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METHOD OF COORDINATES

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

А. С. Смогоржевский

МЕТОД КООРДИНАТ

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INTRODUCTION

The application of algebra to a study of the properties of geometrical figures played an important role in the development of geometry and grew into an independent branch of science – the *analytical geometry*. The rise of analytical geometry is associated with the discovery of its basic method, the method of coordinates.

By *coordinates of a point* we mean the numbers that determine its position on a given line or a given surface or in space. Thus, the position of a point on the earth's surface will be known, if we know its geographical coordinates – the latitude and the longitude.

In order to find the coordinates of a point, one must know the reference points from which measurements are carried out. In case of geographical coordinates, the equator and the zero meridian are reference points.

If reference points are given and the method of using them for finding the coordinates of a point is indicated, we say that a *system of coordinates* is given.

Description of geometrical figures through equations (see Sec. 4) is a characteristic feature of the method of coordinates and allows the use of algebraic means for carrying out geometrical studies and for solving geometrical problems.

By imparting algebraic character to geometrical studies, the method of coordinates transfers to geometry the most important feature of algebra – the uniformity of methods for solving problems. While in arithmetic and elementary geometry one has to look for, as a rule, a special way for solving every problem in algebra and analytical geometry, the solution to all the problems is found according to a common plan, which can be easily applied to any problem. It can be said that analytical geometry occupies the same position with respect to elementary geometry as algebra with respect to arithmetic. The main importance of the method of coordinates lies in that for solving problems it conveys to geometry methods that originally belong to algebra and hence have much in common. The reader must, however, be cautioned against a complete rejection of application of elementary geometry, since in several cases it helps us to get elegant solutions that are much simpler than those obtained through the method of coordinates. Another salient feature of the method of coordinates lies in the fact that its application saves us from the need to rush to a visual representation of complex spatial configurations.

In practical applications of the concept of coordinates, the coordinates of an object taken arbitrarily as a point may be given only approximately. The given coordinates of an object mean that the point described by these coordinates is either one of the points of this object, or is very close to it.

The size and aim of the book has forced us to restrict ourselves to an account of the basic facts about the method of coordinates and its simplest applications. Considerable attention has been devoted to a description of geometrical figures through equations, which presents considerable difficulties for beginners. Elucidation of this question is accompanied by comprehensively solved problems.

Author

Sec. 1. Coordinates of a point on a straight line

The most elementary case of introducing coordinates is connected with the determination of the position of a point on a straight line. We shall begin a description of the method of coordinates with a consideration of this case.

We mark two arbitrary, but different, points O and E on a straight line (Fig. 1) and take an intercept OE as a unit of length*.

We shall consider that every point on the straight line OE corresponds to a number called the *coordinate* of the given point and determined in the following way: the coordinate of point P on the straight line OE is a positive number equal to the length of the intercept OP, if the point P is on the same side of the point O as the point E; the coordinate of point P on the straight line OE is a negative number equal in absolute value to the length of the intercept OP if points P and E lie on

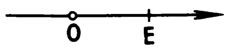

Fig. 1

different sides of the point O; the coordinate of point O is equal to zero.

If these conditions are satisfied, the straight line OE is called the *numerical axis* or the *axis of coordinates*. The point O is called the *origin of coordinates*. The part of numerical axis containing points with positive coordinates is called its positive part; that containing points with negative coordinates is called its negative part.

Each point on a given numerical axis has a definite coordinate; moreover, the coordinates of two different points on one and the same numerical axis are different. On the other hand, every real number is the coordinate of a definite point on the given numerical axis. For example, the coordinate of the point E is $+1$, and the number -1 is the coordinate of a point symmetrical to E with respect to O. Notation $E(1)$, $A\left(-2\frac{1}{3}\right)$, $B(x)$, $C(x_1)$, $D(x_2)$,

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* The points O and E may be chosen so that the intercept OE is equal to an already given unit of length, e. g. 1 cm.

etc. means that the numbers 1, $-2\frac{1}{3}$, x, x_1, x_2 are the coordinates of points E, A, B, C, D, respectively.

The direction, corresponding to the deviation along the numerical axis from the point O towards the point E is called the direction of numerical axis; it is usually indicated by an arrow (Fig. 1).

Sec. 2. Coordinates of a Point in a Plane

We construct two mutually perpendicular numerical axes Ox and Oy in a plane so that their point of intersection is the origin of coordinates for both of them (Fig. 2). We shall call the axes Ox and Oy the x-axis and y-axis, respectively, and the plane in which they are situated – plane Oxy*. We shall consider that the unit of length for both coordinate axes is the same.

The axes Ox and Oy divide the plane Oxy in four quadrants, the order of numbering the quadrants with respect to the direction of coordinate axes is shown in Fig. 2.

Let us consider an arbitrary point P in the plain Oxy and P_x and P_y as the feet of perpendiculars drawn from this point on the axes Ox and Oy, respectively, i. e. its rectangular projections on these axes (Fig. 3). We shall denote the coordinate

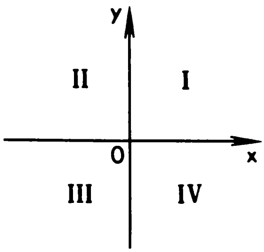

Fig. 2

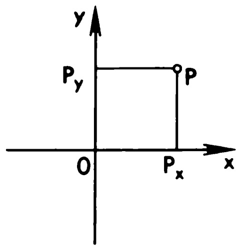

Fig. 3

of point P_x on the axis Ox by x and the coordinate of point P_y on the axis Oy by y. The numbers x and y are called coordinates of the point P, which is denoted in the following

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* Axes Ox and Oy are also called axes of coordinates or coordinate axes.

way: $P(x; y)$. Coordinates of this type are called *rectangular Cartesian coordinates* *.

Thus, the determination of coordinates of a point P *in a plane* leads to a determination of coordinates of *two* points (P_x and P_y) on *numerical axes.*

The coordinate x of the point P_x is called the *abscissa* of the point P. The coordinate y of the point P_y is called the *ordinate* of the point P. If the point P lies on the axis Ox, then its ordinate is equal to zero; if the point P lies on the axis Oy, then its abscissa is equal to zero. Both coordinates of the point O are equal to zero.

In Fig. 4 are shown the signs of coordinates of a point depending on the quadrant in which it lies; on the left side is shown the sign of abscissa, on the right side, the sign of ordinate.

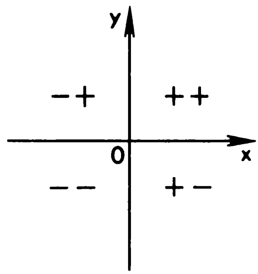

Fig. 4

Let us show how to locate a point P if its coordinates x and y are known. We plot the point P_x by locating it on the axis Ox corresponding to its abscissa x and the point P_y by locating it on the axis Oy corresponding to its ordinate y (see Fig. 3). We draw a perpendicular to the axis Ox through P_x, and a perpendicular to the axis Oy through P_y. These perpendiculars intersect at the desired point P.

The above plotting may be modified in the following way (Fig. 5): we locate the point P_x, by drawing a perpendicular to the axis Ox and cutting off an intercept P_xP on the perpendicular equal in length to the absolute value of the coordinate y, moreover, it is cut off from the point P_x upward if $y > 0$ and downward,

* After the famous XVII century philosopher and mathematician Rene Descartes.

if $y < 0$*. If $y = 0$, the point P coincides with the point P_x.

On the basis of the last construction, it may be said that the coordinates of a point indicate one of the ways leading

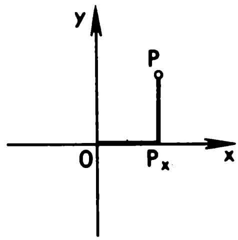

Fig. 5

from the origin of coordinates to the given point: knowing the abscissa x of the point P, we find the part OP_x of this way, while the ordinate y of the point P gives us its second part P_xP.

We shall mention, by the way, that the idea of coordinates is not a conception of mathematicians: it has been borrowed from practice and in its primitive form, the coordinate system is used even by people not acquainted with mathematics. We recall, for example, a fragment from a poem "Who can be happy and free in Russia?" by the famous XIX century Russian poet N. Nekrasov:

"Go straight down the road,
Count the poles until thirty.
Then enter the forest
And walk for a verst.
By then you'll have come
To a smooth little lawn
With two pine-trees upon it.
Beneath these two pine-trees
Lies buried a casket
Which you must discover."**

---

* To be more precise, the point P and the positive part of the axis Oy must lie on the same side of axis Ox if $y > 0$, and on different sides if $y < 0$. In future, we shall not give a detailed description, assuming that the positive part of the axis Ox is on the right side of its negative part, the positive part of the axis Oy is right above its negative part.

