\lambda_1, \quad \eta = -\beta/\lambda_2, \quad \zeta = -\gamma/\lambda_3.$$

The quadric is then CENTRAL.

Transfer the coordinate system to parallel axes through the centre. The equation of the quadric then assumes the form

$$\lambda_1 \xi^2 + \lambda_2 \eta^2 + \lambda_3 \zeta^2 = \delta,$$

where

$$\delta = \frac{\alpha^2}{\lambda_1} + \frac{\beta^2}{\lambda_2} + \frac{\gamma^2}{\lambda_3} - d.$$

(*a*) *When* $\delta = 0$, the quadric is a CONE.

(*b*) *When* $\delta \neq 0$, the quadric is an ELLIPSOID, HYPERBOLOID OF ONE SHEET or HYPERBOLOID OF TWO SHEETS according as 3, 2, or 1 of the numbers $\lambda_1, \lambda_2, \lambda_3$ have the same sign as δ. [We regard as excluded, here and elsewhere, the cases in which the equation of the quadric has no real solutions.]

The cases of EQUALITY AMONG $\lambda_1, \lambda_2, \lambda_3$ may be noted briefly:

(*a*) *When* $\lambda_2 = \lambda_3 \neq \lambda_1$, the quadric is one of REVOLUTION about the line $\eta = \zeta = 0$.

(*b*) *When* $\lambda_1 = \lambda_2 = \lambda_3$, the quadric is a SPHERE.

5. The cases when $\lambda_1 = 0, \lambda_2, \lambda_3 \neq 0$

(*a*) *When* $\alpha = 0$, the equation of the quadric is

$$\lambda_2 \eta^2 + \lambda_3 \zeta^2 + 2\beta\eta + 2\gamma\zeta + d = 0,$$

so that the surface is a CYLINDER 'standing' on the conic in the plane $\xi = 0$ given by that equation. There is a line of centres given by the equations

$$\lambda_2 \eta + \beta = 0, \qquad \lambda_3 \zeta + \gamma = 0.$$

(b) *When* $\alpha \neq 0$, the equation

$$\lambda_1 \xi + \alpha = 0 \quad (\lambda_1 = 0)$$

is not soluble; the quadric is NON-CENTRAL. The equation of the surface may be written in the form

$$\lambda_2(\eta+\beta/\lambda_2)^2+\lambda_3(\zeta+\gamma/\lambda_3)^2+2\alpha\xi+(d-\beta^2/\lambda_2-\gamma^2/\lambda_3) = 0,$$

representing (p. 151) an ELLIPTIC PARABOLOID or a HYPERBOLIC PARABOLOID according as λ_2, λ_3 have like or unlike signs.

When $\lambda_2 = \lambda_3$, the paraboloid is one of REVOLUTION.

6. The cases when $\lambda_1 \neq 0, \lambda_2 = \lambda_3 = 0$

(a) *When* $\beta = \gamma = 0$, the equation of the quadric is

$$\lambda_1 \xi^2 + 2\alpha\xi + d = 0,$$

representing two parallel planes, possibly coincident, each parallel to the plane $\xi = 0$.

(b) *When* β, γ *are not both zero*, the equations

$$\begin{aligned} \lambda_2 \eta + \beta &= 0, \\ \lambda_3 \zeta + \gamma &= 0 \end{aligned} \qquad (\lambda_2 = \lambda_3 = 0)$$

cannot be solved simultaneously, so that the quadric is NON-CENTRAL. The equation of the surface is then

$$\lambda_1 \xi^2 + 2\alpha\xi + 2\beta\eta + 2\gamma\zeta + d = 0.$$

Write

$$\beta\eta + \gamma\zeta = \sqrt{(\beta^2+\gamma^2)}\eta',$$

$$\gamma\eta - \beta\zeta = \sqrt{(\beta^2+\gamma^2)}\zeta',$$

so that $\xi = 0$, $\eta' = 0$, $\zeta' = 0$ are mutually orthogonal planes. The equation becomes

$$\lambda_1 \xi^2 + 2\alpha\xi + 2\sqrt{(\beta^2+\gamma^2)}\eta' + d = 0,$$

representing a PARABOLIC CYLINDER 'standing' on the parabola in the plane $\zeta' = 0$ given by that equation and having its generators in the direction $\xi = \eta' = 0$.

