F. GERRISH

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PURE MATHEMATICS II

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F. GERRISH

PURE MATHEMATICS

VOLUME II

ALGEBRA

TRIGONOMETRY

COORDINATE GEOMETRY

This two-volume text-book on Pure Mathematics has been designed to cover the whole of the mathematics required for Part I of the London B.Sc. General Degree and for any first-year degree courses containing subsidiary pure mathematics. It is the only book of its kind addressed primarily to the General Degree student. It also covers most of the Pure Mathematics required by the recently introduced Diploma in Mathematics of the Mathematical Association.

This book does not assume a complete mastery of earlier work, and much of it is suitable for the pure mathematics required by grammar school pupils preparing for scholarships in Natural Sciences, as well as for the mathematics specialists. However, though Mr Gerrish has aimed to supply a real need of examination students at these levels, he has treated his subject in a serious fashion and this book is in no sense a cram book for examinations. Thus he goes a little beyond the requirements of the examination syllabus.

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Finally the book contains many sets of 'exercises for the reader', often with hints for solution. Each chapter includes a Miscellaneous Exercise for revision purposes. Answers are also provided.

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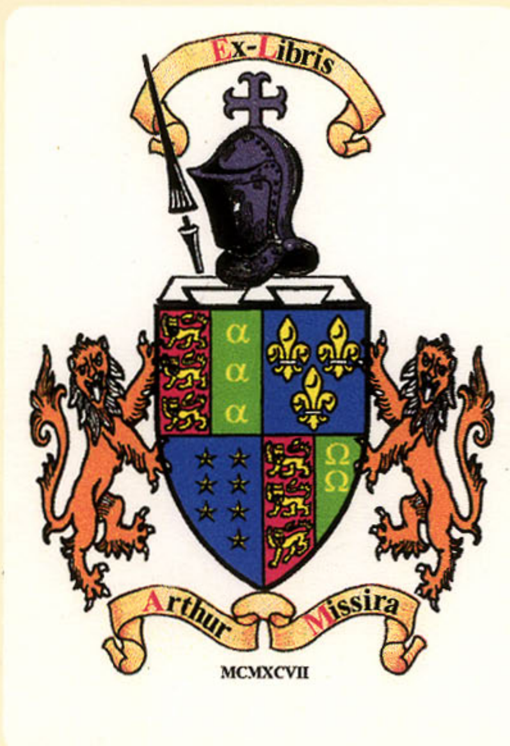

PURE MATHEMATICS

A UNIVERSITY AND COLLEGE COURSE

BY

F. GERRISH, M.A.

VOLUME II

ALGEBRA, TRIGONOMETRY
COORDINATE GEOMETRY

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CONTENTS

GENERAL PREFACE

This two-volume text-book on Pure Mathematics has been designed to cover completely the requirements of the revised regulations for the B.Sc. General Degree (Part I) of the University of London. It presents a serious treatment of the subject, written to fill a gap which has long been evident at this level. The author believes that there is no other book addressed primarily to the General Degree student which covers the ground with the same self-contained completeness and thoroughness, while also indicating the way to further progress. On the principle that 'the correct approach to any examination is from above', the book has been constructed so that those students who do not intend to take the subject Mathematics in Part II of their degree course will find included some useful matter a little beyond the prescribed syllabus (which throughout has been interpreted as an examination schedule rather than a teaching programme); while those who continue with Mathematics will have had sound preparation. As it is the author's experience that many students who begin a degree course have received hasty and inadequate training, a complete knowledge of previous work has *not* been assumed.

Although written for the purpose just mentioned, this book will meet the needs of those taking any course of first-year degree work in which Pure Mathematics is studied, whether at University or Technical College. For example, most of the Pure Mathematics required for a one-year ancillary subject to the London Special Degrees in Physics, Chemistry, etc. is included, and also that for the first of the two years' work ancillary to Special Statistics. The relevant matter for Part I (and some of Part II) of the B.Sc. Engineering Degree is covered. The book provides an introduction to the first year of an Honours Degree in Mathematics at most British universities, and would serve as a basis for the work of the mathematical specialist in the Grammar School. Much of the material is suitable for pupils preparing for scholarships in Natural Sciences.

By a natural division the subject-matter falls conveniently into two volumes which, despite occasional cross-references, can be used independently as separate text-books on Calculus (Vol. I) and on Algebra, Trigonometry and Coordinate Geometry (Vol. II). According to the plan of study chosen, the contents may be dealt with in turn, or else split up into two or even three courses of reading in Calculus, Algebra-Trigonometry and Geometry taken concurrently. Throughout it has been borne in mind that many students necessarily work without much direct supervision, and it is hoped that those of even moderate ability will be able to use this book alone.

A representative selection of worked examples, with explanatory remarks, has been included as an essential part of the text, together with many sets of 'exercises for the reader' spread throughout each chapter and *carefully graded* from easy applications of the bookwork to 'starred' problems (often with hints for solution) slightly above the ultimate standard required. In a normal use of the book there will not be time or need to work through every 'ordinary' problem in each set; but some teachers welcome a wide selection. To each chapter is appended a Miscellaneous Exercise, both backward- and forward-looking in scope, for revision purposes. Answers are provided at the end of each volume. It should be clear that, although practice in solving problems is an important part of the student's training, in no sense is this a cram-book giving drill in examination tricks. However, those who are pressed for time (as so many part-time and evening students in the Technical Colleges unfortunately are) may have to postpone the sections in small print and all 'starred' matter for a later reading.

Most of the problems of 'examination type' have been taken from Final Degree papers set by the University of London, and I am grateful to the Senate for permission to use these questions. Others have been collected over a number of years from a variety of unrecorded (and hence unacknowledged) sources, while a few are home-made.

It is too optimistic to expect that a book of this size will be completely free from typographical errors, or the Answers from mathematical ones, despite numerous proof readings. I shall be grateful if readers will bring to my notice any such corrections or other suggestions for possible improvements.

Finally, I thank the staff of the Cambridge University Press for the way in which they have met my requirements, and for the excellence of their printing work.

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PREFACE TO VOLUME II

Although for convenience of reference the pagination and section numbering are continued from Volume I, this does not imply dependence of the present volume on the first. Apart from occasional backward references, Volume II is a self-contained text-book on Algebra, Trigonometry and Coordinate Geometry of two and three dimensions, in which calculus methods are illustrated when instructive.

Beginning with a chapter which leads from revision of previous algebraic work to an introduction to formal algebra, an early start on determinants can be made in the next, thereby assembling all the equipment necessary for the subsequent geometry (which throughout is real and euclidean). Certain widely-used general theorems on systems of linear equations have received more explicit statement and emphasis than is customary at this level.

In Chapter 12 the passage from finite to infinite series lays the foundations of 'convergence', a subtle subject so often misunderstood and mishandled by beginners. It is hoped that the many somewhat negative cautionary remarks will stimulate rather than shake the reader's confidence.

Chapter 13 sets complex numbers on a logical footing by first briefly retracing the historical steps in their development from 'mystery' through 'diagram' to the concept of 'ordered number-pair'. The subject is often approached from only one of these standpoints, but the present inclusive treatment combines the advantages of all. The opportunity is also taken of setting up a *general* theory of factorisation and polynomial equations. Despite some repetition, it is felt that this can be appreciated only *after* the provisional nature of the conclusions in 10.13, 10.3 has been realised. The following chapter, concerned with trigonometrical applications of the preceding algebraic theories, also introduces some genuine functions of a complex variable.

An anticipated criticism of the book is that complex numbers and the complex exponential have been introduced too late, with a consequent loss of freedom of method in the Calculus section in topics such as the solution of linear differential equations. This delay was intentional, and could almost be claimed as a special feature; for the author believes that (with the exception of confessedly symbolic methods in which 'anything is fair') only confusion of principles can arise by incautiously mixing the real and the complex, especially in calculus techniques.

Since many students come to the course regarding Coordinate Geometry as merely a mixture of 'graphs', Pure Geometry, and easy Calculus, an attempt in Chapter 15 has been made to review the 'known' parts of the subject more as an illustrated account of linear and (real) quadratic algebra. Use is made of oblique axes when appropriate.

In the next three chapters, which contain a fairly detailed treatment of the parabola, ellipse, and hyperbola by means of their 'standard' equations, the emphasis is on parametric methods and the consequent elegant applications of the (real) theory of equations given in Chapter 10. Joachimsthal's ratio equation and the distance quadratic are also used incidentally. Owing to the algebraic analogy between ellipse and hyperbola, the discussion of the latter is centred mainly on properties of the asymptotes.

The unifying influence of a systematic treatment of 'the general conic' by Joachimsthal's method is too valuable to omit. This and the powerful '$s = ks'$ principle' are the themes of Chapter 19.

A short chapter on polar equations, first revising the straight line and circle in this form, develops in more detail their application to the conic, and thus gives the geometrical complement of the calculus methods illustrated for various 'polar' curves in Volume I.

In principle, Chapter 21 returns to the fundamentals of cartesian coordinate geometry, this time for three dimensions. The treatment, designed to emphasise whenever possible the analogies between the two- and three-dimensional cases, while also pointing out important contrasts, revises the methods of linear algebraic geometry, and may well be left until fairly late in the course. Finally the sphere is briefly treated, first geometrically by methods resembling those already used for the circle in Chapter 15, and then trigonometrically.

REFERENCES AND ABBREVIATIONS

In a decimal reference such as 12.73(2),

12	denotes	chapter (Ch. 12),
12.7	denotes	section,
12.73	denotes	sub-section,
12.73(2)	denotes	part.

(ii)	refers to	equation (ii) in the *same* section.
ex. (ii)	refers to	worked example (ii) in the *same* section.
4.64, ex. (ii)	refers to	worked example (ii) in sub-section 4.64.
Ex. 12 (*b*), no. 6	refers to	problem number 6 in Exercise 12 (*b*).
wo	means	with respect to.

In the text, matter in small type (*other than 'ordinary' worked examples*) and in 'starred' worked examples is subsidiary, and may be omitted at a first reading if time is short.

In an exercise

no. 6	refers to	problem number 6 in the *same* Exercise.
a 'starred' problem		*either* depends on matter in small type in the text, or on ideas in a later chapter; *or* is above the general standard of difficulty.
matter in [...]	is	a hint for the solution of a problem.
matter in (...)	is	explanatory comment.

10

ALGEBRA OF POLYNOMIALS

10.1 The remainder theorem and some consequences

10.11 Long division; identities

The reader will be familiar with the process for dividing one polynomial by another of lower degree; we use it here to lead up to an important algebraic theorem.

Example

Find the quotient and remainder obtained by dividing ax^2+bx+c *by* $x-k$.

$$\begin{array}{rlll} x-k\ ) & ax^2+bx & +c & (\ ax+(b+ak) \\ & \underline{ax^2-akx} & & \\ & (b+ak)\,x+c & & \\ & \underline{(b+ak)\,x-k(b+ak)} & & \\ & \qquad ak^2+bk+c & & \end{array}$$

Observe that the remainder is the original expression with x replaced by k.

If only the remainder is required in the above example, we can proceed as follows. Let the quotient be $Q(x)$ (a function of x) and the remainder be R (a constant). Then we have

$$ax^2+bx+c = (x-k)\,Q(x)+R. \qquad \text{(i)}$$

If we now put $x = k$ in this, we obtain

$$ak^2+bk+c = R \qquad \text{(ii)}$$

since the term involving $Q(k)$ is zero.

Remarks

(α) It may be objected that the division process has established relation (i) for all values of x *except* k, so that the substitution $x = k$ is not justified. The validity of (i) for *all* x can be shown by the following method, which also identifies R. We have

$$(ax^2+bx+c)-(ak^2+bk+c) = a(x^2-k^2)+b(x-k),$$

so that the right-hand side has the factor $x-k$. If the other factor is denoted by $Q(x)$, then

$$(ax^2+bx+c)-(ak^2+bk+c) = (x-k)\,Q(x),$$

which is equivalent to (i) with R given by (ii).

(β) We may regard the 'long division' as a process for constructing from the given polynomials $x-k$ and ax^2+bx+c another polynomial $Q(x)$ and a constant R which satisfy (i) *for all values of* x. (A similar principle, justified in 10.13, ex. (v), holds for any pair of polynomials, although when the divisor is of degree $n > 1$, R will be a polynomial of degree $n-1$ or less.)

The relation (i), which holds for *all* values of x, is called an *identity* in x (in contrast to an *equation* in x, which holds only for certain special values of x). The sign $\equiv$ between two polynomials in x is used to denote 'equal for all values of x', and is read 'identically equal to'. Thus (i) would be written

$$ax^2+bx+c \equiv (x-k)\,Q(x)+R. \qquad \text{(i)}'$$

The result of substituting $x = k$ would still be written with the ordinary $=$ sign because (ii) is a relation between special numbers only. However, we may continue to use the sign $=$ even when the expressions are in fact identically equal, unless we specially wish to emphasise their identity.

10.12 The remainder theorem

We now obtain the remainder when the general polynomial

$$p(x) = p_0x^n+p_1x^{n-1}+\ldots+p_{n-1}x+p_n$$

is divided by $x-k$. We have

$$p(x)-p(k) = p_0(x^n-k^n)+p_1(x^{n-1}-k^{n-1})+\ldots+p_{n-1}(x-k). \qquad \text{(iii)}$$

By direct multiplication we can verify that, for any positive integer m and any x and k,

$$x^m-k^m \equiv (x-k)\,(x^{m-1}+x^{m-2}k+x^{m-3}k^2+\ldots+k^{m-1}).$$

(When $x \neq k$, this is equivalent to the sum-formula for the G.P. in the second bracket.) Thus $x-k$ is a factor of the right-hand side of (iii). Denoting the other factor† by $Q(x)$, we have

$$p(x)-p(k) \equiv (x-k)\,Q(x). \qquad \text{(iv)}$$

† Which will be a polynomial in x of degree $n-1$ if $n > 1$.

Hence *the remainder when the polynomial $p(x)$ is divided by $x-k$ is the number $p(k)$ obtained by substituting k for x in this polynomial.*

Also from (iv) we have

THE FACTOR CASE. *If $p(k) = 0$, the polynomial $p(x)$ has $x-k$ for a factor.* Conversely, *if $x-k$ is a factor, then $p(k) = 0$.*

Examples

(i) *Factorise* $2x^3-11x^2+17x-6$.

Calling the polynomial $p(x)$, we seek numbers k for which $p(k) = 0$. Only those numbers k which are factors of the last term 6 need be tried, for any other number could not be the constant term in a factor of the polynomial.†

$$p(1) = 2-11+17-6 \neq 0, \quad \therefore \quad x-1 \text{ is not a factor.}$$

$$p(2) = 16-44+34-6 = 0, \quad \therefore \quad x-2 \text{ is a factor.}$$

To find the other factor, divide the polynomial by $x-2$; the quotient is $2x^2-7x+3$, so

$$p(x) = (x-2)(2x^2-7x+3),$$

$$= (x-2)(x-3)(2x-1)$$

on completing the factorisation by inspection.