** Translated by Juliet M. Soskice.

Fig. 6

Here 30 and 1 are the coordinates of the lawn (in the sense that they define the coordinates of an object – see Inroduction); a verst has been taken as the unit of length (Fig. 6).

Sec. 3.
Basic Problems

Usually, the solution of a complex problem boils down to the solution of a number of simple problems; some of which, encountered more frequently and noted for their extreme simplicity, are called basic. In this section we shall consider two basic geometry problems: the determination of distance between two points, and the determination of the area of a triangle whose vertices are given. Since in analytical geometry a point is defined by its coordinates, the solution to the given problems lies in finding the formulae giving the required quantities through the coordinates of the given points.

PROBLEM 1. Find the distance between two given points.

Let the points $A(x_1;\ y_1)$ and $B(x_2;\ y_2)$ be given in the plane Oxy. We draw perpendiculars AA_x and BB_x from these points on the axis Ox and perpendiculars AA_y and BB_y on the axis Oy (Fig. 7). We denote the length of the intercept AB by d.

Let the straight lines AA_y and BB_x intersect at the point C.

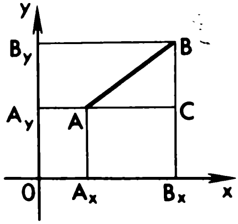

Fig. 7

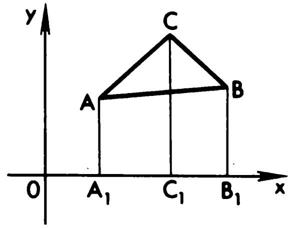

Fig. 8

Since the triangle ABC is rectangular, we have

$$d = AB = \sqrt{AC^2 + CB^2}. \qquad (1)$$

Taking into account, that

$$OA_x = x_1, \quad OB_x = x_2, \quad OA_y = y_1, \quad OB_y = y_2,$$
$$AC = A_xB_x = OB_x - OA_x = x_2 - x_1,$$
$$CB = A_yB_y = OB_y - OA_y = y_2 - y_1,$$

we get from (1):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad (2)$$

It can be proved that this formula is valid for any position of the points A and B.

PROBLEM 2. Determine the area of a triangle from the coordinates of its vertices.

Let the points $A(x_1;\ y_1)$, $B(x_2;\ y_2)$, $C(x_3;\ y_3)$ be the three vertices of the triangle. We draw perpendiculars AA_1, BB_1, CC_1 from these points to the axis Ox (Fig. 8). It is obvious that the area S of the triangle ABC may be expressed in terms of the area of the trapeziums AA_1B_1B, AA_1C_1C, CC_1B_1B:

$$S = \text{area } AA_1C_1C + \text{area } CC_1B_1B - \text{area } AA_1B_1B.$$

Since

$$AA_1 = y_1, \quad BB_1 = y_2, \quad CC_1 = y_3,$$
$$A_1B_1 = x_2 - x_1, \quad A_1C_1 = x_3 - x_1, \quad C_1B_1 = x_2 - x_3,$$

we have

$$\text{area } AA_1C_1C = \frac{1}{2}(y_1 + y_3)(x_3 - x_1),$$

$$\text{area } CC_1B_1B = \frac{1}{2}(y_2 + y_3)(x_2 - x_3),$$

$$\text{area } AA_1B_1B = \frac{1}{2}(y_1 + y_2)(x_2 - x_1).$$

Hence,

$$S = \frac{1}{2}\Big[(y_1 + y_3)(x_3 - x_1) + \\ + (y_2 + y_3)(x_2 - x_3) - (y_1 + y_2)(x_2 - x_1)\Big],$$

whence after simplification we get

$$S = \frac{1}{2}\Big[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\Big]. \tag{3}$$

We observe that formula (3) is valid, accurate to the sign*, for any positions of the vertices of the triangle, although this is not obvious from the above deduction.

Sec. 4. Equations of Geometrical Figures

We mark in a plane a finite or infinite number of points. These marked points form a plane geometrical figure. This figure may be defined if we can tell the points marked by us in the plane.

The indicated points may be marked by a pencil or in ink, what we exactly do, while describing, for example, a circle with a compass and drawing a straight line with a ruler**. It is possible to tell which of the points are marked, by using the notion of the loci, what we exactly do, while defining a circle as a locus of the points in a plane situated at a given distance from a given point. Finally, for this purpose, we may adopt an

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* That is, the value S found by formula (3) may be negative, but its absolute value equals the value of the area of the triangle.

** Strictly speaking, we mark not the points, but that part of the paper which may be considered as carrying the points of interest.

original approach which is used in analytical geometry, and consists in the following.

We construct axes Ox and Oy in rectangular Cartesian coordinates in a plane. An equation is then given which contains the quantities x and y or one of these quantities* and only those points are chosen whose coordinates x and y satisfy the given equation. The points thus singled out form a certain figure; the given equation is called the equation of this figure.

Thus, an equation in analytical geometry acts as a sieve, rejecting the points not needed by us and retaining the points which form the figure of interest.

Note that in the equation of a figure, quantities x and y are called *variables*, since they, generally speaking, vary as we pass from one point in the figure to another (of course, if the figure contains not less than two points). In addition to variables x and y, an equation may contain constant quantities also, moreover, some or all of them may be denoted by letters.

We write an equation with variables x and y in a general form

$$f(x, y) = 0. \tag{4}$$

Here $f(x, y)$** denotes a mathematical expression containing the quantities x and y or at least one of them.

In accordance with the aforesaid, we shall consider that *equation* (4) *defines a certain figure as a set of points whose rectangular Cartesian coordinates satisfy this equation.*

From this basic position of analytical geometry, it is not difficult to draw the following conclusion: A given point P belongs to the figure F defined by equation (4) if its coordinates satisfy equation (4), otherwise the point P does not belong to the figure F.

Let us consider some simple examples.

Example 1. Equation

$$y - x = 0$$

---

* In terminology adopted in algebra, such an equation is called an equation with two unknowns (or with one unknown if only one of the quantities x or y enters into the equation).

** Read as "function f of x and y". Other letters such as F, φ: $F(x, y)$, $\varphi(x, y)$, etc. may be used in place of f. Some examples of expressions indicated like this are: $y - x$, $x^2 + y^2 - 4$, $x \sin y$, $\frac{x+y}{x-y}$, etc.

or, in other words

$$y = x \tag{5}$$

defines a straight line which is a bisector of the angle formed by the positive parts of the coordinate axes (Fig. 9).

Actually, the point $P(x; y)$ on this straight line is equidistant from the axes of coordinates, its distances from the axes Ox and Oy are equal to y and x, respectively, if the line is in the first quadrant and to $-y$ and $-x$, respectively, if it is in the III quadrant. In both cases, the coordinates of the point P satisfy equation (5). On the other hand, the coordinates of a point not lying on the above-mentioned straight line cannot be equal to each other.

In a similar way, we convince ourselves that the equation

$$y = -x$$

defines a straight line which bisects the angle adjacent to the one formed by the positive parts of the coordinate axes (Fig. 10).

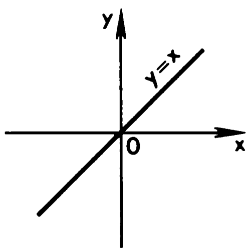

Fig. 9

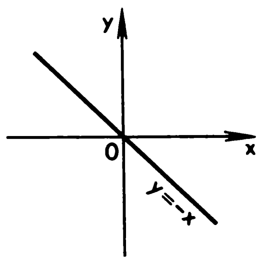

Fig. 10

EXAMPLE 2. Equation

$$y = b \tag{6}$$

defines a straight line parallel to the axis Ox. This straight line lies above the axis Ox if $b > 0$, below the axis Ox if $b < 0$ and coincides with the axis Ox if $b = 0$.

Note that equation (6) does not contain a variable x; it means that it does not impose any restrictions on the value of x; the value of x may be arbitrary.

Let us consider in detail the case, when $b = 0$, i. e., consider the equation

$$y = 0. \tag{7}$$

This equation shows that of all the points in the plane we must single out those and only those points whose distance from the axis Ox is equal to zero, i. e. the points lying on the axis Ox. Consequently, equation (7) defines the axis Ox.

EXAMPLE 3. Equation

$$x = a$$

defines a straight line parallel to the axis Oy. This straight line coincides with the axis Oy if $a = 0$.

EXAMPLE 4. Let the point $M(a; b)$ be the centre of a circle of radius r (Fig. 11). We take any point $P(x; y)$ lying on this circle.