7. Conditions in terms of the general equation

The conditions for the general equation to represent the various types of quadric can all be obtained in terms of the coefficients in that equation. It is doubtful whether there is great value in such analysis, and we do not pursue it. There are, however, two basic conditions which are worthy of note:

(i) The characteristic equation

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

may be written (with a change of sign) in the form

$$\lambda^3-\lambda^2(a+b+c)+\lambda(bc+ca+ab-f^2-g^2-h^2)-$$
$$-(abc+2fgh-af^2-bg^2-ch^2) = 0.$$

Hence the product of the roots λ_1, λ_2, λ_3 is given by the formula

$$\lambda_1\lambda_2\lambda_3 = abc+2fgh-af^2-bg^2-ch^2$$
$$= D$$

in the notation given on p. 161.

It follows that *in order for one (or more) of the values of* λ_1, λ_2, λ_3 *to be zero, it is necessary and sufficient that*

$$D = 0.$$

Thus the discussions of §§ 5, 6 presuppose the condition $D = 0$.

(ii) The equations for the centre (if any) of the quadric are (p. 180)

$$ax_1+hy_1+gz_1+u = 0,$$
$$hx_1+by_1+fz_1+v = 0,$$
$$gx_1+fy_1+cz_1+w = 0.$$

When these conditions are satisfied, the value of S_{11} can be written in the form

$$S_{11} \equiv x_1(ax_1+hy_1+gz_1+u)+y_1(hx_1+by_1+fz_1+v)+$$
$$+z_1(gx_1+fy_1+cz_1+w)+(ux_1+vy_1+wz_1+d)$$
$$\equiv ux_1+vy_1+wz_1+d,$$

so that $$ux_1+vy_1+wz_1+(d-S_{11}) = 0.$$

Eliminate x_1, y_1, z_1 between this equation and the three equations from which we began:

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d-S_{11} \end{vmatrix} = 0.$$

Thus
$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} + \begin{vmatrix} a & h & g & 0 \\ h & b & f & 0 \\ g & f & c & 0 \\ u & v & w & -S_{11} \end{vmatrix} = 0,$$

or (p. 161)
$$\Delta - DS_{11} = 0.$$

Note carefully that the argument has presupposed the existence of x_1, y_1, z_1, so that this relation is established only for *central* quadrics.

(*a*) *The case* $D \neq 0$. If the characteristic roots are not zero, then $\Delta = 0$ implies $S_{11} = 0$, in which case the centre lies on the quadric and the surface is then a cone. Hence *the condition for the general equation of the second degree to represent a cone is*

$$\Delta = 0.$$

(*b*) *The case* $D = 0$. If $D = 0$, then, for a central quadric, it follows that $\Delta = 0$ also. The possibilities when $D = 0$ have already been enumerated.

Thus a procedure for the analysis of the general equation is:

Evaluate D.
If $D \neq 0$, the quadric is central, being a cone if $\Delta = 0$.
If $D = 0$, apply the analysis of §§ 5, 6.

COROLLARY. To prove that, *referred to parallel axes through the centre, the equation of a* **central** *quadric*

$$S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d = 0$$

assumes the form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+\Delta/D = 0.$$

The new coordinates (x', y', z') are connected with the old by means of the relations

$$x = x'+x_1, \quad y = y'+y_1, \quad z = z'+z_1,$$

so that the equation is

$$a(x'+x_1)^2+... = 0,$$

or
$$ax'^2+by'^2+cz'^2+2fy'z'+2yz'x'+2hx'y'+S_{11} = 0,$$

the terms of first degree in x', y', z' vanishing since (x_1, y_1, z_1) is the centre. But (p. 184)

$$S_{11} = \Delta/D,$$

so that, dropping dashes, the equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+\Delta/D = 0.$$