(ii) *Find the values of a and b if $6x^3+ax^2+bx-2$ is divisible (i.e. exactly divisible, without remainder) by $2x^2+x-1$.*

Since $2x^2+x-1 = (x+1)(2x-1)$, both $x+1$ and $2x-1$ are factors of the given polynomial $p(x)$. Hence $p(-1) = 0$ and $p(\frac{1}{2}) = 0$, which give

$$0 = a-b-8, \quad 0 = \tfrac{1}{4}a+\tfrac{1}{2}b-\tfrac{5}{4},$$

and so $a = 7$, $b = -1$.

(iii) *Find the remainder when $p(x)$ is divided by $(x-a)(x-b)$, $a \neq b$, where $p(x)$ has degree greater than* 2.

Since the divisor is quadratic in x, the remainder will in general be linear in x, say $Ax+B$. If $Q(x)$ is the quotient, then (cf. 10.11, Remark (β))

$$p(x) \equiv (x-a)(x-b)Q(x)+Ax+B.$$

Put $x = a$: $\qquad p(a) = Aa+B.$

Put $x = b$: $\qquad p(b) = Ab+B.$

Solving these equations for A and B, we find

$$A = \frac{p(a)-p(b)}{a-b}, \quad B = \frac{ap(b)-bp(a)}{a-b},$$

so the remainder is

$$\frac{\{p(a)-p(b)\}x+ap(b)-bp(a)}{a-b}.$$

† A general result of this kind is proved in 13.62(1).

10.13 Factorisation of a polynomial; 'equating coefficients'

We continue to write

$$p(x) = p_0x^n + p_1x^{n-1} + \ldots + p_n,$$

where $p_0 \neq 0$.

THEOREM I. *If $p(x)$ is zero when x has any one of the n distinct values $a_1, a_2, \ldots, a_n$, then*

$$p(x) \equiv p_0(x-a_1)(x-a_2)\ldots(x-a_n). \qquad \text{(v)}$$

Proof. Since $p(a_1) = 0$ by hypothesis, therefore $x - a_1$ is a factor of $p(x)$. Let the quotient when $p(x)$ is divided by $x-a_1$ be $Q_{n-1}(x)$; it will be a polynomial in x of degree $n-1$ whose first term is p_0x^{n-1}. Then

$$p(x) \equiv (x-a_1)\,Q_{n-1}(x).$$

Since $p(a_2) = 0$ by hypothesis, we have

$$0 = (a_2 - a_1)\,Q_{n-1}(a_2).$$

As $a_2 \neq a_1$ (also by hypothesis), hence $Q_{n-1}(a_2) = 0$ and therefore $x-a_2$ is a factor of $Q_{n-1}(x)$. The other factor $Q_{n-2}(x)$, obtained by division, is a polynomial whose first term is p_0x^{n-2}; thus

$$Q_{n-1}(x) \equiv (x-a_2)\,Q_{n-2}(x),$$

and so

$$p(x) \equiv (x-a_1)(x-a_2)\,Q_{n-2}(x).$$

We can continue step by step to remove the factors $x-a_3, \ldots$, until we reach $x-a_n$, after which the other factor will be $Q_0(x)$ whose only term is $p_0x^0 = p_0$. The result (v) follows.

COROLLARY I (*a*). *A polynomial of degree n in x cannot be zero for more than n distinct values of x.*

Proof. Expression (v) cannot be zero when x takes any value different from $a_1, a_2, \ldots, a_n$; for no factor would then be zero, and $p_0 \neq 0$ by hypothesis.

THEOREM II. *If $p_0x^n + p_1x^{n-1} + \ldots + p_n$ is zero for more than n distinct values of x, then $p_0 = p_1 = \ldots = p_n = 0$, and so $p(x) \equiv 0$.*

Proof. Either all of $p_0, p_1, \ldots, p_n$ are zero,

or there is a first p which is not zero, say p_k $(k < n)$. In this case the expression reduces to

$$p_kx^{n-k} + p_{k+1}x^{n-k-1} + \ldots + p_n \quad (p_k \neq 0),$$

which is a polynomial of degree $n-k$ in x. Hence by Corollary I (*a*), it cannot be zero for more than $n-k$ values of x. However, we are given that it is zero for more than n values of x. Our second alternative thus leads to a contradiction, and so only the first alternative is possible.

Finally, when $p_0 = p_1 = \dots = p_n = 0$, the given polynomial is zero for all x, i.e. $p(x) \equiv 0$.

COROLLARY II (*a*). *If* $p_0x^n + p_1x^{n-1} + \dots + p_n \equiv 0$, *then*

$$p_0 = p_1 = \dots = p_n = 0.$$

For since the polynomial is identically zero, it is zero for more than n distinct values of x.

COROLLARY II (*b*). *If*

$$p_0x^n + p_1x^{n-1} + \dots + p_n = q_0x^n + q_1x^{n-1} + \dots + q_n$$

for more than n distinct values of x, or for all values of x, then

$$p_0 = q_0, \quad p_1 = q_1, \quad \dots, \quad p_n = q_n.$$

For we have

$$(p_0 - q_0)x^n + (p_1 - q_1)x^{n-1} + \dots + (p_n - q_n) = 0,$$

and the results follow from Theorem II or Corollary II (*a*).

Corollary II (*b*) is the basis of the principle† of 'equating coefficients' in an identity between polynomials. It asserts that if two polynomials of degree n are equal for all values of x (i.e. if the polynomials have *identical values* for corresponding values of x), then they agree term by term (i.e. they are *identical in form*). Without this result it would be conceivable that two quite distinct polynomials might take the same values for the same x, as x ranges over the real numbers.

COROLLARY II (*c*). *If*

$$p_0x^n + p_1x^{n-1} + \dots + p_n = q_0x^m + q_1x^{m-1} + \dots + q_m$$

(*where* $m > n$) *for more than m distinct values of x, or for all values of x, then*

$$0 = q_0, \quad \dots, \quad 0 = q_{m-n-1}, \quad p_0 = q_{m-n}, \quad \dots, \quad p_n = q_m.$$

In particular, the polynomial on the right has degree n.

The proof is like that of Corollary II (*b*).

Examples

(i) *Find constants a, b, c, d for which*

$$n^3 \equiv an(n+1)(n+2) + bn(n+1) + cn + d.$$

† Already illustrated in 4.62.

The right-hand side

$$\equiv a(n^3+3n^2+2n)+b(n^2+n)+cn+d$$
$$\equiv an^3+(3a+b)n^2+(2a+b+c)n+d.$$

This is identically equal to n^3 if and only if

$$a = 1,\quad 3a+b = 0,\quad 2a+b+c = 0,\quad d = 0,$$

i.e. $$a = 1,\quad b = -3,\quad c = 1,\quad d = 0.$$

(ii) *Show that x^2 cannot be expressed in the form*

$$a(x+1)(x+2)+b(x+3)(x+4).$$

The expression $\equiv a(x^2+3x+2)+b(x^2+7x+12)$

$$\equiv (a+b)x^2+(3a+7b)x+(2a+12b),$$

which is identical with x^2 if and only if

$$a+b = 1,\quad 3a+7b = 0,\quad 2a+12b = 0.$$

The last two equations give $a = 0$, $b = 0$, and these values fail to satisfy the first. The three conditions cannot be satisfied, and so x^2 cannot be written in the form stated.

(iii) *If a, b, c are all distinct, prove that*

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)} \equiv 1,$$

and deduce relations between a, b, c by equating coefficients of the various powers of x.

When $x = a$, the left-hand side reduces to 1. Hence the polynomial

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}-1$$

is zero when $x = a$. Similarly, it is zero when $x = b$ and when $x = c$. Thus this *quadratic* in x vanishes for *three* distinct values of x. Hence it vanishes for all values of x.

By equating coefficients of x^2, x and the constant terms, we obtain

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)} = 0,$$
$$\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)} = 0,$$
$$\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)}+\frac{ab}{(c-a)(c-b)} = 1.$$

(iv) *If ax^3+bx^2+cx+d contains $(x+1)^2$ as a factor, obtain relations between a, b, c, d.*

If $(x+1)^2$ is a factor, the other factor must be *linear* and of the form $a(x+k)$. Hence

$$ax^3+bx^2+cx+d \equiv a(x+1)^2(x+k)$$
$$\equiv ax^3+a(k+2)x^2+a(2k+1)x+ak,$$

$$\therefore\quad b = a(k+2),\quad c = a(2k+1),\quad d = ak.$$

By eliminating k from the first and third, and from the second and third,

$$b = d + 2a, \quad c = 2d + a.$$

*(v) *Given two polynomials $f(x)$, $g(x)$ of degrees m, n ($m > n$), prove that there is a unique pair of polynomials $Q(x)$, $R(x)$ such that $R(x)$ has degree less than n and*

$$f(x) \equiv g(x)\,Q(x) + R(x). \qquad (a)$$

The process of dividing $f(x)$ by $g(x)$ gives a quotient $Q(x)$ of degree $m-n$ and a remainder $R(x)$ of degree less than n such that (a) holds for all values of x *except possibly those for which* $g(x) = 0$. Thus the polynomial

$$f(x) - \{g(x)\,Q(x) + R(x)\},$$

whose degree certainly does not exceed m, is zero for more than m values of x. Hence by Theorem II it is *identically* zero, which proves (a). Compare 10.11, Remark (β).

It is conceivable that, had we proceeded in a different way, we might have obtained another pair of polynomials $q(x)$, $r(x)$ such that

$$f(x) \equiv g(x)\,q(x) + r(x), \qquad (b)$$

where $r(x)$ has degree less than n. Subtraction of (b) from (a) gives

$$R(x) - r(x) \equiv g(x)\{q(x) - Q(x)\}. \qquad (c)$$

The left side of (c) has degree $<n$, while (unless $q(x) - Q(x) \equiv 0$) the right has degree $\geqslant n$ (the degree of $g(x)$); and by Corollary II (c) this is impossible. Hence $q(x) - Q(x) \equiv 0$, and therefore by (c), $R(x) - r(x) \equiv 0$. This proves the uniqueness.

Exercise 10(*a*)

Factorise

1 $2x^3 + 3x^2 - 1$. 2 $6x^3 - x^2 - 19x - 6$. 3 $x^4 + 2x^3 + x^2 - 4$.

4 Solve $x^3 - 4x^2 + x + 6 = 0$.

5 Find the values of a and b if $6x^3 + 7x^2 + ax + b$ is divisible by $2x - 1$ and by $x + 1$.

6 The remainder when $(x-1)(x-2)$ divides $x^4 + ax^3 + b$ is $x + 1$. Find a and b.

7 A polynomial gives remainder $2x + 5$ when divided by $(x-1)(x+2)$. Find the remainders when it is divided by $x - 1$, $x + 2$ separately.

8 A polynomial gives remainder 2 when divided by $x + 1$ and remainder 1 when divided by $x - 4$. Find the remainder when it is divided by $(x+1)(x-4)$.

*9 A cubic polynomial gives remainders $5x - 7$, $12x - 1$ when divided by $x^2 - x + 2$, $x^2 + x - 1$ respectively. Find the polynomial.

*10 When divided by $x^2 + 1$ a polynomial gives remainder $2x + 3$, and by $x^2 + 2$ gives remainder $x + 2$. Find the remainder when it is divided by $(x^2 + 1)(x^2 + 2)$.

Find values of a, b, c, d for which

11 $n^3 + 5n \equiv an(n-1)(n-2) + bn(n-1) + cn + d$.

12 $x^3 \equiv a(x+2)^3 + b(x+1)^2 + cx + d$.

13 If $x^4 - 6x^3 + ax^2 + 30x + b$ is a perfect square, find a and b and write down the square roots of the polynomial.

14 Express $\dfrac{2x+7}{(x-1)(x+2)}$ in the form $\dfrac{a}{x-1}+\dfrac{b}{x+2}$.

15 Express $\dfrac{5}{(2x+1)(x^2+1)}$ in the form $\dfrac{a}{2x+1}+\dfrac{bx+c}{x^2+1}$.

If a, b, c are all distinct, prove

16 $\dfrac{a(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{b(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{c(x-a)(x-b)}{(c-a)(c-b)}\equiv x.$

17 $\dfrac{(a+b+x)(b+c+x)}{(b-c)(b-a)}+\dfrac{(b+c+x)(c+a+x)}{(c-a)(c-b)}+\dfrac{(c+a+x)(a+b+x)}{(a-b)(a-c)}\equiv 1.$

18 (i) Prove that there cannot be two different quadratic expressions in x which take the values A, B, C when x has the distinct values a, b, c respectively.

(ii) Verify that

$$A\frac{(x-b)(x-c)}{(a-b)(a-c)}+B\frac{(x-c)(x-a)}{(b-c)(b-a)}+C\frac{(x-a)(x-b)}{(c-a)(c-b)}$$

has the required properties, and deduce from (i) that this is the *only* such quadratic.

*19 Write down the (unique) cubic polynomial in x which takes the values A, B, C, D when x has the distinct values a, b, c, d respectively.

20 (i) If x^3+px+q contains a factor $(x-a)^2$, prove $4p^3+27q^2=0$.

(ii) By writing $q=2a^3$, prove the converse of (i).

21 (i) If x^4+px+q contains a factor $(x-a)^2$, prove $27p^4=256q^3$, and express q in terms of a.

(ii) Prove the converse of (i).

10.2 Polynomials in more than one variable

10.21 Extension of the preceding results

If $p(x, y)$ is a polynomial in the two variables x, y, we may arrange it according to powers of x, say as

$$x^n p_0(y)+x^{n-1}p_1(y)+\dots+p_n(y),$$

and consider it to be a polynomial in x whose coefficients are polynomials in y. The preceding theories can then be applied.† Similar considerations hold for more than two variables.

Examples

(i) *If* $ax^2+2hxy+by^2+2gx+2fy+c\equiv a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'$, *prove*

$$a=a',\quad b=b',\quad c=c',\quad f=f',\quad g=g',\quad h=h'.$$

Arranging both expressions as quadratics in x,

$$ax^2+2(hy+g)x+(by^2+2fy+c)\equiv a'x^2+2(h'y+g')x+(b'y^2+2f'y+c').$$

† The proof in 10.12 is valid when $p_0, p_1, \dots, p_n$ are polynomials in y (or in any number of variables); therefore the deductions in 10.13 still hold.

Equating coefficients, we have for *all* y:

$$a = a', \quad 2(hy+g) = 2(h'y+g'), \quad by^2+2fy+c = b'y^2+2f'y+c'.$$

By equating coefficients of like powers of y in the last two identities, we obtain all the results stated.

(ii) *Show that* $2x^2+5xy-3y^2-x+11y-6$ *has linear factors, and find them.*

Since

$$2x^2+5xy-3y^2 \equiv (x+3y)(2x-y),$$

try putting

$$2x^2+5xy-3y^2-x+11y-6 \equiv (x+3y+a)(2x-y+b),$$

where a and b are constants to be determined.

The right-hand side is

$$2x^2+5xy-3y^2+(2a+b)x+(3b-a)y+ab.$$

This will be identical with the given polynomial if and only if a, b can be chosen to satisfy

$$2a+b = -1, \quad 3b-a = 11, \quad ab = -6.$$

The first two equations give $a = -2$, $b = 3$; and these values do satisfy the third. Hence the given polynomial has linear factors

$$(x+3y-2)(2x-y+3).$$

(iii) *Factorise*

$$xy(x+y)+yz(y+z)+zx(z+x)+2xyz.$$

Regarding this as a quadratic polynomial in x whose coefficients are polynomials in y and z, put $x = -y$: the expression becomes

$$0+yz(y+z)-yz(z-y)-2y^2z = 0.$$

Hence $x+y$ is a factor. Similarly, $x+z$ is a factor.