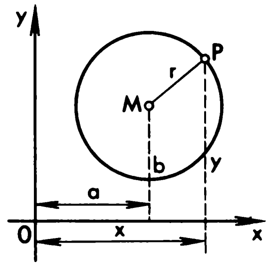

Fig. 11

Since the length of the intercept MP is equal to r, we have from formula (2),

$$r = \sqrt{(x - a)^2 + (y - b)^2}$$

whence

$$(x - a)^2 + (y - b)^2 = r^2. \qquad (8)$$

Consequently, equation (8) is the equation of a circle with radius r, with its centre at the point with coordinates a, b. If, in particular, the centre of the circle coincides with the origin of the coordinates, then $a = b = 0$ and equation (8) takes the form

$$x^2 + y^2 = r^2.$$

Let us consider, for example, the equation

$$x^2 + y^2 = 25, \qquad (9)$$

x	y
0	± 5
± 1	$\pm \sqrt{24} \approx \pm 4 \cdot 9$
± 2	$\pm \sqrt{21} \approx 4 \cdot 6$
± 3	± 4
± 4	± 3
± 5	0

which may be written in the form

$$y = \pm \sqrt{25 - x^2}. \tag{10}$$

Let us find a few points whose coordinates satisfy this equation, and plot them; first of all, we compile a table. In its left column, we shall write arbitrarily chosen values of x, in the right – the corresponding values of the quantity y calculated from formula (10).

The table gives the coordinates of the points belonging to the circle defined by equation (9). These points are plotted in Fig. 12.

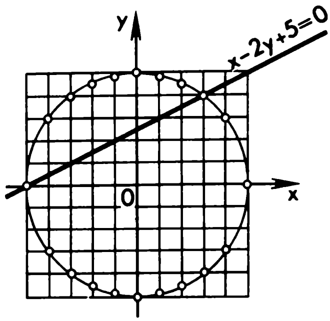

Fig. 12

We could have obtained more points on the given circle, if we had given the variable x not only integral but also fractional values, for example, ± 0.1, ± 0.2, etc.

We observe that both coordinates of any point in a plane are real numbers. Therefore in the given example, there is no point in finding y if $x < -5$ or $x > +5$, since in these cases y will have imaginary values.

Sec. 5. Equation of a Straight Line

Let us consider an equation of the first degree with variables x and y or with one of these variables. Obviously, on simplification such an equation may be expressed in the following form

$$Ax + By + C = 0, \tag{11}$$

where A, B and C are constants. Moreover, at least one of the quantities, A and B, is not equal to zero. We assume that $A \neq 0$.

Let us show that equation (11) represents a straight line. As a basis of the proof, we shall take the obvious fact that the area of a triangle is equal to zero if and only if all its vertices lie on the same straight line.

Let us give two different values y_1 and y_2 to the variable y and find from equation (11) the corresponding values x_1 and x_2 of the variable x. This may be obtained since the coefficient of x in equation (11) is different from zero. The points $L(x_1; y_1)$ and $M(x_2; y_2)$ belong to the figure (11)*. These are different points since $y_1 \neq y_2$. Let us consider another arbitrary point $N(x_3, y_3)$. Substituting successively the coordinates of the points L, M, N in the expression $Ax + By + C$ and calculating its value, we obtain three identities:

$$Ax_1 + By_1 + C = 0,$$

$$Ax_2 + By_2 + C = 0,$$

$$Ax_3 + By_3 + C = a.$$

Right-hand sides of first two identities are equal to zero since the coordinates of the points L and M satisfy equation (11). We denote the right-hand side of the third identity by a, a number which is equal to zero if the point N belongs to the figure (11) and not equal to zero otherwise.

We multiply both sides of the first identity by $y_2 - y_3$, of the second by $y_3 - y_1$, and of the third by $y_1 - y_2$ and add up the equations obtained. As a result, we obtain the following relation in which the coefficient of A, in view of formula (3), is equal to

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* In place of "figure defined by the equation $f(x; y) = 0$", we often say for the sake of brevity "figure $f(x, y) = 0$", or simply indicate the number of the equation which defines the given figure.

$2S$ where S is the area of the triangle LMN:

$$2A \cdot S + B(y_1y_2 - y_1y_3 + y_2y_3 - y_1y_2 + y_1y_3 - y_2y_3) + \\ + C(y_2 - y_3 + y_3 - y_1 + y_1 - y_2) = a(y_1 - y_2).$$

From this after obvious simplifications we get

$$2A \cdot S = a(y_1 - y_2) \tag{12}$$

where, as has been mentioned above, $A \neq 0$, $y_1 - y_2 \neq 0$. If the point N belongs to the figure (11), then $a = 0$, in which case we conclude from equation (12) that S is also equal to zero. Consequently, the point N lies on the straight line LM. We suppose now that N is any arbitrary point on the straight line LM, then $S = 0$. In this case it follows from equation (12) that a also equals zero, consequently, the point N belongs to the figure (11).

Thus, each point on the figure (11) lies on the straight line LM and each point on the straight line LM belongs to the figure (11). Hence, equation (11) defines a straight line, *Q.E.D.*

Let us now show that, conversely, the equation of any straight line can be written in the form (11). Let points $P(x_1;\ y_1)$ and $Q(x_2;\ y_2)$ lie on a given straight line. The equation

$$(x - x_1)(y_2 - y_1) - (y - y_1)(x_2 - x_1) = 0 \tag{13}$$

is of the first degree, hence it defines a straight line. This straight line is PQ since the coordinates of the points P and Q satisfy equation (13).

From the above, it follows that construction of a figure defined by an equation of the first degree is not difficult. Since this figure is, as shown above, a straight line, it is sufficient to find just two of its points, locate them and draw a straight line passing through them.

Let us consider, for example, the equation

$$x + y = 5. \tag{14}$$

It is not difficult to see that points $P(5;\ 0)$ and $Q(0;\ 5)$ belong to the straight line (14). It is drawn on Fig. 13.

We shall consider one more example.

Let the equation

$$y = 3 \tag{15}$$

be given.

We assign two arbitrary values to the variable x, for example, $x = -1$ and $x = 2$. In both cases $y = 3$. Hence, points $P(-1,\ 3)$

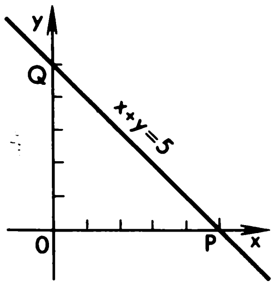

Fig. 13

and $Q(2, 3)$ belong to the straight line (15). This straight line is parallel to the axis Ox which could have been foreseen since equation (15) is a special case of the equation $y = b$ (see Example 2, Sec. 4)

Sec. 6. Method of Coordinates as a Means of Solving Geometrical Problems

As an illustration to the application of the method of coordinates we shall consider the solution of three problems. In each of them, we shall be required to draw a circle which, from the point of view of analytical geometry, is equivalent to writing an equation of the required circle or to a determination of its radius and the coordinates of the centre.

We shall give two solutions to each problem, the first by the method of coordinates and the second by means of elementary geometry. The procedure of solving problems of the first type is characterized by the fact that they follow a common plan and are similar in idea, while the solutions of the second type have much less in common and are based on the applications of different theorems. This fact is of significant importance and shows, albeit in some special examples, that the application of the method of coordinates considerably simplifies the search for ways leading to the solution of a problem.

PROBLEM 1. Plot a circle passing through points $A(1; 1)$, $B(4; 0)$, $C(5; 1)$.

First Solution. The equation of the required circle has the form

$$(x - a)^2 + (y - b)^2 = r^2 \qquad (16)$$

[see formula (8)].

Since points A, B, C lie on the required circle, their coordinates satisfy equation (16). Substituting successively into this equation the coordinates of the given points, we obtain the equalities

$$(1 - a)^2 + (1 - b)^2 = r^2,$$
$$(4 - a)^2 + b^2 = r^2,$$
$$(5 - a)^2 + (1 - b)^2 = r^2,$$

whence we get $a = 3$, $b = 2$, $r = \sqrt{5}$. Consequently, the required circle is defined by the equation

$$(x - 3)^2 + (y - 2)^2 = 5.$$

Second Solution. We draw mid-perpendiculars* to intercepts AB and BC. They intersect at the centre of the required circle.

PROBLEM 2. Through the points $A(4, 1)$ and $B(11, 8)$ draw a circle such that it touches the axis Ox.