EXAMPLES

1. Reduce to standard form the quadrics

(i) $2x^2-4y^2+2z^2+2yz-5zx+2xy-2x-y-2z = 0.$

(ii) $3x^2+y^2+3z^2+2zx+2x-2y+6z+3 = 0.$

(iii) $2x^2+2y^2+5z^2+2yz+2zx+4xy-14x-14y-16z+26 = 0.$

(iv) $x^2+4y^2+z^2+2yz-zx+2xy+x+y+4z-6 = 0.$

2. Prove that the equation

$$x^2+2y^2+3z^2+4zx+4xy+10x+8y+10z+k = 0$$

represents a hyperboloid of one sheet if $k < 14$, a quadric cone if $k = 14$, and a hyperboloid of two sheets if $k > 14$.

ANSWERS

CHAPTER I

§ 1. 1. $(0,0,0), (2,0,0), (0,3,0), (0,0,5), (2,3,5), (0,3,5), (2,0,5), (2,3,0)$.
$(1,0,0)$, $(2,0,2\frac{1}{2})$, $(2,1\frac{1}{2},5)$, $(2,3,2\frac{1}{2})$, $(1,3,0)$, $(0,1\frac{1}{2},0)$.

2. $(1,0,0)$, $(0,4,0)$, $(0,0,3)$, $(0,4,3)$, $(1,0,3)$, $(1,4,0)$.

3. $(0,4,8)$, $(2,0,8)$, $(2,4,0)$.

4. $(4,20,8)$, $(0,20,8)$, $(4,0,8)$, $(4,20,0)$.

7. $x = 3$; $y = 2$, $z = 1$.

§ 6. 1. $\sqrt{200}$, $\sqrt{62}$, $\sqrt{34}$, $\sqrt{146}$, $\sqrt{246}$, $\sqrt{134}$.

2. $2x+18y+2z = 27$.

3. (i) $x^2+y^2+z^2-3x+3y+z-18 = 0$.
(ii) $x+3y+9z+10 = 0$.
(iii) $3x^2+3y^2+3z^2-4x+24y+48z+120 = 0$.

§ 17. 1. $(0,-1/\sqrt{2},1/\sqrt{2})$, $(-1/\sqrt{2},0,1/\sqrt{2})$, $(-1/\sqrt{2},1/\sqrt{2},0)$; $1/\sqrt{2}$.

2. $\frac{1}{2}\pi$; $\cos^{-1}(\frac{5}{7})$.

3. 6, 13; 4.

4. $\cos^{-1}(\frac{14}{15})$.

5. $(\frac{6}{5},\frac{27}{5},7)$, $(\frac{24}{5},\frac{3}{5},-1)$.

7. $\frac{2}{3}$.

CHAPTER II

§ 3. 1. $(\frac{3}{2},-\frac{1}{2},1)$; $(-15,5,-10)$.

2. $(\frac{1}{3},\frac{4}{3},3)$, $(-\frac{4}{3},\frac{5}{3},1)$.

3. $(19,10,9)$, $(-11,-2,-9)$.

4. $(-\frac{1}{2},\frac{8}{3},\frac{35}{6})$, $(-\frac{7}{4},\frac{7}{2},\frac{35}{4})$.

6. $(0,2,\frac{23}{4})$, $(4,0,\frac{13}{4})$, $(\frac{46}{5},-\frac{13}{5},0)$.

§ 8. 1. $(\frac{53}{33},0,0)$, $(0,-53,0)$, $(0,0,-\frac{53}{8})$.

4. $(\frac{1}{9},\frac{2}{9},-\frac{1}{3})$.

5. $(\frac{53}{75},1,\frac{97}{75})$.

6. (i) 3, 1, $\frac{1}{3}$; (ii) 2, $\frac{51}{13}$, $\frac{37}{13}$; (iii) 4, $\frac{46}{13}$, $\frac{90}{13}$.

7. $43x-11y+17z = 52$, $x+3y-z = 4$, $13x-y+7z = 32$,
$31x-7y+9z = 44$.
$(3\rho,-4\rho,12\rho)$, where $\rho = \frac{4}{29}, -\frac{4}{21}, \frac{32}{127}, \frac{44}{229}$.