Next, regarding the expression as a polynomial in y whose coefficients are polynomials in x, z, we may put $y = -z$ and show that $y+z$ is a factor.

Hence $(y+z)(z+x)(x+y)$ is a factor. Since this product and the given polynomial each have total degree 3, any other factor k must be numerical, and so

$$xy(x+y)+yz(y+z)+zx(z+x)+2xyz \equiv k(y+z)(z+x)(x+y).$$

To obtain k, we may *either* substitute numerical values for x, y, z (e.g. $x = 0$, $y = 1$, $z = 1$) in both sides, *or* compare coefficients (e.g. of x^2y). We obtain $k = 1$.

10.22 Symmetric, skew and cyclic functions

In 1.52 (4) we defined a *homogeneous function* of two or more variables, and we now give further useful definitions.

(1) A function (not necessarily a polynomial) of two or more variables is *symmetrical* in these variables if it is unaltered by the interchange of any two.

For example, $x+y$ and x^2+y^2 are symmetrical functions of (x, y); $a+b+c$, $bc+ca+ab$ are symmetrical in (a, b, c); and the expression in 10.21, ex. (iii) is symmetrical in (x, y, z). The polynomials $x-y+z$, $a^2+b^2+2c^2$ are not symmetrical.

Remarks

(α) *The only symmetrical function of* (x, y, z) *of the first degree is a constant multiple of* $x+y+z$. For if $lx+my+nz$ is unaltered by interchange of x, y, then

$$lx+my+nz \equiv ly+mx+nz,$$

i.e.

$$(l-m)x+(m-l)y \equiv 0,$$

so $l = m$. Similarly, we find $l = n$, and the linear function reduces to $l(x+y+z)$.

(β) *The most general symmetric polynomial in* (x, y, z) *consisting of second-degree terms can likewise be shown to have the form*

$$k(x^2+y^2+z^2)+l(yz+zx+xy).$$

(2) A function of two or more variables is *skew* or *alternating* if an interchange of any two of them changes only the sign of the function.

For example, $x-y$ is a skew function of (x, y); and

$$(b-c)(c-a)(a-b),$$

$$xy(x-y)+yz(y-z)+zx(z-x)$$

are skew functions of their respective variables.

The reader should satisfy himself that *the product of two symmetric or of two skew functions is a symmetric function*, while *the product of a symmetric and a skew function is skew.*

Considerations of symmetry, skewness and homogeneity often save algebraic manipulation, and will be helpful in Ch. 11.

Example

Factorise $x^3+y^3+z^3-3xyz$. (Also see Ex. 10 (f), no. 3.)

Regarding this as a polynomial in x, we find that when $x = -(y+z)$ the expression is zero. Hence $x+y+z$ is a factor. The other factor could be found directly by long division, or as follows.

The given expression and the factor $x+y+z$ are both symmetric in (x, y, z), therefore so is the remaining factor.

Since the given polynomial is homogeneous of degree 3 and $x+y+z$ is homogeneous of degree 1, the other factor must be homogeneous of degree 2 in (x, y, z).

By Remark (β) above, the most general symmetric homogeneous polynomial of degree 2 in (x, y, z) has the form

$$k(x^2+y^2+z^2)+l(yz+zx+xy),$$

and so

$$x^3+y^3+z^3-3xyz \equiv (x+y+z)\{k(x^2+y^2+z^2)+l(yz+zx+xy)\}.$$

We can find the constants k, l by substituting numerical values:

$$x=0,\quad y=0,\quad z=1 \quad \text{give} \quad 1=k;$$

$$x=0,\quad y=1,\quad z=1 \quad \text{give} \quad 2=2\{2k+l\}, \qquad \therefore\quad l=-1.$$

Hence $\quad x^3+y^3+z^3-3xyz \equiv (x+y+z)(x^2+y^2+z^2-yz-zx-xy).$

(3) *Cyclic expressions.* Consider again

$$xy(x-y)+yz(y-z)+zx(z-x),$$

and suppose the letters x, y, z are arranged around the circumference of a circle as shown. When the first term $xy(x-y)$ is given, the next can be obtained from it by replacing each letter by the one which follows it on the circle. Repetition of the process gives the third term, and further repetition leads back to the first term. The given sum thus consists of the term $xy(x-y)$ together with the two similar terms obtainable from it by *cyclic interchange* of x, y, z; it can be written

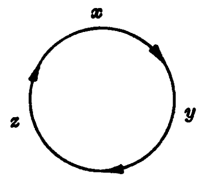

Fig. 123

$$\Sigma xy(x-y).$$

The *number* of letters involved has either to be stated explicitly, or must be clear from the context; otherwise the Σ-notation is ambiguous.

Observe that the complete sum is unaltered if we replace x by y, y by z, and z by x; for this cyclic interchange merely alters the order in which the three terms occur. The expression $(y-z)(z-x)(x-y)$ has the same property.

Definition. An expression in x, y, z which is unaltered by cyclic interchange of these letters is *cyclic* in (x, y, z).

A similar definition can be given when there are more than three letters. For convenience of reference and comparison, it is desirable to write expressions with their terms in cyclic order whenever possible. Thus we prefer to write $bc+ca+ab$ rather than $ab+ac+bc$.

Remark (γ). To test an expression for symmetry or skewness we interchange letters *two at a time*. A cyclic interchange involves change of *all* the letters. Thus a function may be cyclic but not symmetric, e.g. $\Sigma xy(x-y)$, $bc^2+ca^2+ab^2$. See also Ex. 10(*b*), no. 16.

Exercise 10(*b*)

Prove that the following expressions have linear factors, and find them.

1 $2x^2-3xy-2y^2+7x+6y-4.$ 2 $x^2-y^2+2xz-14yz-48z^2.$

3 Find the values of a for which $2x^2-5xy-3y^2-x+ay-3$ has linear factors. [The factors must be of the form $(x-3y+3k)(2x+y-1/k)$.]

4 Show that $x^2+4xy+3y^2+2x-2y+6$ does not possess linear factors.

Using the remainder theorem, together with considerations of degree, symmetry and skewness when helpful, factorise

5 $(x+y+z)^3-(x^3+y^3+z^3)$.

6 $yz(y^2-z^2)+zx(z^2-x^2)+xy(x^2-y^2)$.

7 $x(y-z)^3+y(z-x)^3+z(x-y)^3$.

8 $(x+y+z)^5-(x^5+y^5+z^5)$.

9 Write the expressions in nos. 6, 7 in the Σ-notation.

10 Write in full the following expressions, *assumed to involve three letters*:

(i) Σx^2; (ii) Σx^2y^2; (iii) $\Sigma bc(b-c)$; (iv) Σa^2bc; (v) Σbc^2.

[In (v), cyclic interchange gives $bc^2+ca^2+ab^2$, which is only half the number of terms implied by Σ, meaning 'sum of all terms of the type "letter × another letter squared"'.]

11 Prove that (i) $\Sigma(b-c) = 0$; (ii) $\Sigma bc(b-c) = \Sigma a^2(b-c)$;

(iii) $\Sigma a(b^2-c^2) = (b-c)(c-a)(a-b)$.

12 Prove $\Sigma(b-c)^3 = 3(b-c)(c-a)(a-b)$.

13 Prove $\Sigma a^3(b^2-c^2) = -(b-c)(c-a)(a-b)(bc+ca+ab)$.

14 Prove that $\Sigma a^n(b-c)$ contains the factor $(b-c)(c-a)(a-b)$ for any positive integer $n \geqslant 2$.

*15 (i) By taking $x = b-c$, $y = c-a$, $z = a-b$ in the example in 10.22(2), deduce no. 12. (ii) Similarly, deduce the factors of $\Sigma a^3(b-c)^3$. (iii) Use the result of no. 8 to factorise $(b-c)^5+(c-a)^5+(a-b)^5$.

*16 Prove that a symmetric or skew function of three variables is also cyclic. [If $f(a,b,c)$ is symmetric,

$$f(a,b,c) = f(b,a,c) = f(b,c,a);$$

if it is skew, $f(a,b,c) = -f(b,a,c) = -\{-f(b,c,a)\} = f(b,c,a)$.]

10.3 Polynomial equations: relations between roots and coefficients

10.31 Quadratics: a summary

If the equation $ax^2+bx+c = 0$ has roots α, β (possibly equal), the reader will know that

$$\alpha+\beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

He will have used these symmetrical relations to calculate the values of other symmetric functions of α and β (such as $\alpha^2+\beta^2$, $\alpha/\beta+\beta/\alpha$), and to construct quadratics having prescribed functions of α and β as roots (e.g. 'form the equation whose roots are $3\alpha-\beta$, $3\beta-\alpha$'). We now extend this work to cubic and quartic equations.

10.32 Theory of cubic equations

If the distinct numbers α, β, γ satisfy

$$ax^3+bx^2+cx+d = 0, \tag{i}$$

then by Theorem I of 10.13,

$$ax^3+bx^2+cx+d \equiv a(x-\alpha)(x-\beta)(x-\gamma). \tag{ii}$$

If the unequal numbers α, β satisfy (i), then by the Remainder Theorem $x-\alpha$ and $x-\beta$ are factors of the left-hand side, and by division the other factor is seen to be linear and of the form $a(x-\gamma)$. By the statement 'α, β, β are the roots of (i)' we mean that $\gamma = \beta$, so that the factor $x-\beta$ appears twice in (ii). Similarly, by 'α, α, α are the roots of (i)' we mean that the right-hand side of (ii) is $a(x-\alpha)^3$.

Thus, if α, β, γ (not necessarily distinct) are the roots of (i), then (ii) holds in all cases. Expanding the right-hand side by direct multiplication,

$$ax^3+bx^2+cx+d \equiv ax^3-a(\alpha+\beta+\gamma)x^2+a(\beta\gamma+\gamma\alpha+\alpha\beta)x-a\alpha\beta\gamma.$$

Equating coefficients, we find

$$\alpha+\beta+\gamma = -\frac{b}{a}, \quad \beta\gamma+\gamma\alpha+\alpha\beta = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}. \tag{iii}$$

Conversely, the cubic having roots $x = \alpha, \beta, \gamma$ is

$$(x-\alpha)(x-\beta)(x-\gamma) = 0,$$

i.e. $$x^3-(\alpha+\beta+\gamma)x^2+(\beta\gamma+\gamma\alpha+\alpha\beta)x-\alpha\beta\gamma = 0,$$

in which

	the coefficient of x^3	is	$+1$,
	the coefficient of x^2	is	$-$(sum of roots),
	the coefficient of x	is	(sum of the products of the roots taken in pairs),
and	the constant term	is	$-$(product of roots).

Observe the sequence $+ - + -$ of the signs.

Remark. The relations (iii) do not help us to *solve* the cubic equation, because elimination of (say) β and γ from them leads to

$$a\alpha^3+b\alpha^2+c\alpha+d = 0;$$

they are equivalent to the information that 'α, β and γ are the roots of (i)'.

Examples

(i) *If α, β, γ are the roots of $px^3+qx+r=0$, express in terms of p, q, r:*

$$(a)\ \Sigma\alpha^2,\quad (b)\ \Sigma\frac{1}{\alpha},\quad (c)\ \Sigma\alpha^3,\quad (d)\ \Sigma\alpha^5,\quad (e)\ \Sigma\beta^2\gamma.$$

We have $$\Sigma\alpha = 0,\quad \Sigma\beta\gamma = \frac{q}{p},\quad \alpha\beta\gamma = -\frac{r}{p}.$$

(*a*) $$\Sigma\alpha^2 \equiv (\Sigma\alpha)^2 - 2\Sigma\beta\gamma = 0 - 2\frac{q}{p} = -\frac{2q}{p}.$$

(*b*) $$\Sigma\frac{1}{\alpha} = \frac{\Sigma\beta\gamma}{\alpha\beta\gamma} = -\frac{q}{r}.$$

(*c*) *First method.* Since α is a root of the given equation,

$$p\alpha^3 + q\alpha + r = 0;$$

there are similar relations for β, γ. Adding,

$$p\Sigma\alpha^3 + q\Sigma\alpha + 3r = 0,$$

$$\therefore\quad \Sigma\alpha^3 = -\frac{3r}{p}.$$

Second method. By the example in 10.22 (2),

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma \equiv (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta)$$
$$= 0,\quad \text{since } \Sigma\alpha = 0.$$

$$\therefore\quad \Sigma\alpha^3 = 3\alpha\beta\gamma = -\frac{3r}{p}.$$

(*d*) From the given equation, $p\alpha^3+q\alpha+r=0$, and hence

$$p\alpha^5 + q\alpha^3 + r\alpha^2 = 0.$$

Adding this to the two similar relations for β, γ,

$$p\Sigma\alpha^5 + q\Sigma\alpha^3 + r\Sigma\alpha^2 = 0.$$

$$\therefore\quad p\Sigma\alpha^5 - \frac{3qr}{p} - \frac{2qr}{p} = 0$$

by using (*a*) and (*c*), and so $\Sigma\alpha^5 = 5qr/p^2$.

(*e*) Consider $(\Sigma\alpha)(\Sigma\beta\gamma)$. A term like $\beta^2\gamma$ occurs only once, as the product $\beta.\beta\gamma$. The terms $\alpha\beta\gamma$ arises in three ways, from $\alpha.\beta\gamma$, $\beta.\gamma\alpha$, $\gamma.\alpha\beta$. Hence

$$(\Sigma\alpha)(\Sigma\beta\gamma) \equiv \Sigma\beta^2\gamma + 3\alpha\beta\gamma.$$

$$\therefore\quad 0 = \Sigma\beta^2\gamma - \frac{3r}{p},\quad \text{and}\quad \Sigma\beta^2\gamma = \frac{3r}{p}.$$

(ii) *Form the cubic whose roots are* (*a*) $\beta+\gamma, \gamma+\alpha, \alpha+\beta$; (*b*) $\beta\gamma/\alpha, \gamma\alpha/\beta, \alpha\beta/\gamma$, *where* α, β, γ *are the roots of* $x^3+x^2-24x-16=0$.

We have $$\Sigma\alpha = -1,\quad \Sigma\beta\gamma = -24,\quad \alpha\beta\gamma = 16.$$

(*a*) $\beta+\gamma = \Sigma\alpha - \alpha = -1-\alpha$. Similarly $\gamma+\alpha = -1-\beta$, $\alpha+\beta = -1-\gamma$. We require the cubic whose roots are $-1-\alpha, -1-\beta, -1-\gamma$.

In the given equation put $y = -1-x$, i.e. $x = -1-y$:

$$(-1-y)^3+(-1-y)^2-24(-1-y)-16 = 0,$$

i.e.
$$y^3+2y^2-23y-8 = 0. \qquad \text{(A)}$$

The values of y which satisfy this are related to the values of x which satisfy the given equation by the formula $y = -1-x$. Since these values of x are α, β, γ, hence the values of y are $-1-\alpha$, $-1-\beta$, $-1-\gamma$; therefore (A) is the required cubic. (We could also say that

$$x^3+2x^2-23x-8 = 0$$

is the required equation, because the letter used for the unknown is immaterial when the roots are *given*.)