First Solution. Obviously the required circle lies over the axis Ox; and as it touches the axis Ox, the ordinate of its centre is equal to its radius $b = r$. Hence, the equation of the required circle is of the form

$$(x - a)^2 + (y - r)^2 = r^2$$

or

$$(x - a)^2 + y^2 - 2ry = 0$$

Substituting successively into this equation the coordinates of the points A and B, we get the equations

$$(4 - a)^2 + 1 - 2r = 0,$$
$$(11 - a)^2 + 64 - 16r = 0,$$

whence $a_1 = 7$, $a_2 = -1$, $b_1 = r_1 = 5$, $b_2 = r_2 = 13$. Thus, there are two circles satisfying the conditions of the problem (Fig. 14):

$$(x - 7)^2 + (y - 5)^2 = 25$$

and

$$(x + 1)^2 + (y - 13)^2 = 169.$$

Second Solution. We draw a straight line AB. We denote by C the point of its intersection with the axis Ox. To the

* Mid-perpendicular to an intercept is a straight line passing through its centre and perpendicular to it.

intercepts CA and CB, we plot a geometric mean intercept and cut off on either side of the point C equal intercepts CD and CE along the axis Ox (Fig. 14).

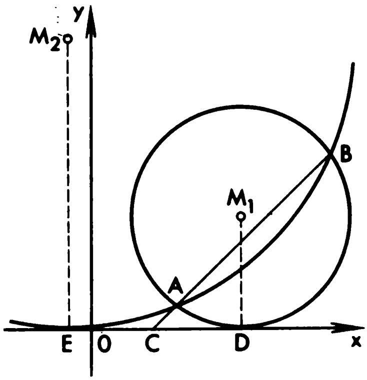

Fig. 14

The circle passing through the points A, B, D satisfies the condition of the problem. Actually, the intercept CD is the tangent to this circle as the geometric mean between the secant CB and external part CA. Similarly, we can make sure that the circle passing through the points A, B and E satisfies the condition of the problem.

PROBLEM 3. Through point $A(2; 1)$ draw a circle touching the coordinate axes.

First Solution. Obviously, the required circle lies in the first quadrant; and, since it touches the axes Ox and Oy, the coordinates of its centre are equal to its radius $a = b = r$. Hence, the equation of the required circle is of the form

$$(x - r)^2 + (y - r)^2 = r^2.$$

Substituting the coordinates of the point A into this equation, we get

$$(2 - r)^2 + (1 - r)^2 = r^2$$

or, after simplification,

$$r^2 - 6r + 5 = 0.$$

Hence, $r_1 = 1$, $r_2 = 5$. Thus, we obtain two circles satisfying the condition of the problem (Fig. 15):

$$(x - 1)^2 + (y - 1)^2 = 1$$

and

$$(x-5)^2+(y-5)^2=25.$$

Second Solution. We solve the problem by the similitude method. We draw a straight line OA and plot in the first quadrant an arbitrary circle, touching the axes Ox and Oy (shown dotted in Fig. 15). Its centre S lies on the bisector of the quadrant angle.

Let the straight line OA intersect the circle thus plotted at the points M and N. We draw straight lines SM and SN, and

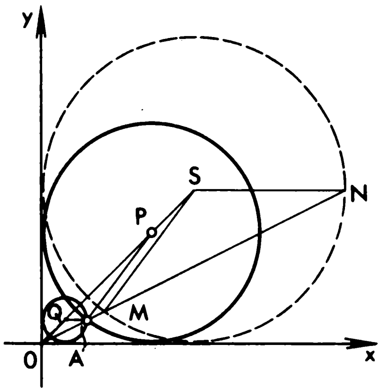

Fig. 15

through the point A we draw straight lines parallel to them and intersecting the bisector OS at points P and Q, respectively: $AP \parallel SM$, $AQ \parallel SN$. The points P and Q are the centres of the required circles. The validity of plotting follows from the similitude theorem.

Sec. 7. Some Applications of the Method of Coordinates

1. *Finding the Common Points of Two Figures.* Let us show how to find common points of figures F and Φ, described by the equations

$$f(x, y)=0, \tag{17}$$

$$\varphi(x, y)=0. \tag{18}$$

Let us suppose that $P(x_1; y_1)$ is one of the required points. Since it belongs to both the given figures, its coordinates satisfy both equation (17) and equation (18). Conversely, if we can find such values x_1 and y_1 of the variables x and y which satisfy equation (17) as well as equation (18), then the point with coordinates x_1 and y_1 will be a common point of the figures F and Φ. Obviously, these values are determined by solving the system of equations (17) and (18).

Thus, the geometrical method of finding common points of two figures boils down to the algebraic method of solving the system of two equations with two unknowns.

Thus, *in order to find common points of two figures, it is necessary to solve their equations simultaneously; each solution gives the coordinates of the common point of these figures.*

For example, solving simultaneously the equations

$$x^2 + y^2 = 25 \tag{19}$$

and

$$x - 2y + 5 = 0, \tag{20}$$

we find the coordinates of the points of intersection of the circle (19) with the straight line (20).

From equation (20) we find that $x = 2y - 5$. Herefrom and from equation (19), we obtain

$$(2y - 5)^2 + y^2 = 25.$$

After simplification, we have

$$y^2 - 4y = 0,$$

whence $y_1 = 0$, $y_2 = 4$. Further, we find that $x_1 = -5$, $x_2 = 3$. Thus, the given circle and the straight line intersect at the points $P(-5; 0)$ and $Q(3; 4)$ (see Fig. 12). It is not difficult to check that the coordinates of the points P and Q satisfy equation (19) as well as equation (20).

2. *Application of the Method of Coordinates to Graphical Solution of Equations.* While we found above the coordinates of common points of two figures by simultaneously solving their equations, we can, conversely, find the roots of two equations with unknowns x and y as the coordinates of common points of the figures defined by the given equations. These considerations form the basis of different practically convenient methods of graphical solutions of problems.

Graphical solution of equations usually gives approximate values of the roots with a low degree of accuracy which, nevertheless, is sufficient in most cases for practical purposes.

Let us consider a couple of examples.

EXAMPLE 1. In order to solve a system of equations of the first degree

$$Ax + By + C = 0,$$

and

$$A_1x + B_1y + C_1 = 0,$$

we draw the straight lines defined by these equations, and by direct measurements find the coordinates of their common point considering, of course, the signs of the coordinates.

EXAMPLE 2. For a graphical solution of the cubic equation

$$x^3 + px + q = 0 \tag{21}$$

we accurately draw on a graph paper the curve

$$y = x^3 \tag{22}$$

called the *cubic parabola*, and draw the straight line

$$y = -px - q \tag{23}$$

having first plotted two of its points.

The abscissae of the common points of these lines will be the roots of equation (21). Actually, denoting the coordinates of the common point of the lines (22) and (23) by ξ, η, we note that the equalities $\eta = \xi^3$ and $\eta = -p\xi - q$ will be identities. Therefore, the equality resulting from them $\xi^3 = -p\xi - q$ or $\xi^3 + p\xi + q = 0$ will also be an identity. Consequently, ξ is a root of equation (21).

The above method permits the determination of only real roots of cubic equations of type (21).

The most labour-consuming part of the solution lies in preparing the graph with the cubic parabola $y = x^3$, but then such a drawing can be used several times, since several straight lines, defined by the equations of type (23), can be drawn on it and consequently, one can solve many equations of the form (21). Moreover, once we have equation (23) of a straight line, there is no need to draw it; it is sufficient to find the coordinates of two points on it, plot these points on the graph, coincide the edge of the ruler with these points and find the abscissae of the points where the edge of the ruler meets the curve (22).

Figure 16 gives a graphical solution of the equations

$$x^3 - x + 0.2 = 0 \tag{24}$$

and

$$x^3 + 2x - 4 = 0 \tag{25}$$

Here, in accordance with the above stated, the cubic parabola $y = x^3$, and the straight lines $y = x - 0.2$ and $y = -2x + 4$ have

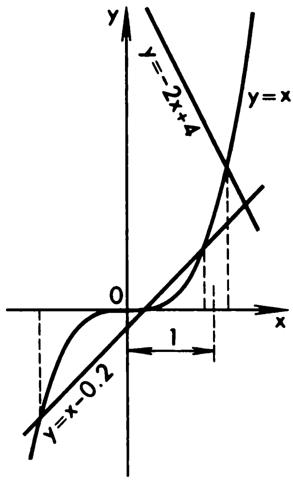

Fig. 16

been drawn. From the graph we find the approximate values of the roots of equation (24): -1.07, $+0.2$, $+0.9$, and the approximate value $+1.2$ of the real root of equation (25). Equation (25) has only one real root since the cubic parabola (22) and the straight line $y = -2x + 4$ have only one common point.

