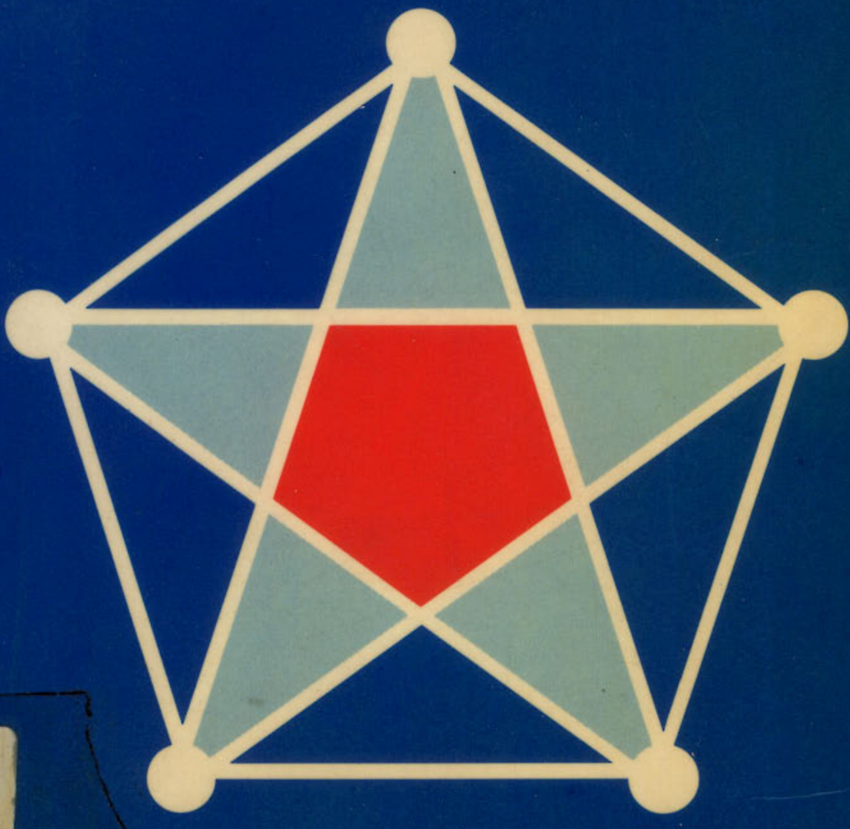
Introduction to GRAPH THEORY

Third Edition



Robin J. Wilson

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Contents

Go forth, my little book! pursue thy way!
Go forth, and please the gentle and the good.

William Wordsworth

Preface to the third edition

In the past few years, graph theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from operational research and linguistics to chemistry and genetics; at the same time it has also emerged as a worthwhile mathematical discipline in its own right. For some time there has been a need for an inexpensive introductory text on the subject, suitable both for mathematicians taking courses in graph theory and also for non-specialists wishing to learn the subject as quickly as possible. It is my hope that this book goes some of the way towards filling this need. The only prerequisites to reading it are a basic knowledge of elementary set theory and matrix theory.

The contents of this book may be conveniently divided into four parts. The first of these, consisting of the first four chapters, provides a basic foundation course containing such topics as connectedness, trees, and Eulerian and Hamiltonian paths and circuits. This is then followed by two chapters on planar graphs and colouring, with special reference to problems relating to the four-colour theorem. The third part (Chapters **7** and **8**) deals with the theory of directed graphs and with transversal theory, relating these fields to such subjects as Markov chains and network flows. The book ends with a chapter on matroid theory which is intended to tie together the material of the previous chapters as well as to introduce some recent developments in the subject.

Throughout the book I have attempted to restrict the text to basic material only, using the exercises as a means for introducing further material of lesser importance. The result of this is that there are about 250 exercises, some of which are designed to test understanding of the text, but many of which are intended to introduce you to new results and ideas. You are urged to read through, and become familiar with, every exercise whether or not you work through all of them in detail. The more difficult exercises are indicated by an asterisk (*).

There are several parts of the book which may be omitted on a first reading, either because of their difficulty or because the material they contain is not referred to later in the book; a star ★ is used to designate the beginning and end of such sections. I have used the symbol // to indicate the end (or absence) of a proof, and bold-face type is used for all definitions. Finally, the number of elements in a set S will be denoted throughout by $|S|$, and the empty set will be denoted by $\emptyset$.

A substantial number of changes have been made in this edition. In particular, many of the exercises have been substantially rewritten, and some of the terminology has been changed to fit in with current usage. Many of these changes have arisen as a result of critical comments by a number of people. I should like to take this opportunity of thanking them for their helpful remarks.

Finally I should like to express my thanks: to my former students, but for whom this book would have been completed a year earlier; to Mr William Shakespeare and others, for their apt and witty comments at the beginning of each chapter; and most of all to my wife, Joy, for many things which have nothing at all to do with graph theory.

R. J. W.

January 1985

The Open University

1
Introduction

The last thing one discovers in writing a book is what to put first.

Blaise Pascal

The object of this introductory chapter is to provide (by means of simple examples) an intuitive background to the material to be presented more formally in succeeding chapters. Terms which appear here in bold-face type are to be thought of more as descriptions than as definitions—the idea is that having met the words in an intuitive setting, you will not find them totally unfamiliar when you meet them again in more formal surroundings. We advise you to read this chapter quickly—and then to forget all about it!

§1 What is a graph?

Let us begin by considering Figs. 1.1 and 1.2 which depict, respectively, part of an electrical network and part of a road map. It is clear that either

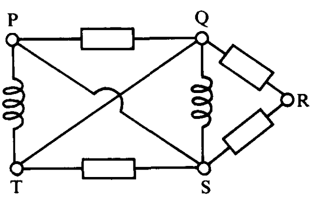


Fig. 1.1

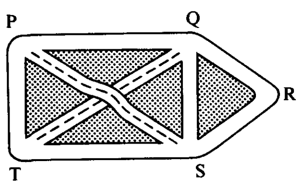


Fig. 1.2

of them can be represented diagrammatically by means of points and lines as in Fig. 1.3. The points P, Q, R, S and T are called **vertices** and the lines are called **edges**; the whole diagram is called a **graph**. (Note that the intersection of the lines PS and QT is not a vertex of the graph since it does not correspond to the meeting of two wires or to a cross-roads.) The **degree** of a vertex is the number of edges which have that vertex as an endpoint, and corresponds in Fig. 1.2 to the number of roads at an intersection; thus the degree of the vertex Q is four.

Clearly the graph in Fig. 1.3 can also represent other situations. For example, if P, Q, R, S and T represent football teams, then the existence of an edge might correspond to the playing of a game between the teams as its endpoints (so that in Fig. 1.3, P has played against S but not against R). In this case, the degree of a vertex is the number of games played by the corresponding team.

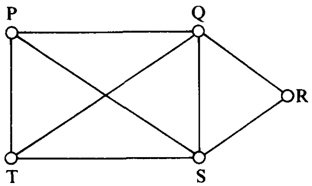


Fig. 1.3

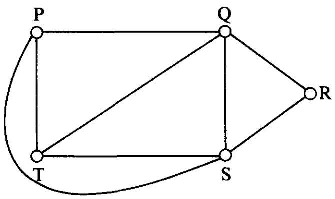


Fig. 1.4

An alternative way of depicting the above situations is given by the graph in Fig. 1.4. Here we have removed the 'crossing' of the lines PS and QT by drawing the line PS outside the rectangle $PQST$. Note that the resulting graph still tells us how the electrical network is wired up, whether there is a direct road from one intersection to another and which football teams have played which. The only information we have lost concerns 'metrical' properties (length of road, straightness of wire, etc.).

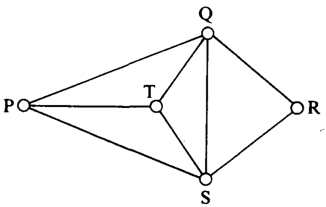


Fig. 1.5

The point we are trying to make is that a graph is a representative of a set of points and of the way they are joined up, and that for our purposes any metrical properties are irrelevant. From this point of view, any two graphs which represent the same situation (such as the ones shown in Figs. 1.3 and 1.4) will be regarded as essentially the same graph. More precisely, we shall say that two graphs are **isomorphic** if there is a one-one correspondence between their vertices which has the property that two vertices are joined by an edge in one graph if and only if the corresponding vertices are joined by an edge in the other. Another graph isomorphic to the graphs in Figs. 1.3 and 1.4 is shown in Fig. 1.5; note that in this graph all idea of space and distance has gone, although we can still tell at a glance which points are joined by a wire or a road.

It is worth pointing out that the graph we have been discussing so far is a particularly 'simple' graph, in the sense that there is never more than one edge joining a given pair of vertices. Suppose, now, that in Fig. 1.5 the roads joining Q and S, and S and T, have too much traffic to carry; then the situation could be eased by building extra roads joining these points, and the resulting diagram would look like Fig. 1.6. (The edges joining Q and S, or S and T, are called **multiple edges**.) If in addition we wish to build a car park at P, then this could be indicated on the graph by drawing an edge from P to itself, usually called a **loop** (see Fig. 1.7). In this book, a graph will in general contain loops and multiple edges; graphs containing no loops or multiple edges (such as the graph in Fig. 1.5) will be referred to as **simple graphs**.

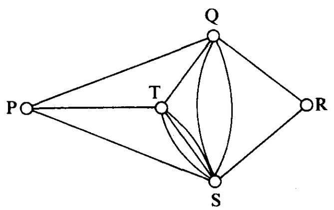


Fig. 1.6

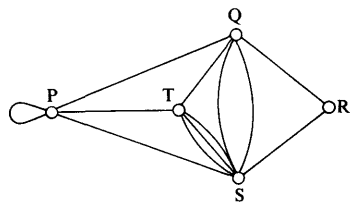


Fig. 1.7

The study of **directed graphs** (or **digraphs**, as we shall usually abbreviate them) arises out of the question, 'what happens if all of the roads are one-way streets?' An example of a digraph is given in Fig. 1.8, the directions of the one-way streets being indicated by arrows. (In this particular example, there would be utter chaos at T, but that does not stop us from studying such situations!) Note that if *not all* of the streets are one-way, then we can obtain a digraph by drawing for each two-way

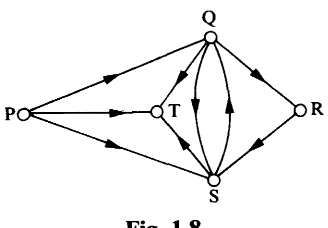


Fig. 1.8

road two directed edges, one in each direction. We shall be discussing digraphs in some detail in Chapter **7**.

Much of graph theory involves the study of walks of various kinds, a **walk** being essentially a sequence of edges, one following on after another; thus, for example, in Fig. 1.5 $P \to Q \to R$ is a 'way of getting from P to R' and is a walk of length two, and similarly $P \to S \to Q \to T \to S \to R$ is a walk of length five. A walk in which no vertex appears more than once is a **path**; for example, $P \to T \to S \to R$ is a path. For obvious reasons, a path of the form $Q \to S \to T \to Q$ is called a **circuit**.

In general, given two vertices v and w in a graph, it is not always possible to find a path connecting them (see Fig. 1.9); such a path will exist only when the graph is 'in one piece'. We can make this clearer by considering the graph whose vertices are the stations of the London Underground and the New York Subway, and whose edges are the various lines joining them; it is obviously impossible to get from Trafalgar Square to Grand Central Station using only edges of the graph. On the other hand, if we confine our attention to the stations and lines of the London Underground, then we can get from any station to any other. A graph in which any two vertices are connected by a path is called a **connected graph**; such graphs will be discussed in Chapter **3**.

Much of Chapters **3** and **4** will be devoted to the study of graphs containing a walk or walks having some particular property. In Chapter **3**, for example, we shall be discussing graphs which contain walks which include every edge or every vertex exactly once, ending up at the initial vertex; such graphs will be called **Eulerian** and **Hamiltonian** graphs respectively. For example, the graph in Fig. 1.5 is Hamiltonian (a possible walk being $P \to Q \to R \to S \to T \to P$) but is not quite Eulerian, since any walk which includes every edge exactly once (e.g. $P \to Q \to T \to P \to S \to R \to Q \to S \to T$) must end up at a vertex different from the initial vertex.

We shall also be interested in connected graphs in which there is only one path connecting each pair of vertices; such graphs are called **trees** (generalizing the idea of a family tree) and will be considered in Chapter **4**. We shall see that a tree can be defined as a connected graph which contains no circuits (see Fig. 1.10).

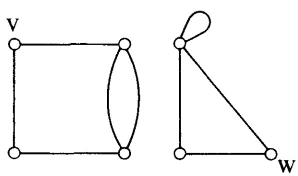


Fig. 1.9

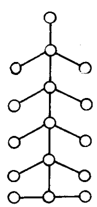


Fig. 1.10

To change the subject a little, you will recall that when we were discussing Fig. 1.3, we pointed out that there are graphs (such as Figs. 1.4 and 1.5) which are isomorphic to the graph under consideration but which contain no crossings. Any graph which can be redrawn in this way without crossings is called a **planar graph**. In Chapter **5** we shall give several criteria for planarity, some of which will involve the properties of subgraphs of the graph in question, and others of which will involve the fundamental notion of **duality**.

Planar graphs also play an important rôle in colouring problems. To motivate such problems, let us return to our 'road-map' graph, and let us suppose that Shell, Esso, BP, and Gulf wish to put up five garages between them at P, Q, R, S and T. Let us further assume that for economic reasons no company wishes to erect two garages at neighbouring corners. Then one solution would be for Shell to build at P, Esso to build at Q, BP at S, and Gulf at T, leaving either Shell or Gulf to build at R. However, if Gulf decides to back out of the whole agreement, then it is clearly impossible for the other three companies to erect the garages in the specified manner.

This problem will be discussed in more colourful language in Chapter **6** where we investigate the question of whether the vertices of a given simple graph can be coloured using k given colours in such a way that every edge of the graph has endpoints of different colours. If the graph happens to be planar, then we shall see that it is always possible to colour its vertices in the above-mentioned way if five colours are available. Moreover, it has recently been proved that the same is true if only four colours are available—this is the famous **four-colour theorem**. (A possibly more familiar version of this theorem is that if we have a map with several countries on it, then it is always possible to colour the countries of the map with four colours in such a way that no two neighbouring countries share the same colour.)

In Chapter **8** we shall investigate various combinatorial problems, including the celebrated **marriage problem** which asks under what conditions a collection of boys, each of whom knows several girls, can be married off in such a way that each boy marries a girl he knows. This problem can be easily expressed in the language of transversal theory, a very important branch of combinatorial mathematics which we discuss in §**26**. It will turn out that these topics are closely related to the problem of finding the number of paths connecting two given vertices in a graph or digraph, subject to the restriction that no two of the paths have an edge in common.

We conclude Chapter **8** with a discussion of network flows and transportation problems. To describe these problems, we suppose that Fig. 1.5 represents part of an electrical network made up of wires of different materials; the problem is then to find out how large a current can safely be passed through the entire network from P to R, given the various currents which each separate wire can take without burning out. Alternatively, we can think of P as a factory and R as a market and the edges of the graph as various channels through which goods can be sent; in this case we want to know how much can be sent from the factory to the market, given the capacities of the various channels.

We end the book with a chapter on the theory of matroids; this chapter is intended to tie together the material of the previous chapters as well as to satisfy the maxim 'be wise—generalize!' In fact, **matroid theory** is essentially the study of sets with 'independence structures' defined on them, generalizing not only properties of linear independence in vector spaces but also several of the results in graph theory obtained earlier in the book. However, as we shall see, matroid theory is far from being 'generalization for generalization's sake'. On the contrary, it gives us a deeper insight into several graph-theoretical problems as well as including among its applications simple proofs of results in transversal theory which are awkward to prove by more traditional methods. Matroid theory has played an important rôle in the development of combinatorial theory in recent years, and we have included it in our book for this reason.

We hope that this introductory chapter has been useful to you in setting the stage and describing some of the things which lie ahead. We now embark upon a formal treatment of the subject.

Exercises 1

(1a) Write down the number of vertices, the number of edges, and the degree of each vertex for:
 (*i*) the graph fig. 1.3;
 (*ii*) the graph in Fig. 1.11.

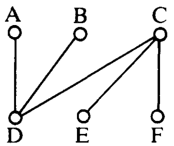


Fig. 1.11

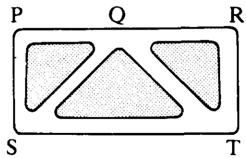


Fig. 1.12

(1b) Draw the graph which represents the road system in Fig. 1.12, and write down the number of vertices, the number of edges and the degree of each vertex.

(1c) Figure 1.13 represents the chemical molecules of methane (CH_4) and propane (C_3H_8).

(*i*) Regarding these diagrams as graphs, what can you say about the vertices representing carbon atoms (C) and hydrogen atoms (H)?
(*ii*) There are two different chemical molecules with the chemical formula C_4H_{10}. Draw the graphs which correspond to these molecules.

methane propane

Fig. 1.13

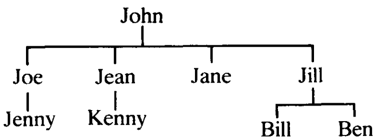


Fig. 1.14

(1d) Draw the graph which corresponds to the family tree in Fig. 1.14.

(1e) John likes Joan, Jean and Jane, Joe likes Jane and Joan, and Jean and Joan like each other. Draw a digraph which illustrates these relationships between John, Joan, Jean, Jane and Joe.

(1f) Snakes eat frogs and birds eat spiders; birds and spiders both eat insects; frogs eat snails, spiders and insects. Draw a digraph which represents this predatory behaviour.

2
Definitions and examples

I hate definitions!

Benjamin Disraeli

In this chapter, the foundations are laid for a proper study of graph theory. §**2** formalizes some of the basic definitions mentioned in Chapter **1** and §**3** provides a variety of examples. Diagrams are used throughout to clarify the material, and the justification for their use is given in §**4**. A description of some typical applications of the theory is deferred until we have more machinery at our disposal (§§**8**, **11**).

§2. Definitions

We shall begin by defining a **simple graph** G to be a pair $(V(G), E(G))$, where $V(G)$ is a non-empty finite set of elements called **vertices** (or **nodes**, or **points**), and $E(G)$ is a finite set of unordered pairs of distinct elements of $V(G)$ called **edges** (or **lines**); $V(G)$ is sometimes called the **vertex-set** and $E(G)$ the **edge-set** of G. For example, Fig. 2.1 represents the simple graph G whose vertex-set $V(G)$ is the set $\{u, v, w, z\}$, and whose edge-set

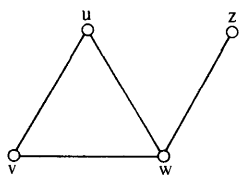


Fig. 2.1

$E(G)$ consists of the pairs $\{u, v\}$, $\{v, w\}$, $\{u, w\}$ and $\{w, z\}$. The edge $\{v, w\}$ is said to **join** the vertices v and w, and will usually be abbreviated to vw.

Note that since $E(G)$ is a set, rather than a family,† there can never be more than one edge joining a given pair of vertices of a simple graph.

It turns out that many of the results which can be proved about simple graphs may be extended without difficulty to more general objects in which two vertices may have more than one edge joining them. In addition, it is often convenient to remove the restriction that any edge must join two *distinct* vertices, and allow the existence of **loops**—i.e., edges joining vertices to themselves. The resulting object, in which loops and multiple edges are allowed, is then called a **general graph**—or, simple, a **graph** (see Fig. 2.2). We emphasize the fact that every simple graph is a graph, but not every graph is a simple graph.

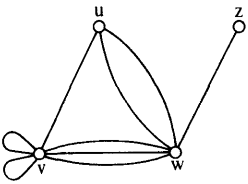


Fig. 2.2

More formally, a **graph** G is defined to be a pair $(V(G), E(G))$, where $V(G)$ is a non-empty finite set of elements called **vertices**, and $E(G)$ is a finite family of unordered pairs of (not necessarily distinct) elements of $V(G)$ called **edges**; note that the use of the word 'family' permits the existence of multiple edges. We shall call $V(G)$ the **vertex-set** and $E(G)$ the **edge-family** of G. Thus in Fig. 2.2, $V(G)$ is the set $\{u, v, w, z\}$ and $E(G)$ is the family consisting of the edges $\{u, v\}$, $\{v, v\}$, $\{v, v\}$, $\{v, w\}$, $\{v, w\}$, $\{v, w\}$, $\{u, w\}$, $\{u, w\}$ and $\{w, z\}$. Any edge of the form $\{v, w\}$ is said to **join** the vertices v and w, and will again be abbreviated to vw. Note that each loop vv joins the vertex v to itself. Although in this book we shall sometimes have to restrict ourselves to simple graphs, we shall wherever possible prove our results for graphs in general.

A subject related to graph theory is the study of digraphs (sometimes called directed graphs or networks, although we shall be using the word 'network' in a slightly different sense). A **digraph** D is defined to be a pair $(V(D), A(D))$, where $V(D)$ is a non-empty finite set of elements called **vertices**, and $A(D)$ is a finite family of *ordered* pairs of elements of $V(D)$ called **arcs**. An arc whose first element is v and whose second element is w is called an **arc from v to w** and is written (v, w), or simply vw; note that two arcs of the form vw and wv are different. Fig. 2.3 represents a digraph whose arcs are uv, vv, vw, wv, wu and wz, the

† We use the word 'family' to mean a collection of elements, some of which may occur several times; for example, $\{a, b, c\}$ is a set, but (a, a, c, b, a, c) is a family.

ordering of the vertices in an arc being indicated by an arrow. If D has no loops, and if the arcs of D are all distinct (so that $A(D)$ is a set, rather than a family), then D is called a **simple digraph**; for example, the digraph in Fig. 1.8 is simple, whereas the digraph in Fig. 2.3 is not.

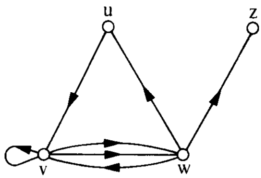


Fig. 2.3

Digraphs will be studied in further detail in Chapter 7. In the meantime, we shall be content to point out that although graphs and digraphs are essentially different objects, a graph can in certain circumstances be thought of as a digraph in which there are two arcs, one in each direction, corresponding to each edge (see Fig. 2.4).

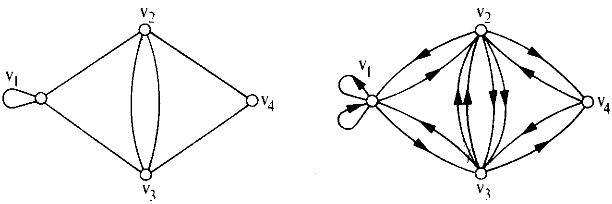


Fig. 2.4

Remark on terminology

The language of graph theory is decidedly non-standard—every author has his own terminology. In this book we are using essentially the terminology of Bondy and Murty.[7] Several graph theorists, however, use the term 'graph' to mean what we have called a simple graph. It is also common, especially when discussing applications, to see the word 'graph' used for what we have called a digraph. To make matters worse, one sometimes sees the term 'graph' used for the object which results if, in the definition of a graph, we remove the restriction that the vertex-set and edge-family must both be finite. (If they are in fact both infinite, then we get what we call an **infinite graph**; we defer a study of infinite graphs until §**16**.) It should be emphasized that any of the above definitions of a graph is perfectly valid, provided that one is always consistent; we repeat that in this book, *all graphs are finite and undirected, loops and multiple edges being allowed unless specifically excluded.*

Before giving examples of some important types of graph (in §3), it will be convenient to introduce a few more simple definitions.

Two vertices v and w of a graph G are said to be **adjacent** if there is an edge joining them (i.e. there is an edge of the form vw); the vertices v and w are then said to be **incident** to such an edge. Similarly, two distinct edges of G are **adjacent** if they have at least one vertex in common. The **degree** (or **valency**) of a vertex v of G is the number of edges incident to v, and is written $\rho(v)$; in calculating the degree of a vertex v, we shall (unless otherwise stated) make the convention that a loop at v contributes two (rather than one) to the degree of v. Any vertex of degree zero is called an **isolated vertex** and a vertex of degree one is an **end-vertex**. Thus the graph in Fig. 2.2 has one end-vertex, one vertex of degree three, one of degree six and one of degree eight.

It is easy to see that *if we add up the degrees of all the vertices of a graph, then the result is an even number*—in fact, twice the number of edges—since each edge contributes exactly two to the sum. This result, known two hundred years ago to Euler, is often called the **handshaking lemma** since it implies that if several people shake hands, the total number of hands shaken must be even—precisely because two hands are involved in each handshake. An immediate corollary of the handshaking lemma is that *in any graph the number of vertices of odd degree must be even*. The analogue of this result for digraphs will be presented in §23.

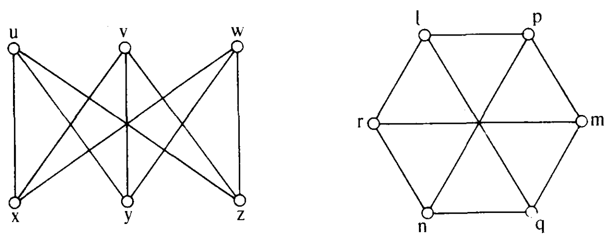
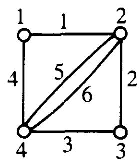


Fig. 2.5

Two graphs G_1 and G_2 are **isomorphic** if there is a one-one correspondence between the vertices of G_1 and those of G_2 with the property that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2. Thus the two graphs shown in Fig. 2.5 are isomorphic under the correspondence $u \leftrightarrow l$, $v \leftrightarrow m$, $w \leftrightarrow n$, $x \leftrightarrow p$, $y \leftrightarrow q$, $z \leftrightarrow r$; note that there are only six vertices—the other points at which edges cross are not vertices. A **subgraph** of a graph G is simply a graph, all of whose vertices belong to $V(G)$ and all of whose edges belong to $E(G)$. Thus the graph in Fig. 2.1 is a subgraph of the graph in Fig. 2.4, but is not a subgraph of either graph in Fig. 2.5 since the latter graphs contain no 'triangle'.

Although it is often very convenient to represent a graph by a diagram of points joined by lines, such a representation may be unsuitable if we wish to store a large graph in a computer. An alternative representation which is useful in such cases is by means of a matrix.

If G is a graph with vertex-set $\{1, 2, \ldots, n\}$, we define its **adjacency matrix** A to be the $n \times n$ matrix whose ij-th entry is the number of edges joining vertex i and vertex j. If, in addition, the edges are labelled $\{1, 2, \ldots, m\}$, we define the **incidence matrix** M to be the $n \times m$ matrix whose ij-th entry is 1 if vertex i is incident to edge j, and 0 otherwise. Fig. 2.6 gives an example of a graph G with its adjacency and incidence matrices.



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Fig. 2.6

★As a change from definitions, we conclude this section by looking at a puzzle which has been popular recently, and which has been marketed under the name of 'Instant Insanity'. It concerns four cubes whose faces are coloured red, blue, green and yellow in such a way that each cube contains at least one face of each colour, as in Fig. 2.7. The problem is to pile these cubes up on top of each other in such a way that each of the four 4×1 sides of the resulting stack shows a face of each colour. Although the cubes can be stacked in thousands of different ways, only one way leads to a solution of the problem.

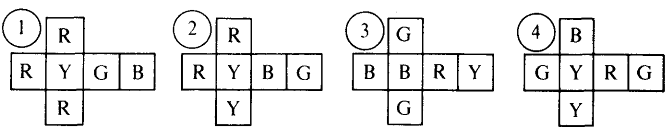


Fig. 2.7

In order to solve this problem, we represent each cube by a graph on four vertices, one vertex corresponding to each colour; in each such graph, two vertices are adjacent if and only if the cube in question has the corresponding colours on *opposite* faces. The graphs corresponding to the cubes of Fig. 2.7 are shown in Fig. 2.8.

We shall find it convenient to superimpose these graphs to form a new graph G (Fig. 2.9). Since every solution of the puzzle has two faces

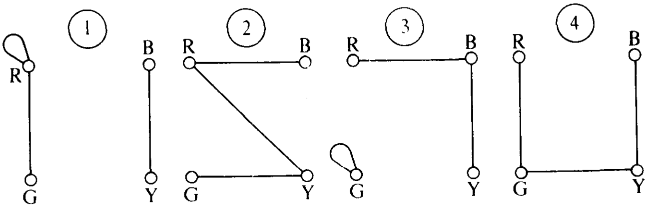


Fig. 2.8

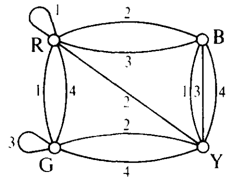


Fig. 2.9

of each colour on each of the two pairs of opposite sides of the 4×1 stack, it is not difficult to see that the required solution is obtained by finding two subgraphs H_1 and H_2 of G which

(*a*) have no edges in common,
(*b*) contain exactly one edge from each cube, and
(*c*) contain only vertices of degree two.

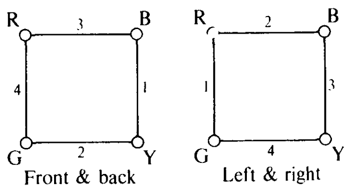


Fig. 2.10

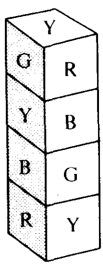


Fig. 2.11

The subgraphs corresponding to our particular example are shown in Fig. 2.10. H_1 and H_2 then represent the colours appearing on the front-and-back and on the left-and-right sides of the 4×1 stack. The solution can now be read off from these subgraphs (Fig. 2.11).★

Exercises 2

(2a) Write down the vertex-set and edge-set of each graph in Fig. 2.5.

(2b) Draw
 (*i*) a simple graph,
 (*ii*) a non-simple graph with no loops,
 (*iii*) a non-simple graph with no multiple edges, each having 5 vertices each having 5 vertices and 8 edges.

(2c) (*i*) Draw a graph on six vertices whose degrees are 5, 5, 5, 5, 3, 3; does there exist a *simple* graph with these degrees?
 (*ii*) How are your answers to part (*i*) changed if the degrees are 5, 5, 4, 3, 3, 2?

(2d) Verify that the handshaking lemma holds for the graphs in Figs 2.1 and 2.5.

(2e) Find, up to isomorphism, all the simple graphs on three or four vertices.

(2f) (*i*) By suitably lettering the vertices, show that the two graphs in Fig. 2.12 are isomorphic.
 (*ii*) Explain why the two graphs in Fig. 2.13 are not isomorphic.

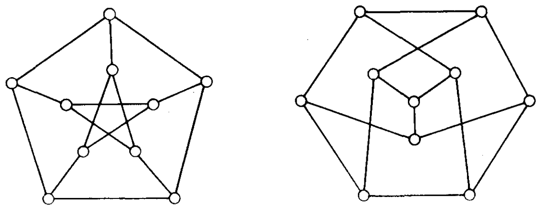


Fig. 2.12

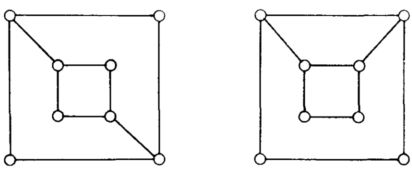


Fig. 2.13

(2g) Classify the following statements as *true* or *false*:
 (*i*) any two isomorphic graphs have the same number of vertices and the same number of edges;
 (*ii*) any two graphs with the same number of vertices and the same number of edges are isomorphic.

(2h) Which of the graphs in Fig. 2.14 are subgraphs of the graphs in Fig. 2.5?

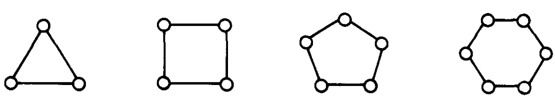


Fig. 2.14

(2i) If G is a graph without loops, what can you say about

(*i*) the sum of the entries in any row or column of the adjacency matrix of G?

(*ii*) the sum of the entries in any row of the incidence matrix of G?

(*iii*) the sum of the entries in any column of the incidence matrix of G?

(*2j) Let G be a simple graph with at least two vertices. Prove that G must contain two or more vertices of the same degree.

(2k) Explain why conditions (*a*), (*b*) and (*c*) on page 13 are relevant to the solution of 'Instant Insanity'.

(*2l) Find a solution of 'Instant Insanity' with the set of cubes in Fig. 2.15. (There are several solutions.)

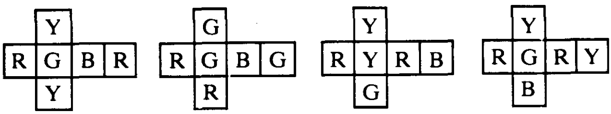


Fig. 2.15

(*2m) If G is a simple graph with edge-set $E(G)$, the **vector space associated with G** is the vector space over the field of integers modulo 2, whose elements are subsets of $E(G)$. The sum $E \oplus F$ of two sets E, F of edges is defined as the set of edges in E or F but not both, and scalar multiplication is defined by $1.E = E$ and $0.E = \emptyset$. Show that this does define a vector space, and find a basis for it.

§3. Examples of graphs

In this section we shall examine some important types of graph. You are advised to become familiar with them since they will appear frequently in examples and exercises.

Null graphs

A graph whose edge-set is empty is called a **null graph** (or **totally-disconnected graph**). We shall denote the null graph on n vertices by N_n; N_4 is shown in Fig. 3.1. Note that in a null graph, every vertex is isolated. Null graphs are not very interesting.

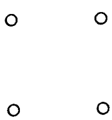


Fig. 3.1

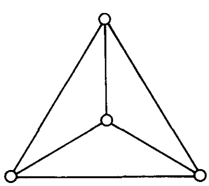


Fig. 3.2

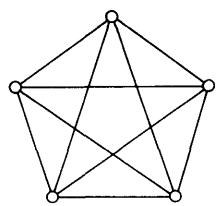


Fig. 3.3

Complete graphs

K

A simple graph in which every pair of distinct vertices are adjacent is called a **complete graph**. The complete graph on n vertices is usually denoted by K_n; K_4 and K_5 are shown in Figs 3.2 and 3.3. You should check that K_n has exactly $\frac{1}{2}n(n-1)$ edges.

Regular graphs

A graph in which every vertex has the same degree is called a **regular graph**; if every vertex has degree r, the graph is called **regular of degree r**. Of particular importance in colouring problems (to be discussed in Chapter **6**) are the **cubic** (or **trivalent**) graphs which are regular graphs of degree three (for example, Figs 2.5 and 3.2). Another well-known example of a cubic graph is the so-called **Petersen graph** shown earlier in Fig. 2.12. Note that every null graph is regular of degree zero, and that the complete graph K_n is regular of degree $n-1$. Note also that if G has n vertices and is regular of degree r, then G has $\frac{1}{2}rn$ edges.

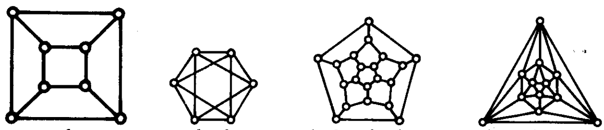


Fig. 3.4

Platonic graphs

Of special interest among the regular graphs are the so-called **Platonic graphs**, the graphs formed by the vertices and edges of the five regular (Platonic) solids—the tetrahedron, cube, octahedron, dodecahedron and icosahedron. The tetrahedral graph is K_4 (see Fig. 3.2), and the graphs of the cube, octahedron, dodecahedron and icosahedron are shown in Fig. 3.4.

Bipartite graphs

Suppose that the vertex-set of a graph G can be split into two disjoint sets V_1 and V_2, in such a way that every edge of G joins a vertex of V_1 to a vertex of V_2 (see Fig. 3.5). G is then said to be a **bipartite graph**, which we denote by $G(V_1, V_2)$ if we wish to specify the two sets involved. An alternative way of thinking of a bipartite graph is in terms of colouring its vertices with two colours, say red and blue—a graph is bipartite if we can colour each vertex red or blue in such a way that every edge has a red end and a blue end. It is worth emphasizing that in a bipartite graph $G(V_1, V_2)$, it is not necessarily true that every vertex of V_1 is joined to every vertex of V_2; if, however, this does happen, and if G is simple, then G is called a **complete bipartite graph**, usually denoted by $K_{r,s}$ where r and s are the numbers of vertices in V_1 and V_2 respectively. For example, Fig. 3.6 represents $K_{4,3}$, and two drawings of $K_{3,3}$ appeared in Fig. 2.5. Note that $K_{r,s}$ has $r+s$ vertices and rs edges. A complete bipartite graph of the form $K_{1,s}$ is called a **star graph**, $K_{1,5}$ being shown in Fig. 3.7.

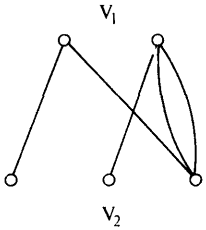


Fig. 3.5

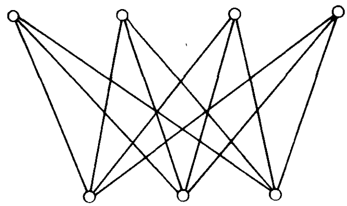


Fig. 3.6

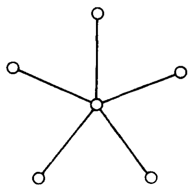


Fig. 3.7

The k-cubes

Of special interest among the bipartite graphs are the k-cubes. The ***k*-cube** Q_k is the graph whose vertices correspond to the sequences $(a_1, a_2, \ldots, a_k)$, where each $a_i = 0$ or 1, and whose edges join those sequences which differ in just one place. Note that the graph of the cube is simply the graph Q_3 (see Fig. 3.8). You should check that Q_k has 2^k vertices and $k2^{k-1}$ edges, and is regular of degree k.

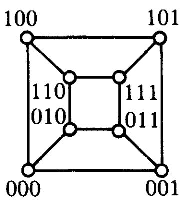


Fig. 3.8

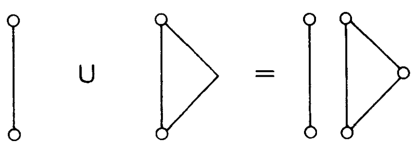


Fig. 3.9

The union of two graphs

There are several ways of combining two graphs to make a larger graph; we shall illustrate one of these. If the two graphs are taken to be $G_1 = (V(G_1), E(G_4))$ and $G_2 = (V(G_2), E(G_2))$, where $V(G_1)$ and $V(G_2)$ are assumed to be disjoint, then their **union** $G_1 \cup G_2$ is defined as the graph with vertex-set $V(G_1) \cup V(G_2)$ and edge-family $E(G_1) \cup E(G_2)$ (see Fig. 3.9).

Deletions and contractions

If e is an edge of a graph G, we denote by $G - e$ the graph obtained from G by deleting the edge e; more generally, if F is any set of edges in G, we denote by $G - F$ the graph obtained by deleting the edges in F. Similarly, if v is a vertex of G, we denote by $G - v$ the graph obtained from G by deleting the vertex v together with the edges incident to v; more generally, if S is any set of vertices in G, we denote by $G - S$ the graph obtained by deleting the vertices in S and all edges incident to any of them. We also denote by $G\backslash e$ the graph obtained by taking an edge e and 'contracting' it—in other words, removing e and identifying its ends v and w in such a way that the resulting vertex is incident to those edges (other than e) which were originally incident to v or w. A **contraction** of G is then defined to be any graph which results from G after a succession of such edge-contractions. Note that K_5 is a contraction of the Petersen graph (by contracting the five edges which connect the inner pentagon to the outer one); we also express this by saying that the Petersen graph is contractible to K_5. Some deletions and contractions are shown in Fig. 3.10.

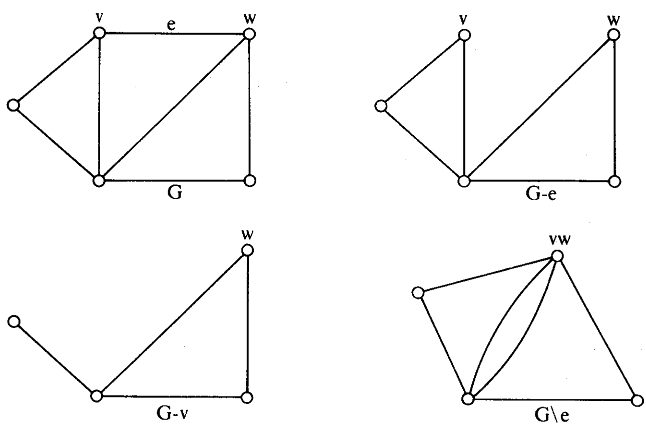


Fig. 3.10

Connected graphs

As you have probably noticed, almost all the graphs we have discussed so far have been 'in one piece', the main exceptions being the null graphs $N_n (n \geq 2)$ and the union of graphs, both of which consist of 'bits which are not joined together'. We can formalize this distinction by defining a graph to be **connected** if it cannot be expressed as the union of two graphs; otherwise it is **disconnected**. It is clear that any disconnected graph G can be expressed as the union of a finite number of connected graphs—each of these connected graphs is called a **(connected) component** of G. (A graph with three components is shown in Fig. 3.11.) When proving results about graphs in general, it is often possible and convenient to obtain the corresponding results for connected graphs, and then apply them to each component separately.

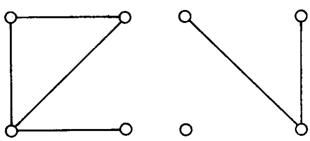


Fig. 3.11

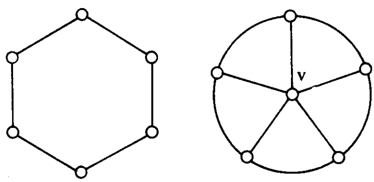


Fig. 3.12

Circuit graphs and wheels

A connected graph which is regular of degree two is called a **circuit graph**, the circuit graph on n vertices being denoted by C_n. The graph obtained from C_{n-1} by joining each vertex to a new vertex v is called the **wheel** on n vertices, and is written W_n. Fig. 3.12 shows C_6 and W_6; W_4 appeared in Fig. 3.2.

The complement of a simple graph

Let G be a simple graph with vertex-set $V(G)$. The **complement** $\bar{G}$ of G is the simple graph which has $V(G)$ as its vertex-set, and in which two vertices are adjacent if and only if they are *not* adjacent in G. It follows that if G has n vertices, then $\bar{G}$ can be constructed by removing from K_n all the edges of G (G being regarded as a subgraph of K_n). Note that the complement of a complete graph is a null graph, and that the complement of a complete bipartite graph is the union of two complete graphs.

Exercises 3

(3a) Draw the following graphs:
 (*i*) the null graph N_5;
 (*ii*) the complete graph K_6;
 (*iii*) the complete bipartite graph $K_{2,4}$;
 (*iv*) the union of the star $K_{1,3}$ and the wheel W_4;
 (*v*) the complement of the circuit graph C_5.

(3b) How many edges has each of the following graphs:
 (*i*) K_{10}; (*ii*) $K_{5,7}$; (*iii*) Q_4; (*iv*) $\overline{W_7}$?

(3c) Draw all simple cubic graphs with at most eight vertices.

(3d) Give an example (if it exists) of each of the following:
 (*i*) a bipartite graph which is regular of degree 5;
 (*ii*) a cubic graph with eleven vertices;
 (*iii*) a Platonic graph which is bipartite;
 (*iv*) a graph (other than K_4, $K_{4,4}$ or Q_4) which is regular of degree 4.