(*b*) $\beta\gamma/\alpha = \alpha\beta\gamma/\alpha^2 = 16/\alpha^2$. Put $y = 16/x^2$, i.e. $x^2 = 16/y$:

$$x\left(\frac{16}{y}-24\right) = 16-\frac{16}{y},$$

so by squaring,
$$64x^2\left(\frac{2}{y}-3\right)^2 = 16^2\left(1-\frac{1}{y}\right)^2,$$

i.e.
$$\frac{4}{y}\left(\frac{2}{y}-3\right)^2 = \left(1-\frac{1}{y}\right)^2,$$

which reduces to
$$y^3-38y^2+49y-16 = 0.$$

The argument used in (*a*) shows that this is the required equation; but see the Remark at the end of 13.53.

10.33 Quartic equations

By reasoning as in 10.32 we find that if $\alpha, \beta, \gamma, \delta$ are the roots of

$$ax^4+bx^3+cx^2+dx+e = 0,$$

then
$$\alpha+\beta+\gamma+\delta = -\frac{b}{a}, \quad \text{i.e.} \quad \Sigma\alpha = -\frac{b}{a},$$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}, \quad \text{i.e.} \quad \Sigma\alpha\beta = \frac{c}{a},$$

$$\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta+\alpha\beta\gamma = -\frac{d}{a}, \quad \text{i.e.} \quad \Sigma\alpha\beta\gamma = -\frac{d}{a},$$

and
$$\alpha\beta\gamma\delta = \frac{e}{a}.$$

Again notice the alternation of the signs.

Applications of the theory given in 10.31–10.33 to coordinate geometry are illustrated in 16.22 (2); 16.26, ex. (i); 16.32; 16.12, ex. (ii); and elsewhere.

Exercise 10(*c*)

1 If α, β are the roots of $x^2-5x+3=0$, calculate

$$\text{(i) } \alpha^2\beta+\alpha\beta^2;\quad \text{(ii) } \frac{1}{\alpha}+\frac{1}{\beta};\quad \text{(iii) } (\alpha-1)(\beta-1);\quad \text{(iv) } \alpha^2+\beta^2;$$

$$\text{(v) } (\alpha-\beta)^2;\quad \text{(vi) } \frac{\alpha}{\beta}+\frac{\beta}{\alpha};\quad \text{(vii) } \alpha^3+\beta^3;\quad \text{(viii) } \alpha^4+\beta^4.$$

2 If α, β are the roots of $3x^2+2x-4=0$, construct the equation whose roots are (i) $1/\alpha$, $1/\beta$; (ii) α^2, β^2; (iii) $\alpha+1/\beta$, $\beta+1/\alpha$.

3 Write down the conditions for both roots of $ax^2+bx+c=0$ to be positive.

4 If α, β, γ are the roots of $x^3+x^2-14x-24=0$, calculate

$$\text{(i) } (\alpha+1)(\beta+1)(\gamma+1);\quad \text{(ii) } \Sigma\frac{1}{\alpha};\quad \text{(iii) } \Sigma\alpha^2;\quad \text{(iv) } \Sigma\frac{\beta\gamma}{\alpha};$$

$$\text{(v) } \Sigma\alpha^3;\quad \text{(vi) } \Sigma\alpha^4;\quad \text{(vii) } \Sigma\alpha^2\beta.$$

5 If α, β, γ are the roots of $x^3+4x^2+x-2=0$, construct the equation whose roots are

$$\text{(i) } \beta+\gamma,\ \gamma+\alpha,\ \alpha+\beta;\quad \text{(ii) } \frac{1}{\beta\gamma},\ \frac{1}{\gamma\alpha},\ \frac{1}{\alpha\beta};\quad \text{(iii) } \alpha(\beta+\gamma),\ \beta(\gamma+\alpha),\ \gamma(\alpha+\beta);$$

$$\text{(iv) } \alpha^2,\ \beta^2,\ \gamma^2;\quad \text{(v) } \alpha^{-2},\ \beta^{-2},\ \gamma^{-2}.$$

6 If α, β, γ are the roots of $6x^3-x^2-6x-2=0$, form the equation whose roots are $\alpha+\frac{1}{2}$, $\beta+\frac{1}{2}$, $\gamma+\frac{1}{2}$. Hence solve the given equation.

7 If $x^3+3Hx+G=0$ has two roots equal, prove $G^2+4H^3=0$. [Let the roots be α, α, β; then $2\alpha+\beta=0$.]

8 If the roots of $x^3+3Hx+G=0$ are in A.P., prove $G=0$. [$\alpha+\gamma=2\beta$.]

9 If $x^3+px^2+qx+r=0$, find the condition for the roots to be in (i) A.P.; (ii) G.P.

10 If α, β, γ are the roots of $x^3+px^2+qx+r=0$, construct the equation whose roots are $\alpha^2-\beta\gamma$, $\beta^2-\gamma\alpha$, $\gamma^2-\alpha\beta$, and comment on the special cases (i) $p=0$, (ii) $q=0$. Also write down the value of $(\alpha^2-\beta\gamma)(\beta^2-\gamma\alpha)(\gamma^2-\alpha\beta)$.

$$\left[\alpha^2-\beta\gamma=\alpha^2+\frac{r}{\alpha}=\frac{\alpha^3+r}{\alpha}=\frac{-p\alpha^2-q\alpha}{\alpha}=-(p\alpha+q).\right]$$

11 If $x^4+px^3+q=0$ has roots α, β, γ, δ, construct the equation whose roots are $\alpha+\beta+\gamma$, $\beta+\gamma+\delta$, $\gamma+\delta+\alpha$, $\delta+\alpha+\beta$.

12 If $x^4+px^2+qx+r=0$ has three equal roots, prove that $p^2+12r=0$ and $9q^2=32pr$; and show that the value of the repeated root is $-3q/4p$.

10.4 Elimination

10.41 Further examples

When a system of equations is given and the number of equations is greater than the number of unknowns, then in general the equations cannot all be satisfied unless the coefficients are related in some way.

There may be more than one such relation. Each is called an *eliminant* of the system of equations. Owing to the importance of elimination,† especially in coordinate geometry, we give some further examples. No general methods can be laid down; but see 11.43.

(i) *Use of an identity.*

Eliminate x, y, z from

$$x+y+z=a, \quad x^2+y^2+z^2=b^2, \quad x^3+y^3+z^3=c^3, \quad xyz=d^3.$$

From the identity (10.22, ex.)

$$x^3+y^3+z^3-3xyz \equiv (x+y+z)(x^2+y^2+z^2-yz-zx-xy),$$

we have $$c^3-3d^3=a(b^2-\Sigma yz);$$

and $$(\Sigma x)^2=\Sigma x^2+2\Sigma yz, \quad \text{so} \quad \Sigma yz=\tfrac{1}{2}(a^2-b^2).$$

Hence $$2(c^3-3d^3)=a(3b^2-a^2).$$

(ii) *Use of theory equations.*

Eliminate m, n from $(y-mx)^2=a^2m^2+b^2$, $(y-nx)^2=a^2n^2+b^2$, $mn=-1$.

The first two equations show that m and n are the roots of the quadratic in t

$$(y-tx)^2=a^2t^2+b^2,$$

i.e. $$(a^2-x^2)t^2+2xyt+(b^2-y^2)=0.$$

The third equation shows that the product of these roots is -1, so

$$\frac{b^2-y^2}{a^2-x^2}=-1,$$

i.e. $$x^2+y^2=a^2+b^2.$$

For the geometrical interpretation see 17.47, (ii).

The method illustrated in ex. (ii) can sometimes be used to *solve* a system of equations as in ex. (iii) following. See also Ex. 10(*d*), nos. 12–14.

(iii) *Solve for x, y, z:*

$$x+ay+a^2z=a^4, \quad x+by+b^2z=b^4, \quad x+cy+c^2z=c^4.$$

These equations show that a, b, c are three roots of the quartic in t

$$t^4-t^2z-ty-x=0.$$

Since the coefficient of t^3 is zero, the sum of the roots is zero, and hence the remaining root is $-(a+b+c)$. Then

$$-x = \text{product of roots} = -abc(a+b+c);$$

$$y = \text{sum of the products of the roots taken in threes}$$
$$= abc-(a+b+c)(bc+ca+ab);$$

$$-z = \text{sum of the products of the roots taken in pairs}$$
$$= bc+ca+ab-(a+b+c)(a+b+c)$$
$$= -(a^2+b^2+c^2+bc+ca+ab).$$

Hence $$x=abc\Sigma a, \quad y=abc-(\Sigma a)(\Sigma bc), \quad z=\Sigma(a^2+bc).$$

† It has already been used in this book.

10.42 Common root of two equations

(1) *The necessary and sufficient condition that the quadratics*

$$ax^2+bx+c=0, \quad a'x^2+b'x+c'=0$$

have a common root is

$$(ab'-a'b)(bc'-b'c)=(ca'-c'a)^2. \tag{i}$$

Necessary. If there is a value of x which satisfies both equations, then by treating them as simultaneous equations in x^2, x and eliminating first x, and then x^2, we obtain

$$(ab'-a'b)x^2=bc'-b'c, \quad (ab'-a'b)x=ca'-c'a.$$

$$\therefore \quad (ab'-a'b)(bc'-b'c)=(ab'-a'b)^2x^2=(ca'-c'a)^2,$$

so that the condition follows.

Sufficient. If $ab'-a'b \neq 0$, the condition (i) can be written

$$\left(\frac{ca'-c'a}{ab'-a'b}\right)^2=\frac{bc'-b'c}{ab'-a'b}.$$

Put $\lambda=(ca'-c'a)/(ab'-a'b)$; then $\lambda^2=(bc'-b'c)/(ab'-a'b)$, and

$$(ab'-a'b)\lambda^2=bc'-b'c, \quad (ab'-a'b)\lambda=ca'-c'a.$$

$$\therefore \quad (ab'-a'b)(a\lambda^2+b\lambda)=a(bc'-b'c)+b(ca'-c'a)$$

$$=-c(ab'-a'b) \quad \text{on simplifying.}$$

Hence $(ab'-a'b)(a\lambda^2+b\lambda+c)=0$; and similarly

$$(ab'-a'b)(a'\lambda^2+b'\lambda+c')=0.$$

Since $ab'-a'b \neq 0$, these show that $x=\lambda$ satisfies both quadratics.

If $ab'-a'b=0$, then the given condition (i) shows $ca'-c'a=0$. Hence $a:b:c=a':b':c'$, and so the quadratics are not independent. In this case the result holds trivially.

Remark. Direct calculation will verify that

$$4\{(ab'-a'b)(bc'-b'c)-(ca'-c'a)^2\}$$

$$\equiv (b^2-4ac)(b'^2-4a'c')-(2ac'+2a'c-bb')^2.$$

The condition (i) is therefore equivalent to

$$(b^2-4ac)(b'^2-4a'c')=(2ac'+2a'c-bb')^2. \tag{ii}$$

(2) When the given equations are of degree higher than the second, we eliminate the highest powers step by step until two quadratics with a common root are obtained. For example, if

$$f \equiv ax^3+bx^2+cx+d = 0$$

and $$g \equiv px^2+qx+r = 0 \tag{iii}$$

have a common root, then this root also satisfies $p.f-ax.g = 0$, i.e.

$$(pb-aq)x^2+(pc-ar)x+pd = 0. \tag{iv}$$

Conversely, a common root of $g = 0$ and $p.f-ax.g = 0$ is also a common root of $g = 0$ and $f = 0$. Hence $f = 0$, $g = 0$ have a common root if and only if (iii) and (iv) have a common root, and a condition like (i) will express this fact.

10.43 Repeated roots

With the notation of 10.13, suppose that

$$p(x) \equiv (x-\alpha)^r g(x) \quad (1 < r \leqslant n),$$

where $g(\alpha) \neq 0$. Then by the Remainder Theorem $x-\alpha$ is *not* a factor of the polynomial $g(x)$. We say that the polynomial $p(x)$ has a *repeated factor* $x-\alpha$ *of order* r; and that the equation $p(x) = 0$ has *an r-fold root* $x = \alpha$, r *equal roots* α, *a root* $x = \alpha$ *of multiplicity* r, or *a root* $x = \alpha$ *of order* r. If $r = 1$, we call $x-\alpha$ a *simple factor* of $p(x)$, and $x = \alpha$ a *simple root* of $p(x) = 0$.

If $p(x) = 0$ *has a root* $x = \alpha$ *of order* r, *then* $x = \alpha$ *is also a root of* $p'(x) = 0$, *of order* $r-1$.

Proof.
$$p'(x) \equiv r(x-\alpha)^{r-1}g(x)+(x-\alpha)^r g'(x)$$
$$\equiv (x-\alpha)^{r-1}\{rg(x)+(x-\alpha)g'(x)\}.$$

Since $g(\alpha) \neq 0$, the contents of the last bracket are non-zero when $x = \alpha$. Hence $x = \alpha$ is a root of $p'(x) = 0$, of order exactly $r-1$.

Conversely, if $p'(x) = 0$ *has a root* $x = \alpha$ *of order* $r-1$, *then provided* $x = \alpha$ *satisfies* $p(x) = 0$, *it is a root of* $p(x) = 0$ *of order* r.

Proof. Suppose $x = \alpha$ is a root of $p(x) = 0$ of order s. Then the preceding theorem shows that α is a root of $p'(x) = 0$ of order $s-1$. Hence $r-1 = s-1$, i.e. $r = s$.

These two results show that *the necessary and sufficient condition for* $p(x) = 0$ *to have a repeated root is that* $p(x) = 0$, $p'(x) = 0$ *have a common root.*

Exercise 10(*d*)

Eliminate t from

1 $x = t - 1/t$, $y = t^2 + 1/t^2$. 2 $x = a(1-t^2)/(1+t^2)$, $y = 2bt/(1+t^2)$.

3 Eliminate x, y from $x+y = a$, $x^2+y^2 = b^2$, $x^4+y^4 = c^4$.

4 If $a^2+x^2 = b^2+y^2 = ay-bx = 1$, prove $a^2+b^2 = 1$.
[$(a^2+x^2)+(b^2+y^2)-2(ay-bx) = 0$, i.e. $(x+b)^2+(y-a)^2 = 0$, $\therefore$ $x = -b$ and $y = a$.]

5 If $a^3x+b^3y = xy$, $b^3x-a^3y = x^2-y^2$, and $x^2+y^2 = 1$, prove $a^2+b^2 = 1$. [Solve the first two equations for a^3, b^3, using the third.]

6 Eliminate x, y, z from $xy = a^2$, $yz = b^2$, $zx = c^2$, $x^2+y^2+z^2 = d^2$.

7 (i) Eliminate x, y from $y+lx = al^3+2al$, $y+mx = am^3+2am$, and $y+nx = an^3+2an$. *(ii) Interpret geometrically.