3. *Some Instances of Analysis of the Figure Described by an Equation.* Generally speaking, study of the figure described by an equation is a complex problem requiring the application of the methods of higher mathematics. However, in certain cases, this problem permits a simple solution. If, for example, a figure is defined by an equation of the first degree, then, as we already know, it represents a straight line. In the example given below, we give a deduction of the equation of a parabola and study some of its properties.

A *parabola* is a curve whose points are equidistant from a given point (*focus*) and a given straight line (*directrix*).

Let the focus F of the parabola have coordinates $x = 0$, $y = a (a > 0)$, and let its directrix l be defined by the equation $y = -a$ (Fig. 17). If $P(x; y)$ is any arbitrary point on this

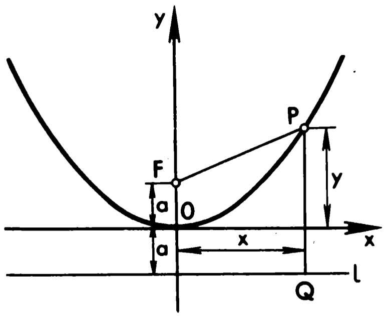

Fig. 17

parabola and Q the foot of the perpendicular drawn from P to l, then

$$FP = QP. \tag{26}$$

Obviously, $QP = y + a$. Using formula (2), we find $FP = \sqrt{x^2 + (y - a)^2}$. Thus, the equality (26) may be written in the form

$$\sqrt{x^2 + (y - a)^2} = y + a$$

whence we get

$$x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2$$

and after simplification

$$x^2 = 4ay. \tag{27}$$

We shall consider some properties of the parabola (27). From equation (27), we see that $y = 0$ if $x = 0$ and $y > 0$ if $x \neq 0$. From this, we conclude that the parabola (27) passes through the origin of coordinates and all its other points lie above the axis Ox.

The parabola (27) is symmetrical with respect to the axis Oy. Actually, if point $A(x_1; y_1)$ lies on the given parabola, then the equality $x_1^2 = 4ay_1$ will become an identity, and hence the equality $(-x_1)^2 = 4ay_1$ will also be an identity. Consequently, the point $B(-x_1; y_1)$, symmetrical to point A with respect to the axis Oy, also lies on the given parabola. The axis of symmetry of the parabola is usually called the *axis of the parabola.*

Let us consider the equation

$$y = kx + m \tag{28}$$

This is an equation of the first degree. Hence, it represents a straight line. Let us find the abscissae of the points of intersection of the parabola (27) with the line (28), omit y from equations (27) and (28) and determine x from the equation thus obtained.

Substituting the expression $kx + m$ for y in (27), we get

$$x^2 = 4a(kx + m)$$

or

$$x^2 - 4akx - 4am = 0 \tag{29}$$

whence

$$x = 2ka \pm 2\sqrt{k^2a^2 + am}. \tag{30}$$

The roots of equation (29) may be either real and different, or imaginary, or real and equal. In the first case we get two points of intersection, in the second, not a single one. Of maximum interest is the third case, when both points of intersection coincide and the straight line (28) will be a tangent to the parabola (27). In this case $k^2a^2 + am = 0$, consequently, $m = -k^2a$, and the equation of the tangent assumes the form

$$y = kx - k^2a. \tag{31}$$

The coordinates of the point of contact M are found from (30) and (27) or (31): $x = 2ka$, $y = k^2a$.

We shall indicate a simple construction of a tangent to the parabola. We shall denote by N the foot of the perpendicular drawn from the point of contact M to the axis Oy (Fig. 18). We plot a point N_1, symmetrical to N, with respect to the origin of coordinates O, and draw a straight line MN_1. The point N_1 lies on the straight line (31), since its coordinates $x = 0$, $y = -k^2a$ satisfy equation (31). Thus the straight lines (31) and MN_1 have two common points M and N_1. Consequently, the straight line MN is the required tangent.

This method is not valid for drawing a tangent at the point O. Let us show that the axis Ox is the tangent to the parabola at the point O. Solving simultaneously the equations $x^2 = 4ay$

and $y = 0$, we find $x_1 = x_2 = 0$; hence both points of intersection of the parabola (27) and the axis Ox coincide with the point O.

We shall also consider the plotting of a normal to the parabola, i. e. a perpendicular to the tangent passing through the point of contact. Let N_2 be the point of intersection of the normal MN_2 with the axis Oy (Fig. 18). From the right-angled triangle MN_1N_2 we have: $NN_1 \cdot NN_2 = MN^2$. Since $MN = 2ka$, $NN_1 = 2k^2a$, then $NN_2 = 2a$. Having plotted the point N_2 in accordance with the last equality, we draw a straight line MN_2; it will be the required normal.

Let us draw another straight line MM' parallel to the axis Oy (Fig. 19). Since the distance from M to l is equal to

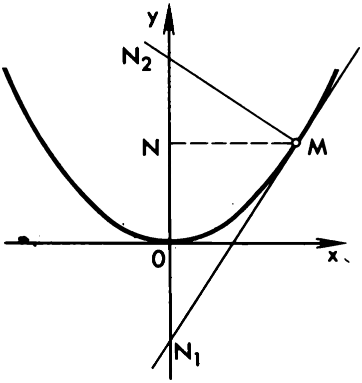

Fig. 18

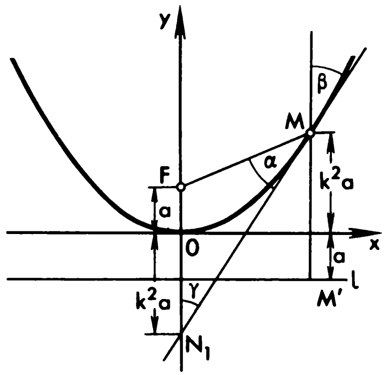

Fig. 19

$k^2a + a$, MF is also equal to $k^2a + a$. On the other hand, $N_1F = N_1O + OF = k^2a + a$. Therefore $MF = N_1F$ and triangle FMN_1 is an isosceles triangle. Hence (see notation on Fig. 19), $\angle\alpha = \angle\gamma$. Since the axis Oy and the straight line MM' are parallel, $\angle\gamma = \angle\beta$. Therefore,

$$\angle\alpha = \angle\beta \tag{32}$$

A concave mirror whose surface may be described by the rotation of a parabola around its axis*, possesses, as can be seen from equation (32), the following properties: it brings the rays parallel to the axis to a focus, if a source of light is at the focus, the rays diverging from it will become parallel to the axis of the mirror

* Such a surface is called a *paraboloid of rotation*.

upon reflection from its surface. Hence, it follows that the reflecting surface of the mirrors of telescopes and projectors should be given the form of a paraboloid of rotation.

Sec. 8. Polar Coordinates

In analytical geometry, one makes use of not only rectangular Cartesian coordinates, but also many other systems of coordinates. Of widest application among these is the system of polar coordinates which differs from others in its extreme simplicity. We shall consider this system in the present section.

While choosing a system of coordinates, one must take into account the nature of the figures to be studied and the problems to be solved, since the success in solution depends, to a considerable extent, on the correlation of the means of solution with the data of the problem. In particular, for a number of problems, the simplest solutions are obtained by using the system of polar coordinates.

Let us go over to the definition of polar coordinates of a point.

Let the point O (*pole*) and semiaxis Ox (*polar axis*) passing through O be given in a plane. Let us take any arbitrary point P in the given plane, draw the intercept OP and consider the length ρ of this intercept and the angle $xOP = \varphi$ (Fig. 20).

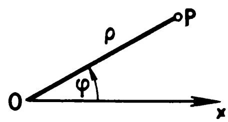

Fig. 20

The quantities ρ and φ are called *polar coordinates* of the point P, ρ is called the *polar radius* of this point, φ – its *polar angle*. Not only the angle φ, but also the angle $\varphi + 2k\pi$ where k is any arbitrary whole number, may be regarded as the polar angle of the point P.*

Let us take the polar axis as the positive part of the axis Ox of the rectangular Cartesian system in the plane under consideration, and the point O as the origin of coordinates, and draw $PP_x \perp Ox$ (Fig. 21). If the point P lies in the first

* Throughout this book, a radian is chosen as a measure of an angle.

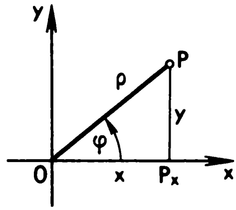

Fig. 21

quadrant, we get from the right-angled triangle OPP_x

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi \tag{33}$$

where x and y are the rectangular Cartesian coordinates of the point P. It is easy to see that formulae (33) are also valid in case when P is any point in the plane Oxy.

From the right-angled triangle OPP_x we also find that

$$\rho = +\sqrt{x^2 + y^2}, \ \tan \varphi = y/x. \tag{34}$$

Formulae (33) and (34) express the relation between the rectangular Cartesian and the polar coordinates of a point.