(3e) The **complete tripartite graph** $K_{r,s,t}$ consists of three sets of vertices (of sizes r, s and t), with an edge joining two vertices if and only if they lie in different sets. Draw the graphs $K_{2,2,2}$ and $K_{3,3,2}$, and write down the number of edges of $K_{3,4,5}$.

(3f) Let G be a graph with n vertices and m edges, and let v be a vertex of G of degree k and e be an edge of G. How many vertices and edges have $G-v$, $G-e$ and $G\backslash e$?

(3g) A simple graph which is isomorphic to its complement is called **self-complementary**.
 (*i*) Show that if G is self-complementary, then G has $4k$ or $4k+1$ vertices, where k is an integer.
 (*ii*) Find all self-complementary graphs with four and five vertices.
 (*iii*) Find a self-complementary graph with eight vertices.

(3h) The **line graph** $L(G)$ of a simple graph G is the graph whose vertices are in one-one correspondence with the *edges* of G, two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G are adjacent.
 (*i*) Show that K_3 and $K_{1,3}$ have the same line graph.
 (*ii*) Find an expression for the number of edges of $L(G)$ in terms of the degrees of the vertices of G.
 (*iii*) Show that if G is regular of degree k, then $L(G)$ is regular of degree $2k-2$.
 (*iv*) Show that $L(K_5)$ is the complement of the Petersen graph.

(*3i) Show that, in a gathering of six people, either there are three people who all know each other or there are three people none of whom knows either of the other two.

(*3j) An **automorphism** φ of a simple graph G is a one-one mapping of the vertex-set of G onto itself with the property that $\varphi(v)$ and $\varphi(w)$ are adjacent if and only if v and w are. The **automorphism group** $\Gamma(G)$ of G is the group of automorphisms of G under composition.
 (*i*) Show that the groups $\Gamma(G)$ and $\Gamma(\bar{G})$ are isomorphic.
 (*ii*) Find the groups $\Gamma(K_n)$, $\Gamma(K_{r,s})$ and $\Gamma(C_n)$.
 (*iii*) Use the results of parts (*i*) and (*ii*) and exercise 3h(*iv*) to find the automorphism group of the Petersen graph.

§4. Embeddings of graphs

Up to now we have been using diagrams to represent graphs, a vertex being represented by a point or small circle, and an edge by a line or curve. Such diagrams are very useful for investigating the properties of particular graphs, and it is natural to ask what it actually means to 'represent' a graph by a diagram, and whether all graphs can be so represented. If you are quite happy drawing pictures and are not concerned with the justification for doing so, you may omit this section for the time being, secure in the knowledge that everything you do is all right—but you may need to refer back here when you reach Chapter **5**.

★What we should like to be able to do is to draw graphs in some space—the plane or Euclidean 3-space, for example—in such a way that there are no 'crossings' (a term which will be defined formally later on, but whose intuitive meaning is clear). For example, Fig. 4.1 represents K_4, but it contains a crossing; we may wish to find a representation (e.g. Fig. 3.2) which contains no crossings. We shall see in fact that *every* graph can be drawn without crossings in 3-space, but that such a drawing is *not* always possible in the plane. In particular, as we shall show in §**12**, K_5 and $K_{3,3}$ (Figs 3.3 and 2.5) cannot be drawn in the plane without crossings.

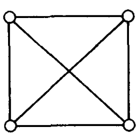


Fig. 4.1

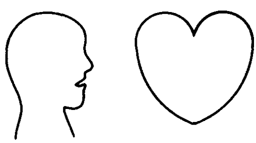


Fig. 4.2

Before defining an embedding of a graph, we remind you that a **Jordan curve** (or **Jordan arc**) in the plane is a continuous curve which does not intersect itself, and a **closed Jordan curve** is one whose end-points coincide (see Fig. 4.2). Jordan curves can similarly be defined in 3-space, or on the surface of such bodies as the sphere and the torus. Later on, we shall be using a form of the famous **Jordan curve theorem** which states that if $\mathscr{C}$ is a closed Jordan curve in the plane, and if x and y are two distinct points of $\mathscr{C}$, then any Jordan curve connecting x and y must either lie completely inside $\mathscr{C}$ (except, of course, for the points x and y), lie completely outside $\mathscr{C}$ (with the same exceptions), or intersect $\mathscr{C}$ at some point other than x and y (see Fig. 4.3). (For further details about the Jordan curve theorem and related topics see, for example, Apostol.[2])

We are now ready to define an embedding of a graph in a given space; the spaces we have in mind are those in which Jordan curves can be defined, but we shall be primarily concerned with the plane and 3-

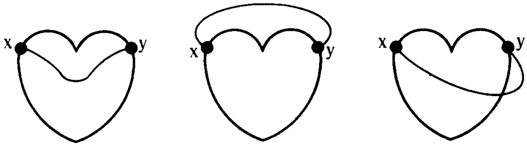


Fig. 4.3

space. A graph G **can be embedded** (or **has an embedding**) in a given space if it is isomorphic to a graph drawn in the space with points representing vertices of G and Jordan curves representing edges in such a way that there are no crossings. In this definition, a **crossing** is said to occur if either

(*i*) the Jordan curves corresponding to two edges intersect at a point which corresponds to no vertex, or

(*ii*) the Jordan curve corresponding to an edge passes through a point which corresponds to a vertex which is not incident to that edge.

(Case (*ii*) is illustrated in Fig. 4.4; note that the vertex v is not incident to the edges e_1 and e_2.)

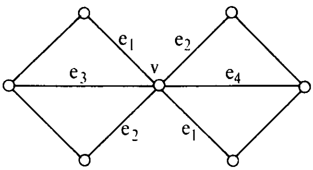


Fig. 4.4

We shall now prove the principal result of this section—that every graph can be drawn without crossings in 3-space.

THEOREM 4A. *Every graph can be embedded in Euclidean 3-space.*

Proof. We shall give an explicit construction for the embedding. First place the vertices of the graph at distinct points of the x-axis; then for each edge, choose a plane through the x-axis in such a way that distinct edges of the graph correspond to distinct planes. (This can always be done since there are only finitely many edges.)

The desired embedding is then obtained as follows: for each loop of the graph we draw in the corresponding plane a circle passing through the relevant vertex; for each edge joining two distinct vertices we draw in the corresponding plane a semicircle connecting these two vertices. Clearly none of these curves can intersect since they lie in different planes. The result now follows immediately.//

Theorem 4A gives us the justification we were seeking for using diagrams to depict graphs; we simply take a three-dimensional representation and project it down onto the plane, making sure that no two vertices are projected into the same point. In general, of course, such a method will lead to crossings, but in some cases we will get diagrams with no crossings. This can arise only when the graph in question can be embedded in the plane; such a graph is called a **planar graph**. Planar graphs will be studied in some detail in Chapter **5** but we have met several examples already, e.g. K_4, the null graphs, the Platonic graphs, the circuit graphs, the wheels and the star graphs.

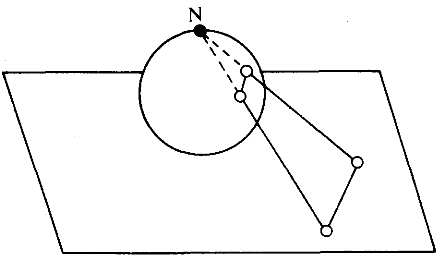


Fig. 4.5

We conclude this section by proving a simple result which will be needed later on. The proof will involve the following definition: if G is a graph embedded in some space, then a point x of the space is said to be **disjoint from G** if x represents neither a vertex of G nor a point which lies on an edge of G.

THEOREM 4B. *A graph is planar if and only if it can be embedded on the surface of a sphere.*

Proof. Let G be a graph embedded on the surface of a sphere. Place the sphere on a plane in such a way that the 'north pole' N (the point diametrically opposite the point of contact) is disjoint from G. The desired planar representation is then obtained by stereographic projection from N (see Fig. 4.5). The converse is similar and will be left as an exercise.//

Exercises 4

(4a) Show, by drawing, that the following graphs are planar:
- (*i*) the null graph N_5;
- (*ii*) the star graph $K_{1,4}$;
- (*iii*) the wheel W_5;
- (*iv*) the graph of the octahedron;
- (*v*) the complete bipartite graph $K_{2,3}$.

(4b) Show how the graph in Fig. 4.6 can be embedded in the plane.

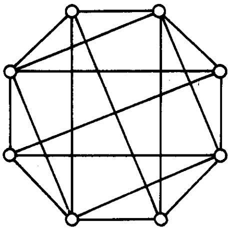


Fig. 4.6

(4c) (*i*) Prove that every subgraph of a planar graph is planar.
(*ii*) Assuming that K_5 and $K_{3,3}$ are non-planar, determine which complete graphs and complete bipartite graphs are planar.

(4d) Verify Theorem 4B for the complete bipartite graph $K_{2,10}$.

(4e) Verify that K_5 and $K_{3,3}$ can each be embedded on the surface of a torus (see Fig. 14.1).

(*4f) By placing the vertices at the points $(1, 1, 1)$, $(2, 2^2, 2^3)$, $(3, 3^2, 3^3)$, . . . , prove that any simple graph can be embedded in Euclidean 3-space in such a way that all of its edges are represented by straight lines.

(*4g) Let G be a planar graph with vertex-set $\{v_1, \ldots, v_n\}$, and let $p_1, \ldots, p_n$ be any n distinct points in the plane. Give a heuristic argument to show that G can be embedded in the plane in such a way that the point p_i represents the vertex v_i for each i. ★

3
Paths and circuits

. . . *So many paths that wind and wind,*
While just the art of being kind
Is all the sad world needs.

Ella Wheeler Wilcox

Now that we have a reasonable armoury of graphs at our disposal, we can start looking at their properties. In order to do this, we need some definitions which describe ways of 'going from one vertex to another'. In §5, we shall give these definitions, and prove some results on connectedness. §6 and §7 are devoted to a rather more detailed study of two particular types of graph, those which contain walks which include every edge, and those which contain circuits which include every vertex. We conclude this chapter with some applications of paths and circuits.

§5. More definitions

Given any graph G, a **walk** in G is a finite sequence of edges of the form

$$v_0v_1, v_1v_2, \ldots, v_{m-1}v_m$$

(also denoted by $v_0 \to v_1 \to v_2 \to \ldots \to v_m$). It is clear that a walk has the property that any two consecutive edges are either adjacent or identical; however, an arbitrary sequence of edges of G which has this property is not necessarily a walk (e.g. consider a star graph, and take its edges in any order). A walk trivially determines a sequence of vertices $v_0, v_1, \ldots, v_m$; we call v_0 the **initial vertex** and v_m the **final vertex** of the walk, and speak of a **walk from $\mathbf{v_0}$ to $\mathbf{v_m}$**. Note that if v_0 is any vertex, then the 'trivial walk' which contains no edges is a walk from v_0 to v_0. The number of edges in a walk is called its **length**; for example, in Fig. 5.1, $v \to w \to x \to y \to z \to z \to y \to w$ is a walk of length seven from v to w.

The concept of a walk is too general for our purposes, so we shall impose some further restrictions. A walk in which all the edges are distinct is called a **trail**; if, in addition, the vertices $v_0, v_1, \ldots, v_m$ are distinct (except, possibly, $v_0 = v_m$), then the trail is called a **path**. A path or trail is **closed** if $v_0 = v_m$, and a closed path containing at least one edge is called a **circuit**. Note that in particular any loop or any pair of multiple edges form a circuit.

Remark. This is another instance of widely differing terminology by various authors. A walk appears in the literature as an edge-sequence, route, path or edge-progression; a trail appears as a path, semi-simple path, or chain; a path as a chain, arc, simple path or simple chain; a closed trail as a cyclic path, re-entrant path or circuit; and a circuit as a cycle, elementary cycle, circular path or simple circuit!

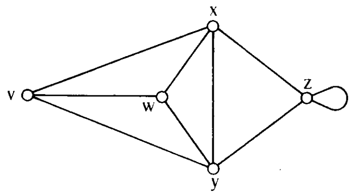


Fig. 5.1

In order to clarify the above concepts, let us again consider Fig. 5.1. We see that $v \to w \to x \to y \to z \to z \to x$ is a trail, $v \to w \to x \to y \to z$ is a path, $v \to w \to x \to y \to z \to x \to v$ is a closed trail, and $v \to w \to x \to y \to v$ is a circuit. A circuit of length three (such as $v \to w \to x \to v$) is called a **triangle**.

It is interesting to note that if G is any bipartite graph, then every circuit of G has even length. We shall prove this result here, leaving the proof of the converse result to the reader (exercise *5g*).

THEOREM 5A. *If $G(V_1, V_2)$ is a bipartite graph, then every circuit has even length.*

Proof. Let $v_1 \to v_2 \to \ldots \to v_m \to v_1$ be a circuit in $G(V_1, V_2)$, and assume (without loss of generality) that $v_1 \in V_1$. Then since $G(V_1, V_2)$ is bipartite, $v_2 \in V_2$, $v_3 \in V_1$, and so on. It follows that v_m is in V_2, and hence that the circuit has even length.//

We can also give an alternative, and possibly more useful, definition of a connected graph. A graph G is said to be **connected** if, given any pair of vertices v, w of G, there is a path from v to w. An arbitrary graph can split up into disjoint connected subgraphs called **(connected) components** by defining an equivalence relation on the vertex-set of G, two vertices being equivalent (or **connected**) if there is a path from one to the other; we leave it to you to verify that the connectedness of vertices is in fact an equivalence relation, and that each of the required components can be obtained by taking the vertices in an equivalence class and the edges incident to them. Clearly a connected graph has only one component; a graph with more than one component is called **disconnected**. We shall now prove that these definitions are consistent with the ones given in §3.

THEOREM 5B. *A graph is connected in the above sense if and only if it is connected in the sense of* §3.

Proof. $\Rightarrow$ Let G be a graph which is connected in the above sense. If G is the union of two (disjoint) subgraphs, and if v and w are two vertices, one from each subgraph, then any path from v to w must contain an edge which is incident to a vertex of each subgraph; since no such edge exists, we have a contradiction.

$\Leftarrow$ Now suppose that G is connected in the sense of §3, and suppose that there is no path connecting a given pair of vertices v and w; if we define connected components as above, then v and w will lie in different components. We can then express G as the union of two graphs, one of which is the component containing v and the other of which is the union of the remaining components; this establishes the required contradiction.//

Now that we know what connectedness means, it is natural to try to find out something about connected graphs. One direction of interest is to investigate bounds for the number of edges of a simple graph on n vertices with a given number of components. If such a graph is connected, it seems reasonable to expect that the graph has fewest edges when it has no circuits—such a graph is called a **tree**—and most edges when it is a complete graph; this would imply that the number of edges must lie between $n-1$ and $\frac{1}{2}n(n-1)$. We shall, in fact, prove a stronger theorem which includes this result as a special case.

THEOREM 5C. *Let G be a simple graph on n vertices; if G has k components, then the number m of edges of G satisfies*

$$n-k \leq m \leq \tfrac{1}{2}(n-k)(n-k+1).$$

Proof. To prove that $m \geq n-k$, we use induction on the number of edges of G, the result being trivial if G is a null graph. If G contains as few edges as possible (say m_0), then the removal of any edge of G must increase the number of components by one, and the graph which remains will have n vertices, $k+1$ components, and m_0-1 edges. It follows from the induction hypothesis that $m_0-1 \geq n-(k+1)$, from which we immediately deduce that $m_0 \geq n-k$, as required.

To prove the upper bound, we can assume that each component of G is a complete graph. Suppose, then, that there are two components C_i and C_j with n_i and n_j vertices respectively, where $n_i \geq n_j > 1$. If we replace C_i and C_j by complete graphs on n_i+1 and n_j-1 vertices, then the total number of vertices remains unchanged, and the number of edges is increased by

$$\tfrac{1}{2}\{(n_i+1)n_i - n_i(n_i-1)\} - \tfrac{1}{2}\{n_j(n_j-1)-(n_j-1)(n_j-2)\} = n_i - n_j + 1,$$

which is positive. It follows that in order to attain the maximum number of edges, G must consist of a complete graph on $n-k+1$ vertices and $k-1$ isolated vertices. The result now follows immediately.//

COROLLARY 5D. *Any simple graph with n vertices and more than* $\frac{1}{2}(n-1)(n-2)$ *edges is connected.*//

Another approach used in the study of connected graphs is to ask the question, 'how connected is a connected graph?' One possible interpretation of this question is to ask how many edges or vertices must be removed from the graph in order to disconnect it. We conclude this section with some definitions which are useful when discussing such a question.

A **disconnecting set** of a connected graph G is a set of edges of G whose removal disconnects G; for example, in the graph of Fig. 5.2, the sets $\{e_1, e_2, e_5\}$ and $\{e_3, e_6, e_7, e_8\}$ are both disconnecting sets of G, the

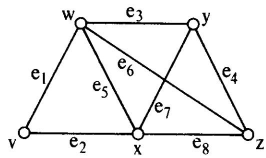


Fig. 5.2

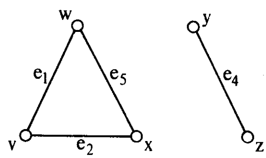


Fig. 5.3

disconnected graph left after removal of the second of these being shown in Fig. 5.3. We further define a **cutset** to be any disconnecting set, no proper subset of which is a disconnecting set; thus, in the example just given, only the second disconnecting set is actually a cutset. It is clear that the removal of the edges in a cutset always leaves a graph with exactly two components. If a cutset contains only one edge e, we shall call e a **bridge** or an **isthmus** (see Fig. 5.4). These definitions can clearly be extended to disconnected graphs: if G is any graph, then a **disconnecting**

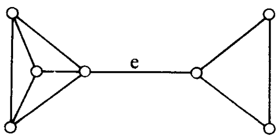


Fig. 5.4

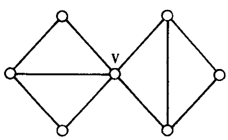


Fig. 5.5

set of G is a set of edges whose removal increases the number of components of G; a **cutset** of G is then simply a disconnecting set, no proper subset of which is a disconnecting set.

If G is connected, we define the **edge-connectivity** $\lambda(G)$ of G to be the size of the smallest cutset in G; in other words, $\lambda(G)$ is the smallest number of edges we can delete in order to disconnect G. For example, if G is the graph of Fig. 5.2, then $\lambda(G)=2$, corresponding to the cutset $\{e_1, e_2\}$. We also say that G is ***k*-edge-connected** if $\lambda(G) \geqq k$, so that the graph of Fig. 5.2 is 2-edge-connected, but not 3-edge-connected.

It is also very useful to define the analogous concepts for the removal of vertices, rather than edges. A **separating set** of a connected graph G is a set of vertices of G whose deletion disconnects G (recall that when we delete a vertex we also remove its incident edges); for example, in the graph of Fig. 5.2, the sets $\{w, x\}$ and $\{w, x, y\}$ are separating sets of G. If a separating set contains only one vertex v, we call v a **cut-vertex**, or **articulation vertex** (see Fig. 5.5). These definitions extend immediately to disconnected graphs, just as above.

If G is connected, we define the **(vertex-) connectivity** $\kappa(G)$ of G to be the size of the smallest separating set in G; in other words, $\kappa(G)$ is the smallest number of vertices we can delete in order to disconnect G. For example, if G is the graph of Fig. 5.2, then $\kappa(G)=2$, corresponding to the separating set $\{w, x\}$. We also say that G is ***k*-connected** if $\kappa(G) \geqq k$, so that the graph of Fig. 5.5 is 1-connected, but not 2-connected. It can be proved that if G is any connected graph, then $\kappa(G) \leqq \lambda(G)$ (see, for example, Bondy and Murty[7]).

Finally, we remark that there is a striking and unexpected similarity between the properties of circuits and those of cutsets; the reader will recognise this if he looks at such exercises as 5k, 5l, 5m, 6h and 9k. The reason for this will be revealed in Chapter **9**, when everything will suddenly become clear!

Exercises 5

(5a) In the Petersen graph, find
   (*i*) a trail of length 5;
   (*ii*) a path of length 9;
   (*iii*) circuits of lengths, 5, 6, 8 and 9;
   (*iv*) cutsets with 3, 4 and 5 edges.

(5b) The **girth** of a graph is the length of its shortest circuit. Find the girths of (*i*) K_9; (*ii*) $K_{7,5}$; (*iii*) C_8; (*iv*) W_8; (*v*) Q_5; (*vi*) the Petersen graph; (*vii*) the graph of the dodecahedron.

(5c) Find $\kappa(G)$ and $\lambda(G)$ for each of the following graphs G: (*i*) C_6; (*ii*) W_6; (*iii*) $K_{4,7}$; (*iv*) Q_4.

(5d) (*i*) Let G be a connected graph with smallest degree k. Show that $\lambda(G) \leqslant k$.
   (*ii*) Construct a graph G with smallest degree k for which $\kappa(G) < \lambda(G) < k$.

(5e) (*i*) Prove that a graph G is 2-connected if and only if each pair of vertices of G are contained in a common circuit.
(*ii*) Write down a corresponding statement for a 2-edge-connected graph.

(5f) Prove that if G is a simple graph, then G and $\bar{G}$ cannot both be disconnected.

(5g) Prove the converse of Theorem 7A—that if every circuit of a graph G has even length, then G is bipartite.

(5h) Let G be a connected graph with vertex-set $\{v_1, v_2, \ldots, v_n\}$ and m edges.
(*i*) If $\mathbf{A}$ is the adjacency matrix of G, prove that the number of walks of length 2 from v_i to v_j is the ij-th entry of the matrix $\mathbf{A}^2$.
(*ii*) Deduce that $2m = \text{trace}(\mathbf{A}^2)$, the sum of the diagonal entries of $\mathbf{A}^2$.

(5i) In a connected graph, the **distance** $d(v,w)$ from v to w is the length of the shortest path from v to w.
(*i*) If $d(v,w) \geqslant 2$, show that there exists a vertex z such that $d(v,z) + d(z,w) = d(v,w)$.
(*ii*) In the Petersen graph, show that $d(v,w) = 1$ or 2 for any distinct vertices v and w.

(*5j) **Turán's extremal theorem**: Let G be a simple graph on $2k$ vertices which contains no triangles. Show (by induction on k) that G has at most k^2 edges, and give an example of a graph for which this upper bound is achieved.

(*5k) (*i*) Prove that, if two distinct circuits of a graph G each contain an edge e, then G has a circuit which does not contain e.
(*ii*) Prove a similar result with 'circuit' replaced throughout by 'cutset'.

(*5l) (*i*) Prove that if C is a circuit and C^* is a cutset of a connected graph G, then C and C^* have an even number of edges in common.
(*ii*) Prove that if S is any set of edges of G with an even number of edges in common with each cutset of G, then S may be split up into edge-disjoint circuits.

(*5m) A set E of edges of a graph G is said to be **independent** if E contains no circuit of G. Prove that
(*i*) every subset of an independent set is independent;
(*ii*) If I and J are independent sets of edges with $|J| > |I|$, then there exists an edge e which is in J but not in I with the property that $I \cup \{e\}$ is an independent set.

Show also that (*i*) and (*ii*) still hold if we replace the word 'circuit' by 'cutset'.

§6. Eulerian graphs

A connected graph G is called **Eulerian** if there exists a closed trail which includes every edge of G; such a trail is then called an **Eulerian trail**. Note that the definition requires each edge to be traversed once and once only. G is **semi-Eulerian** if we remove the restriction that the trail must be closed; thus every Eulerian graph is semi-Eulerian. Figs 6.1, 6.2 and 6.3 show graphs which are non-Eulerian, semi-Eulerian and Eulerian,

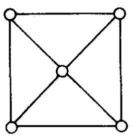


Fig. 6.1

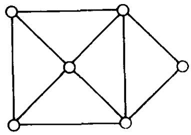


Fig. 6.2

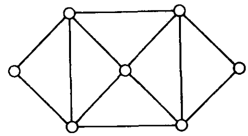


Fig. 6.3

respectively. Note that the assumption that G is connected is merely a technicality introduced in order to avoid the trivial case of a graph containing several isolated vertices.

Problems on Eulerian graphs frequently appear in books on recreational mathematics—a typical problem might ask whether a given diagram can be drawn without lifting one's pencil from the paper and without repeating any lines. The name 'Eulerian' arises from the fact that Euler was the first person to solve the famous Königsberg bridges problem which asked whether it is possible to cross each of the seven bridges in Fig. 6.4 exactly once and return to your starting-point. This is equivalent to asking whether the graph in Fig. 6.5 has an Eulerian trail (it hasn't!). A translation of Euler's paper, and a discussion of various related topics, may be found in Biggs, Lloyd and Wilson.[5]

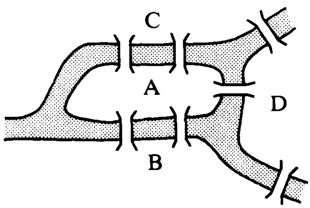


Fig. 6.4

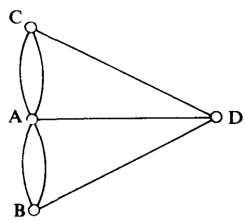


Fig. 6.5

One question which immediately arises is 'can one find necessary and sufficient conditions for a graph to be Eulerian?' Before providing a complete answer to this question in Theorem 6B, we prove a simple lemma.

LEMMA 6A. *If G is a graph in which the degree of every vertex is at least two, then G contains a circuit.*

Proof. If G contains any loops or multiple edges, the result is trivial; we can therefore suppose that G is a simple graph. Let v be any vertex of G. We can construct a walk $v \to v_1 \to v_2 \to \ldots$ inductively by choosing v_1 to be any vertex adjacent to v, and for $i \geqq 1$ choosing v_{i+1} to be any vertex adjacent to v_i except v_{i-1} (the existence of such a vertex v_{i+1} being guaranteed by our hypothesis). Since G has only finitely many vertices, we must eventually choose a vertex which has been chosen before. If v_k is the first such vertex, then that part of the walk which lies between the two occurrences of v_k is the required circuit.//

THEOREM 6B. *A connected graph G is Eulerian if and only if the degree of every vertex of G is even.*

Proof.$\Rightarrow$ Suppose that P is an Eulerian trail of G. Whenever P passes through any vertex, there is a contribution of two towards the degree of that vertex. Since every edge occurs exactly once in P, every vertex must have even degree.

$\Leftarrow$ The proof is by induction on the number of edges of G. Suppose the degree of each vertex is even. Since G is connected, every vertex has degree at least two, and so by the above lemma, G contains a circuit C. If

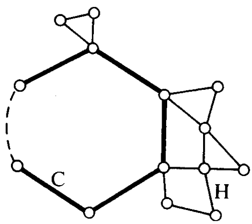


Fig. 6.6

C contains every edge of G, the proof is complete. If not, we remove from G the edges of C to form a new (possibly disconnected) graph H which has fewer edges than G and in which every vertex still has even degree. By the induction hypothesis, each component of H has an Eulerian trail. Since each component of H has at least one vertex in common with C, by connectedness, we obtain the required Eulerian trail of G by following the edges of C until a non-isolated vertex of H is reached, tracing the Eulerian trail of the component of H which contains that vertex, and then continuing along the edges of C until we reach a vertex belonging to another component of H, and so on. The whole process terminates when we get back to the initial vertex (see Fig. 6.6).//

The proof just given can be easily modified to prove the following two results. We omit the details.

COROLLARY 6C. *A connected graph is Eulerian if and only if its edge-family can be split up into disjoint circuits.*//

COROLLARY 6D. *A connected graph is semi-Eulerian if and only if there are 0 or 2 vertices of odd degree.*//

We remark that if a semi-Eulerian graph has exactly two vertices of odd degree, then any semi-Eulerian trail (in the obvious sense) must have one of them as initial vertex and the other as final vertex. By the handshaking lemma, a graph cannot have exactly one vertex of odd degree.

To conclude our discussion of Eulerian graphs, we now give an algorithm for constructing an Eulerian trail in a given Eulerian graph. The method is known as **Fleury's algorithm**.

THEOREM 6E. *Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian trail of G. Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules:*

(i) erase the edges as they are traversed, and if any isolated vertices result erase them too;

(ii) at each stage, use a bridge only if there is no alternative.

★*Proof.* We shall show first that at each stage the construction may be carried out. Suppose that we have just reached a vertex v; then if $v \neq u$, the subgraph H which still remains is connected and contains only two vertices of odd degree—namely, u and v. In order to show that the construction can be carried out, we must show that the removal of the next edge does not disconnect H—or equivalently, that v is incident to at most one bridge. But if this is not the case, then there exists a bridge vw with the property that the component K of $H - vw$ containing w does not

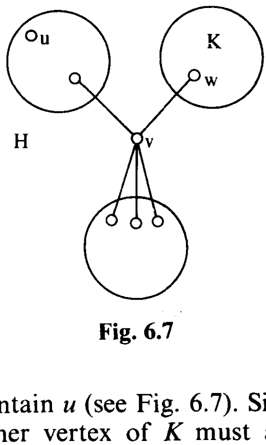


Fig. 6.7

contain u (see Fig. 6.7). Since the vertex w has odd degree in K, some other vertex of K must also have odd degree, giving the required contradiction. If $v = u$, the proof is almost identical, as long as there are still edges incident with u.

It remains only to show that the construction always yields a complete Eulerian trail. But this is clear, since there can be no edges of G remaining untraversed when the last edge incident to u is used, since otherwise the removal of some earlier edge adjacent to one of these edges would have disconnected the graph, contradicting (*ii*).//★

Exercises 6

(6a) Which of the following graphs are Eulerian? semi-Eulerian?
- (*i*) the complete graph K_5;
- (*ii*) the complete bipartite graph $K_{4,3}$;
- (*iii*) the graph of the cube;
- (*iv*) the graph of the octahedron;
- (*v*) the Petersen graph.

(6b)
- (*i*) For which values of n is K_n Eulerian?
- (*ii*) Which complete bipartite graphs are Eulerian?
- (*iii*) Which Platonic graphs are Eulerian?
- (*iv*) For which values of n is the wheel W_n Eulerian?
- (*v*) For which values of k is the k-cube Q_n Eulerian?

(6c) Let G be a connected graph with $k(>0)$ vertices of odd degree.
- (*i*) Show that the minimum number of trails, which have no edges in common and which together include every edge of G, is $\frac{1}{2}k$.
- (*ii*) How many continuous pen-strokes are needed to draw the diagram in Fig. 6.8 without repeating any line?

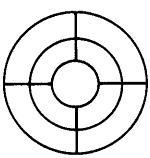


Fig. 6.8

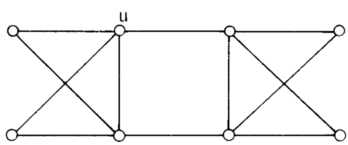


Fig. 6.9

(6d) Use Fleury's algorithm to produce an Eulerian trail for the graph shown in Fig. 6.9.

(6e)
- (*i*) Is it possible for a knight to travel around a chessboard in such a way that every possible move occurs exactly once? (A move between two squares 'occurs' if it is traversed in either direction.)
- (*ii*) Repeat part (*i*) for a king and a rook. Interpret your solutions in graph-theoretical terms.

(6f)
- (*i*) Show that the line graph of a simple Eulerian graph is Eulerian.
- (*ii*) If the line graph of a simple graph G is Eulerian, must G be Eulerian?

(6g) An Eulerian graph is **randomly traceable** from a vertex v if, whenever we start from v and traverse the graph in an arbitrary way never using any edge twice, we eventually obtain an Eulerian trail.
- (*i*) Show that the graph in Fig. 6.10 is randomly traceable.
- (*ii*) Give an example of an Eulerian graph which is not randomly traceable.

(*iii*) Why might a randomly traceable graph be suitable for the layout of an exhibition?

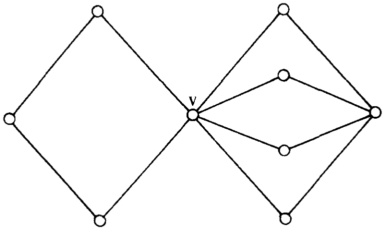


Fig. 6.10

(*6h) Let V be the vector space associated with a graph G (see exercise 2m).

(*i*) Use Corollary 6C to show that if C and D are circuits of G, then their sum $C \oplus D$ can be written as a union of edge-disjoint circuits.

(*ii*) Deduce that the set of such unions of circuits of G forms a subspace W of V (called the **circuit subspace** of G), and find its dimension.

(*iii*) Show that the set of unions of edge-disjoint cutsets of G forms a subspace $\tilde{W}$ of V (called the **cutset subspace** of G), and find its dimension.

§7. Hamiltonian graphs

In the previous section we discussed the problem of whether there exists a closed trail which includes every edge of a given connected graph G. A similar problem we can consider is whether there exists a closed trail which passes exactly once through each *vertex* of G. It is clear that such a trail must be a circuit (excluding the trivial case in which G is the graph N_1). If such a circuit exists, it is called a **Hamiltonian circuit**, and G is called a **Hamiltonian graph**. A graph which contains a path which passes through every vertex is called **semi-Hamiltonian**; note that every Hamiltonian graph is semi-Hamiltonian. Figs 7.1, 7.2 and 7.3 show graphs which are non-Hamiltonian, semi-Hamiltonian and Hamiltonian, respectively.

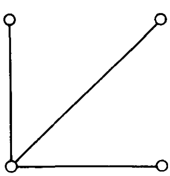


Fig. 7.1

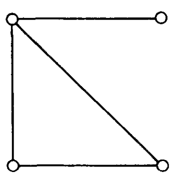


Fig. 7.2

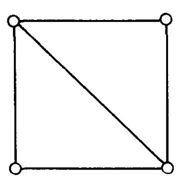


Fig. 7.3

The name 'Hamiltonian circuit' arises from the fact that Sir William Hamilton investigated the existence of such circuits in the dodecahedral graph (although a more general problem had been studied earlier by the Rev. T. P. Kirkman); such a circuit is shown in Fig. 7.4, heavy lines denoting its edges.

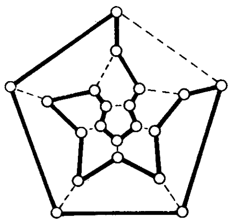


Fig. 7.4

In Theorem 6B we obtained a necessary and sufficient condition for a connected graph to be Eulerian, and it is perhaps reasonable to expect that we can obtain a similar characterization for Hamiltonian graphs. As it happens, the finding of such a characterization is one of the major unsolved problems of graph theory! In fact, little is known in general about Hamiltonian graphs. Most existing theorems have the form, 'if G has enough edges, then G is Hamiltonian'. Probably the most celebrated of these is due to G. A. Dirac, and known, reasonably enough, as **Dirac's theorem**. We shall deduce it from the following more general result, due to O. Ore:

THEOREM 7A. *If G is a simple graph with $n(\geq 3)$ vertices, and if $\rho(v)+\rho(w)\geq n$ for each pair of non-adjacent vertices v and w, then G is Hamiltonian.*

Proof. We shall assume the theorem false, and derive a contradiction. So let G be a *non*-Hamiltonian graph with n vertices, satisfying the given condition on the vertex-degrees. By adding extra edges if necessary, we may assume that G is 'only just' non-Hamiltonian, in the sense that the addition of any further edge results in a Hamiltonian graph. (Note that adding an extra edge does not violate the condition on the vertex-degrees.) It follows that G contains a path $v_1 \to v_2 \to \ldots \to v_n$ which includes every vertex. But since G is non-Hamiltonian, the vertices v_1 and v_n are not adjacent, and so $\rho(v_1)+\rho(v_n)\geq n$. It follows that there must be some vertex v_i adjacent to v_1 with the property that v_{i-1} is adjacent to v_n (see Fig. 7.5). But this gives us the required contradiction, since

$$v_1 \to v_2 \to \ldots \to v_{i-1} \to v_n \to v_{n-1} \to \ldots \to v_{i+1} \to v_i \to v_1$$

is then a Hamiltonian circuit.//

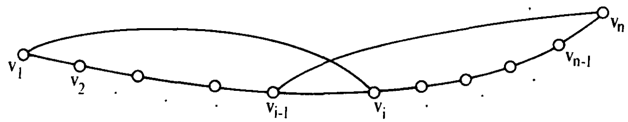


Fig. 7.5

COROLLARY 7B (Dirac 1952). *If G is a simple graph with $n(\geqq 3)$ vertices, and if $\rho(v) \geqq \frac{1}{2}n$ for every vertex v, then G is Hamiltonian.*

Proof. The result follows immediately from Theorem 7A since $\rho(v)+\rho(w) \geqq n$ for every pair of vertices v and w (whether adjacent or not).//

Exercises 7

(7a) Which of the following graphs are Hamiltonian? semi-Hamiltonian?
  (*i*) the complete graph K_5;
  (*ii*) the complete bipartite graph $K_{5,3}$;
  (*iii*) the graph of the octahedron;
  (*iv*) the wheel W_6;
  (*v*) the 4-cube Q_4.

(7b) (*i*) For which values of n is K_n Hamiltonian?
  (*ii*) Which complete bipartite graphs are Hamiltonian?
  (*iii*) Which Platonic graphs are Hamiltonian?
  (*iv*) For which values of n is the wheel W_n Hamiltonian?
  (*v*) For which values of k is the k-cube Q_n Hamiltonian?

(7c) Show that the 'Grötzsch graph' in Fig. 7.6 is Hamiltonian.

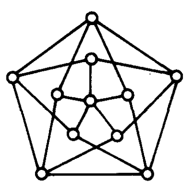


Fig. 7.6

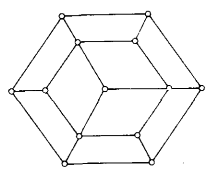


Fig. 7.7

(7d) (*i*) Prove that, if G is a bipartite graph with an odd number of vertices, then G is non-Hamiltonian.
  (*ii*) Deduce that the graph in Fig. 7.7 is non-Hamiltonian.
  (*iii*) Show that if n is odd, it is not possible for a knight to visit all the squares of an $n \times n$ chessboard exactly once and return to its starting point.

(7e) Give an example to show that, in the statement of Dirac's theorem, the condition '$\rho(v) \geqq \frac{1}{2}n$' cannot be replaced by '$\rho(v) \geqq \frac{1}{2}n-1$'.

(7f) (*i*) Let G be a graph with n vertices and $\frac{1}{2}(n-1)(n-2)+2$ edges. Use Theorem 7A to prove that G is Hamiltonian.
  (*ii*) Give an example of a non-Hamiltonian graph with n vertices and $\frac{1}{2}(n-1)(n-2)+1$ edges.

(*7g) Prove that the Petersen graph is non-Hamiltonian.

(*7h) Let G be a Hamiltonian graph and let S be any set of k vertices in G. Prove that the graph $G-S$ has at most k components.

(*7i) (*i*) Show that there exist four Hamiltonian circuits in K_9 with the property that no two of them have an edge in common.

(*ii*) What is the maximum number of edge-disjoint Hamiltonian circuits in K_{2k+1}?

§8. Some applications

Although we are primarily concerned in this book with the *theory* of graphs, it is high time that we mentioned some possible applications. After all, most of the important advances in the subject arose as a result of attempts to solve particular practical problems—Euler and the bridges of Königsberg, Cayley and the enumeration of chemical molecules (§**11**), and Kirchhoff's work on electrical networks (§**11**), to name but three. Much of the present-day interest in the subject is due to the fact that, quite apart from being an elegant mathematical discipline in its own right, graph theory is playing an ever-increasing rôle in such a wide range of subjects as electrical engineering and linguistics, operational research and crystallography, probability and genetics, and sociology, geography and numerical analysis.

It is inappropriate in a book of this size to try to discuss a large number of applications in any kind of detail; for this, we refer the reader to the excellent account in Deo,[11] or to Berge,[4] Bondy and Murty,[7] or Wilson and Beineke.[26] These books include a variety of different applications, often with algorithms or flow-charts, for the solution of particular problems.

In the present section we shall briefly describe three problems which relate to the material of this chapter—namely, the shortest path problem, the Chinese postman problem and the travelling salesman problem. In later sections we shall be discussing the use of graphs and digraphs in, for example, the enumeration of chemical molecules (§**11**), electrical networks (§**11**), timetabling problems (§§**21**, **27**), critical path analysis (§**22**), Markov chains (§**24**), and the problem of finding maximal flows in transportation networks (§**29**).

The shortest path problem

Let us suppose that we have a 'map' of the form shown in Fig. 8.1, in which the letters A–L refer to towns which are connected by roads. If the lengths of these roads are as marked on the diagram, what is the length of the shortest path from A to L?

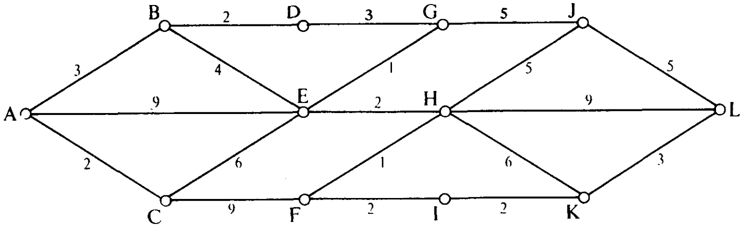


Fig. 8.1

There are several remarks we can make concerning this problem. First of all we note that an upper bound for the answer may easily be obtained by taking any path from A to L and calculating its length. For example, the paths $A \to B \to D \to G \to J \to L$ and $A \to C \to F \to I \to K \to L$ both have total length 18, so that the length of the shortest path cannot possibly exceed 18. Secondly, note that the numbers in the diagram need not necessarily refer to the lengths of the various roads, but could equally well refer to the time taken to travel along the roads, or to the cost involved in doing so. It follows that if we can find a method for solving the shortest path problem in its original formulation, then we can apply the same method to find the quickest or the cheapest route. Note finally that we can express this problem in graph-theoretic terms by regarding the diagram as a connected graph in which a non-negative real number has been assigned to each edge. Such a graph is called a **weighted graph**, and the number assigned to each edge e is called the **weight** of e, denoted by $w(e)$. The problem is then to find a path from A to L with minimum total weight. Note that if we have a weighted graph in which each edge has weight one, then the problem reduces to that of finding the length of the shortest path from A to L in the graph-theoretic sense—that is, the smallest number of edges needed in going from A to L.

There are several methods which can be used to solve this problem. Possibly the simplest of these is to make a model of the map by knotting together pieces of string whose lengths are proportional to the lengths of the roads. In order to find the shortest path, we take hold of the knots corresponding to A and L—and pull tight!

However, there is a more mathematical way of approaching this problem. Essentially the idea is to move across the graph from left to right, associating with each vertex V a number $l(V)$ indicating the shortest distance from A to V. In practice, this means that when we reach a vertex such as J in Fig. 8.1, then $l(J)$ will be either $l(G)+5$ or $l(H)+5$, whichever is the smaller.

To apply the method, we assign A the label 0 and give B, E and C the *temporary labels* $l(A)+3$, $l(A)+9$, $l(A)+2$—that is, 3, 9 and 2. We next take the *smallest* of these, and write $l(C)=2$. C is now permanently labelled 2.