8. Intersections of $12x+8y-17z+17 = 0$ with, respectively,
$5x+y+4z = 4$, $2x-3y = -6$, $3x+4y+4z = 10$.

9. $$x = \frac{(a+b+c)\{(b-a)(b-1)+(c-a)(c-1)\}}{2(a^2+b^2+c^2-bc-ca-ab)}; \text{ etc.}$$

§ 11. 1. $x/a+y/b+z/c = 1$.

2. $\pm 6x \pm 4y \pm 3z = 12$.

3. $(3,-2,4)$, $(1,3,5)$, $(0,0,3)$, $(4,2,2)$, $(1,3,1)$, $(2,0,4)$.

4. $5/\sqrt{3}$, $3/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$;
$6/\sqrt{3}$, $4/\sqrt{3}$, $2/\sqrt{3}$, 0;
$3/\sqrt{2}$, $1/\sqrt{2}$, $1/\sqrt{2}$, $3/\sqrt{2}$.

5. $(1, 1, 1)$.
6. $x+2y-z = 16$.
7. $x-y+2z = 6$.
8. $2x+2y+z = 9, \quad 4x+4y-7z = -27$.
9. $(-2, 8, 0)$, $(\frac{12}{7}, \frac{4}{7}, 0)$, $(\frac{9}{2}, \frac{3}{2}, 0)$.
10. $4x-y+z-8 = 0$, $3y+3z+2 = 0$.
13. $\dfrac{x-2}{3} = \dfrac{y-1}{2} = \dfrac{z-6}{-6}$,
$\dfrac{x-1}{1} = \dfrac{y+7}{8} = \dfrac{z-2}{4}$,
$\dfrac{x-5}{4} = \dfrac{y-3}{10} = \dfrac{z}{-2}$.
14. $\dfrac{x-1}{2} = \dfrac{y-2}{0} = \dfrac{z+3}{3}$
$x-5 = -y = z-2$,
$\dfrac{x-3}{0} = \dfrac{y-4}{2} = \dfrac{z-1}{1}$;
$(3, 2, 0)$.
15. $\dfrac{2x-x_1-x_4}{x_2+x_3-x_1-x_4} = \dfrac{2y-y_1-y_4}{y_2+y_3-y_1-y_4} = \dfrac{2z-z_1-z_4}{z_2+z_3-z_1-z_4}$; etc.
Concur in
$\{\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4), \frac{1}{4}(z_1+z_2+z_3+z_4)\}$.
16. $\dfrac{x-1}{-5} = \dfrac{y-2}{4} = \dfrac{z-3}{1}$.
17. $\dfrac{x}{7} = \dfrac{y-1}{-2} = \dfrac{z-2}{1}$.
18. $\dfrac{x-1}{13} = \dfrac{y-1}{1} = \dfrac{z-1}{-23}$.
19. $(0, 2/\sqrt{5}, 1/\sqrt{5})$.
20. $(3/\sqrt{110}, -1/\sqrt{110}, 10/\sqrt{110})$, $(1/\sqrt{18}, 1/\sqrt{18}, 4/\sqrt{18})$, $(1/\sqrt{11}, -1/\sqrt{11}, 3/\sqrt{11})$.
21. $2x+3y-7z+28 = 0$.
22. $(2, 5, -1)$.
24. $(-3, -1, 1)$.
26. $13x+23y+5z = 0$.
28. $(0, -3, 2)$, $(\frac{3}{4}, 0, \frac{5}{4})$, $(2, 5, 0)$.

MISCELLANEOUS EXAMPLES

1. $4x+3y-12z+7 = 0$.
2. $x-3y = 1$.
3. 6.
4. $ln/\sqrt{(l^2+m^2)}$, $mn/\sqrt{(l^2+m^2)}$, $-\sqrt{(l^2+m^2)}$.
5. $10x-5y+3z = 27$; $(0, 3, 14)$; $\sqrt{134}$.
6. $(5, 4, 1)$; $x/5 = (y-1)/3 = z/1$.
7. $y = z = 0$; $z = x = 0$; $x = y = 0$; $x-y = y+z = 0$.