Let us assume that the equation

$$f(\varphi, \rho) = 0$$

describes a certain figure as a set of points whose polar coordinates satisfy this equation (cf. Sec. 4).

For example, equation

$$\rho = a\varphi \tag{35}$$

where a is a constant positive number, defines an infinite line called the *Archimedean spiral* (Fig. 22).

We draw the semiaxis OL from the point O and denote by A_1, A_2, A_3..., respectively, the points of its intersection with the *Archimedean spiral.* If $\angle xOL = \theta < 2\pi$, then

$$OA_1 = a\theta,$$
$$OA_2 = a(\theta + 2\pi),$$
$$OA_3 = a(\theta + 4\pi), \ ....$$

Hence,

$$A_1A_2 = A_2A_3 = ... = 2\pi a.$$

Thus, the distance between the neighbouring points of intersection

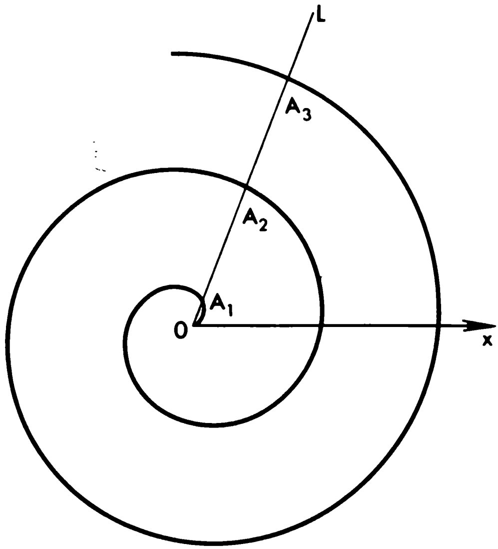

Fig. 22

of the given line is a constant quantity independent of the direction of the semiaxis OL.

From the equation of a figure in rectangular Cartesian coordinates, one can get the equation of the same figure in polar coordinates with the help of formulae (33); the reverse can be accomplished by formulae (34).

For example, having written the equation of the Archimedean spiral in the form

$$\tan\frac{\rho}{a} = \tan\varphi$$

and using formulae (34), we get the following equation of this curve in the rectangular Cartesian coordinates

$$\tan\frac{\sqrt{x^2 + y^2}}{a} = \frac{y}{x}\,. \qquad (36)$$

A comparison of equations (35) and (36) shows that it is preferable to use polar coordinates to study the Archimedean spiral.

Let us consider one more example: Let two circles k and k' be given, each with a diameter a; we denote their centres by M

and M', respectively. If the circle k is stationary and k' rolls around it without slipping, then the point P fixed on k' describes a curve called a *cardioid*. In one of its positions, the point P coincides with a certain point O on the circle k; we regard the respective position of the circle k' as the starting position (shown dotted in Fig. 23).

Let us find the equation of the cardioid in polar coordinates. The point O divides the straight line MO into two semiaxes; we take the one not containing the point M as the polar axis and the point O as the pole.

Let us consider a position of the circle k' other than the starting one and denote the point of contact between the circles k and k' by N. Since k' rolls around k without slipping, $\overset{\smile}{NO} = \overset{\smile}{NP}$. Hence, $OP \parallel MM'$ and $\angle PM'M = \angle OMM' = \angle xOP = \varphi$.

Let us draw $OQ \parallel PM'$. Obviously $OQ = PM' = \frac{1}{2}a$; therefore the triangle MOQ is an isosceles one and $MQ = 2\frac{1}{2}a\cos\varphi = a\cos\varphi$. Further, $\rho = OP = MM' - MQ = a - a\cos\varphi$. Hence, the equation of the cardioid has the form

$$\rho = a(1 - \cos\varphi). \qquad (37)$$

This curve is represented in Fig. 24.

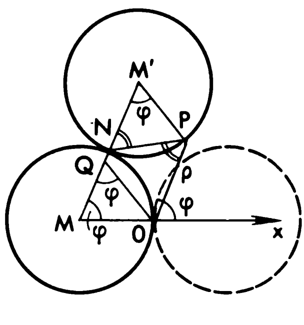

Fig. 23

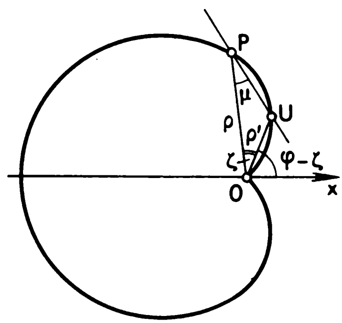

Fig. 24

Let us fix a point $P(\rho, \varphi)$ on the cardioid and consider a point U moving along the cardioid (Fig. 24). Let $OU = \rho'$, $\angle UOP = \zeta$, $\angle OPU = \mu$. Obviously, $\rho' = a[1 - \cos(\varphi - \zeta)]$.

If the point U, moving along the cardioid, comes indefinitely close to the point P, then the straight line PU, rotating around P, tends to a certain limiting position; this limiting position of the straight line PU is the tangent to the cardioid at the point P, and the limiting value of the angle μ is the angle between this tangent and the polar radius OP.

Applying the sine theorem to the triangle OPU, we get

$$\frac{\rho'}{\rho} = \frac{\sin\mu}{\sin(\mu + \zeta)}$$

or

$$\frac{1 - \cos(\varphi - \zeta)}{1 - \cos\varphi} = \frac{\sin\mu}{(\sin\mu + \zeta)}.$$

Subtracting from both sides of the last equality a unity, we get

$$\frac{\cos\varphi - \cos\varphi\cos\zeta - \sin\varphi\sin\zeta}{1 - \cos\varphi} = \frac{\sin\mu - \sin\mu\cos\zeta - \cos\mu\sin\zeta}{\sin(\mu + \zeta)}$$

or

$$\frac{\cos\varphi(1 - \cos\zeta) - \sin\varphi\sin\zeta}{1 - \cos\varphi} = \frac{\sin\mu(1 - \cos\zeta) - \cos\mu\sin\zeta}{(\sin\mu + \zeta)}.$$

Dividing the numerators by $\sin\zeta$ and assuming that

$\frac{1 - \cos\zeta}{\sin\zeta} = \frac{\tan\zeta}{2}$, we get

$$\frac{\cos\varphi\tan\frac{\zeta}{2} - \sin\varphi}{1 - \cos\varphi} = \frac{\sin\mu\tan\frac{\zeta}{2} - \cos\mu}{\sin(\mu + \zeta)}$$

If the point U comes infinitely close to the point P, then ζ

and $\tan\frac{\zeta}{2}$ approach zero in the limiting case and the preceding equality assumes the form

$$\cot\frac{\varphi}{2} = \cot\mu.$$

From here we find the limiting value of μ, $\mu = \frac{1}{2}\varphi$.

Thus the tangent to the cardioid forms with the polar radius of the point of contact an angle equal to half the polar angle of the point of contact.

We shall also show that the straight line PN, passing through the point of contact N of the circles k and k' is the normal to the cardioid at the point P (Fig. 23). Actually, $\angle OPN = \angle PNM' = \frac{\pi}{2} - \frac{\varphi}{2}$. Consequently, the tangent to the point P forms with the straight line PN an angle $\frac{\pi}{2} - \frac{\varphi}{2} + \frac{\varphi}{2} = \frac{\pi}{2}$.

Sec. 9. Examples of Defining Figures by Equations

Examples given in this section will help the reader get a clear idea about the method of defining geometrical figures by equations and will also show that quite complicated figures may be described by relatively simple equations.

EXAMPLE 1. Let us consider the equation*

$$\frac{|x|}{x} + \frac{|y|}{y} = 2. \tag{38}$$

Obviously

$$\frac{|a|}{a} = 1 \text{ if } a > 0,$$

and

$$\frac{|a|}{a} = -1 \text{ if } a < 0.$$

* The absolute value of quantity a is denoted $|a|$.

Therefore, the expression $\frac{|x|}{x} + \frac{|y|}{y}$, where x, y are the coordinates of a certain point P, is equal to 2 if the point P lies in the first quadrant, is equal to zero if the point P lies in the II or IV quadrant, and is equal to -2 if the point P lies in the III quadrant. Finally, this expression is meaningless if the point P lies on one of the coordinate axes, or coincides with the origin of coordinates.

Consequently, equation (38) describes a part of the plane, namely, the first quadrant of the plane Oxy; moreover, this part of the plane does not contain any point lying on the axis Ox or axis Oy (Fig. 25).