The next step is to look at the vertices adjacent to C. We assign F the temporary label $l(C)+9=11$, and we can lower the temporary label at E to $l(C)+6=8$. The smallest temporary label is now 3 (at B), so we write $l(B)=3$.

Now we look at the vertices adjacent to B. We assign D the temporary label $l(B)+2=5$, and we can lower the temporary label at E to $l(B)+4=7$. The smallest temporary label is now 5 (at D), so we write $l(D)=5$.

Continuing in this way, we successively obtain the permanent labels $l(E)=7$, $l(G)=8$, $l(H)=9$, $l(F)=10$, $l(I)=12$, $l(J)=13$, $l(K)=14$, $l(L)=17$. It follows that the shortest path from A to L has length 17. It is shown in Fig. 8.2, with circled numbers representing the labels at the vertices.

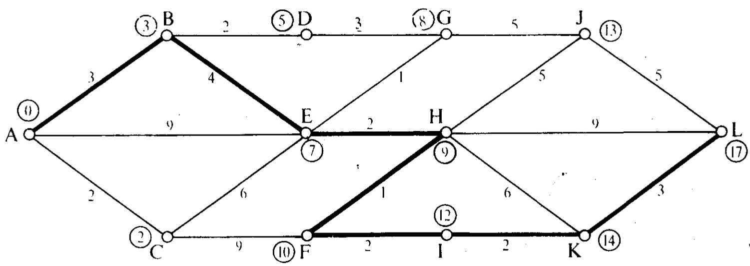


Fig. 8.2

In §**22** we shall see how this method can be adapted to yield the *longest* path in a digraph, and we shall indicate its use in critical path analysis.

The Chinese postman problem

In this problem, discussed by the Chinese mathematician Mei-ko Kwan, a postman wishes to deliver all of his letters in such a way that he covers the least possible total distance and then returns to his starting point. He must obviously traverse each of the roads in his route at least once, but he clearly wishes to avoid covering too many roads more than once.

This problem can be reformulated in terms of weighted graphs, where the graph corresponds to the network of roads, and the weight of each edge corresponds to the length of the corresponding road. In this reformulation, the requirement is to find a closed walk which includes every edge at least once, and has least possible total weight.

It is clear that if the graph in question is Eulerian, then any Eulerian trail is a closed walk of the required type. Such an Eulerian trail can be found by Fleury's algorithm (see §**6**). If the graph is not Eulerian, then the problem is much harder, although a good algorithm for this solution is known. To illustrate the ideas involved, we shall look at a special case—that in which there are exactly two vertices of odd degree (see Fig. 8.3).

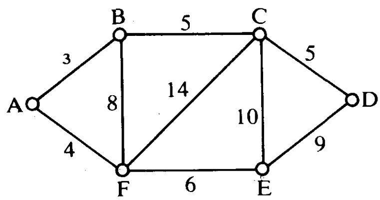


Fig. 8.3

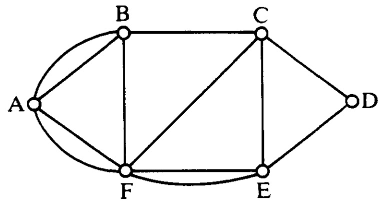


Fig. 8.4

Since vertices B and E are the only vertices of odd degree, we can find a semi-Eulerian trail from B to E covering each edge exactly once. In order to return to the starting point, covering the least possible distance, we now find the shortest path from E to B using the method described above. The solution of the Chinese postman problem is then obtained by taking this shortest path $E \to F \to A \to B$ together with the original semi-Eulerian trail, giving a total distance of $13 + 64 = 77$. Note that if we combine the shortest path and the semi-Eulerian trail, we get an Eulerian graph (see Fig. 8.4).

A fuller discussion of the Chinese postman problem can be found in Bondy and Murty.[7]

The travelling salesman problem

In this problem, a travelling salesman wishes to visit several given cities and return to his starting point, in such a way that he covers the least possible total distance. For example, if there are five cities A, B, C, D and E, and if the distances are as given in Fig. 8.5, then the shortest possible

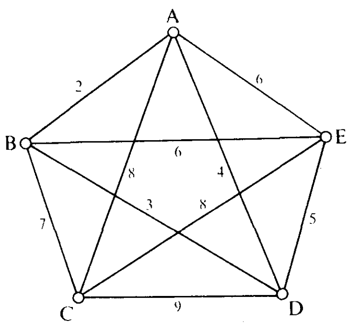


Fig. 8.5

route is $A \to B \to D \to E \to C \to A$, giving a total distance of 26, as can be seen by inspection.

Note that this problem can also be reformulated in terms of weighted graphs; in this case, the requirement is to find a Hamiltonian circuit of least possible total weight in a weighted complete graph. Note also that, just as in the shortest path problem, the numbers can equally well refer to the time taken to travel between the cities, or to the cost involved in doing so. It follows that if we can find an efficient algorithm for solving the travelling salesman problem in its original formulation, then we can apply the same algorithm to find the quickest or the cheapest route.

One possible algorithm would be to calculate the total distance for all of the possible Hamiltonian circuits, but this turns out to be far too complicated for more than about four or five cities. Various other algorithms have been proposed, but they either take too long to apply, or are too complicated to carry out. On the other hand, there are some quite effective procedures which can tell us approximately what the shortest distance is. One of these procedures will be described in §**11**.

Exercises 8

(8a) Use the shortest path algorithm to find a shortest path from A to G in the weighted graph of Fig. 8.6.

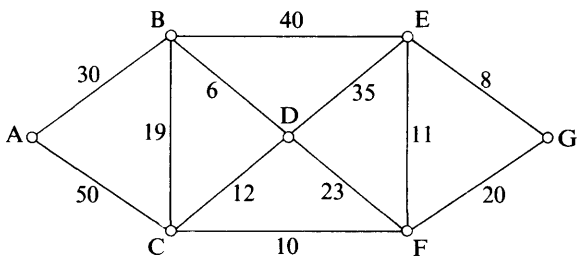
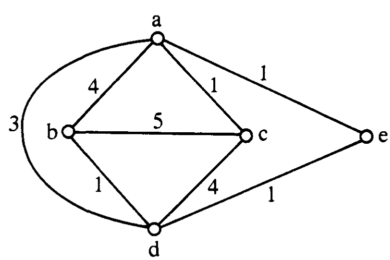


Fig. 8.6



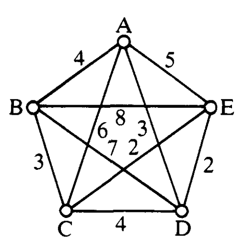


Fig. 8.7

Fig. 8.8

(8b) Use the shortest path algorithm to find the shortest path from L to A in Fig. 8.1, and check that your answer agrees with that given in Fig. 8.2.
(8c) Show how the shortest path algorithm can be adapted to yield the *longest* path from A to L in Fig. 8.1.
(8d) Solve the Chinese postman problem for the weighted graph of Fig. 8.7.
(8e) Solve the travelling salesman problem for the weighted graph of Fig. 8.8.
(8f) Find the Hamiltonian circuit of *greatest* weight in the graph of Fig. 8.5.
(*8g) Find the shortest path from S to each of the other vertices in the weighted graph of Fig. 8.9.

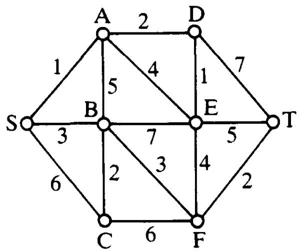


Fig. 8.9

4

Trees

A fool sees not the same tree that a wise man sees.

William Blake

We are all familiar with the idea of a family tree; in this chapter, we shall be studying trees in general, with special reference to spanning trees in a connected graph and (in §**10**) to Cayley's celebrated result on the enumeration of labelled trees. The chapter concludes with a section on some further applications.

§9. Elementary properties of trees

A **forest** is defined to be a graph which contains no circuits, and a connected forest is called a **tree**; for example, Fig. 9.1 shows a forest with four components, each of which is a tree.† Note that trees and forests are by definition simple graphs.

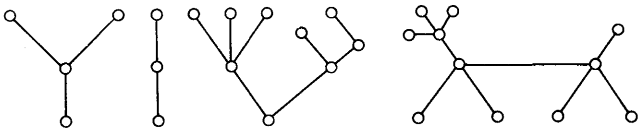


Fig. 9.1

In many ways a tree is the simplest non-trivial type of graph; as we shall see in Theorem 9A, it has several 'nice' properties such as the fact that any two vertices are connected by a unique path. In trying to prove a general result or test a general conjecture in graph theory, it is sometimes convenient to start by trying to prove the corresponding result for a tree; in fact, there are several conjectures which have not been proved for arbitrary graphs but which are known to be true for trees.

†The last tree in Fig. 9.1 is particularly well known for its bark.

The following theorem lists some simple properties of trees:

THEOREM 9A. *Let T be a graph with n vertices. Then the following statements are equivalent:*

(*i*) *T is a tree;*
(*ii*) *T contains no circuits, and has $n-1$ edges;*
(*iii*) *T is connected, and has $n-1$ edges;*
(*iv*) *T is connected, and every edge is a bridge;*
(*v*) *any two vertices of T are connected by exactly one path;*
(*vi*) *T contains no circuits, but the addition of any new edge creates exactly one circuit.*

Proof. If $n=1$, all six results are trivial. We shall therefore assume that $n \geqq 2$.

(*i*)$\Rightarrow$(*ii*). Since T contains no circuits, by definition, it follows that the removal of any edge disconnects T into two graphs, each of which is a tree. It follows by induction that the number of edges in each of these two trees is one fewer than the number of vertices, from which we deduce that the total number of edges of T is $n-1$.

(*ii*)$\Rightarrow$(*iii*). If T is disconnected, then each component of T is a connected graph with no circuits and hence, by the previous part, the number of vertices in each component exceeds by one the number of edges. It follows that the total number of vertices of T exceeds the total number of edges by at least two, contradicting the fact that T has $n-1$ edges.

(*iii*)$\Rightarrow$(*iv*). The removal of any edge results in a graph with n vertices and $n-2$ edges, which must be disconnected by Theorem 5C.

(*iv*)$\Rightarrow$(*v*). Since T is connected, each pair of vertices is connected by at least one path. If a given pair of vertices is connected by two paths, then they enclose a circuit, which contradicts the fact that every edge is a bridge.

(*v*)$\Rightarrow$(*vi*). If T contained a circuit, then any two vertices in the circuit would be connected by at least two paths, contradicting (*v*). If an edge e is added to T, then, since the vertices incident to e are already connected in T, a circuit will be created; the fact that only one circuit is obtained follows from exercise 5k.

(*vi*)$\Rightarrow$(*i*). Suppose that T is disconnected. If to T we add any edge which joins a vertex of one component to a vertex in another, then no circuit will be created.//

COROLLARY 9B. *Let G be a forest with n vertices and k components; then G has $n-k$ edges.*

Proof. Apply the above statement (*iii*) to each component of G.//

Note that by the handshaking lemma, the sum of the degrees of all the n vertices of a tree is equal to twice the number of edges ($=2n-2$); it follows that *if $n \geqq 2$, a tree on n vertices always contains at least two end-vertices.*

Given any connected graph G, we can choose a circuit and remove one of its edges, the resulting graph remaining connected. We repeat this procedure with one of the remaining circuits, continuing until there are no circuits left. The graph which remains will be a tree which connects all the vertices of G; it is called a **spanning tree** of G. An example of a graph and one of its spanning trees appears in Figs 9.2 and 9.3.

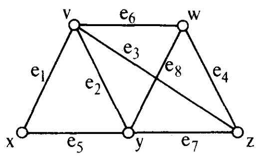


Fig. 9.2

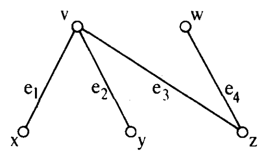


Fig. 9.3

More generally, if G now denotes an arbitrary graph with n vertices, m edges and k components, we can carry out the above procedure on each component of G, the result being called a **spanning forest**. The number of edges removed in the process is called the **circuit rank** (or **cyclomatic number**) of G, and is denoted by $\gamma(G)$; note that $\gamma(G)=m-n+k$, which is a non-negative integer by Theorem 5C. It is convenient also to define the **cutset rank** (or **component rank**) of G to be the number of edges in a spanning forest; it is denoted by $\xi(G)$ and is equal to $n-k$. Some properties of the cutset rank are given in exercise 9j.

Before proceeding, we shall prove a couple of simple results concerning spanning forests. In this theorem, the **complement** of a spanning forest T of a (not necessarily simple) graph G is simply the graph obtained from G by removing the edges of T.

THEOREM 9C. *If T is any spanning forest of a graph G, then*

(*i*) *every cutset of G has an edge in common with T;*

(*ii*) *every circuit of G has an edge in common with the complement of T.*

Proof. (*i*) Let C^* be a cutset of G, the removal of which splits one of the components of G into two subgraphs H and K. Then since T is a spanning forest, T must contain an edge joining a vertex of H to a vertex of K; this edge is the required edge.

(*ii*) Let C be a circuit of G which has no edge in common with the complement of T. Then C must be contained in T, which is a contradiction.//

Closely linked with the idea of a spanning forest T of a graph G is the concept of the fundamental system of circuits associated with T, formed as follows: if we add to T any edge of G not contained in T, then by statement (*vi*) of Theorem 9A we get a unique circuit. The set of all circuits formed in this way (i.e., by adding separately each edge of G not contained in T) is called the **fundamental system of circuits associated with T**. Sometimes we are not interested in the particular spanning forest chosen, and refer simply to a **fundamental system of circuits of G**. In any case, it is clear that the circuits in a given fundamental system must be distinct, and that the number of such circuits must be equal to the circuit rank of G. Fig. 9.4 shows the fundamental system of circuits of the graph shown in Fig. 9.2 associated with the spanning tree of Fig. 9.3.

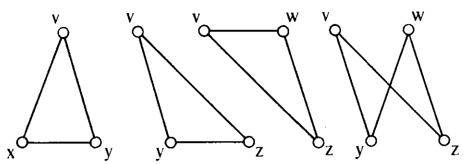


Fig. 9.4

In the light of our remarks at the end of §5, we may hope to be able to define a fundamental system of cutsets of a graph G associated with a given spanning forest T; we shall now show that this is indeed the case. By statement (*iv*) of Theorem 9A, the removal of any edge of T divides the set of vertices of T into two disjoint sets V_1 and V_2. The set of all edges of G joining a vertex of V_1 with one of V_2 is a cutset of G, and the set of all cutsets obtained in this way (i.e. by removing separately each edge of T) is called the **fundamental system of cutsets** associated with T. It is clear that the cutsets in a given fundamental system must be distinct, and that the number of such cutsets must be equal to the cutset rank of G. The fundamental system of cutsets of the graph in Fig. 9.2 associated with the spanning tree of Fig. 9.3 is $\{e_1, e_5\}$, $\{e_2, e_5, e_7, e_8\}$, $\{e_3, e_6, e_7, e_8\}$ and $\{e_4, e_6, e_8\}$.

Exercises 9

(9a) Show, by drawing, that there are (up to isomorphism) three trees on 5 vertices, six trees on 6 vertices and eleven trees on 7 vertices.

(9b) (*i*) Prove that every tree is a bipartite graph.
(*ii*) Which trees are complete bipartite graphs?

(9c) Draw all the spanning trees in the graphs in Fig. 9.5.

(9d) Find the fundamental systems of circuits and cutsets of the graph in Fig. 9.6 associated with the spanning tree shown.

(9e) Calculate the circuit and cutset ranks of (*i*) K_5; (*ii*) $K_{3,3}$; (*iii*) W_5; (*iv*) N_6; (*v*) the Petersen graph.

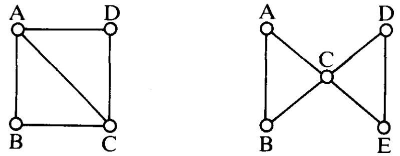


Fig. 9.5

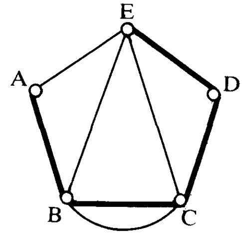


Fig. 9.6

(9f) Let G be a connected graph. What can you say about
  (*i*) an edge of G which appears in every spanning tree?
  (*ii*) an edge of G which appears in no spanning tree?

(9g) If G is a connected graph, a **centre** of G is a vertex v with the property that the maximum of the distances between v and the other vertices of G is as small as possible. By successively removing end-vertices, prove that every tree has either one centre or two adjacent centres.

(*9h) (*i*) Let C^* be a set of edges of a graph G. Show that, if C^* has an edge in common with any spanning forest of G, then C^* contains a cutset.
  (*ii*) Obtain a corresponding result for circuits.

(*9i) Let T_1 and T_2 be spanning trees of a connected graph G.
  (*i*) Show that, if e is any edge of T_1, then there exists an edge f of T_2 such that $(T_1 - \{e\}) \cup \{f\}$ (the graph obtained from T_1 on replacing e by f) is also a spanning tree.
  (*ii*) Deduce that T_1 can be 'transformed' into T_2 by replacing the edges of T_1 one at a time by edges of T_2 in such a way that a spanning tree is obtained at each stage.

(*9j) Show that if H and K are subgraphs of a graph G, and if $H \cup K$ and $H \cap K$ are defined in the natural way, then the cutset rank ξ satisfies:
  (*i*) $0 \leqslant \xi(H) \leqslant |E(H)|$ (the number of edges of H);
  (*ii*) if H is a subgraph of K, then $\xi(H) \leqslant \xi(K)$;
  (*iii*) $\xi(H \cup K) + \xi(H \cap K) \leqslant \xi(H) + \xi(K)$.

(*9k) Let V be the vector space associated with a simple connected graph G, and let T be a spanning tree of G.
  (*i*) Show that the fundamental system of circuits associated with T forms a basis for the circuit subspace W.
  (*ii*) Obtain a corresponding result for the cutset subspace $\tilde{W}$.
  (*iii*) Deduce that the dimensions of W and $\tilde{W}$ are $\gamma(G)$ and $\xi(G)$, respectively.

§10. The enumeration of trees

The subject of graph enumeration is concerned with the problem of finding out how many non-isomorphic graphs there are which possess a given property. The subject was initiated in the 1850s by Cayley, who later applied it to the problem of enumerating alkanes C_nH_{2n+2} with a given number of carbon atoms. As he realized, and as you will see in §**11**, this problem can be expressed as the problem of counting the number of trees in which the degree of every vertex is either four or one.

Many of the standard problems of graph enumeration have been solved. For example, it is possible to calculate the number of graphs, digraphs, connected graphs, trees and Eulerian graphs, containing a given number of vertices and edges; the corresponding results for planar graphs and Hamiltonian graphs have, however, not yet been obtained. Most of the known results can be obtained by using a fundamental enumeration theorem due to Pólya, a good account of which may be found in Harary and Palmer[16]; unfortunately, in almost every case it is impossible to express these results by means of simple formulae. For tables of some known results, you are referred to the Appendix.

This section is devoted primarily to two proofs of a famous result, usually attributed to Cayley, on the number of labelled trees with a given number of vertices. In this context, a labelled graph on n vertices is essentially a graph in which the vertices are 'labelled' with the integers from 1 to n. More precisely, we define a **labelling** of a graph G on n vertices to be a one-one mapping from the vertex-set of G onto the set $\{1, \ldots, n\}$; a **labelled graph** is then a pair (G, φ), where G is a graph and φ is a labelling of G. We shall frequently refer to the integers $1, \ldots, n$ as the **labels** of G, and denote the vertices of G by $v_1, \ldots, v_n$. Furthermore, we shall say that two labelled graphs (G_1, φ_1) and (G_2, φ_2) are **isomorphic** if there exists an isomorphism between G_1 and G_2 which preserves the labelling of the vertices.

In order to clarify these definitions, let us consider Fig. 10.1, which shows various ways of labelling a tree with four vertices. On closer inspection, we see that the second labelled tree is simply the reverse of the first one, and it follows that these two labelled trees must be isomorphic; on the other hand, neither of them is isomorphic to the third labelled tree, as can be seen by looking at the degree of the vertex 3. It follows that the total number of ways of labelling this particular tree must be $\frac{1}{2}(4!) = 12$, since the reverse of any labelling does not result in a new one. Similarly, the total number of ways of labelling the tree shown in Fig. 10.2 must be four, since the central vertex may be labelled in four different ways, and each one determines the labelling. It follows that the

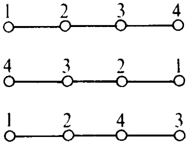


Fig. 10.1

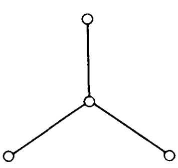


Fig. 10.2

total number of (non-isomorphic) labelled trees on four vertices is sixteen. We now prove **Cayley's theorem** which generalizes this result to labelled trees with n vertices.

THEOREM 10A (Cayley 1889). *There are n^{n-2} distinct labelled trees on n vertices.*

Remark. The proofs we are about to give are due to Prüfer and Clarke; for several other proofs, see Moon.[20]

First proof. We shall establish a one-one correspondence between the set of labelled trees of order n and the set of all ordered symbols $(a_1, a_2, \ldots, a_{n-2})$, where each a_i is an integer satisfying $1 \leqq a_i \leqq n$. Since there are precisely n^{n-2} such symbols, the result will then follow immediately. We shall assume that $n \geqq 3$, since the result is trivial if $n \leqq 2$.

In order to establish the required correspondence, we first let T be a labelled tree of order n, and show how the symbol can be assigned. If b_1 is the smallest label assigned to any of the end-vertices, we let a_1 be the label of the vertex adjacent to the vertex b_1. We then remove the vertex b_1 and its incident edge, leaving a labelled tree of order $n-1$. If we now let b_2 be the smallest label assigned to any of the end-vertices of our new tree, and let a_2 be the label of the vertex adjacent to the vertex b_2, we can then remove the vertex b_2 and its incident edge, as before. Proceeding in this way until there are only two vertices left gives us the required symbol $(a_1, a_2, \ldots, a_{n-2})$. For example, if T is the labelled tree in Fig. 10.3, then $b_1=2$, $a_1=6$; $b_2=3$, $a_2=5$; $b_3=4$, $a_3=6$; $b_4=6$, $a_4=5$; $b_5=5$, $a_5=1$. The required 5-tuple is therefore (6, 5, 6, 5, 1).

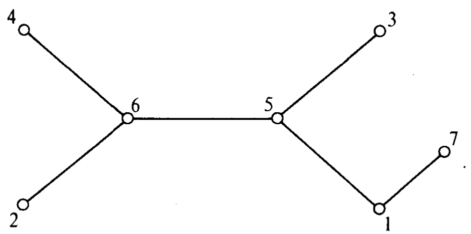


Fig. 10.3

In order to establish the reverse correspondence, we take a symbol $(a_1, \ldots, a_{n-2})$, let b_1 be the smallest number which does *not* appear in it, and join the vertices a_1 and b_1. We then remove a_1 from the symbol, and remove the number b_1 from consideration, and proceed as before. In this way we can build up the tree, edge by edge. For example, if we start with the symbol (6, 5, 6, 5, 1), then $b_1=2$, $b_2=3$, $b_3=4$, $b_4=6$,

$b_5 = 5$, and the corresponding edges are 62, 53, 64, 56, 15; we conclude by joining the last two vertices not yet crossed out—in this case, 1 and 7.

It is a straightforward matter to check that if we start with any labelled tree, find the corresponding symbol, and then find the labelled tree corresponding to that symbol, then we always obtain the tree we started from. The required correspondence is therefore established, and the result follows.//

Second proof. Let $T(n, k)$ denote the number of labelled trees on n vertices in which a given vertex (v, say) has degree k. We shall derive an expression for $T(n, k)$, and the result will then follow on summing from $k=1$ to $k=n-1$.

Let A be any labelled tree in which $\rho(v)=k-1$. The removal from A of any edge wz which is not incident to v leaves two subtrees, one of which contains v and either w or z (let us say, w), and the other of which contains z. If we now join the vertices v and z, we obtain a labelled tree B in which $\rho(v)=k$ (see Fig. 10.4). We shall call a pair (A, B) of labelled trees a **linkage** if B can be obtained from A by the above construction. Our aim is to count the total number of possible linkages (A, B).

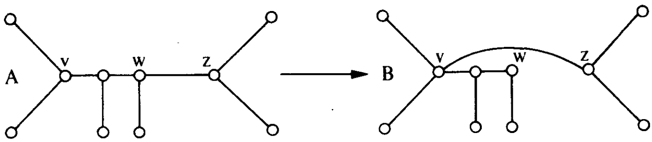


Fig. 10.4

Since A may be chosen in any of $T(n, k-1)$ ways, and B is uniquely defined by the edge wz which may be chosen in $(n-1)-(k-1)=n-k$ ways, the total number of linkages (A, B) is clearly $(n-k)T(n, k-1)$. On the other hand, let B be a labelled tree in which $\rho(v)=k$, and let $T_1, \ldots, T_k$ be the subtrees obtained from B by removing the vertex v and every edge incident to v; then we can obtain a labelled tree A for which $\rho(v)=k-1$ by removing from B just one of these edges (vw_i, say, where w_i lies in T_i) and joining w_i to any vertex u of any other subtree T_j (see Fig. 10.5). It is clear that the corresponding pair (A, B) of labelled trees is a linkage, and that all linkages may be obtained in this way. Since B may be chosen in $T(n, k)$ ways, and the number of ways of joining w_i to vertices in any other T_j is $(n-1)-n_i$ (where n_i denotes the number of vertices of T_i), it follows that the total number of linkages (A, B) is

$$T(n, k)\{(n-1-n_1)+\ldots+(n-1-n_k)\}=(n-1)(k-1)T(n, k),$$

since $n_1+\ldots+n_k=n-1$.

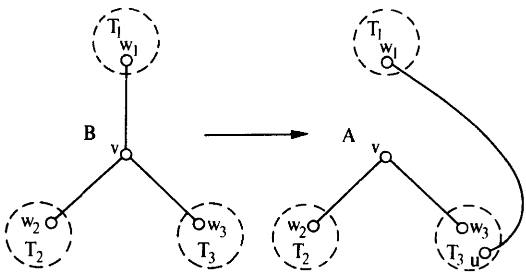


Fig. 10.5

We have thus shown that

$$(n-k)T(n, k-1)=(n-1)(k-1)T(n, k).$$

On iterating this result, and using the obvious fact that $T(n, n-1)=1$, we deduce immediately that

$$T(n, k)=\binom{n-2}{k-1}(n-1)^{n-k-1}.$$

On summing over all possible values of k, it follows that the number $T(n)$ of labelled trees on n vertices is given by

$$T(n)=\sum_{k=1}^{n-1} T(n, k)=\sum_{k=1}^{n-1}\binom{n-2}{k-1}(n-1)^{n-k-1}$$
$$=\{(n-1)+1\}^{n-2}=n^{n-2}.//$$

COROLLARY 10B. *The number of spanning trees of K_n is n^{n-2}.*

Proof. To every labelled tree on n vertices there corresponds (in a unique way) a spanning tree of K_n. Conversely, every spanning tree of K_n gives rise to a unique labelled tree on n vertices.//

We conclude this section by stating an important result which can be used to calculate the number of spanning trees in any connected simple graph. It is usually known as the **matrix-tree theorem** and its proof may be found in Harary:[14]

THEOREM 10C. *Let G be a connected simple graph with vertex-set $\{v_1, \ldots, v_n\}$, and let $\mathbf{M}=(m_{ij})$ be the $n \times n$ matrix in which $m_{ii}=\rho(v_i)$, $m_{ij}=-1$ if v_i and v_j are adjacent, and $m_{ij}=0$ otherwise. Then the number of spanning trees of G is equal to the cofactor of any element of $\mathbf{M}$.//*

Exercises 10

(10a) Verify directly that there are exactly 125 labelled trees on five vertices.

(10b) (*i*) Show that there are exactly $2^{n(n-1)/2}$ labelled simple graphs on n vertices.

(*ii*) How many of these have exactly m edges?

(10c) In the first proof of Cayley's theorem, find:

(*i*) the labelled trees corresponding to the sequences (1, 2, 3, 4) and (3, 3, 3, 3);

(*ii*) the sequences corresponding to the labelled trees in Fig. 10.6.

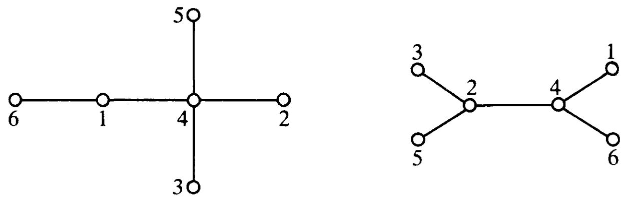


Fig. 10.6

(10d) (*i*) Find the number of trees on n vertices in which a given vertex is an end-vertex.

(*ii*) Deduce that, if n is large, the probability that a given vertex of a tree with n vertices is an end-vertex is approximately e^{-1}.

(10e) How many spanning trees has $K_{2,s}$?

(10f) Let $\tau(G)$ be the number of spanning trees in a connected graph G.

(*i*) Prove that, for any edge e, $\tau(G)=\tau(G-e)+\tau(G\backslash e)$.

(*ii*) Use this result to calculate $\tau(K_{2,3})$.

(*10g) Use the matrix-tree theorem to prove Cayley's theorem.

(*10h) Let $T(n)$ be the number of labelled trees on n vertices.

(*i*) By counting the number of ways of joining a labelled tree on k vertices and one on $n-k$ vertices, prove that

$$2(n-1)T(n)=\sum_{k=1}^{n-1}\binom{n}{k}k(n-k)T(k)T(n-k).$$

(*ii*) Deduce the identity

$$\sum_{k=1}^{n-1}\binom{n}{k}k^{k-1}(n-k)^{n-k-1}=2(n-1)n^{n-2}.$$

§11. More applications

In §**8** we looked at three problems which arise in the area of operational research—the shortest path problem, the Chinese postman problem and the travelling salesman problem. In this section we shall consider three further applications, taken respectively from operational research, organic chemistry and electrical network theory, and each involving the use of trees.

The minimum connector problem

Let us suppose that we wish to build a railway network connecting n given cities in such a way that a passenger can travel from any city to any other. If we assume for economic reasons that the amount of track used must be a minimum, then it is clear that the graph formed by taking the n cities as vertices and the connecting rails as edges must be a tree. The problem is to find an efficient algorithm for deciding which of the n^{n-2} possible trees connecting these cities uses the least amount of track,

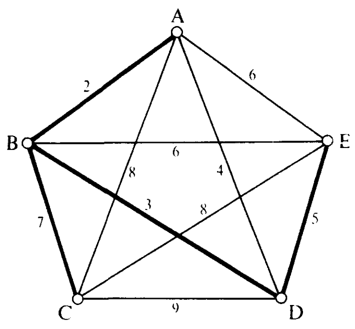


Fig. 11.1

assuming that the distances between the various pairs of cities are known (see Fig. 11.1).

As before, we can reformulate the problem in terms of weighted graphs. We shall denote the weight of the edge e by $w(e)$, and our problem is to find the spanning tree T with least possible total weight $W(T)$. Unlike some of the problems we considered earlier, there is a simple algorithm which provides the solution. It is known as the **greedy algorithm** or **Kruskal's algorithm** and is described in the following theorem:

THEOREM 11A. *Let G be a connected graph with n vertices. Then the following construction gives a solution of the minimum connector problem:*
(i) let e_1 be an edge of G of smallest weight;
(ii) define $e_2, e_3, \ldots, e_{n-1}$ by choosing at each stage an edge (not previously chosen) of smallest possible weight, subject to the condition that it forms no circuit with the previous edges e_i.

The required spanning tree is then the subgraph T of G whose edges are $e_1, \ldots, e_{n-1}$.

Remark. You should verify that if G is the graph shown in Fig. 11.1, then this construction yields: $e_1 = AB$, $e_2 = BD$, $e_3 = DE$, $e_4 = BC$.

Proof. The fact that T is a spanning tree of G follows immediately from statement (*ii*) of Theorem 9A; it remains only to show that the total weight of T is a minimum. In order to do this, we suppose that S is a spanning tree of G with the property that $W(S) < W(T)$. If e_k is the first edge in the above sequence which does not lie in S, then the subgraph of G formed by adding e_k to S contains a unique circuit C containing the edge e_k. Since C clearly contains an edge e lying in S but not in T, it follows that the subgraph obtained from S on replacing e by e_k is still a spanning tree (S', say). But by the construction, $w(e_k) \leqq w(e)$, and so $W(S') \leqq W(S)$, and S' has one more edge in common with T than S. It follows on repeating this procedure that we can change S into T, one step at a time, with the total weight decreasing at each stage; hence $W(T) \leqq W(S)$, giving us the required contradiction.//

An interesting application of the greedy algorithm is its use in obtaining a lower bound for the solution of the travelling salesman problem. If we take any Hamiltonian circuit in a weighted complete graph and remove any vertex v, then we get a semi-Hamiltonian path, and such a path must be a spanning tree. So any solution of the travelling salesman problem must consist of a spanning tree of this type together with two edges incident to v. It follows that if we take the weight of a minimum-weight spanning tree (which is obtained by the greedy algorithm) and add the two smallest weights of edges incident to v, then we get a lower bound for the solution of the travelling salesman problem. For example, if we take the weighted graph of Fig. 11.1 and remove the vertex C, then the remaining weighted graph has the four vertices A, B, D and E. In this case, the minimum-weight spanning tree joining these four vertices is the tree whose edges are AB, BD and DE, with total weight 10, and the two edges of minimum weight incident to C are CB and CA (or CE) with total weight 15, so that the required lower bound for the travelling salesman problem is 25. Since the correct answer in this case is 26, it can be seen that this approach to the travelling salesman problem can yield surprisingly good results.

Enumeration of chemical molecules

One of the earliest examples of the use of trees was in problems relating to the enumeration of chemical molecules. If we have a hydrocarbon (that is, a molecule consisting only of carbon atoms and hydrogen atoms), then we can represent it as a graph in which each carbon atom appears as a vertex of degree four, and each hydrogen atom appears as a vertex of degree one. The graphs of butane and isobutane are shown in Fig. 11.2; note that although they both have the same chemical formula C_4H_{10}, they are different molecules because the atoms are arranged differently within the molecule. These two molecules form part of a general class of molecules known as the *alkanes* or *paraffins*, with chemical formula C_nH_{2n+2}, and it is natural to ask how many different molecules there are having this formula.

In order to answer this question, we notice first that the graph of any molecule with formula C_nH_{2n+2} must be a tree, by Theorem 9A(*iii*), since it is connected and has $n+(2n+2)=3n+2$ vertices and $\frac{1}{2}\{4n+(2n+2)\}=3n+1$ edges. Note also that the molecule is determined completely once we know how the carbon atoms are arranged, since hydrogen atoms can then be added in such a way as to bring the degree of each carbon vertex to four. It follows that we can discard the hydrogen atoms (see Fig. 11.3), and the problem reduces to that of

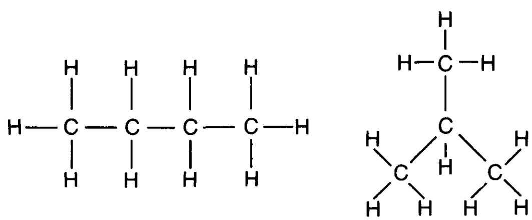


Fig. 11.2

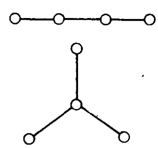


Fig. 11.3

determining the number of trees with n vertices, each of which has degree four or less. This problem was solved by Cayley in 1875, by counting the number of ways in which trees can be built up from their centre(s) (see exercise 9g). The details of this argument are too complicated to describe here, but may be found in Biggs, Lloyd and Wilson.[5] Much of Cayley's work has since been superseded by G. Pólya and others, with the result that many chemical series have been enumerated using graph-theoretical techniques.

Electrical networks

Suppose that we are given the electrical network in Fig. 11.4, and we wish to determine the current in each wire. In order to do this, we assign arbitrary directions to the current in each wire, as in Fig. 11.5, and apply 'Kirchhoff's laws':

(*i*) the algebraic sum of the currents at each vertex is zero;

(*ii*) the total voltage in each circuit is the algebraic sum of the products of the currents i_k and resistances R_k in that circuit.

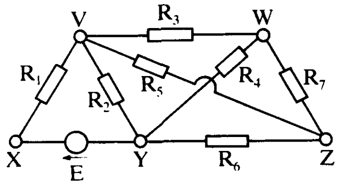


Fig. 11.4

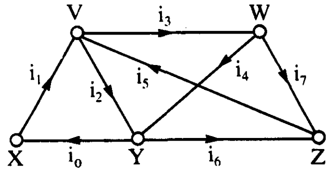


Fig. 11.5

Applying Kirchhoff's second law to the circuits $VXYV$, $VWYV$ and $VWYXV$ gives, respectively:

$$i_1R_1 + i_2R_2 = E; \quad i_3R_3 + i_4R_4 - i_2R_2 = 0; \quad i_1R_1 + i_3R_3 + i_4R_4 = E.$$

Since the last of these three equations is simply the sum of the first two, it gives us no further information. Similarly, if we have the corresponding equations for the circuits $VWYV$ and $WZYW$, we can deduce the equation for the circuit $VWZYV$. It would clearly save a lot of work if we could find a set of circuits which give us all the information we need without any redundancy, and this can be done by using the concept of a fundamental system of circuits, introduced in §9. In our particular example, we can take the fundamental system of circuits shown in Fig. 9.4, and we get the following equations:

for the circuit $VXYV$: $i_1R_1 + i_2R_2 = E$,
for the circuit $VYZV$: $i_2R_2 + i_5R_5 + i_6R_6 = 0$,
for the circuit $VWZV$: $i_3R_3 + i_5R_5 + i_7R_7 = 0$,
for the circuit $VYWZV$: $i_2R_2 - i_4R_4 + i_5R_5 + i_7R_7 = 0$,

and the equations arising from Kirchhoff's first law are:

for the vertex X : $i_0 - i_1 = 0$,
for the vertex V : $i_1 - i_2 - i_3 + i_5 = 0$,
for the vertex W: $i_3 - i_4 - i_7 = 0$,
for the vertex Z : $i_5 - i_6 - i_7 = 0$.

These eight equations can then be solved to give the eight currents $i_0, \ldots, i_7$. For example, if $E = 12$, and if each wire has unit resistance (that is, $R_i = 1$ for each i), then the solution is as given in Fig. 11.6.

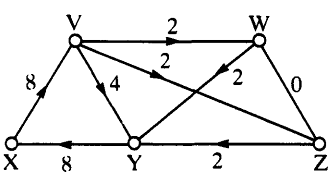


Fig. 11.6

Exercises 11

(11a) Use the greedy algorithm to find a minimum-weight spanning tree in the graphs shown in Figs 11.7 and 11.8.

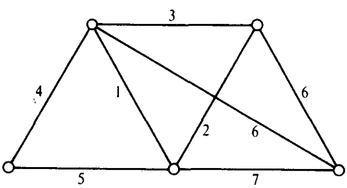


Fig. 11.7

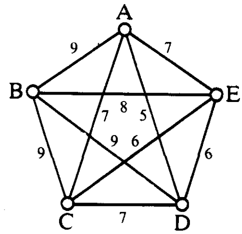


Fig. 11.8

(11b) Show that if every edge of a weighted graph G has the same weight, then the greedy algorithm gives a method for constructing a spanning tree in G.

(11c) Describe an alternative algorithm for the minimum connector problem, involving the removal from the graph of edges of greatest weight.

(11d) (*i*) How would you adapt the greedy algorithm to find a *maximum*-weight spanning tree?

(*ii*) Find such a spanning tree for the weighted graphs in Figs 11.1 and 11.7.

(11e) In the travelling salesman problem on page 55, what lower bounds do you get if you remove the vertices, A, B, D and E, instead of C?

(11f) Show that, for each value of n, the graph associated with the alcohol $C_nH_{2n+1}OH$ is a tree (oxygen has valency two).

(11g) Find the number of chemical molecules with the formulae C_5H_{12} and C_6H_{14}, and draw them.

(11h) Verify the currents in Fig. 11.6 by applying Kirchhoff's laws to the fundamental circuits associated with the spanning tree with edges VX, VW, WZ and YZ.

(*11i) Write down and solve Kirchhoff's equations for the network of Fig. 11.9, in which the numbers refer to the various resistances.

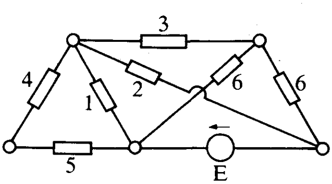


Fig. 11.9

5
Planarity and duality

Flattery will get you nowhere.

Popular saying

We now embark upon a study of topological graph theory, in which the study of graph theory becomes inextricably tied up with topological notions such as planarity, genus, etc. In §**4** it was proved that every graph can be embedded (i.e., drawn without crossings) in three-dimensional space. We now investigate conditions under which a graph can be embedded in the plane and other surfaces. In §**12** we prove the existence of graphs which are not planar, and state Kuratowski's famous characterization of planar graphs. Euler's formula relating the numbers of vertices, edges and faces of a plane graph is then proved in §**13**, and generalized to graphs embedded in other surfaces in §**14**. The following section is devoted to a study of duality, and the chapter concludes with a section on infinite graphs.

§12. Planar graphs

A **plane graph** is a graph drawn in the plane in such a way that no two edges (or rather, the curves representing them) intersect geometrically except at a vertex to which they are both incident; a **planar graph** is one which is isomorphic to a plane graph. In the language of §**4**, this amounts to saying that a graph is planar if it can be embedded in the plane, and that any such embedding is a plane graph; for example, all three graphs in Fig. 12.1 are planar, but only the second and third are plane.

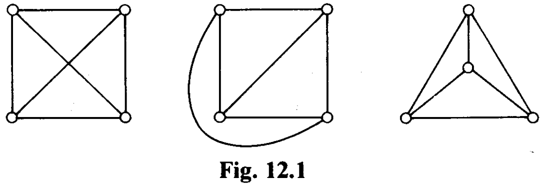


Fig. 12.1

One question which arises from the example just given and from exercise 4f is whether a planar graph can always be drawn in the plane in such a way that all of its edges are represented by straight lines. Although this is clearly false for graphs containing loops or multiple edges, it is in fact true for simple graphs, as was proved by Wagner in 1936. The interested reader should consult Chartrand and Lesniak[10] for further details.

Not all graphs are planar, as the following theorem shows:

THEOREM 12A. *K_5 and $K_{3,3}$ are non-planar.*

Remark. We shall be giving two proofs of this result. The first one, which is presented here, depends on the Jordan curve theorem in the form in which it was given in §4. The second proof, which we defer until the next section, will appear as a corollary of Euler's formula.