8. $\{(a^2+b^2+c^2)/2a,\ (a^2+b^2+c^2)/2b,\ (a^2+b^2+c^2)/2c\}$.
9. 1.
10. (i) $(-1, 12, 12)$, (ii) $x-y = -7$, (iii) $(\pm\sqrt{2}, 10\pm\sqrt{2}, 10\mp 1/\sqrt{2})$.
11. $z^2+yz+zx+x-y+1 = 0$.
12. $x/1 = y/2 = z/3$; $\cos^{-1}(4/\sqrt{133})$, $\cos^{-1}(40/\sqrt{1834})$.
13. $x/7 = (y-2)/(-2) = z-2$.
14. $x-2y-z+11 = 0$; $(x-18)/1 = (y-16)/(-2) = (z+3)/(-1)$.
15. $25x-32y-2z+50 = 0$.
17. $\sum (x-a)(m\nu-n\mu) = 0$.
19. $\{x_1(z_1-a), y_1(z_1+a), z_1^2-a^2\}$; $Ax(z-a)+By(z+a)+C(z^2-a^2) = 0$.
20. $x-2y+z = 0$.
21. $A(lA+mB+nC)-l(A^2+B^2+C^2)$, etc.
22. $4x-4y+8z = 9$, $8x+4y+4z = 41$.
23. 9; $32x+34y+13z = 108$, $4x+11y+5z = 27$.
24. $x = p-\{2a(ap+bq+cr+d)/(a^2+b^2+c^2)\}$, etc.;
$(\frac{1}{12}, -\frac{5}{12}, \frac{2}{3})$, $(-\frac{7}{6}, -\frac{5}{6}, \frac{2}{3})$.
27. $x = ab^2c^2/(b^2c^2+c^2a^2+a^2b^2)$, etc.
29. Circle.
36. $9/\sqrt{2}$.
37. $(-8, 28, 2)$, $(\frac{1}{3}, 3, 2)$, $(7, -2, 2)$, $(\frac{39}{17}, \frac{26}{17}, -\frac{16}{17})$; $(2, 3, 1)$.
38. Same as 27.

CHAPTER III

MISCELLANEOUS EXAMPLES

2. $l = \dfrac{(\mathbf{c}^2)(\mathbf{a}.\mathbf{b})-(\mathbf{a}.\mathbf{c})(\mathbf{b}.\mathbf{c})}{(\mathbf{b}^2)(\mathbf{c}^2)-(\mathbf{b}.\mathbf{c})^2}$, $m = \dfrac{(\mathbf{b}^2)(\mathbf{a}.\mathbf{c})-(\mathbf{a}.\mathbf{b})(\mathbf{c}.\mathbf{b})}{(\mathbf{b}^2)(\mathbf{c}^2)-(\mathbf{b}.\mathbf{c})^2}$.
5. $(\mathbf{l}^2)(\mathbf{m}.\mathbf{n})-(\mathbf{l}.\mathbf{m})(\mathbf{l}.\mathbf{n})$, $(\mathbf{lmn})\mathbf{l}$.
6. $\{(\mathbf{a}.\mathbf{b})\mathbf{a}+\mathbf{b}+(\mathbf{a}\wedge\mathbf{b})\}/\{1+(\mathbf{a}^2)\}$.
7. Compare § 16.
9. $\mathbf{x} = \{k\mathbf{a}-(\mathbf{a}\wedge\mathbf{b})\}/(\mathbf{a}^2)$, valid only if $\mathbf{a}.\mathbf{b} = 0$.
10. $\lambda = (\mathbf{dbc})/(\mathbf{abc})$, etc.
11. $\mathbf{a}-(\mathbf{a}-\mathbf{b}.\mathbf{t})\mathbf{t}$.
12. $\lambda+\mu+\nu = 1$; $(\mathbf{d}-\mu\mathbf{b})/(1-\mu)$, $(\mathbf{d}-\nu\mathbf{c})/(1-\nu)$.
13. $\cos\theta = \{(\mathbf{p}.\mathbf{q})-(\mathbf{p}.\mathbf{n})(\mathbf{q}.\mathbf{n})\}/\sqrt{\{[(\mathbf{p}^2)-(\mathbf{p}.\mathbf{n})^2][(\mathbf{q}^2)-(\mathbf{q}.\mathbf{n})^2]\}}$, taking $(\mathbf{n}^2) = 1$.
15. $(\mathbf{m}.\mathbf{n}) = 0$; 1.
17. $x-2y-2z+3 = 0$.