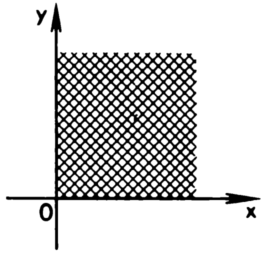

Fig. 25

EXAMPLE 2. Equation

$$\left\{x - \frac{|x|}{x}\right\}^2 + \left\{y - \frac{|y|}{y}\right\}^2 = 4 \tag{39}$$

should be considered separately for each of the four quadrants of the plane Oxy; it may then be written in a simpler form

$$(x - 1)^2 + (y - 1)^2 = 4 \quad \text{for I quadrant} \tag{40}$$

$$(x + 1)^2 + (y - 1)^2 = 4 \quad \text{for II quadrant} \tag{41}$$

$$(x + 1)^2 + (y + 1)^2 = 4 \quad \text{for III quadrant} \tag{42}$$

$$(x - 1)^2 + (y + 1)^2 = 4 \quad \text{for IV quadrant} \tag{43}$$

Equation (40) does not differ in its form from the equation of a circle with radius 2 with the centre $K(1, 1)$, but it describes only that arc of the circle, which lies in the first quadrant, since for other quadrants we have found somewhat different equations. This arc and the arcs of other circles (41), (42), (43) lying in II, III and IV quadrants, respectively, form the figure represented by the equation (39) (Fig. 26).

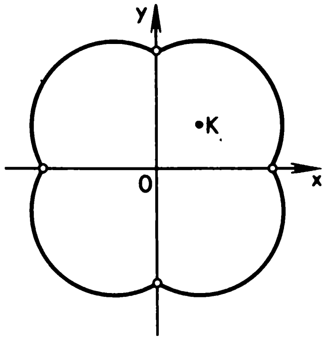

Fig. 26

No point on the axis Ox and the axis Oy is contained in the figure (39), since the expression $\frac{|y|}{y}$ is meaningless if $y = 0$, and the expression $\frac{|x|}{x}$ is also meaningless if $x = 0$.

EXAMPLE 3. Equation

$$|x| + |y| = 2 \tag{44}$$

should be considered separately for each of the four quadrants of the plane Oxy. It may be written in the form

$$\begin{aligned} x + y &= 2 \quad \text{for I quadrant,} \\ -x + y &= 2 \quad \text{for II quadrant,} \\ -x - y &= 2 \quad \text{for III quadrant,} \\ x - y &= 2 \quad \text{for IV quadrant,} \end{aligned}$$

since $|a| = a$ if $a \geqslant 0$ and $|a| = -a$ if $a \leqslant 0$. It is easy to see that equation (44) describes the outline of the square $ABCD$, including its vertices (Fig. 27).

EXAMPLE 4. Equation

$$y = |y| \sin x \tag{45}$$

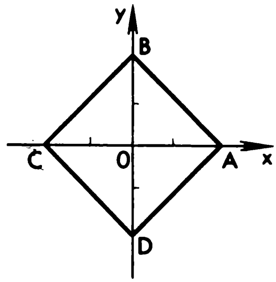

Fig. 27

becomes an identity in the following cases:

(1) if $y = 0$; the variable x in this case may have any arbitrary value;

(2) if y is any arbitrary positive number and $\sin x = 1$, and, consequently, $x = \frac{\pi}{2} + 2k\pi$ where k is any whole number;

(3) if y is any arbitrary negative number and $\sin x = -1$, and, consequently, $x = -\frac{\pi}{2} + 2k\pi$, where k is any whole number. Therefore, the figure (45) consists of the axis Ox and an infinitely large number of semiaxes of two kinds; the semiaxes of the first kind originate from the points on the axis Ox with abscissae $\frac{\pi}{2}$, $\frac{\pi}{2} \pm 2\pi$, $\frac{\pi}{2} \pm 4\pi, \ldots$, are perpendicular to the axis Ox and lie above it; the semiaxes of the second type originate from the points $-\frac{\pi}{2}$, $-\frac{\pi}{2} \pm 2\pi$, $-\frac{\pi}{2} \pm 4\pi, \ldots$, are perpendicular to the axis Ox and lie below it (Fig. 28).

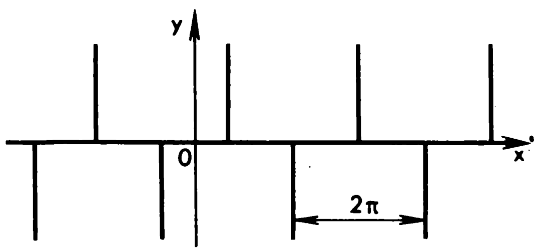

Fig. 28

EXAMPLE 5. Equation

$$\sin(\rho\pi) = 0$$

is equivalent to an infinite number of equations $\rho = 0$, $\rho = \pm 1$, $\rho = \pm 2$, $\rho = \pm 3, \ldots$, and describes a pole and concentric circles with radii 1, 2, 3, ... having their centre at the pole (Fig. 29).

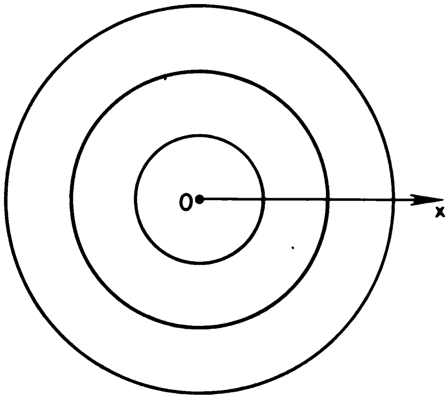

Fig. 29

Negative values of ρ are not to be considered since by definition $\rho \geqslant 0$.

EXAMPLE 6. By $[a]$ we denote the highest integer which does not exceed a*. For example, $[2] = 2$, $[5.99] = 5$, $[-5.99] = -6$, $[\pi] = 3$, $[\sqrt{50}] = 7$, $[-4] = -4$, $[-4.7] = -5$.

Let us consider the equation

$$y = [x]. \tag{46}$$

If $n \leqslant x < n + 1$, where n is an integer, then $y = n$.
Therefore, equation (46) describes a figure, consisting of an infinite number of intercepts, arranged like the steps of a staircase (Fig. 30).

One of these intercepts lies on the axis Ox. The abscissa of its extreme left point is equal to zero. Let us show that it does not have any extreme right point.

Let us suppose that such a point P does exist and that its abscissa is equal to p. Since $[p] = 0$, and obviously $p \neq 0$, $0 < p < 1$. We denote by q the number $p + \frac{1 - p}{2} = 1 + \frac{p}{2}$, and

* The notation $[a]$, which one frequently comes across in mathematical literature, has the same meaning.

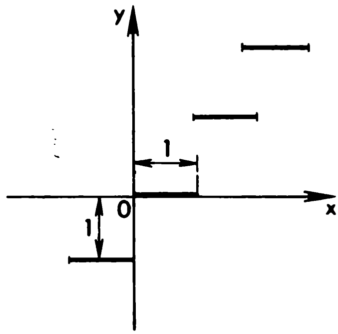

Fig. 30

by Q represent the point on the axis Ox with an abscissa q. Obviously $p < q < 1$ and $[q] = 0$. Hence, the point Q belongs to the same intercept of the figure (46) and is situated on the right side of the point P which is contrary to our supposition.

Similarly, we can prove that any of the above mentioned intercepts of the figure (46) has a left edge, but does not have any extreme position on the right side.

EXAMPLE 7. The figure, described by the equation

$$[x] = [y]$$

consists of an infinite number of squares with their inner points but without their top and right sides. Each of these squares has a side equal to unity, their position is shown in Fig. 31.

Actually, if x and y are any arbitrary numbers satisfying

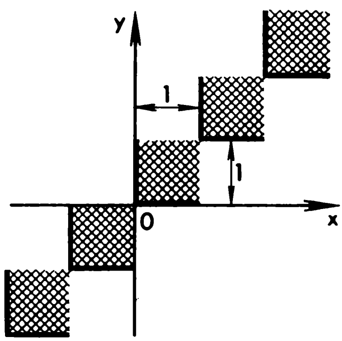

Fig. 31

the inequalities

$$n \leqslant x < n+1 \quad n \leqslant y < n+1$$

where n is an integer, $[x] = [y] = n$.