Proof. Suppose that K_5 is planar. Since K_5 contains a circuit of length five (which we shall take as $v \to w \to x \to y \to z \to v$), any plane embedding can without loss of generality be assumed to contain this circuit drawn in the form of a regular pentagon (as in Fig. 12.2). By the

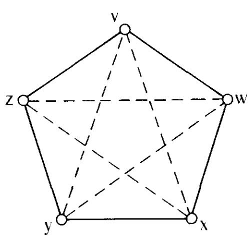


Fig. 12.2

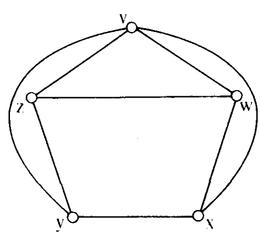


Fig. 12.3

Jordan curve theorem, the edge zw must lie either wholly inside the pentagon or wholly outside it. (The third possibility, namely that the edge has a point in common with the pentagon, does not arise since we are assuming a plane embedding.) We shall deal with the case in which zw lies inside the pentagon—the other case is similar and will be left to the reader. Since the edges vx and vy do not cross the edge zw, they must both lie outside the pentagon; the situation is now as in Fig. 12.3. But the edge xz cannot cross the edge vy and so must lie inside the pentagon, and similarly the edge wy must also lie inside the pentagon. Since the edges wy and xz must then cross, we obtain the required contradiction.

A similar, but easier, argument is used to show that $K_{3,3}$ is non-planar; we simply draw a hexagonal circuit $u \to v \to w \to x \to y \to z \to u$ in Fig. 12.4, and show (using the Jordan curve theorem) that two of the edges ux, vy and wz must both lie inside or outside the hexagon, and hence must cross.//

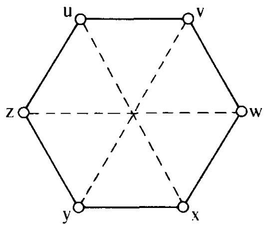


Fig. 12.4

It is clear that every subgraph of a planar graph is planar, and that every graph which contains a non-planar graph as a subgraph must itself be non-planar. From this we immediately deduce that any graph which contains K_5 or $K_{3,3}$ as a subgraph cannot be planar. It turns out that K_5 and $K_{3,3}$ are essentially the only non-planar graphs, in the sense that every non-planar graph 'contains' one of them. To make this statement more precise, we need the concept of 'homeomorphic graphs'.

Two graphs are **homeomorphic** (or **identical to within vertices of degree two**) if they can both be obtained from the same graph by inserting new vertices of degree two into its edges; for example, the graphs shown in Fig. 12.5 are homeomorphic, and so are any two circuit graphs. Note that the homeomorphism of graphs is an equivalence relation.

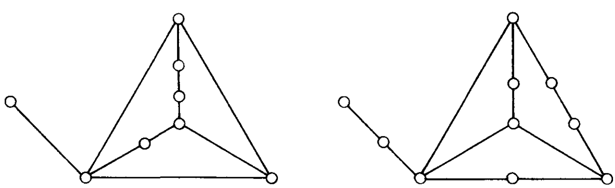


Fig. 12.5

It is clear that the introduction of the term 'homeomorphic' is merely a technicality—the insertion or deletion of vertices of degree two is irrelevant to any considerations of planarity. However, it enables us to state the following important result which is known as **Kuratowski's theorem** and which gives a necessary and sufficient condition for a graph to be planar.

THEOREM 12B (Kuratowski 1930). *A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.*//

The proof of Kuratowski's theorem is rather long and involved, and for this reason we have decided to omit it (see Bondy and Murty,[7] or

Harary[14]). We shall, however, use Kuratowski's theorem to obtain another criterion for planarity; it involves the idea of contractibility, introduced in §3:

THEOREM 12C. *A graph is planar if and only if it contains no subgraph which is contractible to K_5 or $K_{3,3}$.*

★*Sketch of proof.* $\Leftarrow$ Assume first that the graph G is non-planar; then by Kuratowski's theorem, G contains a subgraph H which is homeomorphic to K_5 or $K_{3,3}$. On successively contracting edges of H which are incident to at least one vertex of degree two, we see immediately that H is contractible to K_5 or $K_{3,3}$.

$\Rightarrow$ Now assume that G contains a subgraph H which is contractible to $K_{3,3}$, and let the vertex v of $K_{3,3}$ arise from the contraction of the subgraph H_v of H (see Fig. 12.6). The vertex v is incident in $K_{3,3}$ to three edges e_1, e_2 and e_3; when regarded as edges of H, these edges are incident

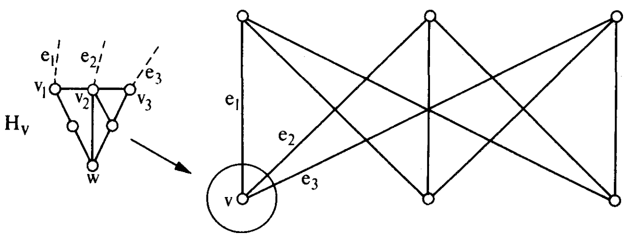


Fig. 12.6

to three (not necessarily distinct) vertices v_1, v_2 and v_3 of H_v. If v_1, v_2 and v_3 are distinct, we can find a vertex w of H_v and three paths from w to these vertices, these paths intersecting only at w. A similar construction can be made if the vertices are not distinct, the paths degenerating in this case to single vertices. It follows that we can replace the subgraph H_v by a vertex w and three paths leading out of it. If this construction is carried out for each vertex of $K_{3,3}$, and the resulting paths joined up with the corresponding edges of $K_{3,3}$, the resulting subgraph will clearly be homeomorphic to $K_{3,3}$, showing (by Kuratowski's theorem) that G is non-planar (see Fig. 12.7).

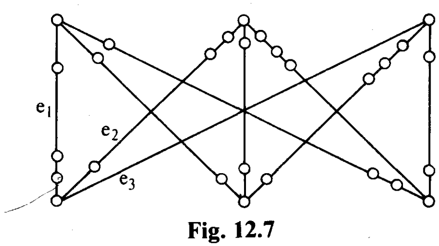


Fig. 12.7

A similar argument can be carried out if G contains a subgraph which is contractible to K_5. In this case the details are more complicated, since the subgraph we obtain by the above process can be homeomorphic to K_5 or $K_{3,3}$. The details can be found in Chartrand and Lesniak.[10]//★

We conclude this section by introducing the 'crossing-number' of a graph. If we try to draw K_5 or $K_{3,3}$ on the plane, then at least one crossing of edges must occur, since these graphs are not planar. However, as Fig. 12.8 shows, it is not necessary to use more than one crossing, and we express this by saying that K_5 and $K_{3,3}$ have crossing-number 1. More generally, the **crossing-number** $cr(G)$ of a graph G is the smallest possible number of crossings occurring when G is drawn in the plane. Thus the crossing-number can be used to measure how 'un-planar' G is; for example, the crossing-number of a planar graph is zero, and $cr(K_5) = cr(K_{3,3}) = 1$. Note that, as in §4, the word 'crossing' always refers to the intersection of just two edges; crossings of three or more edges are not permitted.

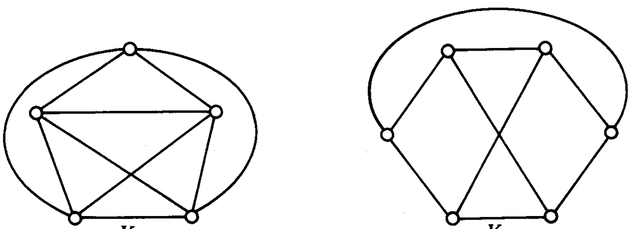


Fig. 12.8

Exercises 12

Some simpler exercises on planar graphs can be found at the end of §4 *(page 23).*

(12a) Three unfriendly neighbours use the same water, oil and treacle wells. In order to avoid meeting, they decide to build non-crossing paths from each of their houses to each of the three wells. Can this be done?

(12b) (*i*) For which values of k is the k-cube Q_k planar?
(*ii*) For which values of r, s and t is the complete tripartite graph $K_{r,s,t}$ planar?

(12c) Prove that the Petersen graph is non-planar
(*i*) by using Theorem 12B;
(*ii*) by using Theorem 12C.

(12d) Give an example of
(*i*) a non-planar graph which is not homeomorphic to K_5 or $K_{3,3}$;
(*ii*) a non-planar graph which is not contractible to K_5 or $K_{3,3}$.
Why does the existence of these graphs not contradict Theorems 12B and 12C?

(12e) (*i*) Prove that the homeomorphism of graphs is an equivalence relation.
(*ii*) If two homeomorphic graphs have n_i vertices and m_i edges ($i=1,2$), show that

$$m_1-n_1=m_2-n_2.$$

(12f) A graph G is **outerplanar** if G can be embedded in the plane in such a way that all of its vertices lie on the exterior boundary.
(*i*) Show that K_4 and $K_{2,3}$ are not outerplanar.
(*ii*) Deduce that if G is an outerplanar graph, then G contains no subgraph homeomorphic or contractible to K_4 or $K_{2,3}$.
(In fact, the converse result also holds, yielding a Kuratowski-type criterion for a graph to be outerplanar.)

(12g) Show that $K_{4,3}$ and the Petersen graph each have crossing-number 2.

(*12h) If r and s are both even, show that

$$cr(K_{r,s}) \leqslant \tfrac{1}{16}rs(r-2)(s-2),$$

and obtain corresponding results when r and/or s is odd. (Hint: place the r vertices along the x-axis with $\frac{1}{2}r$ vertices on each side of the origin, and place the s vertices along the y-axis in a similar way—now count the crossings.)

§13. Euler's formula for plane graphs

In this section we shall prove a theorem relating the numbers of vertices, edges and faces of a given connected plane graph G. Before defining exactly what is meant by a 'face' of G, we recall that a point x of the plane is said to be 'disjoint from G' if x represents neither a vertex of G nor a point which lies on an edge of G.

If x is a point of the plane disjoint from G, we define the **face** (of G) **containing x** to be the set of all points of the plane which can be reached from x by a Jordan curve all of whose points are disjoint from G. Alternatively, we can say that two points x and y of the plane are equivalent if they are both disjoint from G and can be joined by a Jordan curve all of whose points are disjoint from G (Fig. 13.1). This is an equivalence relation on the points of the plane disjoint from G, and the

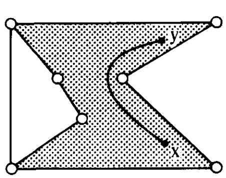


Fig. 13.1

corresponding equivalence classes are called the **faces** of G. Note that one face is unbounded; it is called the **infinite face**. For example, if G is the graph of Fig. 13.2, then G has four faces, f_4 being the infinite face. If you feel that our definition of a face is too pedantic, you may safely rely on your intuition.

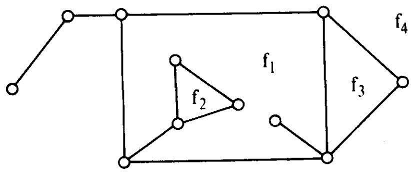


Fig. 13.2

It is important to realize that there is nothing special about the infinite face—in fact, any face can be chosen as the infinite face. To see this, we use Theorem 4B to map the graph onto the surface of a sphere. We now rotate the sphere so that the point of projection (i.e. the north pole) lies inside the face we want as the infinite face, and then project the graph down onto the plane which is tangent to the sphere at the south pole. The chosen face is now the infinite face. Fig. 13.3 shows a representation of the graph of Fig. 13.2 in which the infinite face is f_3. From now on, we shall feel free to talk interchangeably about graphs embedded in the plane and graphs drawn on the surface of a sphere.

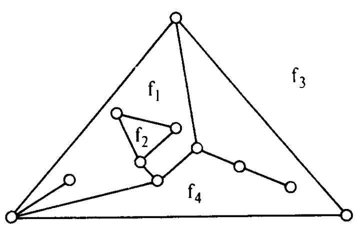


Fig. 13.3

We now state and prove **Euler's formula** which tells us that whatever plane embedding of a graph we take, the number of faces always remains the same and is given by a simple formula; an alternative proof will be outlined in exercise 13k.

THEOREM 13A (Euler 1750). *Let G be a connected plane graph, and let n, m and f denote respectively the number of vertices, edges and faces of G. Then*

$$n-m+f=2.$$

Proof. The proof is by induction on the number of edges of G. If $m=0$, then $n=1$ (since G is connected) and $f=1$ (the infinite face); the theorem is therefore true in this case.

Now suppose that the theorem is true for all graphs with at most $m-1$ edges, and let G be a graph with m edges. If G is a tree, then $m=n-1$ and $f=1$, so that $n-m+f=2$, as required. If G is not a tree, let e be an edge contained in some circuit of G. Then $G-e$ is a connected plane graph with n vertices, $m-1$ edges, and $f-1$ faces, so that $n-(m-1)+(f-1)=2$, by the induction hypothesis. It follows that $n-m+f=2$, as required.//

This result is often called the 'polyhedron formula' since it relates the numbers of vertices, edges and faces of a convex polyhedron; this can easily be seen by projecting the polyhedron out onto the surface of its circumsphere, and then using Theorem 4B. The resulting plane graph is a 3-connected graph in which every face is bounded by a polygon—such a graph is called a **polyhedral graph** (see Fig. 13.4). For convenience we restate Theorem 13A for such graphs.

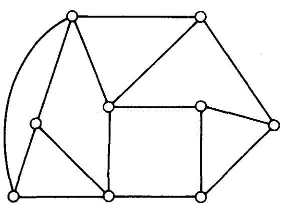


Fig. 13.4

COROLLARY 13B. *Let G be a polyhedral graph; then, with the above notation,*

$$n-m+f=2.//$$

Euler's formula can easily be extended to disconnected graphs:

COROLLARY 13C. *Let G be a plane graph with n vertices, m edges, f faces and k components; then*

$$n-m+f=k+1.$$

Proof. The result follows immediately on applying Euler's formula to each component separately, remembering not to count the infinite face more than once.//

All of the results mentioned so far in this section apply to arbitrary plane graphs; we must now restrict ourselves to simple graphs.

COROLLARY 13D. *(i) If G is a connected simple planar graph with n ($\geqq 3$) vertices and m edges, then $m \leqq 3n-6$.*
(ii) If, in addition, G has no triangles, then $m \leqq 2n-4$.

Proof. (*i*) We can assume without loss of generality that G is a plane graph. Since every face is bounded by at least three edges, it follows on counting up the edges around each face that $3f \leqq 2m$ (the factor 2 arising from the fact that every edge bounds at most two faces). We obtain the required result by combining this inequality with Euler's formula.
(ii) This part follows in the same way, except that the inequality $3f \leqq 2m$ is replaced by $4f \leqq 2m$.//

This corollary can be used to give an alternative proof of Theorem 12A.

COROLLARY 13E. *K_5 and $K_{3,3}$ are non-planar.*

Proof. If K_5 is planar then, applying part (*i*) of Corollary 13D, we obtain $10 \leqq 9$, which is clearly a contradiction. If $K_{3,3}$ is planar then, applying part (*ii*) of Corollary 13D, we obtain $9 \leqq 8$, which is also a contradiction.//

A similar argument is used to prove the following theorem which will be useful when we come to study the colouring of graphs.

THEOREM 13F. *Every simple planar graph contains a vertex whose degree is at most five.*

Proof. Without loss of generality we can assume the graph to be plane and connected, and to contain at least three vertices. If every vertex has degree at least six, then with the above notation we have $6n \leqq 2m$ (i.e., $3n \leqq m$). It then follows immediately from part (*i*) of Corollary 13D that $3n \leqq 3n-6$, an obvious contradiction.//

We conclude this section with a few remarks on the 'thickness' of a graph. In electrical engineering, parts of networks are sometimes printed on one side of a non-conducting plate, and are called 'printed circuits'. Since the wires are not insulated, they cannot cross and the corresponding graphs must be planar. For a general network, it is of importance to know how many printed circuits are needed to complete the entire network; to this end, we define the **thickness** of a graph G (denoted by $t(G)$) to be the smallest number of planar graphs which can be superimposed to form G. Like the crossing-number, the thickness is a measure of how 'un-planar' a graph is; for example, the thickness of a planar graph is one, and of K_5 and $K_{3,3}$ is two.

As we shall see, a lower bound for the thickness of a graph may easily be obtained using Euler's formula; what is rather surprising is that this rather trivial lower bound frequently turns out to be the correct value, as may be verified in special cases by direct construction. In deriving this lower bound, we shall use the symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote respectively the largest integer not greater than x and the smallest integer not less than x (so that, for example, $\lfloor 3 \rfloor = \lceil 3 \rceil = 3$; $\lfloor \pi \rfloor = 3$; $\lceil \pi \rceil = 4$); note that $\lceil x \rceil = -\lfloor -x \rfloor$.

THEOREM 13G. *Let G be a simple graph with $n(\geqq 3)$ vertices and m edges; then the thickness $t(G)$ of G satisfies the following inequalities:*

$$t(G) \geqq \left\lceil \frac{m}{3n-6} \right\rceil; \qquad t(G) \geqq \left\lfloor \frac{m+3n-7}{3n-6} \right\rfloor.$$

Proof. The first part is an immediate application of part (*i*) of Corollary 13D, the brackets arising from the fact that the thickness must be an integer. The second part follows from the first by using the easily-proved relation $\lceil a/b \rceil = \lfloor (a+b-1)/b \rfloor$ (where a and b denote positive integers).//

Exercises 13

(13a) Verify Euler's formula for
- (*i*) the wheel W_8;
- (*ii*) the graph of the octahedron;
- (*iii*) the graph of Fig. 13.4;
- (*iv*) the complete bipartite graph $K_{2,7}$.

(13b) Redraw the graph of Fig. 13.2 with
- (*i*) f_1 as the infinite face;
- (*ii*) f_2 as the infinite face.

(13c)
- (*i*) Use Euler's formula to prove that if G is a connected plane graph of girth 5 then, with the above notation, $m \leqslant \frac{5}{3}(n-2)$.
- (*ii*) Deduce that the Petersen graph is non-planar.
- (*iii*) Obtain an inequality, generalizing that in part (*i*), for connected plane graphs of girth r.

(13d) Let G be a polyhedron (or polyhedral graph), all of whose faces are bounded by pentagons and hexagons.
- (*i*) Use Euler's formula to show that G must have at least 12 pentagonal faces.
- (*ii*) If, in addition, there are exactly three faces meeting at each vertex, prove that G has exactly 12 pentagonal faces.

(13e) Let G be a simple plane graph with less than 12 faces, in which every vertex has degree at least 3.
- (*i*) Use Euler's formula to prove that G has a face bounded by at most four edges.
- (*ii*) Give an example to show that the result of part (*i*) is false if G has 12 faces.

(13f) (*i*) Let G be a simple connected cubic plane graph and let φ_k be the number of k-sided faces. By counting the number of vertices and edges in G, prove that

$$3\varphi_3+2\varphi_4+\varphi_5-\varphi_7-2\varphi_8-\ldots=12.$$

(*ii*) Deduce that G has at least one face bounded by at most five edges.

(13g) Let G be a simple graph with at least 11 vertices and let $\bar{G}$ denote its complement.

(*i*) Prove that G and $\bar{G}$ cannot both be planar. (In fact, a similar result holds if 11 is replaced by 9.)

(*ii*) Find a graph G with 8 vertices such that G and $\bar{G}$ are both planar.

(13h) Find the thickness of

(*i*) the Petersen graph;

(*ii*) the 4-cube Q_4.

(13i) (*i*) Show that the thickness of K_n satisfies $t(K_n)\geqslant\lfloor\frac{1}{6}(n+7)\rfloor$.

(*ii*) Use the results of exercise (13g) to show that equality holds if $n\leqslant 8$, but not if $n=9$ or 10. (In fact, equality holds for *all* n other than 9 or 10.)

(*13j) (*i*) Use part (*ii*) of Corollary 13D to prove that

$$t(K_{r,s})\geqslant\left\lceil\frac{rs}{2(r+s)-4}\right\rceil,$$

and verify that equality holds for $t(K_{3,3})$.

(*ii*) If r is even, show that $t(K_{r,s})\leqslant\frac{1}{2}r$, and deduce from part (*i*) that $t(K_{r,s})=\frac{1}{2}r$ if $s>\frac{1}{2}(r-2)^2$.

(*13k) Let G be a polyhedral graph and let W be the circuit subspace of G.

(*i*) Show that the polygons bounding the finite faces of G form a basis for W.

(*ii*) Deduce Corollary 13B.

§14. Graphs on other surfaces

★In the previous two sections we considered graphs drawn in the plane or (equivalently) on the surface of a sphere. We shall now make a few remarks on the embedding of graphs on other surfaces—for example, the torus. It is easy to see that K_5 and $K_{3,3}$ can be drawn without crossings on the surface of a torus, and it is natural to ask whether there are analogues of Euler's formula and Kuratowski's theorem for graphs drawn on such surfaces.

The torus can be thought of as a sphere with one 'handle' (Fig. 14.1). More generally, a surface is said to be of **genus g** if it is topologically homeomorphic to a sphere with g handles. (If you are unfamiliar with these terms, think of graphs drawn on the surface of a doughnut with g holes in it.) Thus the genus of a sphere is zero, and of a torus is one.

A graph which can be drawn without crossings on a surface of genus g, but not on one of genus $g-1$, is called a **graph of genus g**. Thus K_5 and $K_{3,3}$ are graphs of genus one (also called **toroidal graphs**). We must check that the genus of a graph is well defined.

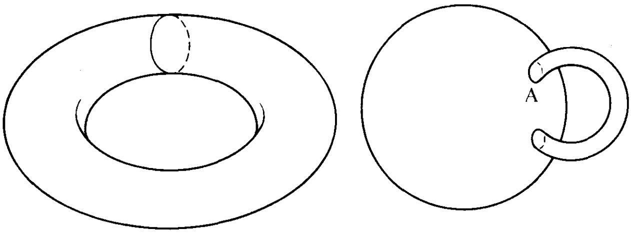


Fig. 14.1

THEOREM 14A. *The genus of a graph is well defined, and does not exceed the crossing-number.*

Proof. To show that the genus is well defined, it is sufficient to find an upper bound for it. This is done by drawing the graph on the surface of a sphere in such a way that the number of crossings is as small as possible, and is therefore equal to the crossing-number c. At each crossing, we construct a 'bridge' (see Fig. 1.2, on page 1) and run one edge over the bridge and the other under it. Since each bridge can be regarded as a handle, we have embedded the graph on the surface of a sphere with c handles. It follows that the genus is well defined, and does not exceed c.//

At the time of writing there is no complete analogue of Kuratowski's theorem for surfaces of genus g, although it is known that there exists, for each value of g, a finite collection of 'forbidden' subgraphs of genus g, corresponding to the forbidden subgraphs K_5 and $K_{3,3}$ for graphs of genus zero. In the case of Euler's formula we are more fortunate, since there is a natural generalization for graphs of genus g. In this generalization, a face of a graph of genus g is defined in the obvious way—namely, in terms of Jordan curves drawn on the surface; we assume that all faces are simply-connected.

THEOREM 14B. *Let G be a connected graph of genus g, with n vertices, m edges and f faces. Then $n-m+f=2-2g$.*

Sketch of proof. We shall outline the main steps in the proof, omitting the details.

(*i*) Without loss of generality, we may assume that G is drawn on the surface of a sphere with g handles. We can also assume that the curves A (see Fig. 14.1) at which the handles meet the sphere are in fact circuits of G (by shrinking those circuits which contain these curves in their interior).

(*ii*) We next disconnect each handle at one end, in such a way that the handle has a free end E and the sphere has a corresponding hole H. We may assume that the circuit corresponding to the end of the handle appears at both the free end E and at the other end, since the additional vertices and edges required for this exactly balance each other, leaving $n-m+f$ unchanged.

(*iii*) We complete the proof by telescoping each of these handles, leaving a sphere with $2g$ holes in it. Note that this telescoping process does not change the value of $n-m+f$. But for a sphere, $n-m+f=2$, and hence for a sphere with $2g$ holes in it, $n-m+f=2-2g$. The result now follows immediately.//

COROLLARY 14C. *The genus $g(G)$ of a simple graph G with n ($\geqq 4$) vertices and m edges satisfies the inequality*

$$g(G) \geqq \lceil \tfrac{1}{6}(m-3n)+1 \rceil.$$

Proof. Since every face is bounded by at least three edges, we have (as in the proof of Corollary 13D) $3f \leqq 2m$. The result follows by substituting this inequality into Theorem 14B, and using the fact that the genus of a graph must be an integer.//

As in the case of the thickness of a graph, little is known about the problem of finding the genus of an arbitrary graph. The usual method is to use Corollary 14c to obtain a lower bound for the genus, and then to try to obtain the required embedding by direct construction.

One case of particular historical importance is that of the genus of the complete graphs. Corollary 14C tells us that the genus of K_n satisfies

$$g(K_n) \geqq \lceil \tfrac{1}{6}(\tfrac{1}{2}n(n-1)-3n)+1 \rceil = \lceil \tfrac{1}{12}(n-3)(n-4) \rceil.$$

Heawood asserted in 1890 that the inequality just obtained is in fact an equality, and this was finally proved in 1968 by Ringel and Youngs after a long and difficult struggle.

THEOREM 14D (Ringel and Youngs 1968).

$$g(K_n) = \lceil \tfrac{1}{12}(n-3)(n-4) \rceil.$$

Remark. This will not be proved here; you should consult Ringel[22] for a discussion and proof of this theorem.//

Further results concerning the embedding of graphs on these surfaces, as well as a discussion of the embedding of graphs on 'non-orientable' surfaces (such as the projective plane and the Möbius strip), can be found in Beineke and Wilson.[3]

Exercises 14

(14a) The surface of a torus can be regarded as a rectangle in which opposite edges have been identified (see Fig. 14.2). Use this representation to find embeddings of K_5 and $K_{3,3}$ on the torus.

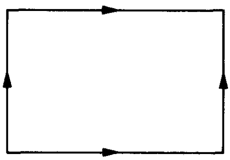


Fig. 14.2

(14b) Using the representation of exercise 14a, show that the Petersen graph has genus 1.

(14c) (*i*) Calculate $g(K_7)$ and $g(K_{11})$.
(*ii*) Give an example of a complete graph of genus 2.

(14d) (*i*) Use Theorem 14D to prove that there is no value of n for which $g(K_n)=7$.
(*ii*) What is the next integer which is not the genus of any complete graph?

(14e) (*i*) Give an example of a planar graph which is regular of degree 4 and every face of which is a triangle.
(*ii*) Show that there is no graph of genus $g \geqslant 1$ with these properties.

(14f) (*i*) Obtain a lower bound analogous to that of Corollary 14C, for a graph containing no triangles.
(*ii*) Deduce that $g(K_{r,s}) \geqslant [\frac{1}{4}(r-2)(s-2)]$. (In fact, Ringel has shown that this is an equality.)

(*14g) (*i*) Let G be a non-planar graph which can be embedded on a Möbius strip. Prove that, with the usual notation, $n-m+f=0$.
(*ii*) Show how K_5 and $K_{3,3}$ can be embedded on the surface of a Möbius strip. ★

§15. Dual graphs

In Theorems 12B and 12C we gave necessary and sufficient conditions for a graph to be planar—namely, that it contains no subgraph which is homeomorphic or contractible to K_5 or $K_{3,3}$. Our aim is now to discuss conditions of a rather different kind. These will involve the concept of duality.

Given a plane graph G, we shall construct another graph G^* called the **(geometric-)dual** of G. The construction is in two stages:
(*i*) inside each face F_i of G we choose a point v_i^*—these points are the vertices of G^*;

(*ii*) corresponding to each edge e of G we draw a line e^* which crosses e (but no other edge of G), and joins the vertices v_i^* which lie in the faces F_i adjoining e—these lines are the edges of G^*.

This procedure is illustrated in Fig. 15.1, the vertices v_i^* being represented by crosses, the edges e of G by solid lines and the edges e^* of G^* by dashed lines. Note that an end-vertex of G gives rise to a loop of G^*, as does any bridge. Note also that if two faces of G have more than one edge in common, then G^* contains multiple edges.

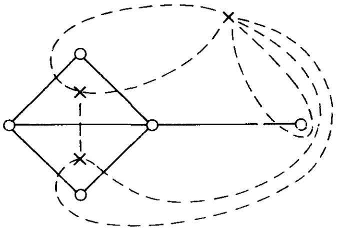


Fig. 15.1

It should be noted that the geometric idea of duality is a very old one. For example, the 'fifteenth book of Euclid', written about 500–600 A.D., remarks that the dual of a cube is an octahedron, and that the dual of a dodecahedron is an icosahedron (see exercise 15b).

It is clear that any two graphs formed from G in this way must be isomorphic; this is why we called G^* '*the* dual of G' instead of '*a* dual of G'. On the other hand, it should be pointed out that if G is isomorphic to H, it does not necessarily follow that G^* is isomorphic to H^*; an example which demonstrates this is given in exercise 15e.

If G is not only plane, but connected as well, then G^* is plane and connected and there are simple relations connecting the numbers of vertices, edges and faces of G and G^*.

LEMMA 15A. *Let G be a plane connected graph with n vertices, m edges and f faces, and let its geometric-dual G^* have n^* vertices, m^* edges and f^* faces. Then $n^* = f$, $m^* = m$ and $f^* = n$.*

Proof. The first two relations are direct consequences of the definition of G^*. The third relation follows immediately on substituting these two relations into Euler's formula applied to both G and G^*.//

Since the dual G^* of a plane graph G is also a plane graph, we can repeat the construction described above to form the dual of G^*, denoted by G^{**}. If G is connected, then the relationship between G^{**} and G is particularly simple, as we now show.

THEOREM 15B. *Let G be a plane connected graph. Then G^{**} is isomorphic to G.*

Proof. The result follows almost immediately from the fact that the construction which gives rise to G^* from G can be reversed to give G from G^*; for example, in Fig. 15.1 the graph G is the dual of the graph G^*. We need to check only that a face of G^* cannot contain more than one vertex of G—it certainly contains at least one—and this follows immediately from the relations $n^{**} = f^* = n$, where n^{**} denotes the number of vertices of G^{**}.//

If G now denotes any planar graph, then a dual of G can be defined by taking any plane embedding and forming its geometric-dual, but uniqueness does not in general hold. Since duals have been defined only for planar graphs, it is trivially true to say that a graph is planar if and only if it has a dual. On the other hand, if we are given an arbitrary graph we have no way of telling from the above whether or not it is planar. It is obviously desirable to find a definition of duality which generalizes the geometric-dual and at the same time enables us (in principle, at least) to determine whether or not a given graph is planar.

One such definition exploits the relationship under duality between the circuits and cutsets of a planar graph G. We shall first describe this relationship and then use it to obtain the definition we seek. An alternative definition will be given in exercise *15*k.

THEOREM 15C. *Let G be a planar graph and G^* be a geometric-dual of G. Then a set of edges in G forms a circuit in G if and only if the corresponding set of edges of G^* forms a cutset in G^*.*

Proof. We can assume without loss of generality that G is a connected plane graph. If C is a circuit in G, then C encloses one or more of the finite faces of G, and thus contains in its interior a non-empty set S of vertices of G^*. It follows immediately that those edges of G^* which cross the edges of C form a cutset of G^* whose removal disconnects G^* into two subgraphs, one with vertex-set S and the other containing those vertices which do not lie in S (see Fig. 15.2). The converse implication is similar, and will be omitted.//

COROLLARY 15D. *A set of edges of G forms a cutset in G if and only if the corresponding set of edges of G^* forms a circuit in G^*.*

Proof. The result follows immediately on applying Theorem 15C to G^* and using Theorem 15B.//

Using Theorem 15C as motivation, we can now give an abstract definition of duality. Note that this definition does not invoke any special properties of planar graphs, but concerns only the relationship between two graphs.

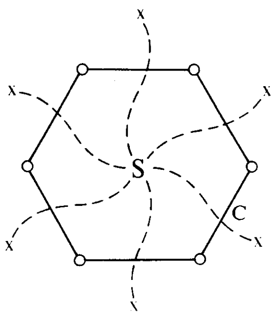


Fig. 15.2

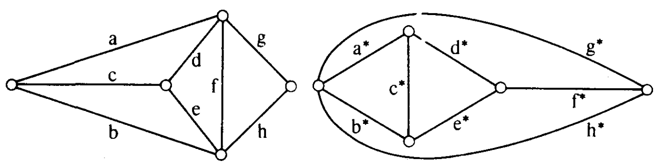


Fig. 15.3

We shall say that a graph G^* is an **abstract-dual** of a graph G if there is a one-one correspondence between the edges of G and those of G^* with the property that a set of edges of G forms a circuit in G if and only if the corresponding set of edges of G^* forms a cutset in G^*. For example, Fig. 15.3 shows a graph and its abstract-dual, with corresponding edges sharing the same letter.

It is clear from Theorem 15C that the concept of an abstract-dual generalizes that of a geometric-dual, in the sense that if G is a planar graph and G^* is a geometric-dual of G, then G^* is an abstract-dual of G. What we should like to be able to do is to obtain analogues for abstract-duals of some of the results on geometric-duals. We shall be content with just one of these here—the analogue for abstract-duals of Theorem 15B.

THEOREM 15E. *If G^* is an abstract-dual of G, then G is an abstract-dual of G^*.*

Remark. Note that we do not require that G should be connected.

Proof. Let C be a cutset of G and let C^* denote the corresponding set of edges of G^*; it will be sufficient to show that C^* is a circuit of G^*. By the first part of exercise 5I, C has an even number of edges in common with any circuit of G, and so C^* must have an even number of edges in common with any cutset of G^*. It follows from the second part of exercise 5I that C^* must be either a single circuit in G^* or an edge-disjoint union of two or more circuits. But the second possibility cannot occur, since one can show similarly that circuits in G^* correspond to edge-disjoint unions of cutsets in G, and so C would then be an edge-disjoint union of two or more cutsets, rather than just a single cutset.//

Although the definition of an abstract-dual seems at first sight rather strange, it turns out to have the properties required of it. We saw in Theorem 15C that a planar graph has an abstract-dual (e.g. any geometric-dual), and we now show that the converse result is true—namely, that any graph which has an abstract-dual must be planar. In other words, we now have an abstract definition of duality which generalizes the geometric-dual and which characterizes planar graphs. It will turn out, in fact, that the definition of an abstract-dual is a natural consequence of the study of duality in matroid theory (see §**32**).

THEOREM 15F. *A graph is planar if and only if it has an abstract-dual.*

Remark. There are several proofs of this result. We shall be presenting a particularly simple one (due to T. D. Parsons) which uses Kuratowski's theorem.

Sketch of proof. As mentioned above, it is sufficient to prove that if G is a graph which has an abstract-dual G^*, then G is planar. The proof is in four steps:

(*i*) We note first that if an edge e is removed from G, then the abstract-dual of the remaining graph may be obtained from G^* by simply contracting the corresponding edge e^*. On repeating this procedure, it follows immediately that if G has an abstract-dual, then so does any subgraph of G.

(*ii*) We next observe that if G has an abstract-dual, and G' is homeomorphic to G, then G' also has an abstract-dual. This follows from the fact that the insertion or removal in G of a vertex of degree two results in the addition or deletion of a 'multiple edge' in G^*.

(*iii*) The third step is to show that neither K_5 nor $K_{3,3}$ has an abstract-dual. If G^* is a dual of $K_{3,3}$, then since $K_{3,3}$ contains only circuits of length four or six and no cutsets with only two edges, it follows that G^* contains no multiple edges, and that every vertex of G^* must have degree at least four. Hence G^* must contain at least five vertices, and thus at least $\frac{1}{2}.5.4 = 10$ edges, which is a contradiction. The argument for K_5 is similar, and will be omitted.

(*iv*) Suppose, now, that G is a non-planar graph which has an abstract-dual G^*. Then by Kuratowski's theorem, G contains a subgraph H homeomorphic to K_5 or $K_{3,3}$. It follows from (*i*) and (*ii*) that H, and hence also K_5 or $K_{3,3}$, must have an abstract-dual, contradicting (*iii*).//

Exercises 15

(15a) Find the duals of the graphs in Fig. 15.4 and verify Lemma 15A for these duals.

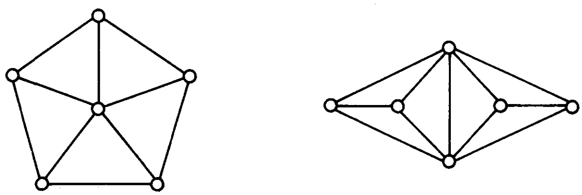


Fig. 15.4

(15b) Show that the dual of the cube graph is the octahedron graph, and that the dual of the dodecahedron graph is the icosahedron graph.

(15c) Show that the dual of a wheel is a wheel.

(15d) Use duality to prove that there exists no plane graph with five faces, each pair of which share an edge in common.

(15e) Show that the graphs in Fig. 15.5 are isomorphic, but that their geometric-duals are non-isomorphic.

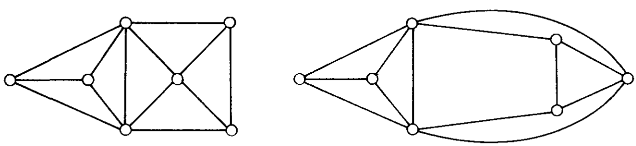


Fig. 15.5

(15f) (*i*) Give an example to show that if G is a disconnected plane graph, then G^{**} is not isomorphic to G.
(*ii*) Prove the result of part (*i*) in general.

(15g) Dualize the results of exercises 13d and 13e.

(15h) Prove that if G is a 3-connected plane graph, then its geometric-dual is a simple graph.

(15i) Let G be a connected plane graph. Using Theorem 5A and Corollary 6C, prove that G is bipartite if and only if its dual G^* is Eulerian.

(*15j) (*i*) Give an example to show that if G is a connected plane graph, then any spanning tree in G corresponds to the complement of a spanning tree in G^*.
(*ii*) Prove the result of part (*i*) in general.

(*15k) A graph G^* is a **Whitney-dual** of G if there is a one-one correspondence between $E(G)$ and $E(G^*)$ such that, if H is a subgraph of G with $V(H) = V(G)$, then the corresponding subgraph H^* of G^* satisfies

$$\gamma(H) + \xi(\tilde{H}^*) = \xi(G^*),$$

where $\tilde{H}^*$ is obtained from G^* by deleting the edges of H^*, and γ and ξ are defined as in §9.

(*i*) Show that this generalizes the idea of a geometric-dual.
(*ii*) Prove that if G^* is a Whitney-dual of G, then G is a Whitney-dual of G^*.

(The name 'Whitney dual' arises since H. Whitney proved that a graph is planar if and only if it has such a dual.)

§16. Infinite graphs

★In this section we show how some of the definitions given in previous sections can be extended to infinite graphs. As you may recall, an **infinite graph** G is a pair $(V(G), E(G))$, where $V(G)$ is an infinite set of elements called **vertices**, and $E(G)$ is an infinite family of unordered pairs of elements of $V(G)$ called **edges**. If $V(G)$ and $E(G)$ are both countably infinite, then G is said to be a **countable graph**. Note that we have excluded from these definitions the possibility of $V(G)$ being infinite but $E(G)$ finite (such objects being merely finite graphs together with infinitely many isolated vertices), or of $E(G)$ being infinite but $V(G)$ finite (such objects being essentially finite graphs but with infinitely many loops or multiple edges).

Several of the definitions given earlier ('adjacent', 'incident', 'isomorphic', 'subgraph', 'connected', 'planar', etc.) generalize immediately to infinite graphs. The **degree** of a vertex v of an infinite graph is defined to be the cardinality of the set of edges incident to v, and may be finite or infinite. An infinite graph all of whose vertices have finite degree is called **locally-finite**, two important examples being the infinite square lattice and the infinite triangular lattice, shown in Figs 16.1 and

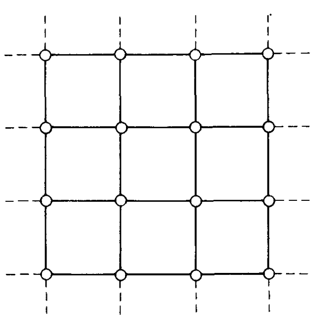


Fig. 16.1

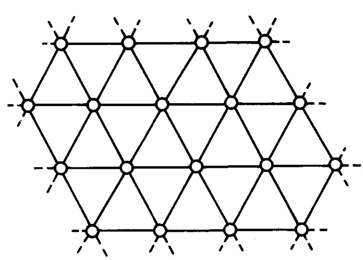


Fig. 16.2

16.2. We similarly define a **locally-countable** infinite graph to be one in which each vertex has countable degree. With these definitions, we now prove the following simple, but fundamental, result.

THEOREM 16A. *Every connected locally-countable infinite graph is a countable graph.*

Proof. Let v be any vertex of such an infinite graph, and let A_1 be the set of vertices adjacent to v, A_2 the set of all vertices adjacent to a vertex of A_1, and so on. By hypothesis, A_1 is countable, and hence so are A_2, $A_3, \ldots$ (using the fact that the union of a countable collection of countable sets is countable). Hence $\{v\}, A_1, A_2, \ldots$ is a sequence of sets whose union is countable. Moreover, this sequence contains every vertex of the infinite graph, by connectedness, and the result follows.//

COROLLARY 16B. *Every connected locally-finite infinite graph is a countable graph.*//

We can also extend to an infinite graph G the concept of a walk, there being essentially three different types:

(*i*) a **finite walk** in G is defined exactly as in §5;
(*ii*) a **one-way infinite walk** in G with **initial vertex** v_0 is an infinite sequence of edges of the form $v_0v_1, v_1v_2, \ldots$;
(*iii*) a **two-way infinite walk** in G is an infinite sequence of edges of the form $\ldots, v_{-2}v_{-1}, v_{-1}v_0, v_0v_1, v_1v_2, \ldots$.

One-way and two-way infinite trails and paths are defined in the obvious way, as are such terms as the length of a path and the distance between vertices. The following result, known as **König's lemma**, tells us that infinite paths are not difficult to come by:

THEOREM 16C (König 1927). *Let G be a connected locally-finite infinite graph. Then for any vertex v of G, there exists a one-way infinite path with initial vertex v.*

Proof. If z is any vertex of G other than v, then there is a non-trivial path from v to z. It follows that there are infinitely many paths in G with initial vertex v. Since the degree of v is finite, there must be infinitely many of these paths which start with the same edge. If vv_1 is such an edge, then we can repeat this procedure for the vertex v_1 and thus obtain a new vertex v_2 and a corresponding edge v_1v_2. By carrying on in this way, we obtain the one-way infinite path $v \to v_1 \to v_2 \ldots$.//

The importance of König's lemma is that it allows us to deduce results about infinite graphs from the corresponding results for finite graphs. The following theorem may be regarded as a typical example:

THEOREM 16D. *Let G be a countable graph, every finite subgraph of which is planar. Then G is planar.*

Proof. Since G is countable, its vertices may be enumerated as $v_1, v_2, v_3, \ldots$. We now construct a strictly increasing sequence $G_1 \subset G_2 \subset G_3 \subset \ldots$ of subgraphs of G, by taking G_k to be the subgraph whose vertices are precisely $v_1, \ldots, v_k$ and whose edges are those edges of G which join two of these vertices. Then, assuming the result that G_i can be embedded in the plane in only a finite number ($m(i)$, say) of topologically distinct ways, we can construct another infinite graph H whose vertices w_{ij} ($i \geqq 1$, $1 \leqq j \leqq m(i)$) correspond to the various embeddings of the graphs G_i, and whose edges join those vertices w_{ij} and w_{kl} for which $k = i+1$ and the plane embedding corresponding to w_{kl} 'extends' (in an obvious sense) the embedding corresponding to w_{ij}. Since H is clearly connected and locally-finite, it follows from König's lemma that H contains a one-way infinite path. Since G is countable, this infinite path gives the required plane embedding of the whole of G.//

It is worth pointing out that if we assume further axioms of set theory (in particular, the uncountable version of the axiom of choice), then various results such as the one just proved can be extended to infinite graphs which are not necessarily countable.