CHAPTER IV

MISCELLANEOUS EXAMPLES

5. (i) 0, (ii) 3, (iii) 6.
8. $\begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix}$.

9. $\begin{pmatrix} \frac{1}{4}\pi\rho & 0 & 0 \\ 0 & \frac{5}{4}\pi\rho & 0 \\ 0 & 0 & \frac{3}{2}\pi\rho \end{pmatrix}$.

CHAPTER V

MISCELLANEOUS EXAMPLES

1. $x^2+y^2+z^2-4x-4y-2z+5 = 0$; $z = 3$, $z = -1$.
2. $x^2+y^2+z^2-2x-2y-2z-6 = 0$.
3. $x^2+y^2+z^2-2x-4y-6z = 0$; $\sqrt{2}$.
4. $(1,3,3)$, $(4,7,-2)$; $(x-1)/3 = (y-3)/4 = (z+2)/5$.
5. $x^2+y^2+z^2-6x-16z-48 = 0$.
6. $(x+4)/11 = (y-3)/10 = (z+7)/2$.
   $(x-18)(x-11\rho+4)+(y-23)(y-10\rho-3)+(z+3)(z-2\rho+7)$, where $\rho = \pm\frac{2}{5}$.
7. $3x-y-3z-2 = 0$, $3x-y-3z+6 = 0$; $8/\sqrt{19}$.
8. $x^2+y^2+z^2 = 25$; $-65 \leqslant \lambda \leqslant 65$; 4.
9. $(1,-1,-1)$; $\sqrt{6}$; $x^2+y^2+z^2-2x+2y+2z-3 = 0$.
10. $x^2+y^2+z^2+2x-6y+4z-11 = 0$; $(1,1,-2)$, $\sqrt{17}$;
    $x = 0$, $2y^2+z^2-4y+4z-11 = 0$.
11. 1; $(-\frac{2}{3}, -\frac{5}{3}, \frac{1}{3})$.
12. $x^2+y^2+z^2-2x+4y+2z+5 = 0$.
13. $x^2+y^2+z^2-2x-4y-6z-11 = 0$; $3x+4y+12z = 8$.
14. $(1,5,0)$, $(\frac{11}{3}, \frac{11}{3}, \frac{4}{3})$.
15. $l+n-d = \pm 2$, $4l+2m+n-d = \pm 1$; $(7,4,1)$, $(3,\frac{4}{3},1)$.
16. $x^2+y^2+z^2-ax-by-cz = 0$.
18. $(4,0,0)$, $(-2,4,-2)$.
19. $(2/\sqrt{3})+(1/\sqrt{2})$, $x+y+z = 6+\sqrt{(3/2)}$;
    $x^2+y^2+z^2-11x+5y-5z+42 = 0$.
20. $(l^2+m^2+n^2)(u^2+v^2+w^2+c) > (lu+mv+nw+p)^2$;
    $x^2+y^2+z^2-2x-2y+2z+2 = 0$.
22. $2x+y-2z+1 = 0$; $(\frac{2}{3}, \frac{1}{3}, \frac{4}{3})$; $\sqrt{2}$.
23. 17/3.
24. $(1,2,3)$; 3.
26. $(1/\sqrt{26}, 3/\sqrt{26}, 4/\sqrt{26})$; $26(\mathbf{r}.\mathbf{p})^2 = r^2$.