8 (i) Eliminate m, n from

$$m^2x-my+a = 0, \quad n^2x-ny+a = 0, \quad m-n = c(1+mn).$$

*(ii) Interpret geometrically.

9 Eliminate l, m, n from the equations in no. 7 and $lm = b$.

10 Eliminate λ, μ from

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda} = 1, \quad \frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu} = 1, \quad \lambda+\mu = a^2+b^2.$$

11 Solve the following equations for x, y, z:

$$x+ay+a^2z = a^3, \quad x+by+b^2z = b^3, \quad x+cy+c^2z = c^3.$$

12 Solve $a+b+c = 5$, $bc+ca+ab = 7$, $abc = 3$.

13 Solve $a+b+c = -2$, $a^2+b^2+c^2 = 30$, $abc = -10$.

14 Solve $x+y+z = 2$, $x^2+y^2+z^2 = 14$, $x^3+y^3+z^3 = 20$.

15 Prove that the necessary and sufficient condition for

$$x^3+2px^2+2qx+r = 0 \quad \text{and} \quad x^2+px+q = 0$$

to have a common root is $r^2-3pqr+p^3r+q^3 = 0$.

Find the necessary and sufficient condition for the following to have a double root.

16 $x^3+px+q = 0$. (Cf. Ex. 10(*a*), no. 20.)

17 $x^4+px+q = 0$. (Cf. Ex. 10(*a*), no. 21.)

18 Find the common root condition for $ax^3+bx+c = 0$, $px^3+qx+r = 0$. [Eliminate x, x^3 in turn.]

*19 If $b^2 \neq c$ and $(x^2+2bx+c)^{r-1}$ is a factor of $p(x)$ and of $p'(x)$, prove that $(x^2+2bx+c)^r$ is a factor of $p(x)$. Explain why the restriction $b^2 \neq c$ is needed.

10.5 The H.C.F. of two polynomials

10.51 The H.C.F. process

The last result in 10.43 shows that, to find the repeated roots (if any) of $p(x) = 0$, we must find all factors common to $p(x)$ and $p'(x)$.

We now show how to find the *highest common factor* (H.C.F.) of any pair of polynomials $f(x)$, $g(x)$. It will be convenient to write $\deg f$ for 'the degree of $f(x)$', and so on.

Suppose $\deg f = m$ and $\deg g = n$, where $m \geqslant n$. Then we can divide $f(x)$ by $g(x)$, obtaining the quotient $q_1(x)$ and remainder $r_1(x)$, where†

$$f(x) \equiv g(x)\,q_1(x) + r_1(x). \qquad \text{(i)}$$

The degree of $r_1(x)$ must be less than that of $g(x)$, and hence $\deg r_1 \leqslant n-1$. We can therefore divide $g(x)$ by $r_1(x)$, getting quotient $q_2(x)$ and remainder $r_2(x)$, where

$$g(x) \equiv r_1(x)\,q_2(x) + r_2(x) \qquad \text{(ii)}$$

and $\deg r_2 < \deg r_1$, i.e. $\deg r_2 \leqslant n-2$.

Similarly, dividing $r_1(x)$ by $r_2(x)$ gives quotient $q_3(x)$ and remainder $r_3(x)$, where

$$r_1(x) \equiv r_2(x)\,q_3(x) + r_3(x) \qquad \text{(iii)}$$

and $\deg r_3 \leqslant n-3$.

Proceeding thus, we eventually obtain

$$r_{s-2}(x) \equiv r_{s-1}(x)\,q_s(x) + r_s(x) \qquad (s)$$

and

$$r_{s-1}(x) \equiv r_s(x)\,q_{s+1}(x) + r_{s+1}(x), \qquad (s+1)$$

where $r_{s+1}(x) \equiv 0$. The stage $(s+1)$ is reached after *at most* $n+1$ applications‡ of the successive division process

$$f \div g, \quad g \div r_1, \quad r_1 \div r_2, \quad \ldots, \quad r_{s-2} \div r_{s-1}, \quad r_{s-1} \div r_s.$$

First, suppose $f(x)$ and $g(x)$ have a common factor $h(x)$, so that

$$f(x) \equiv f_1(x)\,h(x), \quad g(x) \equiv g_1(x)\,h(x).$$

The identity (i) shows that

$$r_1(x) \equiv f(x) - g(x)\,q_1(x) \equiv h(x)\{f_1(x) - g_1(x)\,q_1(x)\},$$

so that $h(x)$ is also a factor of $r_1(x)$. Identity (ii) then shows that $h(x)$ is a factor of $r_2(x)$, and so on. Finally, identity (s) shows that $h(x)$ is a factor of $r_s(x)$. Hence

any common factor of $f(x)$, $g(x)$ is also a factor of $r_s(x)$. (A)

Secondly, suppose that $k(x)$ is a factor of $r_s(x)$. Then identity $(s+1)$ shows that $k(x)$ is also a factor of $r_{s-1}(x)$; hence by identity (s), $k(x)$ is

† Cf. 10.13, ex. (v).

‡ This is seen from the degrees of the successive remainders.

a factor of $r_{s-2}(x)$; etc. Thus from (ii) we see that $k(x)$ is a factor of $g(x)$, and hence by (i) it is a factor of $f(x)$. Therefore

$$\textit{any factor of } r_s(x) \textit{ is a common factor of } f(x) \textit{ and } g(x). \qquad \text{(B)}$$

From (B) it follows that $r_s(x)$ itself must be a common factor of $f(x)$ and $g(x)$; and (A) shows that every common factor of $f(x)$ and $g(x)$ must also be a factor of $r_s(x)$. Consequently,

(*a*) *if* $r_s(x) \equiv$ *constant*, then $f(x)$ and $g(x)$ can have no *algebraic* common factor, and we say that $f(x)$, $g(x)$ are *coprime* polynomials;

(*b*) *if* $r_s(x)$ *is a polynomial*, then $r_s(x)$ is the (algebraic) highest common factor† of $f(x)$ and $g(x)$. It has been found as *the last non-zero remainder in the process of successive division* just described (the H.C.F *process* or *Euclid's algorithm*).

Examples

(i) *Find the* H.C.F. *of*

$$2x^4-4x^3+5x^2+2x-3 \quad \text{and} \quad x^5-x^4+x^3+4x^2-2x+3.$$

We begin by dividing

$$2(x^5-x^4+x^3+4x^2-2x+3) \quad \text{by} \quad 2x^4-4x^3+5x^2+2x-3,$$

in order to avoid introducing fractional coefficients. The introduction or removal of numerical common factors at any stage is permissible since this does not alter the *algebraic* factors.†

$$\begin{array}{rl}
g \equiv & 2x^4-4x^3+5x^2+2x-3 \\
& 2x^4+2x^3-6x^2+18x \\ \hline
& -6x^3+11x^2-16x-3 \\
& -6x^3-6x^2+18x-54 \\ \hline
& 17x^2-34x+51 \\
\text{Remove factor 17:} & x^2-2x+3 \equiv r_2
\end{array}
\qquad
\begin{array}{r}
2x^5-2x^4+2x^3+8x^2-4x+6 \equiv f \\
2x^5-4x^4+5x^3+2x^2-3x \\ \hline
2x^4-3x^3+6x^2-x+6 \\
2x^4-4x^3+5x^2+2x-3 \\ \hline
x^3+x^2-3x+9 \equiv r_1 \\
x^3-2x^2+3x \\ \hline
3x^2-6x+9 \\
3x^2-6x+9 \\ \hline
0 \equiv r_3
\end{array}$$

The H.C.F. is the last non-zero remainder r_2, viz. x^2-2x+3.

(ii) *Test the equation*

$$4x^5-20x^4+25x^3+10x^2-20x-8=0$$

for repeated roots, and hence solve it.

A repeated factor of $f \equiv 4x^5-20x^4+25x^3+10x^2-20x-8$ is a common factor

† Numerical factors are regarded as irrelevant; $\lambda r_s(x)$, where λ is any non-zero constant, would also be called the H.C.F.

of f and $f' \equiv 5(4x^4-16x^3+15x^2+4x-4)$. We therefore begin by applying the H.C.F. process to f and $g \equiv 4x^4-16x^3+15x^2+4x-4.$

$$\begin{array}{rl|l}
g \equiv & 4x^4-16x^3+15x^2+4x-4 & 4x^5-20x^4+25x^3+10x^2-20x-\;\;8 \equiv f \\
 & 4x^4-14x^3+\;\;8x^2+8x & 4x^5-16x^4+15x^3+\;\;4x^2-\;\;4x \\
\hline
 & -\;\;2x^3+\;\;7x^2-4x-4 & -\;\;4x^4+10x^3+\;\;6x^2-16x-\;\;8 \\
 & -\;\;2x^3+\;\;7x^2-4x-4 & -\;\;4x^4+16x^3-15x^2-\;\;4x+\;\;4 \\
\hline
 & 0 \equiv r_2 & -\;\;6x^3+21x^2-12x-12 \\
 & & \text{Remove } -3: \\
 & & 2x^3-\;\;7x^2+\;\;4x+\;\;4 \equiv r_1
\end{array}$$

The H.C.F. is $2x^3-7x^2+4x+4$. By using the Remainder Theorem, we find that it has a factor $x-2$; hence

$$\text{H.C.F.} = (x-2)(2x^2-3x-2) = (x-2)^2(2x+1).$$

From 10.43 it follows that $x-2$, $2x+1$ are repeated factors of orders 3, 2 in f, and that $x = 2$, $x = -\frac{1}{2}$ are roots of $f = 0$ of orders 3, 2. Since f has degree 5, there can be no other roots.

10.52 An important algebraic theorem

The identities (i), (ii), ... in 10.51 show that

$$r_1 \equiv a_0 f + b_0 g, \quad \text{where} \quad a_0 = 1, \quad b_0 = -q_1;$$

$$r_2 \equiv a_1 f + b_1 g, \quad \text{where} \quad a_1 = -q_2, \; b_1 = 1+q_1 q_2;$$

and so on. We thus see that

$$r_s \equiv af+bg, \tag{C}$$

where a, b are polynomials in x.

Writing $f = r_s\phi$, $g = r_s\psi$, then by (C)

$$1 \equiv a\phi + b\psi,$$

where ϕ, ψ have no algebraic common factor. Since $a\phi+b\psi$ has no algebraic factor (being identically 1), hence a, b are coprime.

(*a*) *If* $r_s \equiv$ *constant*, then we may divide (C) by r_s and get the important

THEOREM. *If f, g are coprime polynomials, then other coprime polynomials A, B exist for which $Af+Bg \equiv 1$.*

(*b*) *If r_s is a polynomial in x*, then we have:

The H.C.F. *of two polynomials f, g can be written in the form $af+bg$, where a, b are coprime polynomials.*

Definition. A polynomial is said to be *irreducible* when it has no algebraic factor of lower degree than itself.

COROLLARY. *If f is irreducible and g, h are polynomials such that f is not a factor of g but f is a factor of gh, then f must be a factor of h.*

Proof.† Since f is not a factor of g, the degree of any common factor of f, g is less than the degree of f and is therefore zero since by hypothesis f has no such factor. Hence f, g are coprime and, by the theorem, polynomials A, B exist such that

$$1 \equiv Af+Bg,$$

and so

$$h \equiv Afh+Bgh.$$

Since f is a factor of gh, therefore $gh \equiv fk$ for some polynomial k, and so

$$h \equiv (Ah+Bk)f,$$

i.e. f is a factor of h.

† The result is 'intuitively obvious'.

Remarks

(α) The polynomials a, b in identity (C) are not unique, for clearly

$$a_1 \equiv a+\frac{cg}{r_s} \quad \text{and} \quad b_1 \equiv b-\frac{cf}{r_s}$$

(where c is any polynomial) would also satisfy (C).

(β) There are arithmetical results (important in the Theory of Numbers) analogous to the preceding; they can be formulated by replacing 'polynomial', 'irreducible', 'degree' by 'integer', 'prime', 'magnitude' respectively.

10.53 Theory of partial fractions

An application of the theorem in 10.52 is to prove the possibility of resolving a rational function into partial fractions. In 4.62 we stated and illustrated the practical methods for doing this in any particular example, but we did not attempt a general justification of our statements.

Definitions. The rational function f/g, where f and g are polynomials in x, is called (*a*) *irreducible* if f, g are coprime; (*b*) *proper* if $\deg f < \deg g$.

If f, g have H.C.F. h (found as in 10.51), then $f = hf_1$ and $g = hg_1$, so that (except for those values of x which make $g = 0$) $f/g = f_1/g_1$; and f_1, g_1 are coprime. *We shall therefore always assume that the rational function is irreducible.*

If f/g is not proper, we can express it as the sum of a polynomial and a proper fraction in the form $q+r/g$, by finding the quotient q and remainder r when f is divided by g.

THEOREM. *If p_1, p_2 are coprime polynomials, the (irreducible) rational function $f/(p_1p_2)$ can be expressed uniquely in the form*

$$q+\frac{r_1}{p_1}+\frac{r_2}{p_2},$$

where q is a polynomial and r_1/p_1, r_2/p_2 are irreducible proper fractions.

Proof. Since p_1, p_2 are coprime, polynomials A, B exist such that

$$Ap_1+Bp_2 \equiv 1.$$

$$\therefore \quad \frac{f}{p_1p_2} = \frac{f(Ap_1+Bp_2)}{p_1p_2} = \frac{Bf}{p_1}+\frac{Af}{p_2}.$$

By division, $\quad Bf \equiv q_1p_1+r_1 \quad$ and $\quad Af \equiv q_2p_2+r_2,$

where $\deg r_1 < \deg p_1$ and $\deg r_2 < \deg p_2$. Hence

$$\frac{f}{p_1p_2} = q_1+q_2+\frac{r_1}{p_1}+\frac{r_2}{p_2}, \tag{i}$$

where r_1/p_1, r_2/p_2 are *proper*. They are also *irreducible*; for if (say) r_1/p_1 reduces to r_1'/p_1', then result (i) shows that $f/(p_1p_2)$ is equal to a fraction with denominator $p_1'p_2$, where $\deg(p_1'p_2) < \deg(p_1p_2)$; this contradicts the hypothesis that $f/(p_1p_2)$ is irreducible.

To prove *uniqueness* of the decomposition (i), suppose if possible that

$$q+\frac{r_1}{p_1}+\frac{r_2}{p_2} = \frac{f}{p_1p_2} = q'+\frac{r_1'}{p_1}+\frac{r_2'}{p_2};$$

then $\quad (q-q')p_1p_2+(r_1-r_1')p_2 \equiv (r_2'-r_2)p_1.$

Since p_2 is a factor of the left-hand side, therefore p_2 is a factor of $(r_2'-r_2)p_1$. As p_2 is prime to p_1 by hypothesis, it follows from the corollary in 10.52 that p_2 must be a factor of $r_2'-r_2$. This is impossible unless $r_2'-r_2 \equiv 0$, for p_2 has higher degree than r_2' and r_2, and so certainly $\deg p_2 > \deg(r_2'-r_2)$. Thus $r_2' \equiv r_2$, and similarly we can show $r_1' \equiv r_1$. It then follows that $q' \equiv q$.