We conclude this digression on infinite graphs with a brief discussion on infinite Eulerian graphs. It seems natural to say that a connected infinite graph G is **Eulerian** if there exists a two-way infinite trail which includes every edge of G; such an infinite trail is then called a (two-way) **Eulerian trail**. Note that these definitions require G to be countable. The following theorems give further conditions which are necessary for an infinite graph to be Eulerian.

THEOREM 16E. *Let G be a connected countable graph which is Eulerian. Then*
(i) G has no vertices of odd degree;
(ii) for every finite subgraph H of G, the infinite graph $\bar{H}$ (obtained by deleting from G the edges of H) has at most two infinite connected components;
(iii) if, in addition, every vertex of H has even degree, then $\bar{H}$ has exactly one infinite connected component.

Proof. (*i*) Suppose that P is an Eulerian trail. Then by the argument given in the proof of Theorem 6B, every vertex of G must have either even or infinite degree.

(*ii*) Let P be split up into three subtrails P_-, P_0 and P_+, in such a way that P_0 is a finite trail containing every edge of H (and possibly other edges as well), and P_- and P_+ are both one-way infinite trails. Then the infinite graph K formed by the edges of P_- and P_+, and the vertices incident to them, has at most two infinite components. Since $\bar{H}$ is obtained by adding only a finite set of edges to K, the result follows.

(*iii*) Let the initial and final vertices of P_0 be v and w; we wish to show that v and w are connected in $\bar{H}$. If $v = w$, this is obvious. If not, then the result follows on applying Corollary 6D to the graph obtained by removing from P_0 the edges of H, this graph having exactly two vertices (v and w) of odd degree, by hypothesis.//

It turns out that the conditions given in the previous theorem are not only necessary but also sufficient. We state this result formally in the following theorem; its proof lies beyond the scope of this book, but may be found in Ore.[21]

THEOREM 16F. *Let G be a connected countable graph. Then G is Eulerian if and only if the conditions (i), (ii) and (iii) of Theorem 16E are satisfied.*//

Exercises 16

(16a) Give an example of each of the following:
  (*i*) an infinite graph with infinitely many end-vertices;
  (*ii*) an infinite graph with uncountably many vertices and edges;
  (*iii*) an infinite connected cubic graph;
  (*iv*) an infinite bipartite graph;
  (*v*) an infinite non-planar graph;
  (*vi*) an infinite tree.

(16b) Show by an example that the conclusion of König's lemma is false if we omit the condition that the infinite graph is locally-finite.

(16c) Use the proof of Theorem 4A to show that an infinite graph G can be embedded in Euclidean 3-space if $V(G)$ and $E(G)$ can each be put in one-one correspondence with a subset of the set of real numbers.

(*16d) (*i*) Find an Eulerian trail in the infinite square lattice S.
  (*ii*) Verify that S satisfies the conditions of Theorem 16E.

(*16e) Repeat exercise 16d for the infinite triangular lattice.

(*16f) Show that the infinite square lattice contains both one-way and two-way infinite paths passing exactly once through each vertex.★

6
The colouring of graphs

With colours fairer painted their foul ends.

William Shakespeare (*The Tempest*)

In this chapter we investigate the colouring of graphs and maps, with special reference to the four-colour theorem and related topics. We start in §**17** by discussing under what conditions the vertices of a graph can be painted in such a way that every edge is incident to vertices of different colours. This discussion spills over into the following section where a major theorem are proved. §**19** is devoted to the relationship between the colouring of graphs and the colouring of maps, and both of these are then related in §**20** to problems concerning the colouring of the edges of a graph. All of this material is essentially qualitative, asking *whether* graphs can be coloured under certain circumstances, rather than *in how many ways* the colouring can be done. We conclude with a discussion of this second question (using chromatic polynomials) in §**21**.

§17. The chromatic number

If G is a graph without loops, then G is said to be **k-colourable** if to each of its vertices we can assign one of k colours in such a way that no two adjacent vertices have the same colour. If G is k-colourable, but not $(k-1)$-colourable, we say that G is **k-chromatic**, or that the **chromatic number** of G (denoted by $\chi(G)$) is k. Fig. 17.1 shows a graph which is 4-

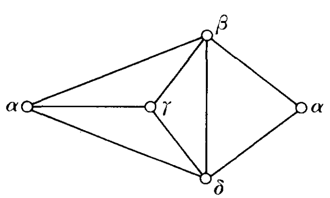


Fig. 17.1

chromatic, and hence k-colourable if $k \geqq 4$; the colours are denoted by Greek letters. For convenience, we shall assume that *all graphs mentioned in* §**17** *and* §**18** *contain no loops;* we may however allow multiple edges, since they are irrelevant to our discussion.

It is clear that $\chi(K_n) = n$, and hence we can easily construct graphs with arbitrarily high chromatic number. At the other end of the scale, it is easy to see that $\chi(G) = 1$ if and only if G is a null graph, and that $\chi(G) = 2$ if and only if G is a non-null bipartite graph. It follows from Theorem 5A and exercise 5g that if G is not a null graph, then $\chi(G) = 2$ if and only if G contains no circuits of odd length. Note, in particular, that every tree with at least two vertices is 2-chromatic, as is any circuit graph with an even number of vertices.

It is not known under what conditions a graph is 3-chromatic, although it is easy to give examples of such graphs. These examples include the circuit graphs with an odd number of vertices, the wheels with an odd number of vertices, and the Petersen graph. The wheels with an even number of vertices are 4-chromatic.

There is little we can say about the chromatic number of an arbitrary graph. If the graph has n vertices, then obviously its chromatic number does not exceed n, and if the graph contains K_r as a subgraph, then its chromatic number cannot be less than r, but these results do not take us very far. If, however, we know the degree of every vertex of the graph, we can usually make significant progress.

THEOREM 17A. *If G is a graph whose largest vertex-degree is ρ, then G is $(\rho+1)$-colourable.*

Proof. The proof is by induction on the number of vertices of G. Let G be a graph with n vertices. Then if we delete any vertex v (and the edges incident to it), the graph which remains is a graph with $n-1$ vertices whose largest vertex-degree is at most ρ. By our induction hypothesis, this graph is $(\rho+1)$-colourable. A $(\rho+1)$-colouring for G is then obtained by colouring v with a different colour from the (at most ρ) vertices adjacent to v.//

By more careful treatment this theorem can be strengthened a little to give the following result which is known as **Brooks' theorem**; its proof will be given in the next section.

THEOREM 17B (Brooks 1941). *If G is a simple connected graph which is not a complete graph, and if the largest vertex-degree of G is $\rho(\geqq 3)$, then G is ρ-colourable.*//

Both of these theorems are useful if the degrees of all the vertices are approximately equal. For example, we can immediately deduce from Theorem 17A that every cubic graph is 4-colourable, and from Theorem

17B that every connected cubic graph (apart from K_4) is in fact 3-colourable. On the other hand, if our graph has a few vertices with rather large degrees, then these theorems tell us very little. This is illustrated very well by the star graph $K_{1,s}$ which by Brooks' theorem is s-colourable, but which is in fact 2-chromatic. There is at present no really effective way of avoiding this situation, although there are techniques which help a little.

This rather depressing situation does not arise if we restrict our attention to planar graphs. In fact, we can prove very easily the rather strong result that every planar graph is 6-colourable.

THEOREM 17C. *Every planar graph is 6-colourable.*

Proof. The proof is very similar to that of Theorem 17A. We prove the theorem by induction on the number of vertices, the result being trivial for planar graphs with fewer than seven vertices. Suppose then that G is a planar graph with n vertices, and that all planar graphs with $n-1$ vertices are 6-colourable. Without loss of generality, G can be assumed to be a simple graph, and so, by Theorem 13F, contains a vertex v with degree at most five. If we delete v, then the graph which remains has $n-1$ vertices and is thus 6-colourable. A 6-colouring of G is then obtained by colouring v with a different colour from the (at most five) vertices adjacent to v.//

As with Theorem 17A this result can be made even stronger by more careful treatment, the result being called the **five-colour theorem**:

THEOREM 17D. *Every planar graph is 5-colourable.*

Proof. The method of proof is similar to that of Theorem 17C, although the details are more complicated. We prove the theorem by induction on the number of vertices, the result being trivial for planar graphs with fewer than six vertices. Suppose then that G is a planar graph with n vertices, and that all planar graphs with less than n vertices are 5-colourable. We can assume that G is a simple plane graph and that, by Theorem 13F, G contains a vertex v with degree at most five. As before, the deletion of v leaves us with a graph with $n-1$ vertices which is thus 5-colourable. Our aim is to colour v in one of our five colours, so completing the 5-colouring of G.

If $\rho(v) < 5$, then v can be coloured with any colour not assumed by the (at most four) vertices adjacent to v, completing the proof in this case. We thus suppose that $\rho(v) = 5$, and that the vertices $v_1, \ldots, v_5$ which are adjacent to v are arranged around v in clockwise order as in Fig. 17.2. If the vertices $v_1, \ldots, v_5$ are all mutually adjacent, then G must contain the non-planar graph K_5 as a subgraph, which is impossible. So at least two of the vertices v_i (say, v_1 and v_3) are not adjacent.

We now contract the two edges vv_1 and vv_3. The resulting graph is a plane graph with less than n vertices, and is thus 5-colourable. We now reinstate the two edges, giving both v_1 and v_3 the colour originally assigned to v. A 5-colouring of G is then obtained by colouring v with a different colour from the (at most four) colours assigned to the vertices v_i.//

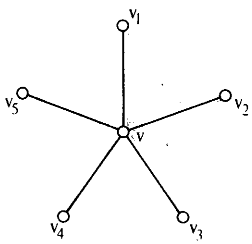


Fig. 17.2

It is natural to ask whether this result can be strengthened further, and this question leads us to what was formerly one of the most famous unsolved problems in the whole of mathematics—the 'four-colour problem'. This problem, in an alternative formulation (see §**19**), was first posed in 1852, and was eventually settled by K. Appel and W. Haken in 1976. Their proof, which took them four years and a substantial amount of computer time, ultimately depends on a complicated extension of the ideas used in the proof of the five-colour theorem. Further information about this proof can be found in Saaty and Kainen,[24] or in Beineke and Wilson.[3] We conclude this section with a formal statement of what is now known as the **four-colour theorem**:

THEOREM 17E. *Every planar graph is 4-colourable.*//

Exercises 17

(17a) Find the chromatic numbers of the graphs in Fig. 17.3.

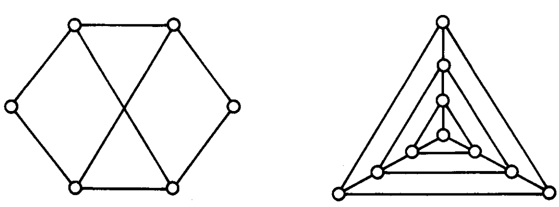


Fig. 17.3

(17b) What is the chromatic number of
 (*i*) each of the Platonic graphs?
 (*ii*) the Petersen graph?
 (*iii*) the complete tripartite graph $K_{r,s,t}$?
 (*iv*) the k-cube Q_k?

(17c) Compare the bound for the chromatic number given by Brooks' theorem, with the correct value, for
 (*i*) the Petersen graph;
 (*ii*) the k-cube Q_k.

(17d) Let G be a simple graph with n vertices, which is regular of degree d. By considering the number of vertices which can be assigned the same colour, prove that $\chi(G) \geqslant n/(n-d)$.

(17e) Let G be a simple planar graph containing no triangles.
 (*i*) Use Euler's formula to show that G contains a vertex whose degree is at most 3.
 (*ii*) Use induction to deduce that G is 4-colourable. (In fact, it can be proved that G is 3-colourable.)

(*17f) Generalize the results of the previous exercise to the cases where
 (*i*) G has girth r;
 (*ii*) G has thickness t.

(17g) Try to prove the four-colour theorem by adapting the above proof of the five-colour theorem. At what point does the proof fail?

(*17h) A graph G is **k-critical** if $\chi(G)=k$ and if the deletion of any vertex yields a graph with smaller chromatic number.
 (*i*) Find all 2-critical and 3-critical graphs.
 (*ii*) Give an example of a 4-critical graph.
 (*iii*) Prove that if G is k-critical, then
  (*a*) every vertex of G has degree at least $k-1$;
  (*b*) G has no cut-vertices.

(*17i) Let G be a countable graph, every finite subgraph of which is k-colourable.
 (*i*) Use König's lemma to prove that G is k-colourable.
 (*ii*) Deduce that every countable planar graph is 4-colourable.

§18. A proof of Brooks' theorem

★In order to avoid disturbing the continuity, we deferred the proof of Brooks' theorem (Theorem 17B). This proof will now be given.

THEOREM 17B. *If G is a simple connected graph which is not a complete graph, and if the largest vertex-degree of G is $\rho(\geqq 3)$, then G is ρ-colourable.*

Proof. The proof will as usual be by induction on the number of vertices of G. Suppose that G has n vertices; then if any vertex of G has degree less than ρ, the proof may be completed by imitating the proof of

Theorem 17A. We can thus suppose without loss of generality that G is regular of degree ρ.

We now choose any vertex v and delete it. The graph which remains is a graph with $n-1$ vertices whose largest vertex-degree is at most ρ. By our induction hypothesis, this graph is ρ-colourable. Our aim is now to colour v with one of the ρ colours. We can suppose that the vertices $v_1, \ldots, v_\rho$ which are adjacent to v are arranged around v in clockwise order, and that they are coloured with distinct colours $c_1, \ldots, c_\rho$, since otherwise there would be a spare colour which could be used to colour v.

We now define H_{ij} $(i \neq j, 1 \leqq i, j \leqq \rho)$ to be the subgraph of G whose vertices are all those vertices coloured c_i or c_j and whose edges are all those edges incident to one vertex coloured c_i and one vertex coloured c_j. If the vertices v_i and v_j lie in different components of H_{ij}, we can interchange the colours of all the vertices in the component of H_{ij} containing v_i. The result of this recolouring is that v_i and v_j both have colour c_j, enabling v to be coloured with colour c_i. We may thus assume that, given any i and j, v_i and v_j are connected by a path which lies entirely in H_{ij}. We shall denote the component of H_{ij} containing v_i and v_j by C_{ij}.

It is clear that if v_i were adjacent to more than one vertex with colour c_j, then there would be a colour (other than c_i) which was not assumed by any of the vertices adjacent to v_i. In this case v_i could be recoloured using this colour, enabling v to be coloured with colour c_i. If this does not happen, then we can use a similar argument to show that every vertex of C_{ij} (other than v_i and v_j) must have degree two; for if w is the first vertex of the path from v_i to v_j which has degree greater than two, then w can be recoloured using a colour different from c_i or c_j, thereby destroying the property that v_i and v_j are connected by a path lying entirely in C_{ij}. We can thus assume that for any i, j, the component C_{ij} consists only of a path from v_i to v_j.

We now remark that two paths of the form C_{ij} and C_{jl} (where $i \neq l$) can be assumed to intersect only at v_j, since if w is another point of intersection, then w can be recoloured using a colour different from c_i, c_j or c_l, contradicting the fact that v_i and v_j are connected by a path.

To complete the proof, we choose two vertices v_i and v_j which are not adjacent, and let w be the vertex with colour c_j which is adjacent to v_i. If C_{il} is a path (for some $l \neq j$), we can interchange the colours of the vertices in this path without affecting the colouring of the rest of the graph. But if we perform this interchange, then w would be a vertex common to the paths C_{ij} and C_{jl}, which is a contradiction. This contradiction establishes the theorem.//

Exercises 18

(18a) Draw diagrams to illustrate the arguments in the last three paragraphs of the above proof.★

§19. The colouring of maps

The four-colour problem arose historically in connexion with the colouring of maps. If we have a map containing several countries, we may wish to know how many colours are needed to colour the various countries in such a way that no two neighbouring countries share the same colour. Possibly the most familiar form of the four-colour theorem is the statement that every map can be coloured using only four colours.

In order to make this statement more precise, we must say exactly what we mean by a 'map'. In the colouring problems we shall be considering, it is necessary to ensure that the two colours on either side of an edge are different, and so we shall need to exclude maps in which there is a bridge. It is convenient, therefore, to define a **map** to be a connected plane graph containing no bridges. (Note that we do not exclude loops or multiple edges when defining a map; the exclusion of bridges corresponds, as we shall see, to the exclusion of loops in §**17**.)

We can now define a map to be ***k*-colourable(f)** if its faces can be coloured with k colours in such a way that no two adjacent faces (i.e., faces whose boundaries have an edge in common) have the same colour. If there is any possibility of confusion, we shall also use 'k-colourable(v)' to mean k-colourable in the usual sense. As an example, we note that the map shown in Fig. 19.1 is 3-colourable(f) and 4-colourable(v).

The **four-colour theorem for maps** may now be stated simply as the statement that *every map is 4-colourable(f)*. We shall prove the equivalence of the two forms of the four-colour theorem in Corollary 19C. In the meantime, we shall investigate the conditions under which a map can be coloured using two colours. It turns out that these conditions take a particularly simple form.

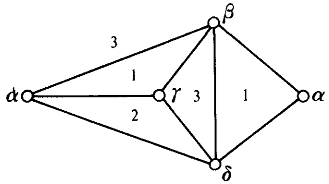


Fig. 19.1

THEOREM 19A. *A map G is 2-colourable(f) if and only if G is an Eulerian graph.*

First proof. ⇒ For any vertex v of G, the faces surrounding v must be even in number since they can be coloured using two colours. It follows that every vertex has even degree and so, by Theorem 6B, G is Eulerian.

$\Leftarrow$ We shall describe a method for actually colouring the faces of G. Choose any face F and colour it red; draw a Jordan curve from a point x in F to a point in any other face, making sure that the curve passes through no vertex of G. If the curve from x to a point in face F' crosses an even number of edges, colour F' red; otherwise colour it blue (see Fig. 19.2). The fact that the colouring is well defined can be shown by taking a 'circuit' consisting of two such Jordan curves and proving that this circuit crosses an even number of edges of G, using the fact that every vertex has an even number of edges incident to it.//

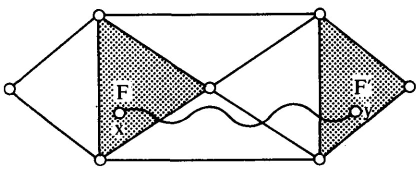


Fig. 19.2

$\Leftarrow$ A simpler proof of Theorem 19A can, and will, be given by translating the problem into one of colouring the vertices of the dual graph. We shall first prove a theorem justifying this procedure, and will then illustrate it by giving our alternative proof of Theorem 19A and by proving the equivalence of the two forms of the four-colour theorem.

THEOREM 19B. *Let G be a planar graph without loops, and let G^* be a geometric-dual of G. Then G is k-colourable(v) if and only if G^* is k-colourable(f).*

Proof. $\Rightarrow$ We may assume that G is plane and connected, so that G^* is a map. If we have a k-colouring(v) for G, then since every face of G^* contains a unique vertex of G, we can k-colour the faces of G^* in such a way that each face inherits the colour of the vertex it contains. The fact that no two adjacent faces of G^* have the same colour follows immediately from the fact that the vertices of G which they contain are adjacent in G and so are differently coloured. Thus G^* is k-colourable(f).

Suppose now that we have a k-colouring(f) of G^*. Then since every vertex of G is contained in a face of G^*, we can k-colour the vertices of G in such a way that each vertex inherits the colour of the face containing it. The fact that no two adjacent vertices of G have the same colour follows immediately by reasoning similar to the above.//

It follows from this result that we can dualize any theorem on the colouring of the vertices of a planar graph to give a theorem on the colouring of the faces of a map, and conversely. As an example of this, consider Theorem 19A.

THEOREM 19A. *A map G is 2-colourable(f) if and only if G is an Eulerian graph.*

Second proof. Since (by exercise 15i) the dual of an Eulerian planar graph is a bipartite planar graph and conversely, it is sufficient to show that a planar graph without loops is 2-colourable(v) if and only if it is bipartite; but this is obvious.//

We can similarly prove the equivalence of the two forms of the four-colour theorem.

COROLLARY 19C. *The four-colour theorem for maps is equivalent to the four-colour theorem for planar graphs.*

Proof.$\Rightarrow$ Let G be a planar graph without loops, and assume without loss of generality that G is plane and connected. Then its geometric-dual G^* is a map, and the 4-colourability(v) of G follows immediately from the fact that this map is 4-colourable(f), using Theorem 19B.
$\Leftarrow$ Conversely, let G be a map and let G^* be its geometric-dual. Then G^* is a planar graph without loops and is therefore 4-colourable(v). It follows immediately that G is 4-colourable(f).//

Duality can also be used to prove the following theorem:

THEOREM 19D. *Let G be a map which is cubic. Then G is 3-colourable(f) if and only if every face is bounded by an even number of edges.*

Proof.$\Rightarrow$ Given any face F of G, the faces of G which surround F must alternate in colour. It follows that there must be an even number of them, and hence that every face is bounded by an even number of edges.
$\Leftarrow$ We shall prove the dual result—if G is a connected plane graph without loops, every face of which is a triangle and every vertex of which has even degree (i.e. G is Eulerian), then G is 3-colourable(v). We shall denote the three colours by α, β and γ.

By Theorem 19A, since G is Eulerian, the faces of G can be coloured with two colours, say red and blue. The required 3-colouring of the vertices of G is then obtained by first colouring the vertices of any red face, the colouring being such that the colours α, β and γ appear in clockwise order, and then colouring the vertices of the surrounding faces, the colours α, β and γ appearing in clockwise order around a face if and only if that face is red (see Fig. 19.3). This colouring of the vertices can be extended to the whole graph, thus proving the theorem.//

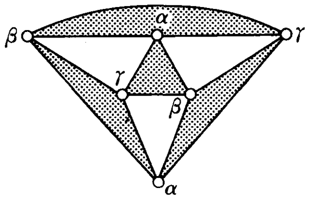


Fig. 19.3

In the above theorem, the map was assumed to be cubic. In fact, we can often remove this condition without loss of generality. Our next theorem is a good example of this:

THEOREM 19E. *In order to prove the four-colour theorem, it is sufficient to prove that every cubic map is 4-colourable(f).*

Proof. By Corollary 19C, it is sufficient to prove that the 4-colourability(*f*) of every cubic map implies the 4-colourability(*f*) of any map.

Let G be any map. Then if G contains any vertices of degree two, these vertices can be removed without affecting the colouring. It therefore remains only to show how one can eliminate any vertices of degree four or more. But if v is such a vertex, then we can stick a 'patch' over v (i.e. draw around v a closed Jordan curve which surrounds no vertex except v) as in Fig. 19.4. Repeating this for every vertex of degree greater than three, we obtain a cubic map which is 4-colourable(*f*) by

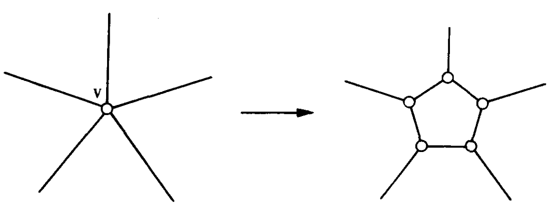


Fig. 19.4

hypothesis. The required 4-colouring of the faces of G may then be obtained by shrinking each patch down to a single vertex and reinstating every vertex of degree two.//

Exercises 19

(19a) Consider the following map, in which the countries are to be coloured red, blue, green and yellow.
  (*i*) Show that country A must be red.
  (*ii*) What colour is country B?

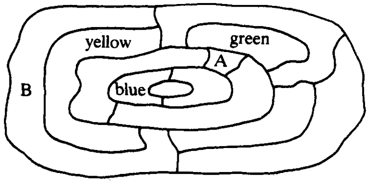


Fig. 19.5

(19b) Find the minimum number of colours needed to colour the faces of each of the Platonic graphs.

(19c) Give an example of a plane graph which is both 2-colourable(f) and 2-colourable(v).

(19d) The plane is divided into a finite number of regions by drawing infinite straight lines in an arbitrary manner. Show (in two different ways) that these regions can be 2-coloured.

(19e) By dualizing the proof of Theorem 17C, prove the six-colour theorem for maps.

(*19f) By dualizing the proof of Theorem 17D, prove the five-colour theorem for maps.

(*19g) Let G be a simple plane graph with fewer than twelve faces, and suppose that every vertex of G has degree at least three.
- (*i*) Use exercise (13e) to prove that G is 4-colourable(f).
- (*ii*) Dualize the result of part (*i*).

(*19h)
- (*i*) Prove that, if a toroidal graph is embedded on the surface of a torus, then its faces can be coloured using seven colours.
- (*ii*) Find a toroidal graph whose faces cannot be coloured with six colours.

§20. Edge-colourings

This section is devoted to a study of the colouring of the edges of a graph. It turns out that the four-colour theorem for planar graphs is equivalent to a theorem concerning edge-colourings of cubic maps.

A graph G is said to be **k-colourable(e)** (or **k-edge-colourable**) if its edges can be coloured with k colours in such a way that no two adjacent edges have the same colour. If G is k-colourable(e) but not $(k-1)$-colourable(e), we say that the **chromatic index** (or **edge-chromatic number**) of G is k, and write $\chi'(G)=k$. Fig. 20.1 shows a graph G for which $\chi'(G)=4$.

It is clear that if ρ denotes the largest vertex-degree of G, then $\chi'(G) \geqq \rho$. The following result, known as **Vizing's theorem**, gives surprisingly sharp bounds for the chromatic index of a simple graph G; its proof may be found in Bondy and Murty[7] or Fiorini and Wilson.[13]

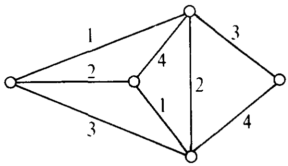


Fig. 20.1

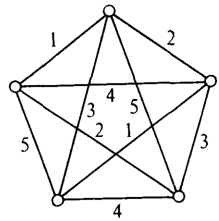


Fig. 20.2

THEOREM 20A (Vizing 1964). *If G is a simple graph whose largest vertex-degree is* ρ, *then* $\rho \leqq \chi'(G) \leqq \rho + 1$.//

It is an unsolved problem to specify exactly which graphs have chromatic index ρ and which have $\rho+1$. However, the results for some particular types of graph can easily be found. For example, $\chi'(C_n)=2$ or 3 depending on whether n is even or odd, and $\chi'(W_n)=n-1$ (if $n \geqq 4$). The corresponding results for complete graphs can also be calculated, as we now show.

THEOREM 20B. $\chi'(K_n)=n$ *if* n *is odd* ($n \neq 1$), *and* $\chi'(K_n)=n-1$ *if* n *is even.*

Proof. If n is odd, then the edges of K_n can be n-coloured by placing the vertices of K_n in the form of a regular n-gon, colouring the edges around the boundary (using a different colour for each edge), and then colouring every remaining edge with the same colour as that used for the boundary which is parallel to it (see Fig. 20.2). The fact that K_n is not $(n-1)$-colourable(e) follows immediately from the observation that the largest possible number of edges of the same colour is $\frac{1}{2}(n-1)$, so that K_n has at most $\frac{1}{2}(n-1)\chi'(K_n)$ edges.

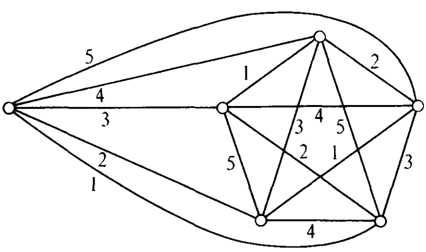


Fig. 20.3

If $n(\geqq 4)$ is even, then K_n can be obtained by joining a complete $(n-1)$-graph K_{n-1} to a single vertex. If the edges of K_{n-1} are then coloured using the method described above, there will be one colour missing at each vertex, and these missing colours will all be different. The colouring of the edges of K_n can thus be completed by colouring the remaining edges with these missing colours (see Fig. 20.3). Finally, if $n=2$, then the result is trivial.//

We now show the connexion between the four-colour theorem and the colouring of the edges of a graph. It is this connexion which accounts for much of the interest in edge-colourings.

THEOREM 20C. *The four-colour theorem is equivalent to the statement that $\chi'(G)=3$ for every cubic map G.*

Proof. $\Rightarrow$ Suppose that we are given a 4-colouring of the faces of G, where the colours are denoted by $\alpha=(1, 0)$, $\beta=(0, 1)$, $\gamma=(1, 1)$, and $\delta=(0, 0)$. A 3-colouring of the edges of G can then be obtained by colouring each edge e with the colour obtained by adding together the colours of the two faces adjoining e, this addition being carried out modulo 2. For example, if e adjoins two faces coloured α and γ, then e is coloured β, since $(1, 0)+(1, 1)=(0, 1)$. Note that the colour δ cannot occur in this edge-colouring since the two faces adjoining each edge must be distinct. Moreover, it is clearly impossible for any two adjacent edges to share the same colour. We thus have the required edge-colouring (see Fig. 20.4).

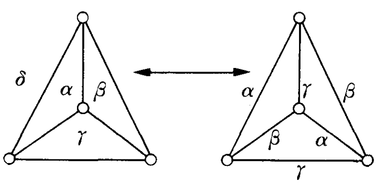


Fig. 20.4

$\Leftarrow$ Suppose now that we are given a 3-colouring of the edges of G; then there will be an edge of each colour at each vertex. The subgraph determined by those edges which are coloured α or β is regular of degree two, and so the faces of this subgraph can be coloured with two colours which we shall call 0 and 1 (using an obvious extension of Theorem 19A to disconnected graphs). In a similar way, the faces of the subgraph determined by those edges which are coloured α or γ can be coloured with the colours 0 and 1. It follows that we can assign to each face of G two coordinates (x, y), where x and y are each 0 or 1. Since the coordinates assigned to two adjacent faces of G must differ in at least one place, it follows that these coordinates $(1, 0)$, $(0, 1)$, $(1, 1)$, $(0, 0)$, give the required 4-colouring of the faces of G.//

We conclude this section with a famous theorem of König on the chromatic index of a bipartite graph.

THEOREM 20D. *If G is a bipartite graph with maximum vertex-degree ρ, then $\chi'(G)=\rho$.*

Remark. The method of proof is somewhat similar to that given in §**18**—namely, we consider a two-coloured subgraph $H_{\alpha\beta}$, and interchange the colours.

Proof. We use induction on the number of edges of G. It is clearly sufficient to prove that if all but one of the edges of G have been coloured with at most ρ colours, then there is a ρ-colouring of the edges of G.

So suppose that each edge of G has been coloured, except for the edge vw. Then there is at least one colour missing at the vertex v, and at least one colour missing at the vertex w. If there is some colour missing from *both* v and w, then the result follows by colouring the edge vw with this colour. If this is not the case, then let α be a colour missing at v, and β be a colour missing at w, and let $H_{\alpha\beta}$ be the connected subgraph of G consisting of the vertex w and all those edges and vertices of G which can be reached from w by a path consisting entirely of edges coloured α or β. Since G is bipartite, the subgraph $H_{\alpha\beta}$ cannot contain the vertex v, and so we can interchange the colours α and β in this subgraph without affecting v or the rest of the colouring. The edge vw can now be coloured α, thereby completing the colouring of the edges of G.//

COROLLARY 20E. $\chi'(K_{r,s})=\max(r, s)$.//

Exercises 20

(20a) Find the chromatic index of the graphs in Fig. 20.5.

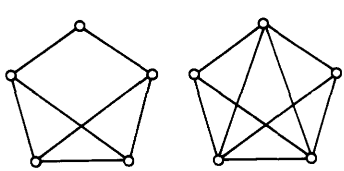


Fig. 20.5

(20b) For each of the following graphs, find (*a*) the lower and upper bounds for $\chi'(G)$ given by Vizing's theorem, and (*b*) the correct value of $\chi'(G)$:
  (*i*) the circuit graph C_7;
  (*ii*) the complete graph K_8;
  (*iii*) the complete bipartite graph $K_{4,6}$.

(20c) What is the chromatic index of each of the Platonic graphs?

(20d) By exhibiting an explicit colouring for the edges of $K_{r,s}$, give an alternative proof of Corollary 20E.

(20e) If G is a cubic Hamiltonian graph, prove that $\chi'(G)=3$.

(20f) (*i*) By considering the possible 3-colourings of the outer 5-circuit, prove that the Petersen graph has chromatic index 4.
(*ii*) Use part (*i*) and exercise 20e to deduce that the Petersen graph is non-Hamiltonian.

(*20g) Let G be a simple graph with an odd number of vertices. If G is regular of degree ρ, prove that $\chi'(G)=\rho+1$.

(*20h) (*i*) Let G be a simple graph which is not a null graph. Prove that $\chi'(G)=\chi(L(G))$, where $L(G)$ is the line graph of G.
(*ii*) By combining part (*i*) with Brooks' theorem, prove Vizing's theorem in the case $\rho=3$.

§21. Chromatic polynomials

We conclude this chapter with a nostalgic glance at vertex-colourings. In this section we shall associate with any graph a function which will tell us, among other things, whether or not the graph is 4-colourable. By investigating this function, we may hope to gain some useful information about the four-colour theorem. Without loss of generality, we shall restrict our attention to simple graphs.

Let G be a simple graph, and let $P_G(k)$ denote the number of ways of colouring the vertices of G with k colours in such a way that no two adjacent vertices have the same colour; P_G will be called (for the time being) the **chromatic function** of G. For example, if G is the graph shown in Fig. 21.1, then $P_G(k)=k(k-1)^2$ since the middle vertex can be coloured in k ways, and the end-vertices can then each be coloured in any of $k-1$ ways. This result can be extended to show that if T is any tree with n vertices, then $P_T(k)=k(k-1)^{n-1}$. Similarly, if G is the complete graph K_3, then $P_G(k)=k(k-1)(k-2)$; this can be extended to give the result $P_G(k)=k(k-1)(k-2)\ldots(k-n+1)$ if G is the graph K_n.

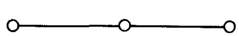


Fig. 21.1

It is clear that *if* $k<\chi(G)$, *then* $P_G(k)=0$, and that *if* $k\geq\chi(G)$, *then* $P_G(k)>0$. Note also that the four-colour theorems is equivalent to the statement: *if G is a simple planar graph, then $P_G(4)>0$.*

If we are given an arbitrary simple graph, it is difficult in general to obtain the chromatic function by inspection. The following theorem and corollary give us a systematic method for obtaining the chromatic function of a simple graph in terms of the chromatic functions of null graphs.

THEOREM 21A. *Let G be a simple graph, and let G_1 and G_2 be the graphs obtained from G by deleting and contracting an edge e. Then*

$$P_G(k) = P_{G1}(k) - P_{G2}(k).$$

(As an illustration of this theorem, let G be the graph shown in Fig. 21.2; the corresponding graphs G_1 and G_2 are shown in Fig. 21.3, and the theorem states that

$$k(k-1)(k-2)(k-3) = k(k-1)(k-2)^2 - k(k-1)(k-2).)$$

Proof. Let $e = vw$. The number of k-colourings of G_1 in which v and w have *different* colours is unchanged if the edge e is drawn joining v and w, and is therefore equal to $P_G(k)$. Similarly, the number of k-colourings of G_1 in which v and w have *the same* colour is unchanged if v and w are identified, and is therefore equal to $P_{G_2}(k)$. The total number $P_{G_1}(k)$ of k-colourings of G_1 is therefore $P_G(k) + P_{G_2}(k)$, as required.//

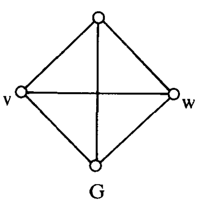


Fig. 21.2

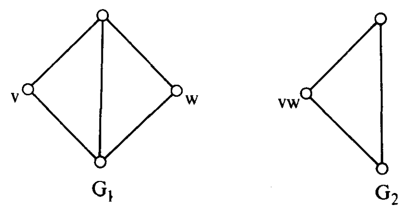


Fig. 21.3

COROLLARY 21B. *The chromatic function of a simple graph is a polynomial.*

Proof. The procedure described in the above theorem may be repeated by choosing edges in G_1 and G_2 and deleting and contracting them in the manner described above, the result being four new graphs. We now repeat the above procedure for these new graphs, and so on. The process terminates when no edges remain—in other words, when each graph is a null graph. Since the chromatic function of a null graph is a polynomial ($= k^r$, where r is the number of vertices), it follows by repeated application of Theorem 21A that the chromatic function of the graph G must be a sum of polynomials and so must itself be a polynomial.//

A worked example to illustrate the procedure just described will be given later in the section. In practice, it is unnecessary to reduce each graph to a null graph—it is enough to reduce each graph to graphs whose chromatic functions you already know, such as trees.

In the light of Corollary 21B, we can now call $P_G(k)$ the **chromatic polynomial** of G. It is easy to see from the proof just given that if G has n

vertices, then $P_G(k)$ is of degree n, since no new vertices are introduced at any stage. Moreover, since the construction yields only one null graph on n vertices, the coefficient of k^n is one. It can also be shown (see exercise 21f) that the coefficient of k^{n-1} is $-m$ where m is the number of edges of G, and that the coefficients alternate in sign. If there are no colours available, then we cannot colour the graph and so the constant term of the chromatic polynomial must be zero.

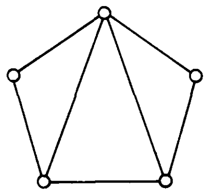


Fig. 21.4

It is high time that we gave an example to illustrate the above ideas. We shall use Theorem 21A to find the chromatic polynomial of the graph G shown in Fig. 21.4 and shall then verify that this polynomial has the form $k^5 - 7k^4 + ak^3 - bk^2 + ck$ (a, b, c positive constants) as the previous paragraph tells us that it must. It is customary at each stage to draw the graph itself, rather than write its chromatic polynomial; for example, instead of writing $P_G(k) = P_{G_1}(k) - P_{G_2}(k)$, where G, G_1 and G_2 denote the graphs of Figs 21.2 and 21.3, it is convenient to write down the 'equation' given in Fig. 21.5.

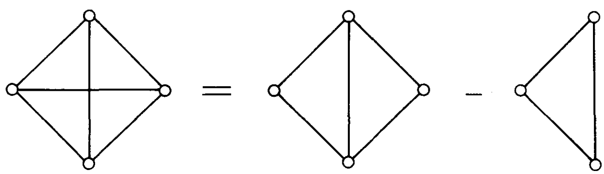
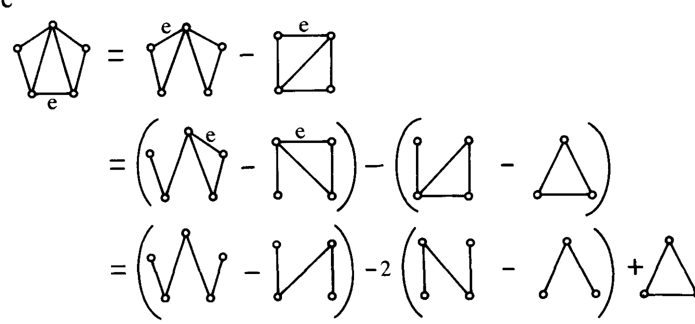


Fig. 21.5

With this convention, and ignoring multiple edges as we proceed, we have



Thus

$$P_G(k) = k(k-1)^4 - 3k(k-1)^3 + 2k(k-1)^2 + k(k-1)(k-2)$$
$$= k^5 - 7k^4 + 18k^3 - 20k^2 + 8k.$$

Note that this result has the required form $k^5 - 7k^4 + ak^3 - bk^2 + ck$, where a, b, c are positive constants.

We conclude this chapter with a few remarks to indicate how a study of chromatic polynomials and colourability is related to such subjects as timetabling. Suppose, for example, that we have to arrange the times at which certain lectures are to be given, knowing that some particular lectures cannot be given at the same time (since there may be students who wish to attend both of them); our aim is to find out whether it is possible to construct a timetable which takes account of this. This is done by constructing a graph whose vertices denote the various lectures and whose edges join those pairs of lectures which cannot be scheduled for the same time. If to each time available for lectures we associate a colour, then a colouring of the vertices of the graph corresponds to a successful scheduling of all the lectures—that is, to a timetable. In this case, a knowledge of the chromatic polynomial of the graph will tell us whether the scheduling is possible, and if so, how many possible ways there are of doing it.

Exercises 21

(21a) Write down the chromatic polynomials of
  (*i*) the complete graph K_6;
  (*ii*) the star graph $K_{1,5}$;
  (*iii*) the path graph P_6.
  In how many ways can these graphs be coloured with 7 colours?

(21b) (*i*) Find the chromatic polynomials of the six connected simple graphs on four vertices.
  (*ii*) Verify that each of the polynomials in part (*i*) has the form

$$k^4 - mk^3 + ak^2 - bk,$$

  where m is the number of edges and a and b are positive constants.

(21c) Find the chromatic polynomials of
  (*i*) the complete bipartite graph $K_{2,5}$;
  (*ii*) the circuit graph C_5.

(*21d) (*i*) Prove that the chromatic polynomial of $K_{2,s}$ is

$$k(k-1)^s + k(k-1)(k-2)^s.$$

  (*ii*) Prove that the chromatic polynomial of C_n is

$$(k-1)^n + (-1)^n(k-1).$$

(21e) Prove that if G is a disconnected simple graph, then P_G is the product of the chromatic polynomials of its components. What can you say about the degree of the lowest non-vanishing term?

(*21f) Let G be a simple graph with n vertices and m edges. Use induction on m, together with Theorem 21A, to prove that

(*i*) the coefficient of k^{n-1} is $-m$;

(*ii*) the coefficients of $P_G(k)$ alternate in sign.

(21g) (*i*) Use the results of exercises 21e and 21f to prove that if $P_G(k) = k(k-1)^{n-1}$, then G must be a tree on n vertices.

(*ii*) Find three graphs with chromatic polynomial

$$k^5 - 4k^4 + 6k^3 - 4k^2 + k.$$

7

Digraphs

By indirections find directions out.

William Shakespeare (*Hamlet*)

This chapter and the following one deal with the theory of digraphs and some of its applications. We begin in §**22** with the basic definitions, and then discuss under what conditions one can 'direct' the edges of a graph in such a way that the resulting digraph is strongly-connected. This is followed by a brief discussion of critical path analysis, and then, in §**23**, by a discussion of Eulerian and Hamiltonian trails and circuits, with particular reference to tournaments. We conclude the chapter with a study of the classification of states of a Markov chain from a digraph point of view.

§22. Definitions

We begin by recalling some of the definitions of §**2**. A **digraph** D is defined to be a pair $(V(D), A(D))$, where $V(D)$ is a non-empty finite set of elements called **vertices**, and $A(D)$ is a finite family of ordered pairs of elements of $V(D)$ called **arcs**; $V(D)$ and $A(D)$ are called the **vertex-set** and **arc-family** of D. Thus Fig. 22.1 represents a digraph whose arcs are uv, vv, vw, vw, wv, wu and zw, the ordering of the vertices in an arc being indicated by an arrow. If D is a digraph, the graph obtained from D by 'removing the arrows' (i.e., by replacing each *arc* of the form vw by a corresponding *edge* vw) is called the **underlying graph** of D (see Fig. 22.2).