EXAMPLE 8. We have seen above that equations (39) and (44) were simplified if we considered not the entire plane Oxy, but only one of its quadrants. We shall likewise apply the method of dividing a plane into parts while considering the equation

$$\left\{x - \left[x + \frac{1}{2}\right]\right\}^2 + \left\{y - \left[y + \frac{1}{2}\right]\right\}^2 = \frac{1}{16}. \qquad (47)$$

We divide the plane Oxy into squares with the help of straight lines

$$x = \pm\frac{1}{2}, \quad x = \pm\frac{3}{2}, \quad x = \pm\frac{5}{2}, \dots \qquad (48)$$

$$y = \pm\frac{1}{2}, \quad y = \pm\frac{3}{2}, \quad y = \pm\frac{5}{2}, \dots \qquad (49)$$

We consider one of these squares, for example, the square Q confined by the lines

$$x = \frac{3}{2}, \quad x = \frac{5}{2}, \quad y = \frac{1}{2}, \quad y = \frac{3}{2}.$$

The coordinates of any point within the square Q satisfy the inequalities

$$\frac{3}{2} < x < \frac{5}{2}, \quad \frac{1}{2} < y < \frac{3}{2}$$

or

$$2 < x + \frac{1}{2} < 3, \quad 1 < y + \frac{1}{2} < 2.$$

Therefore, within the square Q, that is, provided that only such values of variables x and y are considered which are the coordinates of points within the square Q, equation (47) takes the form

$$(x-2)^2 + (y-1)^2 = \frac{1}{16}. \qquad (50)$$

Equation (50) describes a circle of radius $\frac{1}{4}$, whose centre $M(2, 1)$ is also the centre of the square Q. The circle (50) lies completely within Q, therefore the coordinates of any of its points satisfy equation (47).

Reasoning this way, we come to the conclusion that the figure (47) consists of an infinite number of circles, each has a radius $\frac{1}{4}$, and every point with integral coordinates is the centre of one of these circles (Fig. 32).

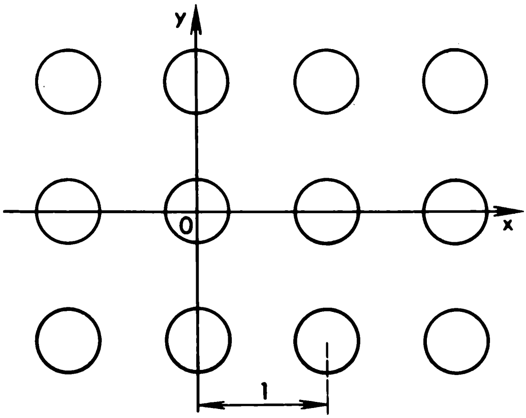

Fig. 32

EXAMPLE 9. Equation

$$\left\{x-\left[x+\frac{1}{2}\right]\right\}^2+\left\{y-\left[y+\frac{1}{2}\right]\right\}^2=\frac{5}{16} \qquad (51)$$

differs from equation (47) only in its right-hand side. Therefore, within the square Q which we have considered in the last example, equation (51) takes the form

$$(x-2)^2+(y-1)^2=\frac{5}{16}.$$

Consequently, it describes a circle with radius $\sqrt{5}/4$, and with centre $M(2, 1)$ within Q. Since $\frac{\sqrt{5}}{4}>\frac{1}{2}$, only that part of the circle lies within the square Q, which is also a part of the figure (51), since the points lying outside Q do not belong to the figure (51). We recommend the reader to make sure that the points of intersection of this circle with the sides of the square Q also belong to the figure (51).

Similarly, we consider other squares, into which we have divided the plane Oxy with the help of straight lines (48) and (49). The figure (51) is shown in Fig. 33.

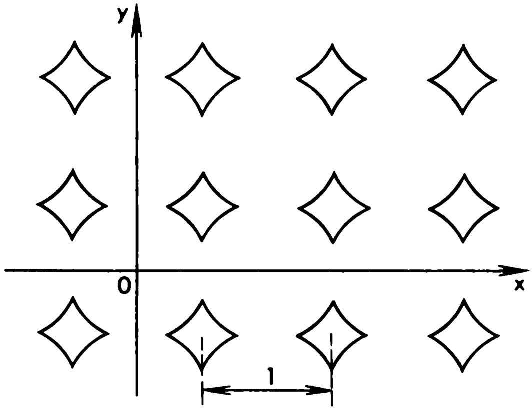

Fig. 33

For the reader, who has carefully studied the above examples, it won't be difficult to plot figures described by the following equations:

(1) $$y = |x|;$$

(2) $$\sin^2(\pi x) + \sin^2(\pi y) = 0;$$

(3) $$\sin(x + y) = 0;$$

(4) $$(x + |x|)^2 + (y + |y|)^2 = 4;$$

(5) $$\left\{x - 2\frac{|x|}{x}\right\}^2 + \left\{y - \frac{2|y|}{y}\right\}^2 = 5;$$

(6) $$\left\{x - \frac{|x|}{x}\right\}^2 + \left\{y + \frac{|y|}{y}\right\}^2 = 4;$$

(7) $$\left\{x - \frac{|x|}{x} - \frac{|y|}{y}\right\}^2 + \left\{y - \frac{|x|}{x} - \frac{|y|}{y}\right\}^2 = 4.$$

CONCLUSION

The idea about the possibility of a systematic application of the method of coordinates in scientific research originated several thousand years ago. It is well known, for example, that the ancient astronomers made use of a special system of coordinates on an imaginary celestial sphere to determine the position of the brightest stars, made maps of the stellar sky, carried out extremely precise observations of the movement of the sun, the moon and the planets with respect to the immovable stars.

Later on, the system of geographical coordinates found wide applications for making a map of the earth's surface to determine the position of ships in the high seas.

However, until XVII century, the method of coordinates found limited practical applications. It was used, as a matter of fact, only for indicating the location of a particular object – immovable (hill, cape) or movable (ship, planet).

The method of coördinates found a new and extremely fruitful application in the book *Geometry* by the famous French philosopher and mathematician Rene Descartes, published in 1637.

Descartes explained the significance of the idea of a variable quantity. While studying the most commonly used curves, Descartes observed that the coordinates of a point moving along a given curve are associated with a particular equation which completely characterises this curve. Thus the method of studying curves through their equations was established, thereby marking the beginning of analytical geometry and facilitating the growth of other mathematical sciences.

"The turning point in mathematics, – F. Engels wrote, – was Descartes' *variable magnitude*. With that came *motion* and hence *dialectics* in mathematics, and *at once, too, of necessity the differential and integral calculus,* which moreover immediately begins." *

The mathematical basis of analytical geometry lies in the peculiar method of defining geometrical figures: a figure is given by an equation. There are two possible means of explaining the gist of this method.

Considering a point with variable coordinates x and y, inter-related by a certain equation, we see that it moves in a plane with a change in its coordinates, but the path traversed by it

* "Dialectics of Nature", Progress Publishers, Six printing, Moscow, 1974, p. 258.

won't be arbitrary since the given equation determines the dependence between quantities x and y. In other words, an equation plays the role of rails guiding the movement of a point along a definite track. For example, the point $P(1; 1)$ of the curve

$$y = x^3 \tag{52}$$

may shift over to a position $P'\left(\frac{3}{2}; 3\frac{3}{8}\right)$ or $P''(2; 8)$, but equation (52) does not allow it to move over to a position Q (2, 7).

It is possible, however, not to associate the description of a figure by an equation with the concept of a moving point, tracing this figure like a tracer bullet, leaving behind a lighted trail, or like the pen of a seismograph, recording a curve reflecting the vibrations of the earth's crust. An equation may be seen as a means of selecting points constituting the figure defined by this equation: only those points in the plane are chosen whose coordinates satisfy the given equation.

The first concept owes its existence to Descartes, and is closely linked with the idea of functional dependence: a curve, defined by an equation, is treated as a graph of the function, and the change in the argument and function depends on the shift of the point describing the graph of the function.

The second concept is simpler in idea and easier to understand. At the same time, it covers a wider range of figures*, and Sec. 4 and partly Secs. 5-9 are devoted to the study of their characteristic properties. Closer to this concept is the method of describing figures by inequalities, to which we can only make a passing reference here, limiting ourselves to the following example: The points whose coordinates satisfy the inequality $x^2 + y^2 \leqslant 25$, belong to a circle with radius 5 and its centre at the origin of coordinates, the points confined within this circumference being included.

* Actually, only after a considerable and hardly justified generalization of the idea of functional dependence can we consider that the figures considered in Examples 1, 4, 7 of Sec. 9 are graphs of certain functions.

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The booklet deals with a fundamental method of analytical geometry and its ability to describe geometrical figures through equations. This method permits geometrical study and solution of geometrical problems by algebraic means, thus making a visual representation of complex spatial configurations superfluous. Considerable attention has been devoted to the question of representing geometrical figures through equations, which is often difficult for those who being to study the method of coordinates.

Detailed examples have been given to illustrate the application of the method.

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