COROLLARY. *If $p_1, p_2, \ldots, p_n$ are polynomials every two of which are coprime, then the (irreducible) rational function $f/(p_1p_2\ldots p_n)$ can be expressed uniquely in the form*

$$q+\frac{r_1}{p_1}+\frac{r_2}{p_2}+\ldots+\frac{r_n}{p_n},$$

where q is a polynomial and $r_1/p_1, \ldots, r_n/p_n$ are proper and irreducible.

This is proved by repeated applications of the preceding theorem: first to $p_1(p_2\ldots p_n)$ to give

$$q_1+\frac{r_1}{p_1}+\frac{f_1}{p_2\ldots p_n};$$

then to $p_2(p_3\ldots p_n)$ to give

$$\frac{f_1}{p_2\ldots p_n}=q_2+\frac{r_2}{p_2}+\frac{f_2}{p_3\ldots p_n};$$

and so on. The uniqueness is proved by the same argument as in the theorem.

Remark. For their application to a given rational function f/g, the preceding results depend on the factorisation of the denominator g into irreducible factors $p_1p_2\ldots p_n$ (cf. 4.63). In 13.61 we shall prove that a polynomial $g(x)$ can be factorised into the product of linear and irreducible quadratic factors like $(x-a)^r$, $\{(x-b)^2+c^2\}^s$. Assuming this, the above corollary shows that

$$\frac{f}{g}=q+\Sigma\frac{r_1}{(x-a)^r}+\Sigma\frac{r_2}{\{(x-b)^2+c^2\}^s}, \tag{ii}$$

where q and the r_1, r_2 are polynomials and $\deg r_1 < r$, $\deg r_2 < 2s$.

By division, any polynomial $\phi(x)$ of degree m can be written in the form

$$\phi(x) \equiv (x-a)\,\phi_1(x)+A_0,$$

where A_0 is constant and $\phi_1(x)$ has degree $m-1$. Similarly

$$\phi_1(x) \equiv (x-a)\,\phi_2(x)+A_1,$$

and so on. Combining all these results, we have†

$$\phi(x) \equiv A_0+A_1(x-a)+A_2(x-a)^2+\ldots+A_m(x-a)^m.$$

Hence the *proper* fraction $r_1/(x-a)^r$ can be expressed as

$$\frac{A_0}{(x-a)^r}+\frac{A_1}{(x-a)^{r-1}}+\ldots+\frac{A_{r-1}}{x-a}. \tag{iii}$$

Similarly, any polynomial $\phi(x)$ can be written

$$\phi(x) \equiv \{(x-b)^2+c^2\}\,\phi_1(x)+A_0x+B_0,$$

where A_0, B_0 are constants. Proceeding likewise with $\phi_1(x)$, etc., we obtain

$$\phi(x) \equiv (A_0x+B_0)+(A_1x+B_1)\{(x-b)^2+c^2\}+\ldots$$
$$+(A_{s-1}x+B_{s-1})\{(x-b)^2+c^2\}^{s-1}$$

† This is the Lemma in 6.41.

if $\phi(x)$ has degree $2s-1$ or less. Hence the *proper* fraction $r_2/\{(x-b)^2+c^2\}^s$ can be decomposed as

$$\frac{A_0x+B_0}{\{(x-b)^2+c^2\}^s}+\frac{A_1x+B_1}{\{(x-b)^2+c^2\}^{s-1}}+\ldots+\frac{A_{s-1}x+B_{s-1}}{(x-b)^2+c^2}. \qquad \text{(iv)}$$

The argument already used in proving the theorem shows that each of the reductions (iii), (iv) is unique.

By combining (ii), (iii) and (iv), we completely prove the statements made in 4.62 (assuming, of course, the facts in 13.61; but see the Remark in that section).

Exercise 10(*e*)

Find the H.C.F. *of*

1 $x^3+3x^2-8x-24$, x^3+3x^2-3x-9.

2 $2x^3+7x^2+10x+35$, $2x^4+7x^3-2x^2-3x+14$.

3 $2x^3-x^2+4x+15$, $x^4+12x-5$.

Test for repeated factors, and hence factorise completely

4 $x^4-9x^2+4x+12$.

5 $x^5-x^3+4x^2-3x+2$.

6 $x^6-3x^5+6x^3-3x^2-3x+2$.

Test for repeated roots, and hence solve completely

7 $12x^4+4x^3-45x^2+54=0$.

8 $x^5-5x^3+5x-2=0$.

*9 Find polynomials A, B of least degree such that

$$A(2x^3-3x^2+4x-1)+B(x^2+2x-3)\equiv 1.$$

Miscellaneous Exercise 10(*f*)

1 Prove $a(x^2-y^2)-2hxy$ always has linear factors.

2 If $3x^2+2\lambda xy+2y^2+2ax-4y+1$ has linear factors, prove that λ must satisfy $\lambda^2+4a\lambda+2(a^2+3)=0$. Is this sufficient?

3 Establish the identity $\Sigma x^3-3xyz\equiv(\Sigma x)(\Sigma x^2-\Sigma yz)$ as follows:

(i) Expand $(x+y)^3$ to show $x^3+y^3\equiv(x+y)^3-3xy(x+y)$.

(ii) Deduce that $x^3+y^3+z^3-3xyz\equiv(x+y)^3+z^3-3xy(x+y+z)$, and factorise the sum of two cubes.

4 (i) Verify that $x^2+y^2+z^2-yz-zx-xy\equiv\frac{1}{2}\{(y-z)^2+(z-x)^2+(x-y)^2\}$.

(ii) If $x+y+z>0$, prove $x^3+y^3+z^3>3xyz$ unless $x=y=z$.

(iii) If $x+y+z=0$, prove $x^3+y^3+z^3=3xyz$.

5 If $x=b+c-a$, $y=c+a-b$, $z=a+b-c$, prove (using no. 4 (i))

$$x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc).$$

6 Factorise (i) $\Sigma(b^3+c^3)(b-c)$; (ii) $\Sigma a(b^4-c^4)$.

7 Use the identity $(\Sigma x)^2\equiv\Sigma x^2+2\Sigma yz$ to prove

(i) $\Sigma(b-c)^2\equiv 2\Sigma(a-b)(a-c)$; (ii) $\Sigma a^2(b-c)^2\equiv 2\Sigma bc(a-b)(a-c)$.

8 Eliminate t from $x=t^2+t^{-2}$, $y=t^3+t^{-3}$.

9 Eliminate x, y from $x-y=a$, $x^2-y^2=b^2$, $x^3-y^3=c^3$.

10 If $px-qy=x^2-y^2$, $py+qx=4xy$, $x^2+y^2=1$, prove $(p+q)^{\frac{2}{3}}+(p-q)^{\frac{2}{3}}=2$. [Solve the first two equations for p, q, using the third.]

11 'The equation $(x-1)^2 = \lambda(x-2\mu)(x-4)$ has equal roots.' (i) If μ has a given value, prove this statement holds for $\lambda = 0$ and one other value. (ii) If λ has a given non-zero value, prove the statement is true for two values of μ only if $\lambda < 1$. If $\lambda = -15$, find the two sets of equal roots.

12 Find the condition for the roots of $ax^2+bx+c = 0$ to divide the distance between the roots of $a'x^2+b'x+c' = 0$ internally and externally in the same ratio.

*13 If $(a_1x^2+b_1x+c_1)/(a_2x^2+b_2x+c_2)$ takes the same value when x has the values given by $a_3x^2+b_3x+c_3 = 0$ $(b_3^2 > 4a_3c_3)$, prove that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

[If the value taken is k, then $(a_2k-a_1)x^2+(b_2k-b_1)x+(c_2k-c_1) = 0$ must have the same roots as $a_3x^2+b_3x+c_3 = 0$. Hence corresponding coefficients are proportional.]

14 What can be said about the coefficients in an equation whose roots are (i) $\alpha, \beta, -\alpha-\beta$; (ii) $0, \alpha, \beta, \gamma$; (iii) $a/b, b/c, c/a$; *(iv) $\tan\theta_1, \tan\theta_2, \tan\theta_3$, where $\theta_1+\theta_2+\theta_3$ is an integral multiple of π?

15 If a line cuts the curve $x = at^2$, $y = at^3$ in three points for which t has the values t_1, t_2, t_3, prove $\Sigma t_2t_3 = 0$.

16 If one root of $x^3+ax+b = 0$ is twice the difference of the other two, prove that the roots are

$$-13b/12a,\ 13b/3a,\ -13b/4a, \text{ and that } 144a^3+2197b^2 = 0.$$

17 If α, β, γ are the roots of $x^3+px+q = 0$, express $\alpha^4+\beta^4+\gamma^4$ in terms of p, q.

18 Obtain the equation whose roots exceed by 3 the roots of

$$x^4+12x^3+49x^2+78x+42 = 0,$$

and hence solve the given equation.

19 If α, β are the roots of $ax^2+2hx+b = 0$, find the quartic equation whose roots are $\pm 1/\alpha$, $\pm 1/\beta$.

20 Eliminate λ, μ, ν from

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda} = 1, \quad \frac{x^2}{a+\mu}+\frac{y^2}{b+\mu}+\frac{z^2}{c+\mu} = 1, \quad \frac{x^2}{a+\nu}+\frac{y^2}{b+\nu}+\frac{z^2}{c+\nu} = 1$$

and $$\lambda\mu\nu = abc.$$

21 If $x^3+3ax^2+3bx+c = 0$ has a repeated root, prove that this root also satisfies $x^2+2ax+b = 0$. Hence show that the repeated root is $(c-ab)/2(a^2-b)$.

22 Prove that $x^4+px+q = 0$ cannot have a repeated root of order 3.

23 Prove that
$$x+\frac{x^2}{2}+\frac{x^3}{3}+\dots+\frac{x^n}{n} = 0$$
cannot have a repeated root. [$p'(x) = 0$ only when $x = -1$ if n is even, and $p'(x) \neq 0$ if n is odd. Clearly $p(-1) \neq 0$.]

24 Find k so that $2x^4-3x^2-2x+k = 0$ has (i) a double root; (ii) a triple root; and solve in each case.

*25 Determine the values of m and c for which the line $y = mx+c$ is (i) an inflexional tangent, (ii) a double tangent, to the curve $y = x^2(x^2+4x-18)$. [In (ii), the equation $x^2(x^2+4x-18)-mx-c = 0$ has two double roots.]

26 If $2ac'+2a'c = bb'$ and if ax^2+bx+c, $a'x^2+b'x+c'$ have a common factor, prove that at least one of these quadratics is a perfect square. [See 10.42 (1), Remark.]

27 If a, b, c, d are constants such that $ad \neq bc$, and f, g, p, q are polynomials in x such that $p \equiv af+bg$, $q \equiv cf+dg$, prove that f, g and p, q have the same H.C.F. What happens if $ad = bc$?

*28 If $p(x,y)$ is a symmetrical polynomial in (x,y) having factor $x-y$, prove that actually $(x-y)^2$ is a factor. [$p(x,y) = \Sigma a_{rs}(x^r y^s + x^s y^r)$. The conditions for p to be zero when $x = y$ are the same as those for $\partial p/\partial x$ to be zero when $x = y$.]

*29 *Cubic equations: Cardan's method*† *of solution.* Every cubic can be reduced (see Ex. 13 (*d*), no. 15) to the standard form

$$x^3+3Hx+G = 0. \tag{a}$$

The identity in no. 3 shows that *if* y, z can be chosen so that

$$y^3+z^3 = G \quad \text{and} \quad yz = -H, \tag{b}$$

then $x = -y-z$ will be a root of (a). Prove that y^3, z^3 must be the roots of

$$t^2-Gt-H^3 = 0. \tag{c}$$

*30 (i) If $G^2+4H^3 > 0$, show that there are *distinct* numbers y, z satisfying (b), so that $x = -y-z$ is a root of (a). Use no. 4 (i) to show that there are no other roots.

(ii) If $G^2+4H^3 = 0$, show that the right-hand side of the identity in no. 3 becomes $(x+2y)(x-y)^2$, so that (a) has three roots, two of which are equal, viz. $-2y, y, y$.

(iii) If $G^2+4H^3 < 0$, numbers satisfying (b) do not exist. The following *trigonometrical method* can then be used.

Put $x = k\cos\theta$ in (a), and choose k so that $k^3 : 3Hk = 4 : -3$. Show that (a) then becomes $\cos 3\theta = G/\{2H\sqrt{(-H)}\}$. Verify that condition (iii) ensures that 3θ can be found from this, and hence that three values of $\cos\theta = x/k$ are obtainable.

† So-called, although discovered by Tartaglia.

11

DETERMINANTS AND SYSTEMS OF LINEAR EQUATIONS

11.1 Linear simultaneous equations

11.11 Two equations in two unknowns

The usual method of solving

$$a_1 x + b_1 y = c_1,$$

$$a_2 x + b_2 y = c_2$$

by elimination shows that

$$(a_1 b_2 - a_2 b_1)\,x = c_1 b_2 - c_2 b_1 \quad \text{and} \quad (a_1 b_2 - a_2 b_1)\,y = a_1 c_2 - a_2 c_1. \quad \text{(i)}$$

If $a_1 b_2 - a_2 b_1 \neq 0$, these give unique values for x and y.

When $a_1 b_2 - a_2 b_1 = 0$, we may assume throughout that a_1 and b_1 (and likewise a_2 and b_2) are not both zero; for if $a_1 = 0 = b_1$, the first equation becomes $0 = c_1$, which is either false or trivial. Two cases arise.

(*a*) If at least one of $c_1 b_2 - c_2 b_1$, $a_1 c_2 - a_2 c_1$ is not zero, (i) gives a contradiction. The given equations therefore have no solution, and are said to be *inconsistent*.

(*b*) If $a_1 b_2 - a_2 b_1 = c_1 b_2 - c_2 b_1 = a_1 c_2 - a_2 c_1 = 0$, then (i) gives no information. Since a_1, b_1 are assumed to be not both zero, suppose $a_1 \neq 0$. The first of the given equations can then be solved for x in terms of y. When y is given the value y_0, let the value obtained for x be x_0; then

$$x_0 = \frac{c_1 - b_1 y_0}{a_1}.$$

Since

$$\begin{aligned} a_2 x_0 + b_2 y_0 - c_2 &= \frac{a_2(c_1 - b_1 y_0)}{a_1} + b_2 y_0 - c_2 \\ &= \frac{a_2 c_1 - a_1 c_2 + (a_1 b_2 - a_2 b_1)\,y_0}{a_1} \\ &= 0 \quad \text{by hypothesis } (b), \end{aligned}$$

any values of x, y satisfying the first equation also satisfy the second. The solution is said to be *indeterminate*. (Roughly, hypothesis (*b*) shows that corresponding coefficients are 'proportional', so that the equations are not distinct.)

Geometrically, the given equations represent straight lines. If $a_1b_2 - a_2b_1 \neq 0$, the lines are not parallel, and the above solution (i) represents their unique point of intersection. In case (*a*), the lines are parallel, while in (*b*) the equations represent the same line.

11.12 Three equations in three unknowns

Here we obtain only the general form of the results corresponding to (i), and postpone discussion of details until 11.4.