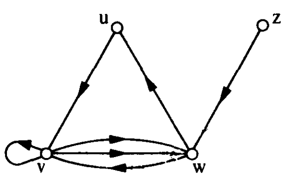


Fig. 22.1

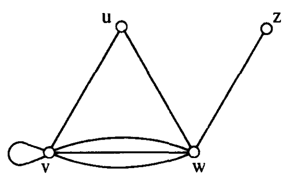


Fig. 22.2

We also say that D is a **simple digraph** if the arcs of D are all distinct, and if there are no 'loops' (arcs of the form vv). Note that the underlying graph of a simple digraph need not be a simple graph (see Fig. 1.8).

We can imitate many of the definitions given in §2 for graphs. For example, two vertices v and w of a digraph D are said to be **adjacent** if there is an arc in $A(D)$ of the form vw or wv; the vertices v and w are then said to be **incident** to any such arc. If D has vertex-set $\{v_1, \ldots, v_n\}$, the **adjacency matrix** of D is the $n \times n$ matrix $\mathbf{A} = (a_{ij})$, where a_{ij} is the number of arcs from v_i to v_j. Two digraphs are **isomorphic** if there is an isomorphism between their underlying graphs which preserves the ordering of the vertices in each arc. Note, in particular, that the digraphs shown in Figs 2.3 and 22.1 are not isomorphic.

There are also natural generalizations to digraphs of some of the definitions given in §5. A **walk** in a digraph D is a finite sequence of arcs of the form $v_0v_1, v_1v_2, \ldots, v_{m-1}v_m$. We shall sometimes write this sequence as $v_0 \to v_1 \to \ldots \to v_m$, and speak of a **walk from** $\mathbf{v_0}$ **to** $\mathbf{v_m}$. In an analogous way we can define directed trails, directed paths and directed circuits or, simply, trails, paths and circuits, if there is no possibility of confusion. Note that although a trail cannot contain a given arc vw more than once, it can contain both vw and wv; for example, in Fig. 22.1, $z \to w \to v \to w \to u$ is a trail.

We are now in a position to define connectedness. More precisely, we shall define here the two most natural and useful types of connected digraph, corresponding to whether or not we wish to take account of the direction of the arcs. These definitions are the natural extensions to digraphs of the definitions of connectedness given in §3 and §5.

A digraph D is said to be **connected** (or **weakly-connected**) if it cannot be expressed as the union of two disjoint digraphs, defined in the obvious way; this is equivalent to saying that the underlying graph of D is a connected graph. Suppose, in addition, that for any two vertices v and w of D there is a path from v to w; then D is called **strongly-connected**. It is clear that every strongly-connected digraph is connected, but the converse is not true—Fig. 22.1 shows a connected digraph which is not strongly-connected since there is no path from v to z.

The distinction between a connected digraph and a strongly-connected one may become clearer if we consider the road map of a city, all of whose streets are one-way. To say that the road map is connected is to say that we can drive from any part of the city to any other, ignoring the direction of the one-way streets as we go. If the map is strongly-connected, then we can drive from any part of the city to any other, always going the 'right way' down the one-way streets.

It is clearly important that a one-way system should be strongly-connected, and a natural question to ask is, 'when can we impose a one-way system on a street map in such a way that we can drive from any part of the city to any other?' If, for example, the city consists of two parts connected only by a bridge, then we can never impose such a one-way system on the city, since whatever direction we give to the bridge, one part of the city will be cut off. (Note that this includes the case in which we have a cul-de-sac.) If, on the other hand, there are no bridges, then we can always impose such a one-way system; this result will be stated formally in Theorem 22A.

For convenience, we shall define a graph G to be **orientable** if every edge of G (regarded as a pair of vertices) can be ordered in such a way that the resulting digraph is strongly-connected. This process of ordering the edges will be described as 'orienting the graph' or 'directing the edges'. For example, if G is the graph shown in Fig. 22.3, then G can be oriented to give the strongly-connected digraph of Fig. 22.4.

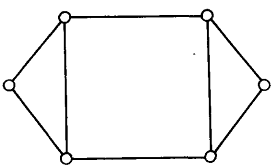


Fig. 22.3

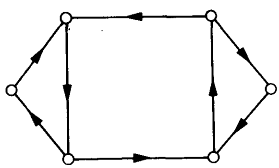


Fig. 22.4

It is easy to see that any Eulerian graph is orientable, since we merely follow any Eulerian trail directing the edges in the direction of the trail as we go. We now give a necessary and sufficient condition (due to H. E. Robbins) for a graph to be orientable.

THEOREM 22A. *Let G be a connected graph. Then G is orientable if and only if each edge of G is contained in at least one circuit.*

Proof. The necessity of the condition is clear. To prove the sufficiency, we choose any circuit C and orient its edges cyclically (in either of the two possible ways). If every edge of G is contained in C, the proof is complete. If not, we choose any edge e which is not in C but which is adjacent to an edge of C. By hypothesis, the edge e is contained in some circuit C' (say) whose edges we may direct cyclically (with the exception of those edges which have already been directed—i.e., those edges of C' which lie also in C). It is not difficult to convince oneself that the resulting digraph is strongly-connected; the situation is illustrated in Fig. 22.5, dashed lines denoting edges of C'. We proceed in this way, at each stage directing at least one new edge, until the whole graph is oriented. Since, at each stage, the digraph remains strongly-connected, the result follows.//

We conclude this section by discussing a 'critical path' problem relating to the scheduling of a series of operations. Suppose that we have

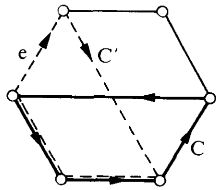


Fig. 22.5

a job to perform (such as the building of a house), and that this job can be divided into a number of smaller operations (such as laying the foundations, putting on the roof, doing the wiring, etc.). Since several of these operations can be performed simultaneously, whereas some may need to be completed before others can be started, it would clearly be useful if we could find an efficient method for determining which jobs should be performed at which times so that the entire job is completed in minimum time.

In order to solve this problem, we construct a 'weighted digraph' (or **network**, as we shall usually call it) in which the arcs represent the length of time taken for each operation. Such a network is given in Fig. 22.6, where the vertex A represents the beginning of the job, and the vertex L represents its completion. Since the entire job cannot be completed until each of the paths from A to L has been traversed, the problem reduces to

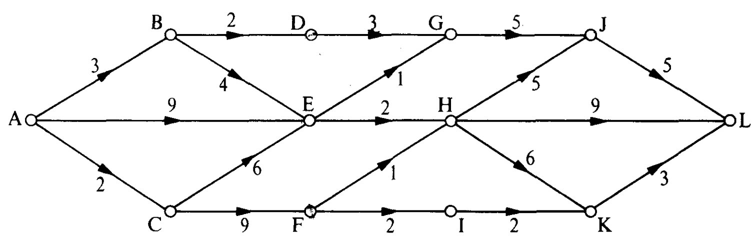


Fig. 22.6

that of finding the longest path from A to L. This may be accomplished by using a technique known as PERT (Programme Evaluation and Review Technique), which is very similar to that used for the shortest path problem (§**8**), except that as we move across the digraph from left to right, we associate to each vertex V a number $l(V)$ indicating the length of the *longest* path from A to V. So for the digraph of Fig. 22.6, we assign:

to vertex A, the number 0;
to vertex B, the number $l(A)+3$—that is, 3;
to vertex C, the number $l(A)+2$—that is, 2;
to vertex D, the number $l(B)+2$—that is, 5;

to vertex E, the number $\max\{l(A)+9, l(B)+4, l(C)+6\}$—that is, 9;
to vertex F, the number $l(C)+9$—that is, 11;
to vertex G, the number $\max\{l(D)+3, l(E)+1\}$—that is, 10;
to vertex H, the number $\max\{l(E)+2, l(F)+1\}$—that is, 12;
to vertex I, the number $l(F)+2$—that is, 13;
to vertex J, the number $\max\{l(G)+5, l(H)+5\}$—that is, 17;
to vertex K, the number $\max\{l(H)+6, l(I)+2\}$—that is, 18;
to vertex L, the number $\max\{l(H)+9, l(J)+5, l(K)+3\}$—that is, 22.

(As in the shortest path problem, we can keep track of these numbers by writing them next to the vertex they represent.) Note that, unlike the problem we considered in §8, there is no 'zig-zagging', since the arcs are all directed from left to right. It follows that the longest path (the so-called **critical path**) has length 22, and is given by Fig. 22.7. The job cannot therefore be completed until time 22.

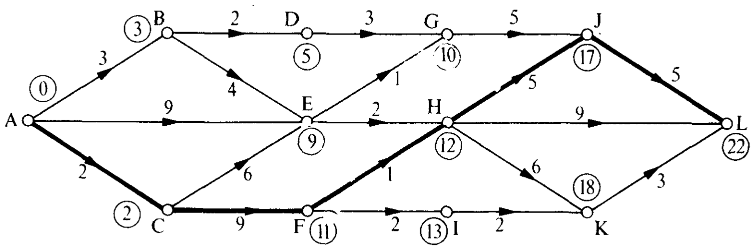


Fig. 22.7

We can now calculate the latest time by which any given operation must be completed if the work is not to be delayed.

Working backwards from L, we see that we must reach K by time $22-3=19$, J by time $22-5=17$, H by time $\min\{17-5, 22-9, 19-6\}=12$, and so on. Given this information, the required schedule can then be worked out.

Exercises 22

(22a) Two of the digraphs in Fig. 22.8 are isomorphic. Which two are they?

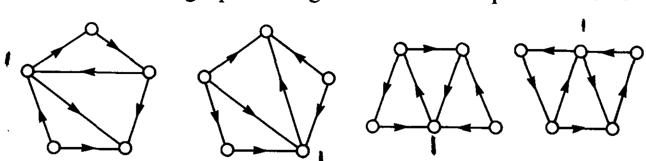


Fig. 22.8

(22b) Let D be a simple digraph with n vertices and m arcs.
(*i*) If D is connected, prove that $n-1 \leqslant m \leqslant n(n-1)$.
(*ii*) Obtain corresponding bounds for m if D is strongly-connected.

(22c) Write down adjacency matrices for the digraphs in Figs 22.1 and 22.4.

(22d) The **converse** $\tilde{D}$ of a digraph D is obtained by reversing the direction of every arc of D.

 (*i*) Give an example of a digraph which is isomorphic to its converse.

 (*ii*) What is the connexion between the adjacency matrices of D and $\tilde{D}$?

(22e) (*i*) Without using Theorem 22A, prove that every Hamiltonian graph is orientable.

 (*ii*) Show, by finding an orientation for each, that K_n $(n \geqslant 3)$ and $K_{r,s}$ $(r, s \geqslant 2)$ are orientable.

 (*iii*) Find orientations for the Petersen graph and the graph of the dodecahedron.

(22f) In the above scheduling problem, calculate the latest times at which we can reach the vertices G, E and B.

(22g) Find the longest path from A to G in the network of Fig. 22.9.

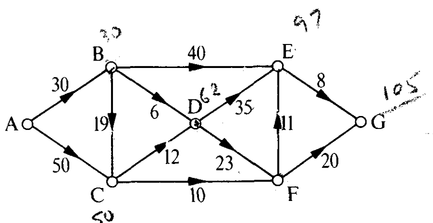


Fig. 22.9

§23. Eulerian digraphs and tournaments

In this section we shall attempt to obtain digraph analogues of some of the results of §§**6** and **7**. This will lead us to the study of Hamiltonian circuits in a particular type of digraph called a tournament.

A connected digraph D is called **Eulerian** if there exists a closed trail which includes every arc of D; such a trail is called an **Eulerian trail**. For example, the digraph shown in Fig. 23.1 is not Eulerian, although its underlying graph is an Eulerian graph. Our first aim is to give a necessary and sufficient condition (analogous to the one given in Theorem 6B) for a connected digraph to be Eulerian. It is easy to see that one necessary condition is that the digraph is strongly-connected.

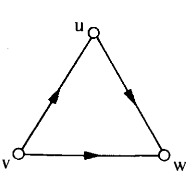


Fig. 23.1

We shall need some preliminary definitions. If v is a vertex of a digraph D, we define the **out-degree** of v (denoted by $\overleftarrow{\rho}(v)$, with the arrow 'pointing away from' v) to be the number of arcs of D of the form vw; similarly, the **in-degree** of v (denoted by $\overrightarrow{\rho}(v)$) is the number of arcs of D of the form wv. It follows immediately that the sum of the in-degrees of all the vertices of D is equal to the sum of their out-degrees, since each arc of D contributes exactly one to each sum; we shall call this result the **handshaking di-lemma**!

For later convenience, we further define a **source** of D to be a vertex whose in-degree is zero, and a **sink** of D to be one whose out-degree is zero; thus, in Fig. 23.1, v is a source and w is a sink. Note that an Eulerian digraph (other than the trivial one containing no arcs) can contain no sources or sinks.

We are now in a position to state the basic theorem on Eulerian digraphs.

THEOREM 23A. *A connected digraph is Eulerian if and only if* $\overrightarrow{\rho}(v) = \overleftarrow{\rho}(v)$ *for each vertex v of D.*

Proof. The proof is entirely analogous to the proof of Theorem 6B and will be left as an exercise.//

We shall leave it to you to define a semi-Eulerian digraph, and to prove results analogous to Corollaries 6C and 6D.

The corresponding study of Hamiltonian digraphs is, as may be expected, rather less successful than the Eulerian case. A digraph D is called **Hamiltonian** if there is a circuit which includes every vertex of D; a digraph which contains a path passing through every vertex is called **semi-Hamiltonian.** Very little is known about Hamiltonian digraphs, and in fact some theorems on Hamiltonian graphs do not seem to generalize easily (if at all) to digraphs. It is natural to ask whether there is a generalization to digraphs of Dirac's theorem (Corollary 7B). One such generalization is due to Ghouila–Houri; its proof is considerably more difficult than that of Dirac's theorem, and may be found in Bondy and Murty.[7]

THEOREM 23B. *Let D be a strongly-connected digraph with n vertices. If $\overrightarrow{\rho}(v) \geqq \frac{1}{2}n$ and $\overleftarrow{\rho}(v) \geqq \frac{1}{2}n$ for each vertex v, then D is Hamiltonian.*//

It seems that results in this direction are not going to come very easily, and so we might consider instead what kinds of digraphs are Hamiltonian. In this respect, certain digraphs are particularly important—namely, the *tournaments*—the results in this case taking a very simple form.

A **tournament** is a digraph in which any two vertices are joined by exactly one arc (see Fig. 23.2). The reason for the name 'tournament' is that the digraph can be used to record the result of a tennis tournament or any other game in which draws are not allowed. In Fig. 23.2, for example, team z beat team w, but was beaten by team v, etc.

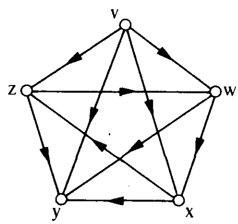


Fig. 23.2

Because of the possibility that a tournament has a source or a sink, tournaments are not in general Hamiltonian. However, the following theorem (due to L. Rédei and P. Camion) shows that every tournament is 'nearly Hamiltonian'.

THEOREM 23C. *(i) Every tournament is semi-Hamiltonian; (ii) every strongly-connected tournament is Hamiltonian.*

Proof. (*i*) The statement is clearly true if the tournament has less than four vertices. We shall prove the result by induction on the number of vertices, and will assume that every tournament on n vertices is semi-Hamiltonian. Let T be a tournament on $n+1$ vertices, and let T' be the tournament on n vertices obtained by removing from T a vertex v and every arc incident to v. Then, by the induction hypothesis, T' has a semi-Hamiltonian path $v_1 \to v_2 \to \dots \to v_n$. There are three cases to consider:

(*1*) if vv_1 is an arc in T, then the required path is

$v \to v_1 \to v_2 \to \dots \to v_n$.

(*2*) if vv_1 is not an arc in T (which means that $v_1 v$ is) and if there exists an i such that vv_i is an arc in T, then choosing i to be the first such, it is clear that the required path is (see Fig. 23.3)

$v_1 \to v_2 \to \dots \to v_{i-1} \to v \to v_i \to \dots \to v_n$.

(*3*) if there is no arc in T of the form vv_i, then the required path is

$v_1 \to v_2 \to \dots \to v_n \to v$.

(*ii*) We shall prove the stronger result that a strongly-connected tournament T on n vertices contains circuits of length $3, 4, \dots, n$.

To show that T contains a circuit of length three, let v be any vertex of T, and let W be the set of all vertices w such that vw is an arc in T, and Z be the set of all vertices z such that zv is an arc. Since T is strongly-connected, W and Z must both be non-empty, and there must be an arc

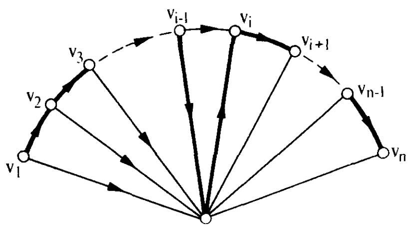


Fig. 23.3

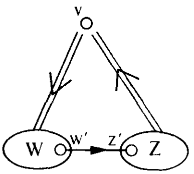


Fig. 23.4

in T of the form $w'z'$ where w' is in W and z' is in Z (see Fig. 23.4). The required circuit of length three is then $v \rightarrow w' \rightarrow z' \rightarrow v$.

It remains only to show that if there is a circuit of length $k(k<n)$, then there is one of length $k+1$. Let $v_1 \rightarrow \ldots \rightarrow v_k \rightarrow v_1$ be such a circuit. Suppose first that there exists a vertex v not contained in this circuit, with the property that there exist arcs in T of the form vv_i and of the form v_jv. Then there must be a vertex v_i such that both $v_{i-1}v$ and vv_i are arcs in T. The required circuit is then

$$v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_{i-1} \rightarrow v \rightarrow v_i \rightarrow \ldots \rightarrow v_k \rightarrow v_1 \text{ (see Fig. 23.5).}$$

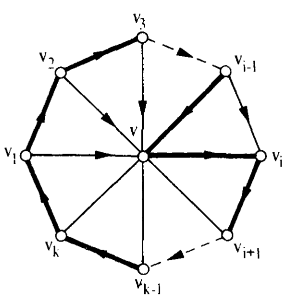


Fig. 23.5

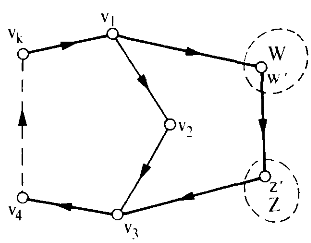


Fig. 23.6

If no vertex exists with the above-mentioned property, then the set of vertices not contained in the circuit may be divided into two disjoint sets W and Z, where W is the set of vertices w such that v_iw is an arc for each i, and Z is the set of vertices z such that zv_i is an arc for each i. Since T is strongly-connected, W and Z must both be non-empty, and there must be an arc in T of the form $w'z'$ where w' is in W and z' is in Z. The required circuit is then

$$v_1 \rightarrow w' \rightarrow z' \rightarrow v_3 \rightarrow \ldots \rightarrow v_k \rightarrow v_1 \text{ (see Fig. 23.6).//}$$

Exercises 23

(23a) Verify the handshaking dilemma for the digraph of Fig. 22.1 and the tournament of Fig. 23.2.

(23b) In the tournament of Fig. 23.7, find
  (*i*) circuits of length 3, 4 and 5;
  (*ii*) an Eulerian trail;
  (*iii*) a Hamiltonian circuit.

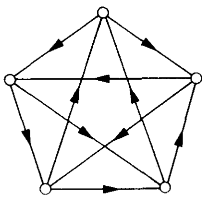


Fig. 23.7

(23c) Prove that a tournament cannot have more than one source or more than one sink.

(23d) Let T be a tournament on n vertices. If Σ denotes a summation over all the vertices of T, prove that

(*i*) $\Sigma\vec{\rho}(v)=\Sigma\overleftarrow{\rho}(v)$; (*ii*) $\Sigma\vec{\rho}(v)^2=\Sigma\overleftarrow{\rho}(v)^2$.

(23e) Let D be the digraph whose vertices are the pairs of integers 11, 12, 13, 21, 22, 23, 31, 32, 33, and whose arcs join ij to kl if and only if $j=k$. Find an Eulerian trail in D and use it to obtain a circular arrangement of nine 1s, nine 2s and nine 3s in which each of the 27 possible triples (111, 233, etc.) occurs exactly once.
(Problems of this kind arise in communication theory.)

(23f) A tournament T is called **irreducible** if it is impossible to split the set of vertices of T into two disjoint sets V_1 and V_2 in such a way that every arc *joining a vertex of* V_1 and a vertex of V_2 is directed from V_1 to V_2.
  (*i*) Give an example of an irreducible tournament.
  (*ii*) Show that a tournament is irreducible if and only if it is strongly-connected.

(23g) A tournament is called **transitive** if the existence of arcs uv and vw implies the existence of the arc uw.
  (*i*) Give an example of a transitive tournament.
  (*ii*) Show that in a transitive tournament the teams can be ranked so that each team beats all the teams which follow it in the ranking.
  (*iii*) Deduce that a transitive tournament with at least two vertices cannot be strongly-connected.

(*23h) The **score** of a vertex of a tournament is its out-degree; the **score-sequence** of a tournament is the sequence formed by arranging the scores of its vertices in non-decreasing order (so that, for example, the score-sequence of the tournament in Fig. 23.2 is (0, 2, 2, 2, 4)). Show that if $(s_1, \ldots, s_n)$ is the score-sequence of a tournament T, then
  (*i*) $s_1+\ldots+s_n=\frac{1}{2}n(n-1)$;
  (*ii*) for any positive integer $k>n$, $s_1+\ldots+s_k\geqslant\frac{1}{2}k(k-1)$, with strict inequality for all k if and only if T is strongly-connected;
  (*iii*) T is transitive if and only if $s_k=k-1$ for each k.

§24. Markov chains

★As the reader has already seen, digraphs turn up in a variety of 'real-life' situations. In this section we describe a not very deep but nonetheless instructive application of digraph theory to the study of finite Markov chains. Another application—the study of flows in networks—will be discussed in the next chapter. The reader who is interested in further applications is referred to Deo,[11] or Wilson and Beineke.[26]

The study of Markov chains has arisen in a wide variety of areas ranging from genetics and statistics to computing and sociology. However, for ease of presentation we shall consider a much more trivial problem, that of the drunkard who is standing directly between his two favourite pubs, 'The Markov Chain' and 'The Source and Sink' (see Fig. 24.1). Every minute he either staggers ten metres towards the first pub

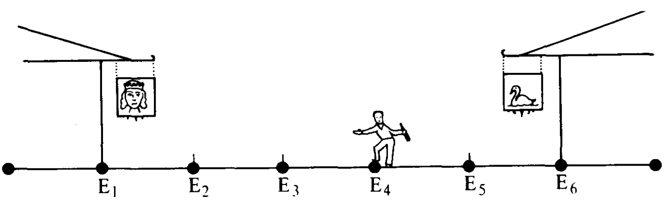


Fig. 24.1

(with probability $\frac{1}{2}$) or towards the second pub (with probability $\frac{1}{3}$) or else he stays where he is (with probability $\frac{1}{6}$)—such a procedure is called a one-dimensional **random walk**. We shall assume also that the two pubs are 'absorbing' in the sense that if he arrives at either of them he stays there. Given the distance between the two pubs and his initial position, there are several questions we can ask. For example, we can ask which pub he is more likely to end up at, and how long he is likely to take getting there.

In order to study the problem of the drunkard in more detail, let us suppose that the two pubs are fifty metres apart and that our friend is initially twenty metres from 'The Source and Sink'. If we denote the various places at which he can stop by $E_1, \ldots, E_6$, where E_1 and E_6 denote the two pubs, then his initial position E_4 can be described by the vector $\mathbf{x} = (0, 0, 0, 1, 0, 0)$, in which the i-th component is the probability that he is initially at E_i. Furthermore, the probabilities of his position after one minute are given by the vector $(0, 0, \frac{1}{2}, \frac{1}{6}, \frac{1}{3}, 0)$, and after two minutes by $(0, \frac{1}{4}, \frac{1}{6}, \frac{13}{36}, \frac{1}{9}, \frac{1}{9})$. It is clearly going to be awkward to calculate directly the probability of his being at a given place after k minutes, and it turns out that the most convenient way of doing this is to introduce the transition matrix.

Let p_{ij} be the probability that he moves from E_i to E_j in one minute; then, for example, $p_{23} = \frac{1}{3}$ and $p_{24} = 0$. These probabilities p_{ij} are called the **transition probabilities**, and the 6×6 matrix $\boldsymbol{P} = (p_{ij})$ is known as the **transition matrix** (see Fig. 24.2); note that every entry of $\boldsymbol{P}$ is non-negative and that the sum of the entries in any row is one. It now follows that if $\mathbf{x}$ is the initial row vector defined above, then the probabilities of his position after one minute are given by the row vector $\mathbf{x}\boldsymbol{P}$, and after k minutes by the vector $\mathbf{x}\boldsymbol{P}^k$. In other words, the i-th component of $\mathbf{x}\boldsymbol{P}^k$ represents the probability that he is at E_i after k minutes have elapsed.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Fig. 24.2

We can generalize these ideas somewhat by defining a **probability vector** to be a row vector whose entries are all non-negative and have unit sum. A **transition matrix** is then defined to be a square matrix, each of whose rows is a probability vector. We can now define a **finite Markov chain** (or simply, a **chain**) as a pair $(\boldsymbol{P},\mathbf{x})$ where $\boldsymbol{P}$ is an $n \times n$ transition matrix and $\mathbf{x}$ is a $1 \times n$ row vector. If every entry p_{ij} of $\boldsymbol{P}$ is regarded as the (transition) probability of getting from a position E_i to a position E_j, and $\mathbf{x}$ is regarded as an initial probability vector, then this definition ties up with the classical definition of a finite discrete stationary Markov chain to be found in books on probability (see, for example, Feller[12]). The positions E_i are usually referred to as the **states** of the chain, and the aim of this section is to describe various ways of classifying them.

For the remainder of this section we shall be primarily concerned with whether or not we can get from a given state to another state, and if so, what is the shortest time in which this can be done. (For example, in the problem of the drunkard, we can get from E_4 to E_1 in three minutes, but it is impossible to get from E_1 to E_4.) It follows that we shall be primarily concerned not with the actual probabilities p_{ij} but with whether or not they are positive, and it is at least reasonable to hope that we may be able to represent the whole set-up by a digraph in which the vertices correspond to the states and in which the arcs tell us whether we can go from one state to another in one minute. More precisely, if each state E_i is represented by a corresponding vertex v_i, then the required digraph is obtained by drawing an arc from v_i to v_j if and only if $p_{ij} \neq 0$; alternatively, the digraph may be defined in terms of its adjacency matrix by replacing each non-zero entry of the matrix $\boldsymbol{P}$ by one. We shall refer to this digraph as the **associated digraph** of the Markov chain; the

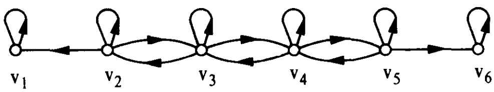


Fig. 24.3

associated digraph of the one-dimensional random walk is shown in Fig. 24.3. As a further example, if we are given a chain whose transition matrix is the matrix of Fig. 24.4, then its associated digraph is as shown in Fig. 24.5.

It is now clear that we can get from a state E_i to a state E_j in a Markov chain if and only if there is a path from v_i to v_j in the associated digraph, and the least possible time taken is then the length of the shortest such path. A Markov chain in which we can get from any state to any other is called an **irreducible chain.** Clearly a Markov chain is irreducible if and only if its associated digraph is strongly-connected. Note that neither of the chains described above is irreducible.

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{12} & 0 & \frac{1}{12} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Fig. 24.4

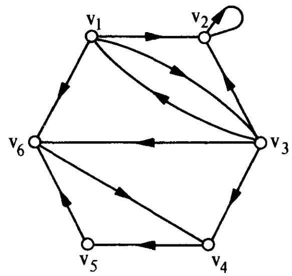


Fig. 24.5

In investigating these matters further, it is usual to make a distinction between those states to which we keep on returning however long we continue, and those which we visit a few times and then never return to. More formally, if on starting at E_i the probability of returning to E_i at some later stage is 1, then E_i is called a **persistent** (or **recurrent) state**; otherwise E_i is called **transient**. For example, in the problem of the drunkard, E_1 and E_6 are trivially persistent, whereas the other states are transient. In more complicated examples, the calculation of the relevant probabilities can become very tricky, and it is often easier to classify the states by analysing the associated digraph of the chain. It is not difficult to see that a state E_i is persistent if and only if the existence of a path from v_i to v_j in the associated digraph implies the existence of a path from v_j to v_i. In Fig. 24.5 there is a path from v_1 to v_4 but no path from v_4 to v_1. It follows that E_1 is transient, and similarly so is E_3; E_2, E_4, E_5 and E_6 are persistent. A state (such as E_2) from which we can get to no other state is called an **absorbing state**.

An alternative way of classifying states is in terms of their periodicity. A state E_i of a Markov chain is called **periodic of period t** ($t \neq 1$) if it is possible to return to E_i only after a period of time which is a multiple of t; if no such t exists, then E_i is called **aperiodic**. Clearly every state E_i for which $p_{ii} \neq 0$ is aperiodic; it follows that every absorbing state is aperiodic. In the problem of the drunkard the absorbing states E_1 and E_6 are not the only aperiodic states—in fact, every state is aperiodic. On the other hand, in the second example, the absorbing state E_2 is the only aperiodic state, since E_1 and E_3 are periodic of period two and E_4, E_5 and E_6 are periodic of period three. In digraph terms, it is easy to see that a state E_i is periodic of period t if and only if in the associated digraph the length of every closed trail containing v_i is a multiple of t.

Finally, for the sake of completeness, we shall call a state of a finite Markov chain an **ergodic** state if it is both persistent and aperiodic, and if every state is ergodic then we shall call the chain an **ergodic chain**. For many purposes ergodic chains are the most important and desirable chains to deal with. An example of such a chain will be given in exercise 24b.

Exercises 24

(24a) (*i*) Suppose that, in the problem of the drunkard, the right-hand pub ejects him as soon as he gets there. Write down the resulting transition matrix and its associated digraph, and re-classify the states.

(*ii*) How would your answers to part (*i*) be changed if both pubs eject him?

(24b) A game is played with a die by 5 people around a circular table. If the player with the die throws an odd number, he passes the die to the player on his left; if he throws a 2 or 4, he passes it two places to his right; if he throws a 6, he keeps the die and throws again.

(*i*) Write down the corresponding transition matrix and its associated digraph.

(*ii*) Show that each state is persistent and aperiodic, and deduce that the corresponding Markov chain is ergodic.

(24c) (*i*) Prove that, if **P** and **Q** are transition matrices, then so is **PQ**.

(*ii*) What is the connection between the associated digraphs of **P** and **Q** and that of **PQ**?

(*24d) (*i*) Prove that every finite Markov chain has at least one persistent state.

(*ii*) Deduce that if a finite Markov chain is irreducible then every state is persistent.

(*iii*) Show how infinite Markov chains can be defined, and construct one in which every state is transient.★

8
Matching, marriage and Menger's theorem

They drew all manner of things—everything that begins with an M—.

Lewis Carroll

The results of this chapter are more combinatorial in nature than those of the preceding chapters, although we shall see that they are in fact very closely connected with graph theory. We begin with a discussion of Philip Hall's well-known 'marriage' theorem in several different contexts, including some of its applications to such topics as the construction of latin squares and timetabling problems. This is followed in §**28** by a theorem due to Menger on the number of disjoint paths connecting a given pair of vertices in a graph. In §**29** we present an alternative formulation of Menger's theorem, known as the max-flow min-cut theorem, which is of fundamental importance in connexion with network flows and transportation problems.

§25. Hall's 'marriage' theorem

The marriage theorem, proved in 1935 by Philip Hall, answers the following question, known as the **marriage problem**: *if we have a finite set of boys each of whom knows several girls, under what conditions can we marry off the boys in such a way that each boy marries a girl he knows?* For example, if there are four boys $\{b_1, b_2, b_3, b_4\}$ and five girls $\{g_1, g_2, g_3, g_4, g_5\}$, and the relationships are as shown in Fig. 25.1, then a possible solution is for b_1 to marry g_4, b_2 to marry g_1, b_3 to marry g_3, and b_4 to marry g_2.

boy	girls known by boy
b_1	g_1 g_4 g_5
b_2	g_1
b_3	g_2 g_3 g_4
b_4	g_2 g_4

Fig. 25.1

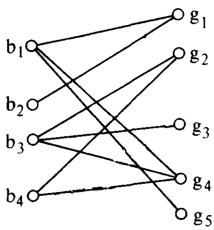


Fig. 25.2

This problem can be represented graphically by taking G to be the bipartite graph in which the vertex-set is divided into two disjoint sets V_1 and V_2 (corresponding to the boys and girls respectively) and in which every edge joins a boy to a girl he knows; Fig. 25.2 shows the graph G corresponding to the situation in Fig. 25.1.

A **complete matching** from V_1 to V_2 in a bipartite graph $G(V_1, V_2)$ is a one-one correspondence between the vertices in V_1 and a subset of the vertices in V_2, with the property that corresponding vertices are joined. It is clear that the marriage problem can be expressed in graph-theoretic terms in the form: *'if $G=G(V_1, V_2)$ is a bipartite graph, when does there exist a complete matching from V_1 to V_2 in G?'*

Returning to 'matrimonial terminology', it is clear that a necessary condition for the solution of the marriage problem is that every k boys know (collectively) at least k girls, for all integers k satisfying $1 \leqq k \leqq m$, where m denotes the total number of boys. We refer to this condition as the **marriage condition**. That it is a necessary condition follows immediately from the fact that if it were not true for a given set of k boys, then we could not marry off the boys in that set, let alone the others.

What is at first sight surprising is that the marriage condition also turns out to be sufficient. This is the content of **Hall's 'marriage' theorem**. Because of its importance we shall give three proofs, the first of which is due to Halmos and Vaughan.

THEOREM 25A (P. Hall 1935). *A necessary and sufficient condition for a solution of the marriage problem is that every set of k boys collectively know at least k girls* ($1 \leqq k \leqq m$).

Remark. Although this theorem is couched in the somewhat frivolous terms of the marriage problem, it applies equally well to more serious problems. For example, it gives a necessary and sufficient condition for the solution of the **personnel assignment problem** in which various applicants must be assigned to jobs for which they are variously qualified. A simple example of this problem is given in exercise 25b.

Proof. The condition is obviously necessary, as was pointed out above. To prove sufficiency, we shall use induction, and assume that the theorem is true if the number of boys is less than m. (The theorem is clearly true if $m=1$.) Suppose then that there are m boys; there are two cases to consider:

(*i*) Suppose first that every k boys (where $k<m$) collectively know at least $k+1$ girls (so that the condition is always true 'with one girl to spare'). Then if we take any boy and marry him to any girl he knows, the original condition remains true for the other $m-1$ boys. These $m-1$ boys can now be married off by induction, completing the proof in this case.

(*ii*) Suppose now that there is a set of k boys ($k<m$) who collectively know exactly k girls. Then these k boys can be married off by induction, leaving $m-k$ boys. But any collection of h of these $m-k$ boys ($h \leqq m-k$) must know at least h of the remaining girls, since otherwise these h boys together with the above collection of k boys would collectively know fewer than $h+k$ girls, contrary to our assumption. It follows that the original condition applies to the $m-k$ boys. They can therefore be married off by induction in such a way that everyone is happy and the proof is complete.//

We can also state Hall's theorem in the language of matchings in a bipartite graph; we remind you that the number of elements in a set S is denoted by $|S|$.

COROLLARY 25B. *Let $G=G(V_1, V_2)$ be a bipartite graph, and for every subset A of V_1, let $\varphi(A)$ be the set of those vertices of V_2 which are adjacent to at least one vertex in A. Then a complete matching from V_1 to V_2 exists if and only if $|A| \leqq |\varphi(A)|$ for each subset A of V_1.*

Proof. The proof of this corollary is simply a translation into graph-theoretic terminology of the above proof.//

Exercises 25

(25a) Suppose that three boys a, b, c know four girls w, x, y, z as in the following table:

boy	girls known by boy
a	w, y, z
b	x, z
c	x, y

(*i*) Draw the bipartite graph corresponding to this table of relationships.
(*ii*) Find five different solutions of the corresponding marriage problem.
(*iii*) Check the marriage condition for this problem.

(25b) A building contractor advertises for a bricklayer, a carpenter, a plumber and a toolmaker. He has five applicants—one for the job of bricklayer, one for that of carpenter, one for those of bricklayer and plumber, and two for those of plumber and toolmaker.
 (*i*) Draw the corresponding bipartite graph.
 (*ii*) Check whether the marriage condition holds for this problem. Can all of the jobs be filled by qualified people?

(25c) Explain why the graph in Fig. 25.3 has no complete matching from V_1 to V_2. When does the marriage condition fail?

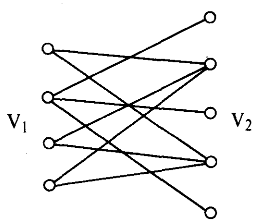


Fig. 25.3

(25d) (The 'harem problem'.) Let B be a set of boys, and suppose that each boy in B wishes to marry more than one of his girl friends. Find a necessary and sufficient condition for the harem problem to have a solution. (Hint: replace each boy by several identical copies of himself, and then use Hall's theorem.)

(25e) Prove that if $G = G(V_1, V_2)$ is a bipartite graph in which the degree of every vertex in V_1 is not less than the degree of every vertex in V_2, then G has a complete matching.

(*25f) (*i*) Use the marriage condition to show that if every boy has r ($\geqslant 1$) girl friends and every girl has r boy friends, then the marriage problem has a solution.
 (*ii*) Use the result of part (*i*) to prove that if G is a bipartite graph which is regular of degree r, then G has a complete matching. Deduce that the chromatic index of G is r.

(*25g) Suppose that the marriage condition is satisfied, and that each of the m boys knows at least t girls. Show, by induction on m, that the marriages can be arranged in at least $t!$ ways if $t \leq m$, and in at least $t!/(t-m)!$ ways if $t > m$.

§26. Transversal theory

This section is devoted to an alternative proof of Hall's theorem, given in the language of transversal theory. We shall leave the translation of this proof into matching or marriage terminology as an exercise.

You will remember that in our example in the previous section (see Fig. 25.1) the sets of girls known by the four boys were $\{g_1, g_4, g_5\}$, $\{g_1\}$,

$\{g_2, g_3, g_4\}$, $\{g_2, g_4\}$, and that a solution of the marriage problem was obtained by finding four distinct g's, one from each of these sets of girls. In general, if E is a non-empty finite set, and $\mathscr{S}=(S_1, \ldots, S_m)$ is a family of (not necessarily distinct) non-empty subsets of E, then a **transversal** (or **system of distinct representatives**) of $\mathscr{S}$ is a set of m distinct elements of E, one from each set S_i.

To take another example, suppose $E=\{1, 2, 3, 4, 5, 6\}$, and let $S_1=S_2=\{1, 2\}$, $S_3=S_4=\{2, 3\}$, $S_5=\{1, 4, 5, 6\}$. Then it is impossible to find five distinct elements of E, one from each subset S_i; in other words, the family $\mathscr{S}=(S_1, \ldots, S_5)$ has no transversal. Note, however, that the subfamily $\mathscr{S}'=(S_1, S_2, S_3, S_5)$ has a transversal—for example, $\{1, 2, 3, 4\}$. We call a transversal of a subfamily of $\mathscr{S}$ a **partial transversal** of $\mathscr{S}$; in this example $\mathscr{S}$ has several partial transversals ($\{1, 2, 3, 6\}$, $\{2, 3, 6\}$, $\{1, 5\}$, $\emptyset$, etc.). It is clear that any subset of a partial transversal is a partial transversal.

A natural question to ask is, 'under what conditions does a given family of subsets of a set have a transversal?' The connexion between this problem and the marriage problem is easily seen by taking E to be the set of girls, and S_i to be the set of girls known by boy b_i ($1 \leqq i \leqq m$). A transversal in this case is then simply a set of m girls, one corresponding to (and known by) each boy. It follows that Theorem 25A gives a necessary and sufficient condition for a given family of sets to have a transversal. We restate Hall's theorem in this form, and give an alternative proof due to R. Rado.

THEOREM 26A. *Let E be a non-empty finite set, and $\mathscr{S}=(S_1, \ldots, S_m)$ be a family of non-empty subsets of E. Then $\mathscr{S}$ has a transversal if and only if the union of any k of the subsets S_i contains at least k elements* ($1 \leqq k \leqq m$).

★*Proof.* The necessity of the condition is clear. To prove the sufficiency, we shall show that if one of the subsets (S_1, say) contains more than one element, then we can remove an element from S_1 without altering the condition. By repeating this procedure, we can eventually reduce the problem to the case in which each subset contains only one element, the proof then being trivial.

It remains only to show the validity of the 'reduction procedure'. So, suppose that S_1 contains elements x and y, the removal of either of which invalidates the condition. Then there are subsets A and B of $\{2, 3, \ldots, m\}$ with the property that $|P| \leqq |A|$ and $|Q| \leqq |B|$, where

$$P=\bigcup_{j\in A} S_j \cup (S_1-\{x\}) \quad \text{and} \quad Q=\bigcup_{j\in B} S_j \cup (S_1-\{y\}).$$

Then

$$|P \cup Q| = \Big|\bigcup_{j\in A\cup B} S_j \cup S_1\Big| \quad \text{and} \quad |P\cup Q| \geqslant \Big|\bigcup_{j\in A\cap B} S_j\Big|.$$

The required contradiction now follows since

$$\begin{aligned} |A|+|B| &\geqq |P|+|Q| \\ &= |P\cup Q|+|P\cap Q| \\ &\geqq \left|\bigcup_{j\in A\cup B} S_j \cup S_1\right| + \left|\bigcup_{j\in A\cap B} S_j\right| \\ &\geqq (|A\cup B|+1)+|A\cap B| \text{ (by Hall's condition)} \\ &= |A|+|B|+1.// \end{aligned}$$

The beauty of this proof lies in the fact that essentially only one step is involved, in contrast to the Halmos–Vaughan proof which involves the consideration of two separate cases. It is, however, more awkward to express this proof in the intuitive and appealing language of matrimony!★

Before proceeding to some applications of Hall's theorem, we shall find it convenient to state two corollaries; these will be needed in §**33**. In marriage terminology the first of these corollaries gives us a condition under which at least t boys can marry girls known to them.