CHAPTER VI

MISCELLANEOUS EXAMPLES

4. $x = a+l\{l(\xi-a)+m(\eta-b)+n(\zeta-c)\}$, etc.;
   $$x^2\sin^2\theta+y^2\cos^2\theta+xy\left(\frac{1+\lambda^2}{1-\lambda^2}\right)\sin 2\theta+z^2-2zc\left(\frac{1+\lambda^2}{1-\lambda^2}\right)+c^2 = 0.$$
6. See § 8.
10. $\sum x \sum ax+\sum a(b+c)x+abc = 0$.
11. $(x-a\cos\theta)/a\sin\theta = (y-a\sin\theta)/(-a\cos\theta) = \pm z/c$.
    $a = c$ (if positive).
12. (ii) $(-1,-\frac{1}{3})$, $(-\frac{1}{2},1)$; $(-\frac{7}{5},-2,\frac{3}{5})$.
13. The most obvious one is $x+y+z = \sqrt{(a^2+b^2+c^2)}$.

15. See § 5.
17. $x = lp/a\Delta$, etc., where $\Delta = l^2/a+m^2/b+n^2/c$.
    Direction having ratios (bcl, cam, abn).
22. See § 5.
26. $(\pm a/\sqrt{2}, b/2, c/2)$.
29. See § 8.
36. See § 1, at the end.
37. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}\sin^2\alpha = \cos^2\alpha$.
43. $a(hx-fz)^2+b(hy-gz)^2+c(z-h)^2 = 0$.
44. $(cx-az)^2+c^2y^2-r^2(z-c)^2 = 0$.
45. $8z^2+4yz-zx-5xy = 0$.
46. $(a^2-r^2)x^2+a^2y^2+(az-cx)^2 = 0$.
47. There is only one circle, lying in the plane $z = 0$. The two cones are, so to speak, parallel.
48. $b = c$; $x^2+y^2+z^2+\{(a-b)/b\}x = 0$.

CHAPTER VIII

PART I

MISCELLANEOUS EXAMPLES

1. $4,\ 4\pm 2\sqrt{3}$.
2. $t = 1, 2, 5$; ratios $(0, 1, -1)$, $(1, 1, 1)$, $(-2, 1, 1)$.
3. Direction ratios $(1, -2, 0)$, $(1, 1, 1)$, $(5, -16, 12)$.
4. $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$, $(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.
5. Values 1, $\frac{1}{3}(1\pm 2i\sqrt{2})$; vectors $(1/\sqrt{2}, 0, 1/\sqrt{2})$, $(1, \mp i\sqrt{2}, -1)$.

10. $$\begin{pmatrix} 20ma^2 & 0 & -2ma^2 \\ 0 & 20ma^2 & -4ma^2 \\ -2ma^2 & -4ma^2 & 20ma^2 \end{pmatrix}.$$

12. Inertia tensor:
$$\begin{pmatrix} \frac{2}{5}ma^2 & 0 & -\frac{3}{8}ma^2 \\ 0 & \frac{7}{5}ma^2 & 0 \\ -\frac{3}{8}ma^2 & 0 & \frac{7}{5}ma^2 \end{pmatrix}.$$
    Direction ratios $(3, 0, 1)$, $(0, 1, 0)$, $(1, 0, -3)$.

13. $4ma^2$, $\frac{11}{3}ma^2$.
14. Direction ratios $(1, -1, 0)$, $(1, 1, 1)$, $(1, 1, -2)$; lengths $1/\sqrt{2}$, $1/\sqrt{3}$, $1/\sqrt{6}$.
15. Elliptic cylinder, axis of 'symmetry' $(-2, 1, 2)$; axes of elliptic section through origin, $(2, 2, 1)$, $(1, -2, 2)$.
22. 2 and 3.

PART II

EXAMPLES

1. (i) $3x'^2-3y'^2 = 2z'$,
   (ii) $x'^2+2y'^2+4z'^2 = 1$,
   (iii) $x'^2+2y'^2 = 1$,
   (iv) $x'^2+3y'^2 = 2z'$.

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PRINTED IN GREAT BRITAIN
AT THE UNIVERSITY PRESS, OXFORD
BY VIVIAN RIDLER
PRINTER TO THE UNIVERSITY