Elimination of z from the first and second of

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3$$

gives $$(a_1c_2 - a_2c_1)x + (b_1c_2 - b_2c_1)y = d_1c_2 - d_2c_1,$$

and elimination of z from the second and third gives

$$(a_2c_3 - a_3c_2)x + (b_2c_3 - b_3c_2)y = d_2c_3 - d_3c_2.$$

Now eliminate y from these last two equations:

$$\{(b_2c_3 - b_3c_2)(a_1c_2 - a_2c_1) - (b_1c_2 - b_2c_1)(a_2c_3 - a_3c_2)\}x$$
$$= (b_2c_3 - b_3c_2)(d_1c_2 - d_2c_1) - (b_1c_2 - b_2c_1)(d_2c_3 - d_3c_2).$$

On simplifying we find that the coefficient of x is

$$c_2(a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1),$$

and that the right-hand side reduces to c_2 times a similar expression. The solution for x therefore has denominator

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1. \qquad \text{(ii)}$$

We should find similarly that this is also the denominator in the expressions for y and z.

11.13 Structure of the solutions

Instead of proceeding to higher eliminations, we notice the following properties of the expression (ii).

(α) There are six terms. In each term there is just one a, one b and one c; and in each term the suffixes 1, 2, 3 all occur, without repetition. The *signs* preceding the terms are alternately $+$, $-$.

(β) The suffixes of the letters a, b, c in each term form one of the six permutations of the numbers 1, 2, 3. For example, consider $a_2b_1c_3$: if we interchange the suffixes in pairs until they are in natural order 1, 2, 3, we find that the *number* of interchanges is *odd*. Similarly, for the term $a_3b_1c_2$ the number is *even*. It is easily verified that all the terms for which the number is odd are preceded by the sign $-$, while those for which it is even† have $+$. (For a given term it can be proved that the number of interchanges is either always odd or else always even, no matter how the rearrangement to natural order is carried out.) Hence *the sign of each term in* (ii) *is decidable by the parity of the number of interchanges of its suffixes from the actual to the natural order.*

11.2 Determinants

11.21 Determinants of order 2

The symbol
$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$
called a *second-order determinant*, is defined to mean $a_1b_2 - a_2b_1$. The following five properties‡ of Δ are easily verified.

(1) *The value of Δ is unaltered by interchange of rows and columns.* That is, if
$$\Delta' = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$
then $\Delta' = \Delta$. We call Δ' the *transpose* of Δ.

By use of this result, any property proved for *rows* extends at once to *columns*, and conversely. The following will therefore be stated for rows only.

(2) *Interchange of two rows alters only the sign of Δ.* That is,
$$\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = -\Delta.$$

(3) *If two rows are identical, then $\Delta = 0$.*

(4) *Multiplication of any one row by k multiplies Δ by k.* For example,
$$\begin{vmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{vmatrix} = k\Delta.$$

† 0 is reckoned as even.

‡ With the wording used, they hold without modification for third-order determinants (11.22).

(5) *Addition to any row of a multiple of another row does not alter the value of* Δ.

For example, by adding k times the second row to the first row, we obtain

$$\begin{vmatrix} a_1+ka_2 & b_1+kb_2 \\ a_2 & b_2 \end{vmatrix} = \Delta.$$

11.22 Determinants of order 3

The symbol

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

called a *third-order determinant*, is defined to mean

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \quad \text{(i)}$$

which (by using the definition of the second-order determinants) is equal to

$$a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)$$
$$= a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1 \quad \text{(ii)}$$

on rearranging. This is the expression (ii) of 11.12, and is sometimes referred to as the *expansion* of Δ.

The numbers a_1, b_1, c_1, a_2, ..., c_3 are called the *elements* of Δ; with our notation, the suffix denotes the row and the letter denotes the column in which a particular element lies.

The remarks in 11.13 show that *the expansion of a third-order determinant consists of* 6 *terms, each of which involves an element from each row and each column but no two elements from the same row or the same column.* The sign before each term is determined by the suffixes according to the rule in 11.13 (β). We may write the expansion of Δ shortly as

$$\Sigma \pm a_i b_j c_k.$$

The definition expressed by (i) above can be readily generalised to define determinants of fourth and higher orders (see 11.7). *Observe how the second-order determinants in* (i) *are constructed from* Δ: the coefficient of a_1 is the determinant obtained from Δ after deleting the row and column containing a_1; the determinant in the middle term is got by omitting the row and column containing b_1 from Δ; and similarly for the last term. Since the elements a_1, b_1, c_1 appear as multipliers of the second-order determinants, the expression (i) is consequently referred to as the *expansion of* Δ *from the first row.*

The diagonal running from top left to bottom right is called the *leading diagonal*, and the product $a_1 b_2 c_3$ of the elements in it is called the *leading term* in the expansion of Δ.

Examples

(i)
$$\begin{vmatrix} 4 & 5 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 2 \end{vmatrix} = 4\begin{vmatrix} 4 & 7 \\ 6 & 2 \end{vmatrix} - 5\begin{vmatrix} 2 & 7 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix}$$
$$= 4(8-42)-5(4-21)+3(12-12)$$
$$= -51.$$

(ii)
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$$
$$= a(bc-f^2)-h(ch-fg)+g(hf-bg)$$
$$= abc+2fgh-af^2-bg^2-ch^2.$$

The reader should now do Ex. 11 *(a), nos.* 1–10.

11.23 Other expansions of a third-order determinant

We may arrange the expression (ii) of 11.22 according to elements from any row of Δ and the corresponding second-order determinants formed by deletion as described above. Thus, grouping by elements of the second row,

$$\Delta = -a_2(b_1c_3-b_3c_1)+b_2(a_1c_3-a_3c_1)-c_2(a_1b_3-a_3b_1)$$
$$= -a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \qquad \text{(iii)}$$

which is the *expansion of Δ from the second row.* Similarly,

$$\Delta = a_3(b_1c_2-b_2c_1)-b_3(a_1c_2-a_2c_1)+c_3(a_1b_2-a_2b_1)$$
$$= a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \qquad \text{(iv)}$$

the expansion from the third row.

Likewise, we may arrange (ii) by elements of any one column; e.g.

$$\Delta = a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$
$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \qquad \text{(v)}$$

is the expansion from the first column.

The reader should not attempt to memorise these results because they are all easily obtainable from the *definition* of Δ in expanded form.

11.24 Properties of Δ

We now show that the properties (1)–(5) in 11.21 hold for third-order determinants. In view of the first, all subsequent properties hold for columns as well as for rows.

Proof of (1).

$$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \quad \text{by definition,}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= \Delta \quad \text{by the line preceding (v) in 11.23.}$$

Proof of (2). Interchange of two rows is equivalent to interchanging two of the suffixes 1, 2, 3. Results (i), (iii), (iv) show† that the sign of the determinant is changed in every case.

Alternatively, we may start from the definition; e.g. interchange of the first and third rows gives

$$\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = a_3 \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} - b_3 \begin{vmatrix} a_2 & c_2 \\ a_1 & c_1 \end{vmatrix} + c_3 \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$$

$$= a_3(b_2c_1 - b_1c_2) - b_3(a_2c_1 - a_1c_2) + c_3(a_2b_1 - a_1b_2)$$

$$= a_1(b_3c_2 - b_2c_3) - b_1(a_3c_2 - a_2c_3) + c_1(a_3b_2 - a_2b_3)$$

on arranging by elements of the first row of Δ; and this is clearly $-\Delta$. A similar direct proof would hold for other row interchanges.

Proof of (3). By interchange of the two identical rows we obtain $-\Delta$, by (2). However, interchange of identical rows clearly leaves the same determinant Δ as before. Therefore $-\Delta = \Delta$, so $\Delta = 0$.

Proof of (4). Since each term in the expansion of Δ contains exactly one element from each row, multiplication of each element in a given row by k causes every term in the expansion to be multiplied by k. Hence the value of the new determinant is $k\Delta$.

Proof of (5). We expand the new determinant by the row which

† Using the corresponding property already proved for second order when necessary.

has not been mentioned. Thus, if we add k times the second row to the first row, we get

$$\begin{vmatrix} a_1+ka_2 & b_1+kb_2 & c_1+kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \qquad \text{(vi)}$$

The row 'not mentioned' is the third; using an equation like (iv) to expand from it, we see that the new determinant is equal to

$$a_3\begin{vmatrix} b_1+kb_2 & c_1+kc_2 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1+ka_2 & c_1+kc_2 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1+ka_2 & b_1+kb_2 \\ a_2 & b_2 \end{vmatrix}$$

$$= a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

by using Property (5) for each of the second-order determinants. Again by equation (iv), this expression is Δ.

Remarks

(α) There are many extensions of Property (5). Three are indicated in Ex. 11 (a), no. 22; but see the Remark about random manipulations after no. 29.

(β) The operation by which determinant (vi) above is obtained from Δ can be denoted by $\boldsymbol{r}_1 \to \boldsymbol{r}_1 + k\boldsymbol{r}_2$. With this notation, properties (4) and (5) can be combined in a single statement:

If $\quad \boldsymbol{r}_i \to k_1\boldsymbol{r}_1 + k_2\boldsymbol{r}_2 + k_3\boldsymbol{r}_3,\quad$ *then* $\quad \Delta \to k_i\Delta \quad (i = 1 \text{ or } 2 \text{ or } 3).$

11.25 Examples

Properties (2)–(5) give ways of simplifying a determinant when direct application of the definition would be clumsy owing to the large numbers or heavy algebra involved.

(i) *Evaluate*

$$\begin{vmatrix} 35 & 29 & 86 \\ 36 & 31 & 87 \\ 38 & 32 & 89 \end{vmatrix}.$$

By $\boldsymbol{r}_3 \to \boldsymbol{r}_3 - \boldsymbol{r}_2$, we get

$$\begin{vmatrix} 35 & 29 & 86 \\ 36 & 31 & 87 \\ 2 & 1 & 2 \end{vmatrix}.$$

By $\boldsymbol{r}_2 \to \boldsymbol{r}_2 - \boldsymbol{r}_1$, this becomes

$$\begin{vmatrix} 35 & 29 & 86 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}.$$

(These two steps may be condensed into one by saying '$\boldsymbol{r}_3 \to \boldsymbol{r}_3 - \boldsymbol{r}_2$, followed by $\boldsymbol{r}_2 \to \boldsymbol{r}_2 - \boldsymbol{r}_1$'.)

The last determinant is now easily expanded, according to the definition, as

$$35\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 29\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + 86\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 35.3 - 29.0 + 86(-3)$$

$$= -153.$$

(ii) *Prove*
$$\begin{vmatrix} b+c & c+a & a+b \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

By $r_1 \to r_1 + r_2$, the determinant is equal to

$$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$ on removing the factor $(a+b+c)$ from the first row,

$= 0$ since the last determinant has two rows identical.

(iii) *Evaluate*
$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}.$$

Direct expansion would be easy, but the following method has the advantage of giving the result in factorised form.

By $c_2 \to c_2 - c_1$, followed by $c_3 \to c_3 - c_1$, the determinant is equal to

$$\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^3 & y^2+xy+x^2 & z^2+xz+x^2 \end{vmatrix}$$

by removing the factor $y-x$ from column 2, and $z-x$ from column 3. Expanding† from the first row, the only non-zero term is

$$(y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y^2+xy+x^2 & z^2+xz+x^2 \end{vmatrix}$$

$= (y-x)(z-x)(z^2+xz-y^2-xy)$

$= (y-x)(z-x)(z-y)(z+y+x)$ on factorising the last bracket,

$= (y-z)(z-x)(x-y)(x+y+z)$ on arranging cyclically.

† Before expanding a determinant it is helpful to get zeros in a row or column, as here. Also see ex. (v) below.

(iv) *Without expanding either determinant, prove*

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

Introducing a factor a in row 1, b in row 2, and c in row 3 of the left-hand side, we have

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

on removing the factor abc from the first column of the middle determinant.

(v) '*Triangular*' *determinants.*

$$\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & 0 \\ b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3,$$

the product of the elements in the leading diagonal. Similarly, a determinant whose elements *below* the leading diagonal are all zero is thus readily evaluated.

Remarks

(α) In expanding a determinant from a given row (or column), the amount of calculation is reduced if row- and column-operations can be used to introduce one or more zeros into that row (or column), or indeed elsewhere also.

(β) *A determinant having a complete row or column of zeros has the value* 0.

Exercise 11(a)

Evaluate the following determinants by direct expansion.

1 $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 5 & 6 & 5 \end{vmatrix}.$ 2 $\begin{vmatrix} 4 & 1 & -2 \\ 0 & 3 & 4 \\ 2 & 1 & 3 \end{vmatrix}.$ 3 $\begin{vmatrix} 3 & -5 & 2 \\ 4 & 1 & -3 \\ -2 & 5 & 1 \end{vmatrix}.$

4 $\begin{vmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{vmatrix}.$ 5 $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$ 6 $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}.$

7 Show that

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(n-1)x & \cos nx & \cos(n+1)x \\ \sin(n-1)x & \sin nx & \sin(n+1)x \end{vmatrix} = (1-2a\cos x+a^2)\sin x.$$

Verify the following equations by expanding each side.

8 $\begin{vmatrix} x & y & z \\ a & b & c \\ ax & by & cz \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ yz & zx & xy \end{vmatrix}.$ 9 $\begin{vmatrix} a^2 & b^2 & c^2 \\ ax & by & cz \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ bc & ca & ab \end{vmatrix}.$

10 Verify that
$$\begin{vmatrix} a+\lambda & h & g \\ h & b+\lambda & f \\ g & f & c+\lambda \end{vmatrix}$$
$= \lambda^3 + (a+b+c)\lambda^2 + (bc+ca+ab-f^2-g^2-h^2)\lambda + (abc+2fgh-af^2-bg^2-ch^2)$.

Using the properties of determinants, evaluate the following.

11 $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 5 & 7 & 9 \end{vmatrix}$. 12 $\begin{vmatrix} 6 & 9 & 15 \\ 1 & 6 & 3 \\ 3 & 12 & 5 \end{vmatrix}$. 13 $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 5 & 6 & 7 \end{vmatrix}$.

14 $\begin{vmatrix} 13 & 14 & 15 \\ 6 & 7 & 8 \\ 1 & 2 & 3 \end{vmatrix}$. 15 $\begin{vmatrix} 101 & 19 & 1 \\ 102 & 20 & 2 \\ 103 & 20 & 2 \end{vmatrix}$. 16 $\begin{vmatrix} b-c & c-a & a-b \\ c & a & b \\ b & c & a \end{vmatrix}$.

17 $\begin{vmatrix} a-b & a+b & a \\ b-c & b+c & b \\ c-a & c+a & c \end{vmatrix}$. 18 $\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & x^2 \\ 1 & x^2 & x^4 \end{vmatrix}$. 19 $\begin{vmatrix} 1 & x & x^2 \\ 1 & x^2 & x^4 \\ 1 & x^3 & x^6 \end{vmatrix}$.

20 Prove
$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (a+b+c)(a^2+b^2+c^2-bc-ca-ab).$$
Deduce an identity by comparing this result with that of no. 6.