COROLLARY 26B. *If E and $\mathscr{S}$ are as before, then $\mathscr{S}$ has a partial transversal of size t if and only if the union of any k of the subsets S_i contains at least $k+t-m$ elements.*

Sketch of proof. The result follows on applying Theorem 26A to the family $\mathscr{S}'=(S_1\cup D, \ldots, S_m\cup D)$, where D is any set disjoint from E and containing $m-t$ elements. Note that $\mathscr{S}$ has a partial transversal of size t if and only if $\mathscr{S}'$ has a transversal.//

COROLLARY 26C. *If E and $\mathscr{S}$ are as before, and if X is any subset of E, then X contains a partial transversal of $\mathscr{S}$ of size t if and only if, for every subset A of $\{1, \ldots, m\}$,*

$$\left|\left(\bigcup_{j\in A} S_j\right)\cap X\right| \geqq |A|+t-m.$$

Sketch of proof. The result follows on applying the previous corollary to the family $\mathscr{S}_X=(S_1\cap X, \ldots, S_m\cap X)$.//

Exercises 26

(26a) Decide which of the following families of subsets of $E=\{1, \ldots, 5\}$ have transversals, and list all the partial transversals of those which have no transversal:

(*i*) $(\{1\}, \{2, 3\}, \{1, 2\}, \{1, 3\}, \{1, 4, 5\})$;
(*ii*) $(\{1, 2\}, \{2, 3\}, \{4, 5\}, \{4, 5\})$;
(*iii*) $(\{1, 3\}, \{2, 3\}, \{1, 2\}, \{3\})$;
(*iv*) $(\{1, 3, 4\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\})$.

(26b) Repeat exercise 26a for the set $\{T, H, E, O, R, Y\}$ and the following families of subsets:
(*i*) $(\{T\}, \{H, T\}, \{E, O\}, \{E, R\}, \{H, E\})$;
ii) $(\{H\}, \{H, T\}, \{E, T\}, \{E, H\})$;
(*iii*) $(\{T, H\}, \{H, O, R\}, \{T, Y\}, \{H, R\})$;
(*iv*) $(\{H, O\}, \{H, O\}, \{H, T\}, \{H\})$.

(26c) Let E be the set of letters in the word MATROIDS. Show that the family (STAR, ROAD, MOAT, RIOT, RIDS, DAMS, MIST) of subsets of E has exactly eight transversals.

(26d) Let E be the set $\{1, 2, \ldots, 50\}$. How many distinct transversals has the family $(\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{50, 1\})$?

(26e) Verify the statements of Corollaries 26B and 26C when $E = \{a, b, c, d, e\}$, $\mathscr{S} = (\{a, c, e\}, \{b, d\}, \{b, d\}, \{b, d\})$, and $X = \{a, b, c\}$.

(26f) Let $E = \{\square, \bigcirc, \diamond, \triangle, \bigstar\}$ and $\mathscr{S} = (\{\square, \bigcirc, \diamond\}, \{\diamond, \triangle\}, \{\triangle\}, \{\triangle\}, \{\square, \bigcirc, \bigstar\})$.
(*i*) List all the subsets of E for which the marriage condition is not satisfied.
(*ii*) Verify the statement of Corollary 26B.

(26g) Rewrite
(*i*) the statements of Corollaries 26B and 26C in marriage terminology;
(*ii*) the Halmos–Vaughan proof of Hall's theorem in the language of transversal theory.

(*26h) Let E and $\mathscr{S}$ have their usual meanings, let T_1 and T_2 be transversals of $\mathscr{S}$, and let x be an element of T_1.
(*i*) Show that there exists an element y of T_2 such that $(T_1 - \{x\}) \cup \{y\}$ (the set obtained from T_1 on replacing x by y) is also a transversal of $\mathscr{S}$.
(*ii*) Compare this result with exercise 9i.
(This result will be in Chapter **9**).

(*26i) The **rank** $\rho(A)$ of a subset A of E is the number of elements in the largest partial transversal of $\mathscr{S}$ contained in A.
(*i*) Show that
(*a*) $0 \leqslant \rho(A) \leqslant |A|$;
(*b*) if $A \subseteq B \subseteq E$, then $\rho(A) \leqslant \rho(B)$;
(*c*) if $A, B \subseteq E$, then $\rho(A \cup B) + \rho(A \cap B) \leqslant \rho(A) + \rho(B)$.
(*ii*) Compare these results with exercise 9j.
(This result will also be needed in Chapter **9**.)

(*26j) Let $\mathscr{S}$ be a family consisting of m non-empty subsets of E, and let A be a subset of E. By applying Hall's theorem to the family consisting of $\mathscr{S}$ together with $|E| - m$ copies of $E - A$, prove that there exists a transversal of $\mathscr{S}$ containing A if and only if
(*i*) $\mathscr{S}$ has a transversal;
(*ii*) A is a partial transversal of $\mathscr{S}$.
(A simpler proof, using matroid theory, will be given in §**33**.)

(*26k) Let E be a countable set, and let $\mathscr{S} = (S_1, S_2, \ldots)$ be a countable family of non-empty *finite* subsets of E.
(*i*) Defining a transversal of $\mathscr{S}$ in the natural way, show (using König's lemma) that $\mathscr{S}$ has a transversal if and only if the union of any k subsets S_i contains at least k elements, for all finite k.

(*ii*) By considering the example $E=\{1, 2, 3, \ldots\}$, $S_1=E$, $S_2=\{1\}$, $S_3=\{2\}$, $S_4=\{3\}$, ..., show that the result of part (*i*) is false if not all of the S_i are finite.

§27. Applications of Hall's theorem

★In this section we apply Hall's theorem to problems concerning the construction of Latin squares, the elements of a (0, 1)-matrix, and the existence of a common transversal of two families of subsets of a given set. We shall see that the last of these applications is of relevance in timetabling problems.

Latin squares

An $m \times n$ **latin rectangle** is an $m \times n$ matrix $M=(m_{ij})$ whose entries are integers satisfying:
(*i*) $1 \leqq m_{ij} \leqq n$;
(*ii*) no two entries in any row or in any column are equal.
Note that (*i*) and (*ii*) imply that $m \leqq n$; if $m=n$, then the latin rectangle is called a **latin square**. For example, Figs 27.1 and 27.2 show a 3×5 latin rectangle and a 5×5 latin square. We can ask the following question: given an $m \times n$ latin rectangle with $m<n$, when can we adjoin $n-m$ new rows in such a way that a latin square is produced? Surprisingly, the answer is 'always'!

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$$

Fig. 27.1

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 4 & 3 & 5 & 2 & 1 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$$

Fig. 27.2

THEOREM 27A. *Let M be an $m \times n$ latin rectangle with $m<n$. Then M can be extended to a latin square by the addition of $n-m$ new rows.*

Proof. We shall prove that M can be extended to an $(m+1) \times n$ latin rectangle; by repeating the procedure involved, we eventually obtain a latin square.

Let $E=\{1, 2, \ldots, n\}$, and $\mathscr{S}=(S_1, \ldots, S_n)$, where S_i denotes the set consisting of those elements of E which do *not* occur in the i-th column of M. If we can prove that $\mathscr{S}$ has a transversal, then the proof is complete, since the elements in this transversal will form the additional row. By Hall's theorem, it is sufficient to show that the union of any k of the S_i

contains at least k distinct elements. But this is obvious, since such a union contains $(n-m)k$ elements altogether (including repetitions), and if there were fewer than k distinct elements, then at least one of them would have to appear more than $n-m$ times. Since each element occurs exactly $n-m$ times, we have the required contradiction.//

(0, 1)-Matrices

An alternative way of studying transversals of a family $\mathscr{S}=(S_1, \ldots, S_m)$ of non-empty subsets of a set $E=\{e_1, \ldots, e_n\}$ is to study the **incidence matrix** of the family—the $m \times n$ matrix $A=(a_{ij})$ in which $a_{ij}=1$ if $e_j \in S_i$, and $a_{ij}=0$ otherwise. We shall call such a matrix, in which every entry is 0 or 1, a **(0, 1)-matrix**. If we define the **term rank** of A to be the largest number of 1's of A, no two of which lie in the same row or column, then $\mathscr{S}$ has a transversal if and only if the term rank of A is m. Moreover, the term rank of A is precisely the number of elements in a partial transversal of largest possible size. We now prove, as a second application of Hall's theorem, a famous result on (0, 1)-matrices known as the **König-Egerváry theorem**.

THEOREM 27B (König–Egerváry 1931). *The term rank of a* (0, 1)-*matrix A is equal to the minimum number μ of rows and columns which together contain all the* 1*'s of A.*

Remark. As an illustration of the theorem, consider the matrix of Fig. 27.3 which is the incidence matrix of the second family $\mathscr{S}$ described on page 000. Clearly the term rank and μ are both four.

Proof. It is obvious that the term rank cannot exceed μ. To prove equality, we can suppose without loss of generality that all of the 1's of A are contained in r rows nd s columns (where $r+s=\mu$), and that the order of the rows and columns is such that A contains, in the bottom left-hand corner, an $(m-r)\times(n-s)$ submatrix consisting entirely of zeros (Fig. 27.4). If $i \leqq r$, define S_i to be the set of integers $j \leqq n-s$ such that $a_{ij}=1$. It

$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{matrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Fig. 27.3

$$\begin{array}{cc} \begin{matrix} n-s & s \end{matrix} & \\ \left(\begin{array}{c|c} M & B \\ \hline 0 & N \end{array}\right) & \begin{matrix}]\,r \\ ]\,m-r \end{matrix} \end{array}$$

Fig. 27.4

is a straightforward exercise to check that the union of any k of the S_i contains at least k integers, and hence that the family $\mathscr{S}=(S_1, \ldots, S_r)$ has a transversal. It follows that the submatrix M of A contains a set of r 1's, no two of which lie in the same row or column; similarly, the matrix N contains a set of s 1's with the same property. Hence A contains a set of $r+s$ 1's, no two of which lie in the same row or column. This shows that μ cannot exceed the term rank, as required.//

We have just proved the König–Egerváry theorem using Hall's theorem. It is even easier to prove Hall's theorem using the König–Egerváry theorem (see exercise 27e). It follows that the two theorems are, in some sense, equivalent. Later on in this chapter we shall be proving Menger's theorem and the max-flow min-cut theorem, both of which can also be shown to be equivalent to Hall's theorem.

Common transversals

We conclude this section with a brief discussion of common transversals. If E is a non-empty finite set and $\mathscr{S}=(S_1, \ldots, S_m)$ and $\mathscr{T}=(T_1, \ldots, T_m)$ are two families of non-empty subsets of E, it is of interest to know when there exists a **common transversal** for $\mathscr{S}$ and $\mathscr{T}$—i.e., a set of m distinct elements of E which form a transversal of both $\mathscr{S}$ and $\mathscr{T}$. In timetabling problems, for example, if E denotes the set of times at which lectures may be given, the sets S_i denote the times that m given professors are willing to lecture, and the sets T_i denote the times that m lecture rooms are available, then the finding of a common transversal of $\mathscr{S}$ and $\mathscr{T}$ enables us to assign each professor to an available lecture room at a time suitable to him.

We can in fact give a necessary and sufficient condition for two families to have a common transversal; note that Theorem 27C reduces to Hall's theorem if we put $T_j=E$ for $1 \leqq j \leqq m$.

THEOREM 27C. *Let E be a non-empty finite set, and let $\mathscr{S}=(S_1, \ldots, S_m)$ and $\mathscr{T}=(T_1, \ldots, T_m)$ be two families of non-empty subsets of E. Then $\mathscr{S}$ and $\mathscr{T}$ have a common transversal if and only if, for all subsets A and B of $\{1, 2, \ldots, m\}$,*

$$\left|\left(\bigcup_{i\in A} S_i\right)\cap\left(\bigcup_{j\in B} T_j\right)\right| \geqq |A|+|B|-m.$$

Sketch of proof. Consider the family $\mathscr{U}=\{U_i\}$ of subsets of $E\cup\{1, \ldots, m\}$ (assuming E and $\{1, \ldots, m\}$ to be disjoint), where the indexing set is also $E\cup\{1, \ldots, m\}$ and where $U_i=S_i$ if $i\in\{1, \ldots, m\}$ and $U_i=\{i\}\cup\{j: i\in T_j\}$ if $i\in E$.

It is not difficult to verify that $\mathscr{S}$ and $\mathscr{T}$ have a common transversal if and only if $\mathscr{U}$ has a transversal. The result then follows on applying Hall's theorem to the family $\mathscr{U}$.//

It is not known under what conditions there exists a common transversal for three families of non-empty subsets of a set, and the problem of finding such conditions seems to be very difficult. Many attempts to solve this problem use matroid theory. In fact, as we shall see in the next chapter, several problems in transversal theory (for example, exercise 26j and Theorem 27C) become much simpler when looked at from this viewpoint. Further results in transversal theory may also be found in Mirsky[18] or Bryant and Perfect.[9]

Exercises 27

(27a) Give an example of a 5×8 Latin rectangle and a 6×6 Latin square.

(27b) Find two ways of completing the following Latin rectangle to a 5×5 Latin square.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix}.$$

(27c) (*i*) Use the result of exercise 25g to prove that if $m<n$, then an $m \times n$ latin rectangle can be extended to an $(m+1) \times n$ latin rectangle in at least $(n-m)!$ ways.

(*ii*) Deduce that the number of $n \times n$ latin squares is at least $n!(n-1)! \ldots 1!$.

(27d) Verify the König–Egerváry theorem for the following matrices

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

(*27e) By regarding a (0, 1)-matrix as the incidence matrix of a family of subsets, show how the König–Egerváry theorem can be used to prove Hall's theorem.

(27f) Let $E=\{a, b, c, d, e\}$ and let $\mathscr{S}=(\{a, c, e\}, \{a, b\}, \{c, d\})$ and $\mathscr{T}=(\{d\}, \{a, e\}, \{a, b, d\})$.

(*i*) Find a common transversal of $\mathscr{S}$ and $\mathscr{T}$.

(*ii*) Verify the condition of Theorem 27C.

(27g) Repeat exercise 27f for the families $\mathscr{S}=(\{a, b, d\}, \{c, e\}, \{a, e\})$ and $\mathscr{T}=(\{c, d\}, \{b\}, \{b, c, e\})$.

(*27h) Let G be a finite group and H be a subgroup of G. Use Theorem 27C to show that if

$$G=x_1H\cup x_2H\cup \ldots \cup x_mH=Hy_1\cup Hy_2\cup \ldots \cup Hy_m$$

are left and right coset decompositions of G with respect to H, then there exist elements $z_1, \ldots, z_m$ in G such that

$$G=z_1H\cup z_2H\cup \ldots \cup z_mH=Hz_1\cup Hz_2\cup \ldots \cup Hz_m.$$

§28. Menger's theorem

We now discuss a theorem which turns out to be closely related to Hall's theorem, and which has very far-reaching practical applications. This theorem is due to Menger and concerns the number of paths connecting two given vertices v and w in a graph G. We might need, for example, to find the maximum number of paths from v to w, no two of which have an edge in common—such paths are called **edge-disjoint paths**. Alternatively, we may want to find the maximum number of paths from v to w, no two of which have a vertex in common (except, of course, v and w)—these are called **vertex-disjoint paths**. (In the graph of Fig. 28.1, there are clearly four edge-disjoint paths and two vertex-disjoint ones.)

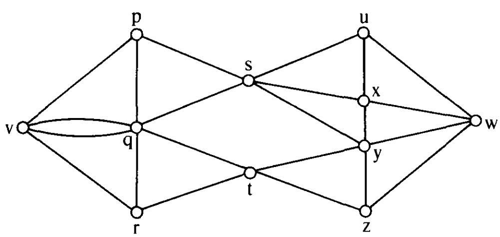


Fig. 28.1

Analogously, we can ask for the maximum number of vertex-disjoint or arc-disjoint paths from a vertex v to a vertex w in a digraph. In this case we can, without loss of generality, take v to be a source and w to be a sink. We shall be concentrating primarily on graphs, the corresponding discussion for digraphs being left to you.

In order to investigate these problems, we shall need some further definitions. We shall assume throughout that G is a connected graph and that v and w are given distinct vertices of G. A **vw-disconnecting set** of G is a set E of edges of G with the property that any path from v to w includes an edge of E; note that a vw-disconnecting set is a disconnecting set of G. Similarly, a **vw-separating set** of G is a set S of vertices (not including v or w) with the property that any path from v to w passes through a vertex of S. In Fig. 28.1, for example, the sets $E_1 = \{ps, qs, ty, tz\}$ and $E_2 = \{uw, xw, yw, zw\}$ are vw-disconnecting sets, and $V_1 = \{s, t\}$ and $V_2 = \{p, q, y, z\}$ are vw-separating sets.

In order to count the number of edge-disjoint paths from v to w, we first note that if E is a vw-disconnecting set containing k edges, then the number of edge-disjoint paths cannot possibly exceed k, since otherwise some edge in E would be included in more than one path. If, moreover, E is a vw-disconnecting set of smallest possible size, then it turns out that the number of edge-disjoint paths is actually equal to k, and that consequently there is exactly one edge of E in each such path. This result is known as the edge-form of **Menger's theorem**, although it was in fact first proved by Ford and Fulkerson in 1955.

THEOREM 28A. *The maximum number of edge-disjoint paths connecting two distinct vertices v and w of a connected graph G is equal to the minimum number k of edges in a vw-disconnecting set.*

Remark. The proof we are about to give is non-constructive, in the sense that if we are given G, it will not provide us with a systematic way of obtaining k edge-disjoint paths, or even of finding the value of k. An algorithm which can be used to solve these problems will be given in the next section.

Proof. As we have just pointed out, the maximum number of edge-disjoint paths connecting v and w cannot exceed the minimum number of edges in a vw-disconnecting set. We shall use induction on the number of edges of G to prove that these numbers are actually equal. Suppose that the number of edges of G is m, and that the theorem is true for all graphs with fewer than m edges. There are two cases to consider:

(*i*) We suppose first that there exists a vw-disconnecting set E of minimum size k, with the property that not all of its edges are incident to v, and not all of them are incident to w; for example, in the graph of Fig. 28.1, the set E_1 defined above would be such a vw-disconnecting set. The removal from G of the edges in E leaves two disjoint subgraphs V and W containing v and w, respectively. We now define two new graphs G_1 and G_2 as follows: G_1 is obtained from G by contracting every edge of V (i.e., by shrinking V down to v) and G_2 is obtained similarly by contracting every edge of W. (The graphs G_1 and G_2 obtained from Fig. 28.1 are shown in Fig. 28.2; the dashed lines denote edges of E_1.) Since G_1 and G_2 have fewer edges than G, and since E is clearly a vw-disconnecting set of minimum size for both G_1 and G_2, the induction hypothesis tells us that there are k edge-disjoint paths in G_1 from v to w, and similarly for G_2. The required k edge-disjoint paths in G are then obtained by combining these paths in the obvious way.

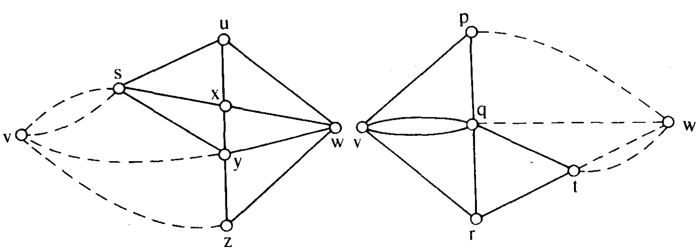


Fig. 28.2

(*ii*) We now suppose that every vw-disconnecting set of minimum size k consists only of edges which are all incident to v or all incident to w; for example, in Fig. 28.1, the set E_2 is such a vw-disconnecting set. We can assume without loss of generality that every edge of G is contained in a vw-disconnecting set of size k, since otherwise its removal would not affect the value of k and we could use the induction hypothesis to obtain k edge-disjoint paths. It follows that if P is any path from v to w, then P must consist either of a single edge or of two edges, and can thus contain at most one edge of any vw-disconnecting set of size k. By removing from G the edges of P, we obtain a graph which contains at least $k-1$ edge-disjoint paths (by the induction hypothesis). These paths, together with P, give the required k paths in G.//

We turn now to the other problem mentioned at the beginning of the section—namely, to find the number of vertex-disjoint paths from v to w. (It was actually this problem which Menger himself solved, although his name is usually given to both Theorem 28A and Theorem 28B). What is at first sight rather surprising is that not only does its solution have a form very similar to Theorem 28A, but also the proof of Theorem 28A goes through with only minor changes, mainly involving the replacement of such terms as 'edge-disjoint' and 'incident' by 'vertex-disjoint' and 'adjacent'. We now state the vertex-form of Menger's theorem—its proof will be omitted.

THEOREM 28B (Menger 1927). *The maximum number of vertex-disjoint paths connecting two distinct non-adjacent vertices v and w of a graph G is equal to the minimum number of vertices in a vw-separating set.*//

Using Theorems 28A and 28B we can immediately deduce the following necessary and sufficient conditions for a graph to be k-connected and k-edge-connected:

COROLLARY 28C. *A graph G is k-edge-connected if and only if any two distinct vertices of G are connected by at least k edge-disjoint paths.*//

COROLLARY 28D. *A graph G with at least $k+1$ vertices is k-connected if and only if any two distinct vertices of G are connected by at least k vertex-disjoint paths.*//

As we pointed out earlier, the above discussion can be modified to give the number of vertex-disjoint or arc-disjoint paths in a digraph in terms of disconnecting sets and separating sets. In this case, a vw-disconnecting set is a set A of arcs with the property that every path from v to w includes an arc in A. Once again the corresponding theorem takes a form very similar to Theorem 28A, and the proof goes through almost word for word. We state it formally as the **integrity theorem**; the reason for this name will become apparent in the following section.

THEOREM 28E (Integrity theorem). *The maximum number of arc-disjoint paths from a vertex v to a vertex w in a digraph D is equal to the minimum number of arcs in a vw-disconnecting set.*//

As an example of the integrity theorem, we let D be the digraph shown in Fig. 28.3. It is straightforward to verify that there are six arc-disjoint paths from v to w; a corresponding vw-disconnecting set is indicated by dashed lines.

As the reader can see, these diagrams are likely to become very cumbersome as the number of arcs joining pairs of adjacent vertices increases. This can be overcome by drawing just one arc and writing next

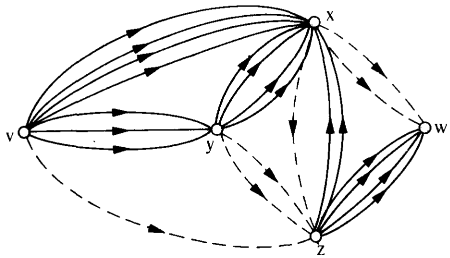


Fig. 28.3

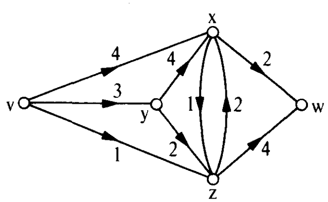


Fig. 28.4

to it the number of arcs there should be (see Fig. 28.4). This seemingly innocent remark turns out to be fundamental in the study of network flows and transportation problems, which will be discussed in the following section.

We end this section by proving that Hall's theorem can be deduced from Menger's theorem. We shall prove the version of Hall's theorem that appears in Corollary 25B.

THEOREM 28F. *Menger's theorem implies Hall's theorem.*

Proof. Let $G=G(V_1, V_2)$ be a bipartite graph. We have to prove that if $|A| \leq |\varphi(A)|$ for each subset A of V_1 (using the notation of Corollary 25B), then there exists a complete matching from V_1 to V_2. This is done by applying the vertex-form of Menger's theorem (Theorem 28B) to the graph obtained by adjoining to G a vertex v adjacent to every vertex in V_1 and a vertex w adjacent to every vertex in V_2 (see Fig. 28.5). Since a complete matching from V_1 to V_2 exists if and only if the number of vertex-disjoint paths from v to w is equal to the number of vertices in $V_1(=k$, say), it is enough to show that every vw-separating set contains at least k vertices.

Let S be a vw-separating set, consisting of a subset A of V_1 and a subset B of V_2. Since $A \cup B$ is a vw-separating set, there can be no edges joining a vertex of $V_1 - A$ to a vertex of $V_2 - B$, and hence $\varphi(V_1 - A) \subseteq B$. It follows that $|V_1 - A| \leq |\varphi(V_1 - A)| \leq |B|$, and so $|S| = |A| + |B| \geq |V_1| = k$, as required.//

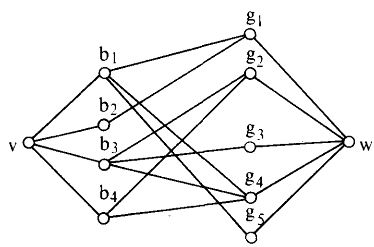


Fig. 28.5

Exercises 28

(28a) (*i*) Verify the edge-form of Menger's theorem (Theorem 28A) for the graphs in Fig. 28.6.
(*ii*) Verify the vertex-form of Menger's theorem (Theorem 28B) for the same graphs.

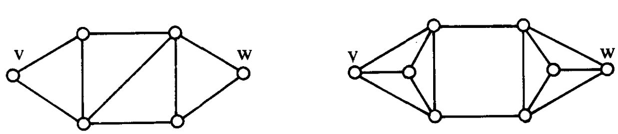


Fig. 28.6

(28b) Verify Theorems 28A and 28B for the Petersen graph in the following two cases:
(*i*) when v and w are adjacent vertices;
(*ii*) when v and w are not adjacent.

(28c) Prove Theorem 28B in detail.

(28d) Verify Corollary 28C for the following graphs:
(*i*) W_5; (*ii*) $K_{3,4}$; (*iii*) Q_3.

(28e) Verify Corollary 28D for the following graphs:
(*i*) $K_{3,5}$; (ii) $K_{3,3,3}$; (*iii*) the graph of the octahedron.

(28f) Verify the integrity theorem for the digraphs in Fig. 28.7.

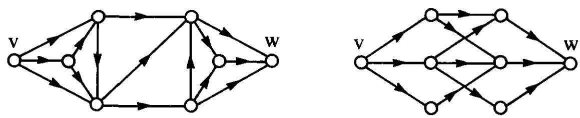


Fig. 28.7

§29. Network flows

Our society today is largely governed by networks—transportation, communication, the distribution of goods, etc.—and the mathematical analysis of such networks has become a subject of fundamental importance. In this section we shall attempt to show by means of simple examples that network analysis is essentially equivalent to the study of digraphs.

A manufacturer of home computers wants to send several boxes of computers to a given market. We shall assume that there are various channels through which the boxes can be sent, and that these channels are as shown in Fig. 29.1 (with v representing the manufacturer and w

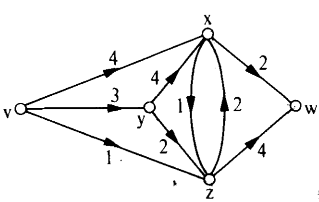


Fig. 29.1

←v(8) →w(6)
→x(10) (6)
←y (3)
←x(3) →y
(2) 6
→(2) (4)

8 3 6 6 23
2 4 6 3 23

the market). The numbers appearing on the diagram refer to the maximum loads which may be passed through the corresponding channels. The manufacturer is clearly interested in finding the maximum number of boxes he can send through the network without exceeding the permitted capacity of any channel.

Fig. 29.1 can also be used to describe other situations. For example, if each arc of the digraph represents a one-way street and the number associated with each street refers to the maximum possible flow of traffic (in vehicles per hour) along that street, then we may want to find the greatest possible number of vehicles which can travel from v to w in one hour. Alternatively, we can regard the diagram as depicting an electrical network, the problem then being to find the maximum current which can safely be passed through the network given the currents at which the individual wires burn out.

Using these examples as motivation, we may now define a **network** N to be a weighted digraph—that is, a digraph to each arc a of which has been assigned a non-negative real number $\psi(a)$ called its **capacity**. Equivalently, a network may be defined as a pair (D, ψ) where D is a digraph and ψ is a function from the arc-set of D to the set of non-negative real numbers. The **out-degree** $\overleftarrow{\rho}(x)$ of a vertex x is then defined to be the sum of the capacities of the arcs of the form xz, and the **in-degree** $\overrightarrow{\rho}(x)$ is similarly defined. For example, in the network of Fig. 29.1, $\overleftarrow{\rho}(v)=8$ and $\overrightarrow{\rho}(x)=10$. It is clear that the analogue of the handshaking di-lemma then takes the following form: *the sum of the out-degrees of the vertices of a network is equal to the sum of the in-degrees.* In the following, we shall always assume (unless otherwise stated) that the digraph D contains exactly one source v and one sink w. The general case of several sources and sinks (corresponding in the first example above to more than one manufacturer and market) may be easily reduced to this special case (see exercise 29d).

Given a network $N=(D, \psi)$, we define a **flow in N** to be a function φ which assigns to each arc a of D a non-negative real number $\varphi(a)$ (called the **flow in a**) in such a way that

(*i*) for any arc a, $\varphi(a) \leq \psi(a)$;

(*ii*) with respect to the network (D, φ), the out-degree and in-degree of any vertex (other than v or w) are equal.

Informally this means that the flow in any arc cannot exceed its capacity, and that the 'total flow' into any vertex (other than v or w) is equal to the 'total flow' out of it. Fig. 29.2 gives a possible flow for the network of Fig. 29.1. Another flow is the **zero flow** in which the flow in every arc is zero, any other flow being called a non-zero flow. For convenience, we shall say that an arc a for which $\varphi(a)=\psi(a)$ is called **saturated**; in Fig. 29.2, the arcs vz, xz, yz, xw and zw are saturated, the remaining arcs being called **unsaturated**.

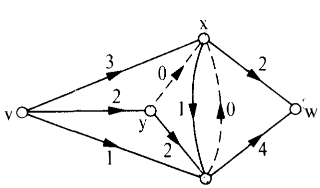


Fig. 29.2

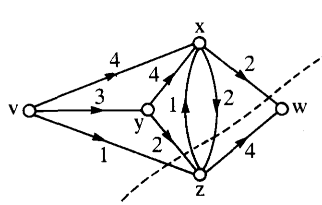


Fig. 29.3

It follows from the handshaking di-lemma that the sum of the flows in the arcs incident to v is equal to the sum of the flows in the arcs incident to w; this sum is called the **value of the flow**. Prompted by the examples considered at the beginning of this section, we shall be primarily interested in those flows whose value is as large as possible—the so-called **maximum flows**; you can easily check that the flow of Fig. 29.2 is a maximum flow for the network of Fig. 29.1, and that its value is six. Note that in general a network can have several different maximum flows but that their values must all be equal.

The study of maximum flows in a network $N=(D, \psi)$ is closely tied up with the concept of a **cut**, which is simply a set A of arcs of D with the property that every path from v to w includes an arc in A. In other words, a cut in a network is merely a vw-disconnecting set in the corresponding digraph D. The **capacity of a cut** is then defined to be the sum of the capacities of the arcs in the cut. We shall be concerned mainly with those cuts whose capacity is as small as possible, the so-called **minimum cuts**; in Fig. 29.3, an example of a minimum cut is provided by the arcs vz, xz, yz and xw, the capacity of this cut being six.

It is clear that the value of any flow cannot exceed the capacity of any cut, and hence that the value of any maximum flow cannot exceed the capacity of any minimum cut. What is not immediately clear is that these last two numbers are always equal. This famous result is known as the **max-flow min-cut theorem**, and was first proved by Ford and Fulkerson in 1955. We shall present two proofs. The first one shows that the max-flow min-cut theorem is essentially equivalent to Menger's theorem, whereas the second one is a direct proof.

THEOREM 29A (Max-flow min-cut theorem). *In any network, the value of any maximum flow is equal to the capacity of any minimum cut.*

Remark. In applying this theorem it is frequently simplest to find a flow and a cut with the property that the value of the flow is equal to the capacity of the cut. It follows from the theorem that the flow must be a maximum flow and that the cut must be a minimum cut. Note that if all the capacities are integers, then the value of a maximum flow will also be an integer. This important fact turns out to be very useful in certain applications of network flows.

First proof. We shall suppose, to begin with, that the capacity of every arc is an integer. In this case, the network can be regarded as a digraph $\tilde{D}$ in which the capacities represent the number of arcs connecting the various vertices (see Figs 28.3 and 28.4). The value of a maximum flow then corresponds to the total number of arc-disjoint paths from v to w in $\tilde{D}$, and the capacity of a minimum cut refers to the minimum number of arcs in a vw-disconnecting set of $\tilde{D}$. The result now follows immediately from the integrity theorem (Theorem 28E).

The extension of this result to networks in which all the capacities are rational numbers is effected simply by multiplying all these capacities by a suitable integer d to make them integral (e.g. the least common multiple of the denominators of the capacities). We then have the case described in the previous paragraph, and the result follows on dividing by d.

Finally, if some of the capacities are irrational numbers, then the theorem is proved by approximating these capacities as closely as we please by rationals and using the result of the previous paragraph. By carefully choosing these rationals, we can always ensure that the value of any maximum flow and the capacity of any minimum cut are each altered by an amount which we can make as small as we wish. The precise details of this argument will be left as an exercise. In practical examples, of course, such irrational capacities would rarely occur since the capacities would generally be given in decimal form.//

Second proof. We now give a direct proof of the max-flow min-cut theorem. Note, as remarked above, that *since the value of any maximum flow cannot exceed the capacity of any minimum cut, it is sufficient to prove the existence of a cut whose capacity is equal to the value of a given maximum flow.*

Let φ be a maximum flow. We define two sets V and W of vertices of the network as follows: if G denotes the underlying graph of the digraph D of the network, then a vertex z of the network is contained in V if and only if there exists in G a path $v = v_0 \to v_1 \to v_2 \to \ldots \to v_{m-1} \to v_m = z$, with the property that each edge $v_i v_{i+1}$ corresponds either to an arc $v_i v_{i+1}$ which is unsaturated, or to an arc $v_{i+1} v_i$ which carries a non-zero flow. (Note that v is trivially contained in V.) The set W then consists of all those vertices which do not lie in V. For example, in Fig. 29.2, the set V consists of the vertices v, x and y, and the set W consists of the vertices z and w.

We shall now show that W is non-empty, and that in particular it contains the vertex w. If this is not so, then w is in V, and hence there exists in G a path $v \to v_1 \to v_2 \to \ldots \to v_{m-1} \to w$ of the above type. We now choose a positive number ε satisfying the following two conditions:
(*i*) ε must not exceed any of the amounts needed to saturate the arcs of the first type;
(*ii*) ε must not exceed the flow in any of the arcs of the second type.
It is now easy to see that if we increase by ε the flow in the arcs of the first type and decrease by ε the flow in the arcs of the second type, then the effect will be to increase the value of the flow to $\varphi + \varepsilon$. But this contradicts our assumption that φ is a maximum flow, and it therefore follows that w is contained in W.

To complete the argument, we let E denote the set of all arcs of the form xz, where x is in V and z is in W. Clearly E is a cut. Moreover, it is easy to see that every arc xz of E is saturated, since otherwise z would also be an element of V. Similarly, one can show that every arc zx carries a zero flow. It follows that the capacity of E must be equal to the value of φ, and that E is therefore the required cut.//

★The max-flow min-cut theorem provides a useful check on the maximality or otherwise of a given flow, as long as the network is fairly simple. In practice, of course, the networks one has to deal with are large and complicated, and it will in general be difficult to find a maximum flow by inspection. We conclude this section with a method for finding a maximum flow in any network with integral capacities. The extension of this method to networks with rational capacities is trivial and will be omitted.

Suppose then that we are given a network $N=(D, \psi)$. The finding of a maximum flow in N involves three steps:

Step 1. We first find by inspection a flow φ whose value is non-zero (if one exists). For example, if N is the network of Fig. 29.4, then a suitable flow would be the flow shown in Fig. 29.5. It is worth pointing out that the larger we can make the value of our initial flow φ, the easier the subsequent steps will be.

Step 2. We next construct from N a new network N' obtained by reversing the direction of the flow φ. More precisely, any arc a for which $\varphi(a)=0$ appears in N' with its original capacity, but any arc a for which

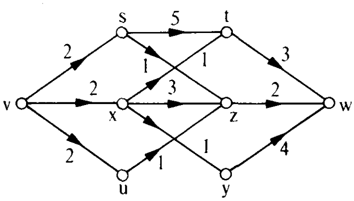


Fig. 29.4

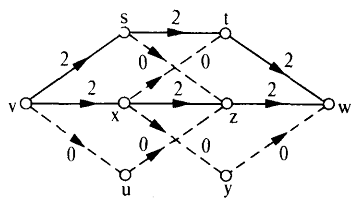


Fig. 29.5

$\varphi(a)\neq 0$ is replaced by an arc a with capacity $\psi(a)-\varphi(a)$ together with an arc in the direction opposite to a with capacity $\varphi(a)$. In our particular example, the network N' takes the form shown in Fig. 29.6; note that v is no longer a source and w no longer a sink.

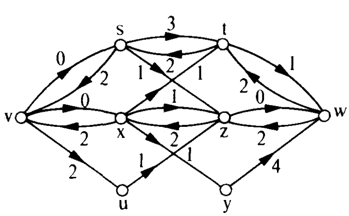


Fig. 29.6

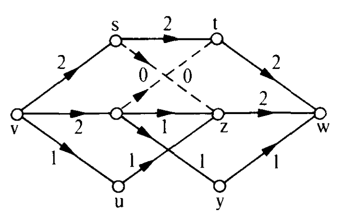


Fig. 29.7

Step 3. If in the network N' we can find a non-zero flow from v to w, then this flow can be added to the original flow φ to give a flow φ' of larger value in N. We can now repeat Step 2 using our new flow φ' in place of φ in the construction of the network N'. On continuing this procedure, we will eventually end up with a network N' which contains no non-zero flow. The corresponding flow φ will then be a maximum flow, as can easily be shown. In Fig. 29.6, for example, there is a non-zero flow in which the flow in the arcs vu, uz, zx, xy and yw is 1 and in the remaining arcs is zero. Adding this to the flow of Fig. 29.5 results in the flow shown in Fig. 29.7, which may easily be shown to be maximum by repeating Step 2. We have thus obtained the required maximum flow.★

In this section we have been able only to scratch the surface of this very diverse and important subject. If you wish to pursue these topics further, you should consult Lawler.[17]

Exercises 29

(29a) Consider the network in Fig. 29.8.
  (*i*) List all the cuts in this network, and find a minimum cut.
  (*ii*) Find a maximum flow, and verify the max-flow min-cut theorem for this network.

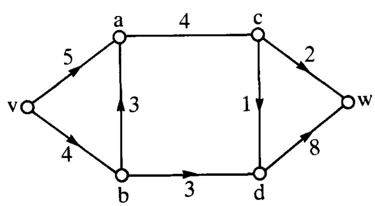


Fig. 29.8

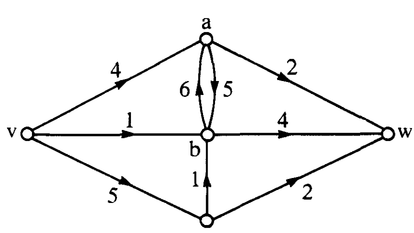


Fig. 29.9

(29b) Repeat exercise 29a for the network in Fig. 29.9.
(29c) Show that the flows of Figs 29.2 and 29.7 are maximum flows for the networks of Figs 29.1 and 29.4, and verify the max-flow min-cut theorem in each case.

(29d) (*i*) Show how the analysis of the flows in a network with several sources and sinks can be reduced to the standard case by the addition of a new 'source vertex' and a new 'sink vertex'.
(*ii*) Illustrate your answer to part (*i*) with reference to the network in Fig. 29.10.

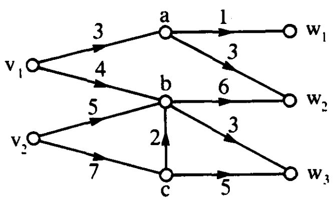


Fig. 29.10

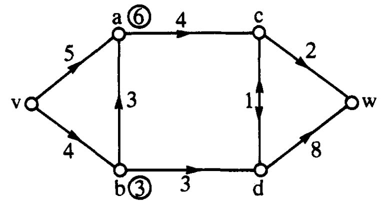


Fig. 29.11

(29e) (*i*) How would you reduce to the standard case a network problem in which
(*a*) some arcs are replaced by edges with a flow in either direction?
(*b*) some vertices are assigned 'capacities', giving the maximum flow permitted through those vertices?
(*ii*) Illustrate your answers to part (*i*) with reference to the network in Fig. 29.11.

(29f) Verify the max-flow min-cut theorem for the network of Fig. 22.6.

(29g) Find a flow with value 20 in the network of Fig. 29.12.

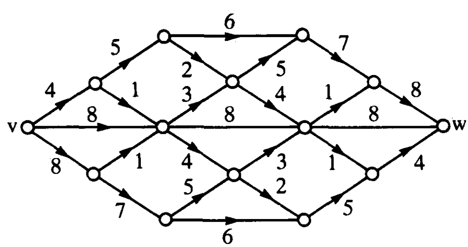


Fig. 29.12

(*29h) Show how the max-flow min-cut theorem can be used to prove
(*i*) Hall's theorem;
(*ii*) Theorem 27C on common transversals.

9
Matroid theory

The first of earthly blessings, independence.

Edward Gibbon

In this chapter we shall investigate the rather unexpected similarity between certain results in graph theory and their analogues in transversal theory (for example, exercises 9i and 26h, or exercises 9j and 26i). In order to do this it is convenient to introduce the idea of a matroid, first studied in 1935 by Whitney. As we shall see, a matroid is essentially a set with an 'independence structure' defined on it, where the notion of independence generalizes not only that of independence in graphs (as defined in exercise 5m) but also that of linear independence in vector spaces. The link with transversal theory is then provided by exercise 26h. In §**32** we shall show how to define duality in matroids in such a way as to explain the similarity between the properties of circuits and cutsets in a graph. It will follow from this that the rather unintuitive definition of an abstract-dual of a graph (§**15**) arises as a natural consequence of matroid duality. In the final section, we shall show how matroids can be used to give 'easy' proofs of results in transversal theory, and will conclude with matroid proofs of two deep results in graph theory. Throughout this chapter we shall be content to state results without proof where convenient. The omitted proofs may be found in Welsh.[25]

§30. Introduction to matroids

In §**9** we defined a spanning tree in a connected graph G to be a connected subgraph of G which contains no circuits and which includes every vertex of G. It is clear that a spanning tree cannot contain any other spanning tree as a proper subgraph. It can also be shown (see exercise 9i) that if B_1 and B_2 are spanning trees of G and e is any edge of B_1, then we can find an edge f in B_2 with the property that $(B_1 - \{e\}) \cup \{f\}$ (i.e., the graph obtained from B_1 on replacing e by f) is also a spanning tree of G.

Analogous results hold also in the theory of vector spaces and in transversal theory. If V is a vector space and if B_1 and B_2 are bases of V, then given any element e of B_1, we can find an element f of B_2 with the property that $(B_1-\{e\})\cup\{f\}$ is also a basis of V. The corresponding result in transversal theory appears in exercise 26h. Using these three examples as motivation, we can now given our first definition of a matroid.

A **matroid** M is a pair $(E, \mathscr{B})$, where E is a non-empty finite set and $\mathscr{B}$ is a non-empty collection of subsets of E (called **bases**) satisfying the following properties:

($\mathscr{B}i$) no base properly contains another base;

($\mathscr{B}ii$) if B_1 and B_2 are bases and if e is any element of B_1, then there is an element f of B_2 with the property that $(B_1-\{e\})\cup\{f\}$ is also a base.

By repeatedly using propery ($\mathscr{B}ii$), it is a straightforward exercise to show that any two bases of a matroid M contain the same number of elements (see exercise 30e); this number is called the **rank** of M.

As we indicated above, a matroid can be associated in a natural way with any graph G by letting E be the set of edges of G and taking as bases the edges of the spanning forests of G; for reasons which will appear later, this matroid is called the **circuit matroid** of G and is denoted by $M(G)$. Similarly, if E is a finite set of vectors in a vector space V, then we can define a matroid on E by taking as bases all linearly independent subsets of E which span the same subspace as E; a matroid obtained in this way is called a **vector matroid**. We shall consider such matroids in further detail later.

A subset of E will be called **independent** if it is contained in some base of the matroid M. It follows that the bases of M are precisely the maximal independent sets (i.e. those independent sets which are contained in no larger independent set), and hence that any matroid is uniquely defined by specifying its independent sets. In the case of a vector matroid, a subset of E is independent if and only if its elements are linearly independent when regarded as vectors in the vector space. Similarly, if G is a graph, then the independent sets of $M(G)$ are simply those sets of edges of G which contain no circuit—in other words the edge-sets of the forests contained in G.

Since a matroid can be completely described by listing its independent sets, it seems reasonable to ask whether there is a simple definition of a matroid in terms of its independent sets. One such definition will now be given; you can find a proof of the equivalence of this definition and the above one in Welsh.[25]

A **matroid** M is a pair $(E, \mathscr{I})$, where E is a non-empty finite set, and $\mathscr{I}$ is a non-empty collection of subsets of E (called **independent sets**) satisfying the following properties:

($\mathscr{I}i$) any subset of an independent set is independent;

($\mathscr{I}ii$) if I and J are independent sets with $|J|>|I|$, then there is an element e contained in J but not in I, such that $I\cup\{e\}$ is independent.

(Note that with this definition, a **base** is defined to be any maximal independent set; property ($\mathscr{I}\,ii$) can then be used repeatedly to show that any independent set can be extended to a base.)

If $M=(E,\mathscr{I})$ is a matroid defined in terms of its independent sets, then a subset of E is said to be **dependent** if it is not independent; a minimal dependent set is called a **circuit**. Note that if $M(G)$ is the circuit matroid of a graph G, then the circuits of $M(G)$ are precisely the circuits of G. It is clear that since a subset of E is independent if and only if it contains no circuits, a matroid can be defined in terms of its circuits. One such definition, generalizing to matroids the result of exercise 5k, is given in exercise 30g.

Before proceeding to some examples of matroids, it will be convenient to give one further definition of a matroid. This definition, in terms of a **rank function** ρ, is essentially the one given by Whitney in his pioneering paper of 1935.

If $M=(E,\mathscr{I})$ is a matroid defined in terms of its independent sets, and if A is a subset of E, then the size of the largest independent set contained in A is called the **rank** of A and is denoted by $\rho(A)$. Note that the previously-defined rank of M is then equal to $\rho(E)$. Since a subset A of E is independent if and only if $\rho(A)=|A|$, it follows that a matroid may be defined in terms of its rank function, as we now show.

THEOREM 30A. *A matroid may be defined as a pair (E,ρ), where E is a non-empty finite set, and ρ is an integer-valued function defined on the set of subsets of E and satisfying:*

(ρi) $0\leqq\rho(A)\leqq|A|$, *for every subset A of E;*

(ρii) *if $A\subseteq B\subseteq E$, then $\rho(A)\leqq\rho(B)$;*

(ρiii) *for any $A,B\subseteq E$, $\rho(A\cup B)+\rho(A\cap B)\leqq\rho(A)+\rho(B)$.*

Remark. Note that this is the extension to matroids of the results of exercise 9j and 26i.

Proof. We assume first that $M=(E,\mathscr{I})$ is a matroid defined in terms of its independent sets; we wish to prove properties ($\rho\, i$)–($\rho\, iii$). Clearly ($\rho\, i$) and ($\rho\, ii$) are trivial. To prove ($\rho\, iii$), we let X be a base (i.e., a maximal independent subset) of $A\cap B$. Since X is an independent subset of A, X can be extended to a base Y of A, and then (in a similar way) to a base Z of $A\cup B$. Since $X\cup(Z-Y)$ is clearly an independent subset of B, it follows that

$$\begin{aligned}\rho(B)\geqq\rho(X\cup(Z-Y))&=|X|+|Z|-|Y|\\&=\rho(A\cap B)+\rho(A\cup B)-\rho(A),\end{aligned}$$

as required.

Conversely, let $M=(E,\rho)$ be a matroid defined in terms of a rank function ρ, and define a subset A of E to be independent if and only if $\rho(A)=|A|$. It is then a straightforward matter to prove property ($\mathscr{I}\,i$). To prove ($\mathscr{I}\,ii$), let I and J be independent sets with $|J|>|I|$, and suppose that $\rho(I \cup \{e\}) = k$ for each element e which lies in J but not in I. If e and f are two such elements, then

$$\rho(I \cup \{e\} \cup \{f\}) \leqq \rho(I \cup \{e\}) + \rho(I \cup \{f\}) - \rho(I) = k;$$

it follows that $\rho(I \cup \{e\} \cup \{f\}) = k$. We now continue this procedure, adding one new element of J at a time. Since at each stage the rank has value k, we conclude that $\rho(I \cup J) = k$, and hence (by ρ *ii*) that $\rho(J) \leqq k$, which is a contradiction. It follows that there exists an element f which is in J but not in I with the property that $\rho(I \cup \{f\}) = k + 1$.//

We conclude this section with two simple definitions. A **loop** of a matroid $M = (E, \rho)$ is an element e of E satisfying $\rho(\{e\}) = 0$, and a **pair of parallel elements** of M is a pair $\{e, f\}$ of elements of E which are not loops and which satisfy $\rho(\{e, f\}) = 1$. You should verify that if M is the circuit matroid of a graph G, then the loops and parallel elements of M correspond to loops and multiple edges of G.

Exercises 30

(30a) Let $E = \{a, b, c, d, e\}$. Find matroids on E for which
- (*i*) E is the only base;
- (*ii*) the empty set is the only base;
- (*iii*) the bases are those subsets of E containing exactly three elements.

For each matroid, write down the independent sets, the circuits (if there are any) and the rank function.
(This question will be answered in the next section.)

(30b) Let G_1 and G_2 be the graphs shown in Fig. 30.1. Write down the bases, circuits and independent sets of the circuit matroids $M(G_1)$ and $M(G_2)$.

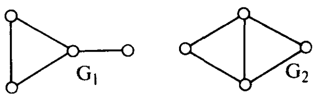


Fig. 30.1

(30c) Let M be the matroid on the set $E = \{a, b, c, d\}$ whose bases are $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$ and $\{c, d\}$. Write down the circuits of M, and deduce that there is no graph G which has M as its circuit matroid.

(30d) Let $E = \{1, 2, 3, 4, 5, 6\}$ and $\mathscr{S} = (S_1, S_2, S_3, S_4, S_5)$, where $S_1 = S_2 = \{1, 2\}$, $S_3 = S_4 = \{2, 3\}$, $S_5 = \{1, 4, 5, 6\}$.
- (*i*) Write down the partial transversals of $\mathscr{S}$, and check that they form the independent sets of a matroid on E.
- (*ii*) Write down the bases and circuits of this matroid.

(30e) Use properties ($\mathscr{B}$ *i*) and ($\mathscr{B}$ *ii*) to prove that
- (*i*) any two bases of a matroid on a set E have the same number of elements;
- (*ii*) if $A \subseteq E$, then any two maximal independent subsets of A have the same number of elements.

(30f) Show how the definition of a fundamental system of circuits in a graph may be extended to matroids.

(*30g) Show that a matroid M can be defined as a pair $(E, \mathscr{C})$, where E is a non-empty finite set, and $\mathscr{C}$ is a collection of non-empty subsets of E (called **circuits**) satisfying the following properties:

(*i*) no circuit properly contains another circuit;

(*ii*) if C_1 and C_2 are two distinct circuits each containing an element e, then there exists a circuit in $C_1 \cup C_2$ which does not contain E.

(30h) (*i*) Use the result of exercise 5k to show that the cutsets of a graph satisfy conditions (*i*) and (*ii*) of the previous exercise.

(*ii*) Write down the bases of the corresponding matroids for the graphs of Fig. 30.1.

(*30i) State and prove a matroid analogue of the greedy algorithm (Theorem 11A).

§31. Examples of matroids

In this section we shall examine several important types of matroid.

Trivial matroids

Given any non-empty finite set E, we can define on it a matroid whose only independent set is the empty set. This matroid is called the **trivial matroid** on E, and is clearly a matroid of rank zero.

Discrete matroids

At the other extreme is the **discrete matroid** on E, in which every subset of E is independent. Note that the discrete matroid on E has only one base, namely E itself, and that the rank of any subset A is simply the number of elements in A.

Uniform matroids

Both of the previous examples are special cases of the **k-uniform matroid** on E, whose bases are those subsets of E which contain exactly k elements. It follows that the independent sets are those subsets of E containing not more than k elements, and that the rank of any subset A is either $|A|$ or k, whichever is smaller. Note that the trivial matroid on E is 0-uniform and that the discrete matroid is $|E|$-uniform.

Before developing the examples described in the previous section, it will be convenient to formalize the idea of isomorphism between matroids. Two matroids $M_1 = (E_1, \mathscr{I}_1)$ and $M_2 = (E_2, \mathscr{I}_2)$ are said to be **isomorphic** if there is a one-one correspondence between the sets E_1 and E_2 which preserves independence—in other words, a set of elements of E_1 is independent in M_1 if and only if the corresponding set of elements of E_2 is independent in M_2. As an example, note that the circuit matroids of the three graphs in Fig. 31.1 are all isomorphic. We emphasize the fact that, although matroid isomorphism preserves circuits, cutsets and the number of edges in a graph, it does *not* in general preserve connectedness, the number of vertices, or their degrees.

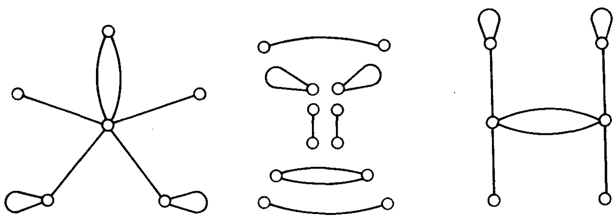


Fig. 31.1

Using the above definition of isomorphism, we can now define graphic, transversal and representable matroids.

Graphic matroids

As we saw in the previous section, we can define a matroid $M(G)$ on the set of edges of a graph G by taking the circuits of G as the circuits of the matroid. $M(G)$ is then called the **circuit matroid** of G and its rank function is simply the cutset rank ξ (see exercise 9j). It is a reasonable question to ask whether a given matroid M is the circuit matroid of some graph—in other words, whether there exists a graph G such that M is isomorphic to $M(G)$. Such matroids are called **graphic matroids**, and a characterization of them will be given in the next section. As an example of a graphic matroid, consider the matroid M on the set $\{1, 2, 3\}$ whose independent sets are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, and $\{1, 3\}$; clearly M is isomorphic to the circuit matroid of the graph shown in Fig. 31.2. It can be shown, however, that non-graphic matroids exist. A simple example is the 2-uniform matroid on a set of four elements, as you were asked to show in exercise 30c.

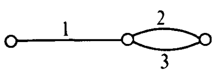


Fig. 31.2

Cographic matroids

Given a graph G, the circuit matroid $M(G)$ is not the only matroid which can be defined on the set of edges of G. Because of the similarity between the properties of circuits and of cutsets in a graph, it is reasonable to hope that a matroid can be constructed by taking the cutsets of G as circuits of the matroid. We saw in exercise 30h that this construction does in fact define a matroid, and we shall refer to it as the **cutset matroid** of G, written $M^*(G)$. Note that a set of edges of G is independent if and only if it contains no cutset of G. We shall call a matroid M **cographic** if there exists a graph G such that M is isomorphic to $M^*(G)$; the reason for the name 'cographic' will appear in the next section.

Planar matroids

A matroid which is both graphic and cographic is called a **planar matroid**. We shall indicate the connexion between planar matroids and planar graphs in the next section.

Representable matroids

Since the definition of a matroid is partly motivated by the idea of linear independence in vector spaces, it is of interest to investigate those matroids which arise as vector matroids associated with some set of vectors in a vector space over a given field. More precisely, given a matroid M on a set E, we shall say that M is **representable over a field F** if there exist a vector space V over F and a map φ from E to V, with the property that a subset A of E is independent in M if and only if φ is one-one on A and $\varphi(A)$ is linearly independent in V. (Note that this amounts to saying that if we ignore loops and parallel elements, then M is isomorphic to a vector matroid defined in some vector space over F.) Of particular importance are those matroids which are representable over the field of integers modulo two—such matroids will be called **binary matroids**. For convenience, we often say simply that M is a **representable matroid** if there exists some field F such that M is representable over F. It turns out that some matroids are representable over every field (the so-called **regular matroids**), some representable over no field and some representable only over some restricted class of fields.

It is not difficult to show that if G is a graph, then its circuit matroid $M(G)$ is a binary matroid. To see this, we associate with each edge of G the corresponding row in the incidence matrix of G, regarded as a vector each of whose components is zero or one. Note that if a set of edges of G form a circuit, then the sum (modulo two) of the corresponding vectors is zero.

An example of a binary matroid which is neither graphic nor cographic is the Fano matroid, described at the end of this section.

Transversal matroids

Our next example provides the link between matroid theory and transversal theory. We recall from exercises 26h, 26i, and 30d that if E is a non-empty finite set and if $\mathscr{S}=(S_1, \ldots, S_m)$ is a family of non-empty subsets of E, then the partial transversals of $\mathscr{S}$ may be taken as the independent sets of a matroid on E. Any matroid obtained in this way (for a suitable choice of E and $\mathscr{S}$) is called a **transversal matroid** and is denoted by $M(S_1, \ldots, S_m)$. For example, the graphic matroid M described above is a transversal matroid on the set $\{1, 2, 3\}$, since its independent sets are the partial transversals of the family $\mathscr{S}=(S_1, S_2)$, where $S_1=\{1\}$ and $S_2=\{2, 3\}$. Note that the rank of a subset A of E is the size of the largest partial transversal contained in A. An example of a matroid which is not transversal will be given in exercise 31e.

It can be proved that every transversal matroid is representable over some field, but is binary if and only if it is graphic. Further results on transversal matroids will be discussed in §**33**.

Restrictions and contractions

In graph theory it is often possible to investigate the properties of a graph by looking at its subgraphs or by considering the graph obtained by contracting some of its edges. We shall find it useful to define the corresponding notions in matroid theory. If M is a matroid defined on a set E and if A is a subset of E, then the **restriction** of M to A (denoted by $M \times A$) is the matroid whose circuits are precisely those circuits of M which are contained in A. Similarly, we define the **contraction** of M to A (denoted by $M\,.\,A$) as the matroid whose circuits are obtained by taking the minimal members of the collection $\{C_i \cap A\}$, where the C_i denote circuits of M. (A simpler definition will be given in exercise 32g.) We leave it to you to verify that these are in fact matroids, and that they correspond to the deletion and contraction of edges in a graph. A matroid obtained from M by a succession of restrictions and contractions is called a **minor** of M.

Bipartite and Eulerian matroids

We conclude this section by showing how bipartite and Eulerian matroids may be defined. Since the usual definitions of bipartite and

Eulerian graphs as given in §3 and §6 are unsuitable for matroid generalization, we must find alternative characterizations of these graphs. In the case of bipartite graphs, exercise 5g comes to our rescue—a **bipartite matroid** is a matroid, every circuit of which contains an even number of elements. For Eulerian graphs we use Corollary 6c and define a matroid on a set E to be an **Eulerian matroid** if E can be expressed as the union of disjoint circuits. In the next section we shall see that Eulerian matroids and bipartite matroids are (in a sense to be made precise) dual concepts, as one might expect from exercise 15i.

The Fano matroid

The **Fano matroid** F is the matroid defined on the set $E=\{1, 2, 3, 4, 5, 6, 7\}$ whose bases are all those subsets of E with three elements except $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{3, 4, 6\}$, $\{4, 5, 7\}$, $\{5, 6, 1\}$, $\{6, 7, 2\}$ and $\{7, 1, 3\}$. This matroid can be represented geometrically by Fig. 31.3, the bases being precisely those sets of three elements which do *not* lie on a line. It can be shown that F is binary and Eulerian, but is not graphic, cographic, transversal or regular.

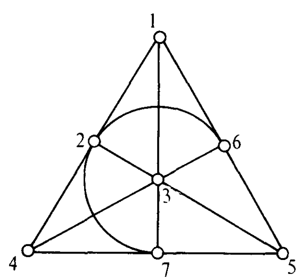


Fig. 31.3

Exercises 31

(31a) Let $E=\{a, b\}$. Show that up to isomorphism there are exactly four matroids on E, and list their bases, independent sets and circuits.

(31b) Let $E=\{a, b, c\}$. Show that up to isomorphism there are exactly eight matroids on E, and list their bases, independent sets and circuits.

(31c) Let E be a set of n elements. Show that, up to isomorphism,
  (*i*) the number of matroids on E is at most 2^{2^n};
  (*ii*) the number of transversal matroids on E is at most 2^{n^2}.

(31d) Let G_1 and G_2 be the graphs of Fig. 30.1.
  (*i*) Are $M(G_1)$ and $M(G_2)$ transversal matroids?
  (*ii*) Are $M^*(G_1)$ and $M^*(G_2)$ transversal matroids?

(31e) Show that $M(K_4)$ is not a transversal matroid.

(31f) Show that every uniform matroid is a transversal matroid.
(31g) Show that the graphic matroids $M(K_5)$ and $M(K_{3,3})$ are not cographic.
(31h) Describe the circuits of the Fano matroid.
(31i) Let M be a matroid on a set E, and let $A \subseteq B \subseteq E$. Prove that
  (*i*) $(M \times B) \times A = M \times A$;
  (*ii*) $(M\ .\ B)\ .\ A = M\ .\ A$.
(*31j) Prove that, if M satisfies any of the following properties, then so does any minor of M:
  (*i*) graphic; (*ii*) cographic; (*iii*) binary; (*iv*) regular.

§32. Matroids and graph theory

We come now to a study of duality in matroids, our aim being to show how several of the results which appeared earlier in the book seem far more natural when looked at in this light. We shall see, for example, that the rather artificial definition of an abstract-dual of a planar graph (see **§15**) arises as a direct consequence of the corresponding definition of a matroid-dual. The point we shall be trying to get across is that not only do various concepts in matroid theory generalize their counterparts in graph theory—they frequently simplify them as well.

We start by recalling from our examination of cographic matroids that we can form a matroid $M^*(G)$ on the set of edges of a graph G by taking as circuits of $M^*(G)$ the cutsets of G. In view of Theorem 15c it would seem sensible to choose our definition of the dual of a matroid in such a way as to make this matroid the dual of the circuit matroid $M(G)$ of G.

This may be achieved as follows: if $M = (E, \rho)$ is a matroid defined in terms of its rank function, we define the **dual matroid** of M, denoted by M^*, to be the matroid on E whose rank function ρ^* is given by the expression

$$\rho^*(A) = |A| + \rho(E - A) - \rho(E), \text{ for } A \subseteq E.$$

We must first verify that ρ^* actually is the rank function of a matroid on E.

THEOREM 32A. *$M^* = (E, \rho^*)$ is a matroid on E.*

Proof. We verify the properties ($\rho\, i$) and ($\rho\, iii$) of **§30**, for the function ρ^*. The proof of ($\rho\, ii$) is equally straightforward, and will be left as an exercise.

To prove ($\rho\, i$), we note first that $\rho(E - A) \leqq \rho(E)$, and hence that $\rho^*(A) \leqq |A|$. Also, by ($\rho\, iii$) applied to the function ρ, we have $\rho(E) + \rho(\emptyset) \leqq \rho(A) + \rho(E - A)$, and hence that

$$\rho(E) - \rho(E - A) \leqq \rho(A) \leqq |A|.$$

It follows immediately that $\rho^*(\mathrm{A}) \geqq \mathrm{O}$.

To prove (ρ *iii*), we have, for any $A, B \subseteq E$,

$$\begin{aligned}\rho^*(A \cup B) + \rho^*(A \cap B) &= |A \cup B| + |A \cap B| + \rho(E - (A \cup B)) \\ &\qquad + \rho(E - (A \cap B)) - 2\rho(E) \\ &= |A| + |B| + \rho((E-A) \cap (E-B)) \\ &\qquad + \rho((E-A) \cup (E-B)) - 2\rho(E) \\ &\leqq |A| + |B| + \rho(E-A) + \rho(E-B) - 2\rho(E) \\ &\qquad \text{(by } (\rho\ iii)\text{, applied to } \rho) \\ &= \rho^*(A) + \rho^*(B), \text{ as required.//}\end{aligned}$$

Although the above definition seems highly contrived, it turns out that the bases of M^* can be described very simply in terms of the bases of M, as we now show:

THEOREM 32B. *The bases of M^* are precisely the complements of the bases of M.*

Remark. This result is often used to define M^*.

Proof. We shall show that if B^* is a base of M^*, then $E - B^*$ is a base of M; the converse result is obtained by simply reversing the argument.

Since B^* is independent in M^*, $|B^*| = \rho^*(B^*)$, and hence $\rho(E - B^*) = \rho(E)$. It thus remains only to prove that $E - B^*$ is independent in M. But this follows immediately from the fact that $\rho^*(B^*) = \rho^*(E)$, on using the above expression for ρ^*.//

As an immediate consequence of the above definition, we observe that, in contrast to the duality of planar graphs, *every matroid has a dual and this dual is unique*. It also follows immediately from Theorem 32B that *the double-dual M^{**} is equal to M*. In fact, as we shall see, this completely trivial result is the natural generalization to matroids of the (non-trivial) results of Theorems 15B and 15E.

We shall now show that the cutset matroid $M^*(G)$ of a graph G is the dual of the circuit matroid $M(G)$:

THEOREM 32C. *If G is a graph, then $M^*(G) = (M(G))^*$.*

Proof. Since the circuits of $M^*(G)$ are the cutsets of G, we must check that C^* is a circuit of $(M(G))^*$ if and only if C^* is a cutset of G.

Suppose first that C^* is a cutset of G. If C^* is independent in $(M(G))^*$, then C^* can be extended to a base B^* of $(M(G))^*$. It follows that $C^* \cap (E - B^*)$ is empty, contradicting the result of Theorem 9C since $E - B^*$ is a spanning forest of G. It follows that C^* is a dependent set in $(M(G))^*$, and thus contains a circuit of $(M(G))^*$.

If, on the other hand, D^* is a circuit of $(M(G))^*$, then D^* is not contained in any base of $(M(G))^*$. It follows that D^* intersects every base of $M(G)$—i.e., every spanning forest of G. Hence, by the result of exercise 9h, D^* contains a cutset. The result follows.//

Before proceeding further, it will be convenient to introduce some more terminology. We shall say that a set of elements of a matroid M form a **cocircuit** of M if they form a circuit of M^*. Note that in view of Theorem 32C *the cocircuits of the circuit matroid of a graph G are precisely the cutsets of G*. We can similarly define a **cobase** of M to be a base of M^*, with corresponding definitions for **corank, co-independent set**, etc. We shall also say that a matroid M is **cographic** if and only if its dual M^* is graphic, and in view of Theorem 32C this definition agrees with the one given in the previous section. The reason for introducing this 'co-notation' is that we may now restrict ourselves to a single matroid M without having to bring in M^*. To illustrate this, we shall prove the analogue for matroids of Theorem 9C.

THEOREM 32D. *Every cocircuit of a matroid intersects every base.*

Proof. Let C^* be a cocircuit of a matroid M, and suppose that there exists a base B of M with the property that $C^* \cap B$ is empty. Then C^* is contained in $E-B$, and so C^* is a circuit of M^* which is contained in a base of M^*. This contradiction establishes the result.//

COROLLARY 32E. *Every circuit of a matroid intersects every cobase.*

Proof. Apply the result of Theorem 32D to the matroid M^*.//

Note that by taking a matroid point of view, the two results in Theorem 9C turn out to be dual forms of a single result. Thus, instead of proving two results in graph theory (as we had to in §9), it is sufficient to prove a single result in matroid theory and then use duality. Not only does this represent a considerable saving of time and effort, it also gives us greater insight into several of the problems we have encountered earlier in the book. One example of this is the often-mentioned similarity between the properties of circuits and cutsets. Another is a deeper understanding of duality in planar graphs.

As a further example of the simplification introduced by matroid theory, let us look again at exercise 5k. A straightforward proof of this result would involve two separate operations—a proof for circuits and a different proof for cutsets. If, however, we prove the matroid analogue of the result for circuits (as stated in exercise 30g), then we can simply apply it to the matroid $M^*(G)$, and immediately obtain the corresponding result for cutsets. Conversely, we can use duality to deduce the result for circuits from the result for cutsets.

Let us now turn our attention to planar graphs, and in particular to the problem of showing how the definitions of a geometric-dual and an abstract-dual of a graph arise as consequences of duality in matroids. It can also be shown that the Whitney-dual of a graph, introduced in exercise 15k, arises as a consequence of matroid duality. The equation given in exercise 15k is simply a restatement of the expression for ρ^* given at the beginning of this section.

We shall start with the abstract-dual:

THEOREM 32F. *If G^* is an abstract-dual of a graph G, then $M(G^*)$ is isomorphic to $(M(G))^*$.*

Proof. Since G^* is an abstract-dual of G, there is a one-one correspondence between the edges of G and those of G^* with the property that circuits in G correspond to cutsets in G^* and conversely. It follows immediately from this that the circuits of $M(G)$ correspond to the cocircuits of $M(G^*)$, and hence, by Theorem 32C, that $M(G^*)$ is isomorphic to $M^*(G)$, as required.//

COROLLARY 32G. *If G^* is a geometric-dual of a connected planar graph G, then $M(G^*)$ is isomorphic to $(M(G))^*$.*

Proof. This result follows immediately from Theorems 32F and 15C.//

Note that (as remarked before) a planar graph can have several different duals, whereas a matroid can have only one. The reason for this is the easily-checked fact that if G^* and G^x are two (possibly non-isomorphic) duals of G, then the circuit matroids of G^* and G^x are isomorphic matroids.

We conclude this section by giving an answer to the question, 'under what conditions is a given matroid M graphic?' It is not difficult to find necessary conditions. For example, it follows from our discussion of representable matroids (§**31**) that such a matroid must be binary. Furthermore, by exercise 31j and our discussion of the Fano matroid F, it is clear that M cannot contain as a minor any of the matroids $M^*(K_5)$, $M^*(K_{3,3})$, F or F^*. It was shown in the following deep theorem by Tutte that these necessary conditions are in fact sufficient. The proof of this result is too difficult to be given here (see Welsh[25]).

THEOREM 32H (Tutte 1958). *A matroid M is graphic if and only if it is binary and contains no minor isomorphic to $M^*(K_5)$, $M^*(K_{3,3})$, F or F^*.//*

On applying Theorem 32H to M^*, and using the fact that the dual of a binary matroid is binary, we obtain necessary and sufficient conditions for a matroid to be cographic.

COROLLARY 32I. *A matroid M is cographic if and only if it is binary and contains no minor isomorphic to $M(K_5)$, $M(K_{3,3})$, F or F^*.//*

Tutte also proved that *a binary matroid is regular if and only if it contains no minor isomorphic to F or F**. By combining this result with the results of Theorem 32H and Corollary 32I, we immediately deduce the following matroid analogue of Kuratowski's theorem (Theorem 12C).

THEOREM 32J. *A matroid is planar if and only if it is regular and contains no minor isomorphic to $M(K_5)$, $M(K_{3,3})$ or their duals.*//

Exercises 32

(32a) (*i*) Show that the dual of a discrete matroid is a trivial matroid.
(*ii*) What is the dual of the k-uniform matroid on a set of n elements?

(32b) Find the duals of the eight matroids on the set $E=\{a, b, c\}$, obtained in exercise 31b.

(32c) Verify property (ρ *ii*) of §**30** for the function ρ^*.

(32d) Verify the result of Theorem 32C for the graph K_3.

(32e) What are the cocircuits and cobases of
(*i*) the 3-uniform matroid on a set of nine elements;
(*ii*) the circuit matroids of the graphs in Fig. 30.1;
(*iii*) the circuit matroid of the graph in Fig. 31.2;
(*iv*) the Fano matroid?

(32f) Show by an example that the dual of a transversal matroid is not necessarily transversal.

(32g) Show that the contraction matroid $M \, . \, A$ is the matroid whose cocircuits are precisely those cocircuits of M which are contained in A.

(*32h) Show that if C is any circuit and C^* is any cocircuit in a matroid, then $|C \cap C^*| \neq 1$.
(This is the generalization to matroids of exercise 51.)

(*32i) Let M be a binary matroid on a set E.
(*i*) Prove that if M is an Eulerian matroid, then M^* is bipartite.
(*ii*) Use induction on $|E|$ to prove the converse result.
(*iii*) By considering the 5-uniform matroid on a set of eleven elements, show that the word 'binary' cannot be omitted.
(This exercise generalizes exercise 15i.)

§33. Matroids and transversal theory

We showed in the previous section that there is a close connexion between results in matroid theory and in graph theory. The connexion between matroid theory and transversal theory will now be described. Our first aim is to show how the proofs of several of the earlier results on transversal theory may be considerably simplified by taking a matroid-theoretic point of view.

You will recall that if E is a non-empty finite set and $\mathscr{S}=(S_1, \ldots, S_m)$ is a family of non-empty subsets of E, then the partial transversals of $\mathscr{S}$ may be taken as the independent sets of a matroid on E, denoted by $M(S_1, \ldots, S_m)$. In this matroid, the rank of a subset A of E is simply the size of the largest partial transversal of $\mathscr{S}$ contained in A.

Our first example of the use of matroids in transversal theory is a proof of the result of exercise 26j that a family $\mathscr{S}$ of subsets of E has a transversal containing a given subset A if and only if (*i*) $\mathscr{S}$ has a transversal, and (*ii*) A is a partial transversal of $\mathscr{S}$. It is clear that both of these conditions are necessary. To prove that they are sufficient, it is enough to observe that since A is a partial transversal of $\mathscr{S}$, A must be an independent set in the transversal matroid M determined by $\mathscr{S}$ and so can be extended to a base of M. Since $\mathscr{S}$ has a transversal, every base of M must be a transversal of $\mathscr{S}$, and the result follows immediately. If you have worked through exercise 26h, you will realize how much simpler this argument is.

Before showing how matroid theory can be used to simplify the proof of Theorem 27C on the existence of a common transversal of two families of subsets of a set E, we shall prove a natural extension to matroids of Hall's theorem. We recall that if $\mathscr{S}$ is a family of subsets of E, then Hall's theorem gave a necessary and sufficient condition for $\mathscr{S}$ to have a transversal. If we also have a matroid structure defined on E, then it is reasonable to ask whether there is a corresponding condition for the existence of an **independent transversal**—i.e., a transversal of $\mathscr{S}$ which is also an independent set in the matroid. The following theorem, known as **Rado's theorem**, answers this question.

THEOREM 33A (Rado 1942). *Let M be a matroid on a set E, and let $\mathscr{S} = (S_1, \ldots, S_m)$ be a family of non-empty subsets of E. Then $\mathscr{S}$ has an independent transversal if and only if the union of any k of the subsets S_i contains an independent set of size at least k, for $1 \leqq k \leqq m$.*

Remark. If M is the discrete matroid on E, then this theorem reduces to Hall's theorem as stated in Theorem 26A.

Proof. We shall imitate the proof of Theorem 26A. As before, the necessity of the condition is clear, and it is thus sufficient to prove that if one of the subsets (S_1, say) contains more than one element, then we can remove an element from S_1 without altering the condition. By repeating this procedure, we eventually reduce the problem to the case in which each subset contains only one element, the proof then being trivial.

To show the validity of the reduction procedure, we suppose that S_1 contains elements x and y, the removal of either of which invalidates the condition. Then there are subsets A and B of $\{2, 3, \ldots, m\}$ with the property that

$$\rho(P) \leqq |A| \quad \text{and} \quad \rho(Q) \leqq |B|,$$

where

$$P = \bigcup_{j \in A} S_j \cup (S_1 - \{x\}), \quad \text{and} \quad Q = \bigcup_{j \in B} S_j \cup (S_1 - \{y\}).$$

Then

$$\rho(P \cup Q) = \rho(\bigcup_{j \in A \cup B} S_j \cup S_1) \quad \text{and} \quad \rho(P \cap Q) \geqslant \rho(\bigcup_{j \in A \cap B} S_j).$$

The required contradiction now follows, since

$$\begin{aligned} |A| + |B| &\geqq \rho(P) + \rho(Q) \\ &\geqq \rho(P \cup Q) + \rho(P \cap Q) \\ &\geqq \rho\left(\bigcup_{j \in A \cup B} S_j \cup S_1\right) + \rho\left(\bigcup_{j \in A \cap B} S_j\right) \\ &\geqq (|A \cup B| + 1) + |A \cap B| \text{ (by Hall's condition)} \\ &= |A| + |B| + 1.// \end{aligned}$$

By imitating the proof of Corollary 26B, we immediately obtain the following result:

COROLLARY 33B. *With the above notation, $\mathscr{S}$ has an independent partial transversal of size t if and only if the union of any k of the subsets S_i contains an independent set of size at least $k + t - m$.*//

We can now give a matroid-theoretic proof of Theorem 27C on the existence of a common transversal of two families of subsets of a given set.

THEOREM 27C. *Let E be a non-empty finite set, and let $\mathscr{S} = (S_1, \ldots, S_m)$ and $\mathscr{S} = (T_1, \ldots, T_m)$ be two families of non-empty subsets of E. Then $\mathscr{S}$ and $\mathscr{S}$ have a common transversal if and only if, for all subsets A and B of $\{1, 2, \ldots, m\}$,*

$$\left|\left(\bigcup_{i \in A} S_i\right) \cap \left(\bigcup_{j \in B} T_j\right)\right| \geqq |A| + |B| - m.$$

Proof. Let $M = (E, \rho)$ be the matroid whose independent sets are precisely the partial transversals of the family $\mathscr{S}$. Then $\mathscr{S}$ and $\mathscr{T}$ have a common transversal if and only if $\mathscr{T}$ has an independent transversal. But by Theorem 33A, this is so if and only if the union of any k of the sets T_i contains an independent set of size at least k (for $1 \leqq k \leqq m$)—in other words, if and only if the union of any k of the sets T_i contains a partial transversal of $\mathscr{S}$ of size k. The result now follows from Corollary 26C.//

We conclude this section with some results on the union of matroids. If $M_1, M_2, \ldots, M_k$ are matroids on the same set E, then we can define a new matroid $M_1 \cup M_2 \cup \ldots \cup M_k$, called their **union**, by taking as independent sets all possible unions of an independent set in M_1, an independent set in $M_2, \ldots$, and an independent set in M_k. The rank of this matroid is given by the following theorem, whose proof may be found in Welsh[25]:

THEOREM 33C. *If $M_1, \ldots, M_k$ are matroids on a set E with rank functions $\rho_1, \ldots, \rho_k$ respectively, then the rank function ρ of $M_1 \cup \ldots \cup M_k$ is given by*

$$\rho(X) = \min_{A \subseteq X}\{\rho_1(A) + \ldots + \rho_k(A) + |X - A|\}.//$$

This result yields two deep results in graph theory, as we now show:

COROLLARY 33D. *Let $M = (E, \rho)$ be a matroid. Then M contains k disjoint bases if and only if, for each subset A of E,*

$$k\rho(A) + |E - A| \geqq k\rho(E).$$

Proof. M contains k disjoint bases if and only if the union of k copies of the matroid M has rank at least $k\rho(E)$. The result now follows immediately from Theorem 33C.//

COROLLARY 33E. *Let $M = (E, \rho)$ be a matroid. Then E can be expressed as the union of k independent sets if and only if, for each subset A of E, $k\rho(A) \geqq |A|$.*

Proof. In this case, the union of k copies of the matroid M has rank $|E|$. It thus follows immediately from Theorem 33C that $k\rho(A) + |E - A| \geqq |E|$, as required.//

If we apply these last corollaries to the circuit matroid $M(G)$ of a graph G, we immediately obtain necessary and sufficient conditions for G to contain k edge-disjoint spanning forests, and for G to split up into k forests. It turns out, in fact, that these results are not at all easy to obtain by more direct methods, and we have thus once again demonstrated the power of the theory of matroids in solving problems in graph theory.

THEOREM 33F. *A graph G contains k edge-disjoint spanning forests if and only if, for any subgraph H of G,*

$$k(\xi(G) - \xi(H)) \leqq m(G) - m(H),$$

where $m(H)$ and $m(G)$ denote the number of edges of H and G respectively.//

THEOREM 33G. *A graph G may be split up into k forests if and only if, for any subgraph H of G, $k\xi(H) \geqq m(H)$.*//

Exercises 33

(33a) Verify Rado's theorem when M is the Fano matroid, as described on page 146, and $\mathscr{S}=(\{1\}, \{1, 2\}, \{2, 4, 5\})$.

(33b) Verify Corollary 33D when M is the 3-uniform matroid on a set of 8 elements.

(33c) Verify Corollary 33E when M is the 4-uniform matroid on a set of 9 elements.

(*33d) By modifying the Halmos–Vaughan proof of Hall's theorem (see §**25**), give an alternative proof of Theorem 33A.

(33e) Show that a matroid M is a transversal matroid if and only if M can be expressed as the union of matroids of rank 1.

(33f) Dualize the results of Theorems 33F and 33G to obtain two further results in graph theory.

Postscript

Only a little more
I have to write,
Then I'll give o'er
And bid the world Goodnight.

Robert Herrick

Although we have now almost reached the end of the book, we have by no means reached the end of the subject. It is our hope that many of you will wish to continue your graph-theoretic studies, and for this reason, we thought it might be helpful if we suggested possible directions for further reading.

If you are interested primarily in 'pure' graph theory, you should consult the books by Chartrand and Lesniak,[10] Berge,[4] Bondy and Murty,[7] Harary,[14] and Ore.[21] Also worth reading are the specialist books of Harary and Palmer[16] on enumeration, Moon[19,20] on tournaments and trees, Ringel[22] on the genus of graphs, Fiorini and Wilson[13] on edge-colourings of graphs, and Saaty and Kainen[24] on general colouring problems. Surveys on a wide variety of topics in graph theory may also be found in Beineke and Wilson.[3] For a discussion of the various applications of graph theory, you are recommended to look at the surveys in Wilson and Beineke,[26] and the books of Berge,[4] Harary, Norman and Cartwright[15] (digraphs), Lawler[17] (operations research), and Roberts[23] (social sciences). For a general all-round introduction to combinatorial theory, including techniques of enumeration and block designs, there are good books by Bogart,[6] Anderson,[1] and Brualdi.[8] For material on transversal theory and matroids you should consult Lawler,[17] Mirsky,[18] Bryant and Perfect,[9] and Welsh.[25]

Sooner or later, you may need to refer to mathematical journals rather than to books. There are a large number of journals which frequently include papers in graph theory and related fields, and there are even some—for example, the *Journal of Graph Theory*, the *Journal of Combinatorial Theory*, the *European Journal of Combinatorics*, *Ars Combinatoria*, and *Discrete Mathematics*—which are aimed at specialists in these fields.

Finally, if you are interested in the history of graph theory, there is an excellent account in Biggs, Lloyd and Wilson[5] (although I say it myself!). In this book, as well as the historical and biographical material, there are about forty lengthy extracts from the most important papers in the history of the subject. You would be well advised to spend some time reading through these original papers.

Here is my journey's end.

William Shakespeare (*Macbeth*)

Appendix

This table lists the number of graphs and digraphs of various types with a given number n of vertices. Numbers greater than one million are given to one significant figure.

Types of graph	$n=1$	2	3	4	5	6	7	8
Simple graphs	1	2	4	11	34	156	1044	12 346
Connected simple graphs	1	1	2	6	21	112	853	11 117
Eulerian simple graphs	1	0	1	1	4	8	37	184
Hamiltonian simple graphs	1	0	1	3	8	48	383	6020
Trees	1	1	1	2	3	6	11	23
Labelled trees	1	1	3	16	125	1296	16 807	262 144
Connected simple planar graphs	1	1	2	6	20	105	?	?
Simple digraphs	1	3	16	218	9608	$\sim 2\times 10^6$	$\sim 9\times 10^8$	$\sim 2\times 10^{12}$
Connected simple digraphs	1	2	13	199	9364	$\sim 2\times 10^6$	$\sim 9\times 10^8$	$\sim 2\times 10^{12}$
Strongly-connected simple digraphs	1	1	5	83	5048	$\sim 1\times 10^6$	$\sim 7\times 10^8$	$\sim 2\times 10^{12}$
Tournaments	1	1	2	4	12	56	456	6880
Matroids (on n elements)	2	4	8	17	38	?	?	?

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Of making many books there is no end; and much study is a weariness of the flesh.

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Index of symbols

I copied all the letters in a big round hand.

W. S. Gilbert

A	adjacency matrix
$A(D)$	arc-family of D
B	base of M
C_n	circuit graph
$cr(G)$	crossing-number of G
D	digraph
E	non-empty finite set
$E(G)$	edge-set of G
f	number of faces
F	Fano matroid
g	genus
G	graph
$\bar{G}$	complement of G
G^*	dual of G
$G(V_1, V_2)$	bipartite graph
$G_1 \cup G_2$	union of graphs
I	independent set of M
k	number of components
K_n	complete graph,
$K_{r,s}$	complete bipartite graph
$K_{r,s,t}$	complete tripartite graph
$L(G)$	line graph of G
m	number of edges
M	matroid
M^*	dual matroid
$M.A$	contraction matroid
$M \times A$	restriction matroid
$M(G)$	circuit matroid
$M(S_1, \ldots, S_m)$	transversal matroid
n	number of vertices
N_n	null graph
$P_G(k)$	chromatic polynomial of G
Q_k	k-cube,
$t(G)$	thickness of G
T	tree
$u, v\ w, z$	vertices of G
$v_0 \rightarrow \ldots \rightarrow v_m$	walk
V	vector space associated with G
$V(D)$	vertex-set of D
$V(G)$	vertex-set of G
W	circuit subspace of G
$\tilde{W}$	cutset subspace of G
W_n	wheel
α, β, γ	colours
$\gamma(G)$	circuit rank of G
$\Gamma(G)$	automorphism group of G
$\kappa(G)$	connectivity of G
$\lambda(G)$	edge-connectivity of G
$\xi(G)$	cutset rank of G
ρ	largest vertex-degree of G
ρ	rank function of M
ρ^*	rank function of M^*
$\rho(v)$	degree of v
$\overleftarrow{\rho}\ (v)$	in-degree of v
$\vec{\rho}\ (v)$	out-degree of v
$\chi(G)$	chromatic number of G
$\chi'(G)$	chromatic index of G
$\mathscr{B}$	bases of M
$\mathscr{C}$	circuits of M
$\mathscr{I}$	independent sets of M
$\mathscr{S}$	family of subsets

Index of definitions

I've got a little list.

W. S. Gilbert

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