21 Prove
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-a)(a-b).$$

22 Prove
$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1+la_1 & c_1+mb_1+na_1 \\ a_2 & b_2+la_2 & c_2+mb_2+na_2 \\ a_3 & b_3+la_3 & c_3+mb_3+na_3 \end{vmatrix}.$$
(This shows that to each column we may add multiples of the *preceding* columns.)
What are the values of

(i) $$\begin{vmatrix} a_1+\lambda a_2+\mu a_3 & b_1+\lambda b_2+\mu b_3 & c_1+\lambda c_2+\mu c_3 \\ a_2+\nu a_3 & b_2+\nu b_3 & c_2+\nu c_3 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

(ii) $$\begin{vmatrix} a_1+\lambda b_1 & b_1 & c_1+\mu b_1 \\ a_2+\lambda b_2 & b_2 & c_2+\mu b_2 \\ a_3+\lambda b_3 & b_3 & c_3+\mu b_3 \end{vmatrix}?$$

23 Solve
$$\begin{vmatrix} 3-x & 4 & 2 \\ 4 & 2-x & 3 \\ 2 & 3 & 4-x \end{vmatrix} = 0.$$
[$\mathbf{c}_1 \to \mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_3$, and remove $9-x$; then $\mathbf{r}_2 \to \mathbf{r}_2-\mathbf{r}_1$, $\mathbf{r}_3 \to \mathbf{r}_3-\mathbf{r}_1$.]

24 Without expanding, prove that

$$\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix} = abc \begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix},$$

and hence evaluate the first determinant.

25 Prove
$$\begin{vmatrix} a+x & b+x & c+x \\ a+y & b+y & c+y \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)(x-y).$$

[Use the result of no. 21.]

26 If A, B, C are the angles of a triangle, prove that

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 0;$$

and by expanding, obtain a relation between $\cos A$, $\cos B$, $\cos C$. [Use $a = b\cos C + c\cos B$, etc.]

27 *Prove*
$$\begin{vmatrix} a_1+x_1 & b_1+y_1 & c_1+z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

State this result as a general property.

28 (i) If each element of Δ consists of the sum of two terms, prove that Δ is equal to the sum of 8 determinants. [See no. 27.]

(ii) If each element of Δ consists of the sum of three terms, how many determinants are there in the expression for Δ as a sum of determinants?

29 By using no. 28 (i), show that

$$D \equiv \begin{vmatrix} a_1+\lambda b_1 & b_1+\mu c_1 & c_1+\nu a_1 \\ a_2+\lambda b_2 & b_2+\mu c_2 & c_2+\nu a_2 \\ a_3+\lambda b_3 & b_3+\mu c_3 & c_3+\nu a_3 \end{vmatrix} = (1+\lambda\mu\nu) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Remark. We may be tempted to say that D is formed from Δ by the operations $\mathbf{c}_1 \to \mathbf{c}_1 + \lambda\mathbf{c}_2$, $\mathbf{c}_2 \to \mathbf{c}_2 + \mu\mathbf{c}_3$, and $\mathbf{c}_3 \to \mathbf{c}_3 + \nu\mathbf{c}_1$. A choice of λ, μ, ν such that $\lambda\mu\nu = -1$ makes $D = 0$, and appears to prove that $\Delta = 0$; but the compound manipulations of columns just described are a misuse of Property (5) and have *not* left the value of Δ unaltered. It is essential in applying Property (5) or its extensions to *leave at least one row or one column* UNALTERED AT EACH STEP.

11.3 Minors and cofactors

11.31 Definitions and notation

The expansions of Δ from its various rows or columns (written out in 11.23) all follow a pattern similar to the original definition: the three terms each consist of an element from the particular row or column from which the expansion is being made, multiplied by the second-order determinant obtained from Δ by deleting the row and column which contain that element, and prefixed by the sign + or −. In some cases the signs run − + − instead of the standard + − +. We now introduce a notation which will make all this systematic.

Definitions

(*a*) The *minor* of an element of Δ is the second-order determinant obtained from Δ by deleting the row and column in which that element lies. For example, the minors of b_1, a_2 are respectively

$$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}.$$

(*b*) The *cofactor* of the element in the ith row and jth column is the minor of that element multiplied by $(-1)^{i+j}$. For example, the cofactor of b_1, which is the element in the *first* row, *second* column, is

$$(-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix};$$

that of c_3 is

$$(-1)^{3+3}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = +\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Cofactors are 'signed minors'.

Notation. We denote the cofactors of $a_1, b_1, c_1, a_2, \ldots, c_3$ in Δ by the corresponding capital letters $A_1, B_1, C_1, A_2, \ldots, C_3$. The various expansions of Δ can now be written concisely and uniformly: those from the first, second and third rows are

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1 \qquad \text{(i)}'$$

$$= a_2A_2 + b_2B_2 + c_2C_2 \qquad \text{(iii)}'$$

$$= a_3A_3 + b_3B_3 + c_3C_3 \qquad \text{(iv)}'$$

and for columns we have

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 \qquad \text{(v)}'$$

$$= b_1B_1 + b_2B_2 + b_3B_3$$

$$= c_1C_1 + c_2C_2 + c_3C_3.$$

The reader should verify the last two as in 11.23. These six results are summarised in the following rule.

The expansion of a determinant from any row (or column) is the sum of the elements of that row (column) each multiplied by the corresponding cofactor.

11.32 Expansion by alien cofactors

Consider the determinant

$$\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

It has two rows identical, and consequently is zero. By expanding from the first row according to the definition, we have

$$0 = a_2\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_2\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$
$$= a_2A_1 + b_2B_1 + c_2C_1,$$

where A_1, B_1, C_1 are cofactors in

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

In this way *a total of* 12 *expressions each equal to zero can be formed by taking the sum of the elements in any row* (*or column*) *each multiplied by the cofactor of the corresponding element from a* DIFFERENT *row* (*or column*). These results, which may be called 'expansions' of Δ by 'alien' cofactors, are useful when employed in conjunction with those of 11.31 for expansion of Δ by 'true' cofactors.

Examples

(i)
$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

If we interchange rows and columns, we obtain the *same* determinant; Δ is said to be a *symmetric* determinant, meaning that it is identical with its transpose.

The cofactors are

$$A = bc - f^2, \quad B = ac - g^2, \quad C = ab - h^2,$$
$$F = gh - af, \quad G = hf - bg, \quad H = fg - ch.$$

Owing to the symmetry, we shall get only three different expansions by true cofactors, and six expansions by alien cofactors:

$$\Delta = aA + hH + gG = hH + bB + fF = gG + fF + cC;$$
$$0 = aH + hB + gF = aG + hF + gC$$
$$= hA + bH + fG = hG + bF + fC$$
$$= gA + fH + cG = gH + fB + cF.$$

This determinant arises frequently in coordinate geometry, and the use of the above relations often saves much algebraic manipulation. The reader is invited to verify some of them by direct substitution (see 11.22, ex. (ii)).

(ii) *The proof of Property* (5) *in* 11.24 *can be expressed more concisely in the notation for cofactors.*

$$\begin{vmatrix} a_1+ka_2 & b_1+kb_2 & c_1+kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= (a_1+ka_2)A_1+(b_1+kb_2)B_1+(c_1+kc_2)C_1$$

$$= (a_1A_1+b_1B_1+c_1C_1)+k(a_2A_1+b_2B_1+c_2C_1)$$

$$= \Delta+k.0 = \Delta.$$

No. 27 of Ex. 11 (*a*) can be done in a similar way.

11.4 Determinants and linear simultaneous equations

We now return to the subject of systems of linear equations (11.1), and apply our knowledge of determinants to the problem of solving them.

11.41 Cramer's rule

Consider again the system

$$a_1x+b_1y+c_1z = d_1,$$

$$a_2x+b_2y+c_2z = d_2,$$

$$a_3x+b_3y+c_3z = d_3.$$

Multiply these equations by the cofactors of a_1, a_2, a_3 in

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and add:

$$(a_1A_1+a_2A_2+a_3A_3)x+(b_1A_1+b_2A_2+b_3A_3)y+(c_1A_1+c_2A_2+c_3A_3)z$$
$$= d_1A_1+d_2A_2+d_3A_3.$$

The coefficient of x is Δ, while by alien cofactors the coefficients of y, z are zero. Hence

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

because the sum $d_1A_1+d_2A_2+d_3A_3$ is the expansion from the first column of the determinant obtained from Δ by replacing its first column by the elements d_1, d_2, d_3 appearing on the right-hand sides of the given equations. We may therefore write

$$\Delta x = \Delta^{(1)},$$

where $\Delta^{(1)}$ denotes Δ with its first column modified as just indicated.

Multiplying the given equations by the cofactors of b_1, b_2, b_3 in Δ and adding gives

$$\Delta y = \Delta^{(2)};$$

and similarly

$$\Delta z = \Delta^{(3)}.$$

If $\Delta \neq 0$, the solution of the given equations is unique and is†

$$x = \frac{\Delta^{(1)}}{\Delta}, \quad y = \frac{\Delta^{(2)}}{\Delta}, \quad z = \frac{\Delta^{(3)}}{\Delta}.$$

The case $\Delta = 0$ will be considered later (11.42).

Example

Solve

$$2x+4y-3z = 5,$$
$$3x-8y+6z = 4,$$
$$8x-2y-9z = 12.$$

Here

$$\Delta = \begin{vmatrix} 2 & 4 & -3 \\ 3 & -8 & 6 \\ 8 & -2 & -9 \end{vmatrix} = 2\times 3\times \begin{vmatrix} 2 & 2 & -1 \\ 3 & -4 & 2 \\ 8 & -1 & -3 \end{vmatrix}$$

$$= 6\times \begin{vmatrix} 0 & 0 & -1 \\ 7 & 0 & 2 \\ 9 & -7 & -3 \end{vmatrix} \quad \text{by } c_1 \to c_1 - c_2, \text{ followed by } c_2 \to c_2 + 2c_3,$$

$$= 6\times 49;$$

and

$$\Delta^{(1)} = \begin{vmatrix} 5 & 4 & -3 \\ 4 & -8 & 6 \\ 12 & -2 & -9 \end{vmatrix} = 2\times 3\times 2\times \begin{vmatrix} 5 & 2 & -1 \\ 2 & -2 & 1 \\ 12 & -1 & -3 \end{vmatrix}$$

$$= 12\times \begin{vmatrix} 7 & 0 & 0 \\ 2 & -2 & 1 \\ 12 & -1 & -3 \end{vmatrix} \quad \text{by } r_1 \to r_1 + r_2,$$

$$= 12\times 7(6+1) = 12\times 49.$$

$$\therefore \quad x = \Delta^{(1)}/\Delta = 2.$$

† Strictly, what we have just proved is that, *if* there are numbers x, y, z which satisfy the three given equations, *then* these numbers also satisfy $\Delta x = \Delta^{(1)}$, $\Delta y = \Delta^{(2)}$, $\Delta z = \Delta^{(3)}$. What we are now asserting is the converse of this, viz. that if there are numbers x, y, z which satisfy these three derived equations, then they also satisfy the given equations; see Ex. 11 (*b*), no. 16. The checking of solutions is thus not only a practical precaution but a logical necessity.

Similarly
$$\Delta^{(2)} = \begin{vmatrix} 2 & 5 & -3 \\ 3 & 4 & 6 \\ 8 & 12 & -9 \end{vmatrix} = 3\times 49, \quad \text{so} \quad y = \tfrac{1}{2};$$
and
$$\Delta^{(3)} = \begin{vmatrix} 2 & 4 & 5 \\ 3 & -8 & 4 \\ 8 & -2 & 12 \end{vmatrix} = 2\times 49, \quad \text{so} \quad z = \tfrac{1}{3}.$$

11.42 The case when $\Delta = 0$: inconsistency and indeterminacy

The proof of Cramer's rule shows that, if the given equations are satisfied by (x, y, z), then
$$\Delta x = \Delta^{(1)}, \quad \Delta y = \Delta^{(2)}, \quad \Delta z = \Delta^{(3)}.$$
When $\Delta = 0$ it follows that, if the system possesses a solution, we must have
$$\Delta^{(1)} = \Delta^{(2)} = \Delta^{(3)} = 0. \tag{i}$$

If one or more of the determinants $\Delta^{(1)}$, $\Delta^{(2)}$, $\Delta^{(3)}$ is non-zero, there can be no common solution; the equations are *inconsistent*.

Assuming that the consistency requirements (i) are satisfied, two cases arise.

(1) *At least one cofactor in Δ is non-zero.*

Suppose $C_3 \neq 0$,† and consider the two equations in two unknowns‡ x, y:
$$\left.\begin{aligned} a_1x + b_1y &= d_1 - c_1z, \\ a_2x + b_2y &= d_2 - c_2z. \end{aligned}\right\} \tag{ii}$$
Since $C_3 \neq 0$, these can be solved for x and y (as in 11.11) in terms of z. When z is given the particular value z_0, let the values obtained for x, y be x_0, y_0.

We may actually solve (ii) for x and y, and show by direct substitution that the values x_0, y_0, z_0 do satisfy the third equation of the system.§ More elegantly, however, we can prove this as follows.

Since (x_0, y_0, z_0) satisfies (ii),
$$a_1x_0 + b_1y_0 + c_1z_0 - d_1 = 0$$
and
$$a_2x_0 + b_2y_0 + c_2z_0 - d_2 = 0.$$
Suppose that
$$a_3x_0 + b_3y_0 + c_3z_0 - d_3 = u.$$
To show that these values satisfy the third equation of the system we have to prove $u = 0$.

Multiply the above equations by the cofactors of the elements in the last column of
$$\Delta^{(3)} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$
and add:
$$(a_1C_1 + a_2C_2 + a_3C_3)x_0 + (b_1C_1 + b_2C_2 + b_3C_3)y_0 + \Delta z_0 - \Delta^{(3)} = C_3u.$$
Hence
$$-\Delta^{(3)} = C_3u,$$

† This can always be *made* the case if necessary, by renaming the unknowns and rearranging the equations.

‡ Associated with the elements in C_3.

§ The reader may try this for himself.

the coefficients of x_0 and y_0 being zero by 'alien cofactors', and $\Delta = 0$ by hypothesis. Also $C_3 \neq 0$ by hypothesis, and $\Delta^{(3)} = 0$ by the assumed consistency of the equations. Therefore $u = 0$, which proves that the solution of the first two equations for x, y in terms of z will be also satisfy the third.

Since the value of z is arbitrary, we say that the solution is *indeterminate* with one 'degree of choice', or that there are ∞^1 solutions.

(2) *All cofactors in Δ are zero, but at least one element of Δ is non-zero.*

We may suppose $a_1 \neq 0$. Then the first of the given equations can be solved for x in terms of y and z. When y, z are given the values y_0, z_0, let the value obtained for x be x_0; then

$$x_0 = \frac{d_1 - b_1 y_0 - c_1 z_0}{a_1},$$

and so

$$a_2 x_0 + b_2 y_0 + c_2 z_0 - d_2 = \frac{a_2(d_1 - b_1 y_0 - c_1 z_0) + a_1 b_2 y_0 + a_1 c_2 z_0 - a_1 d_2}{a_1}$$

$$= \frac{(a_2 d_1 - a_1 d_2) + C_3 y_0 - B_3 z_0}{a_1}$$

$$= \frac{a_2 d_1 - a_1 d_2}{a_1}$$

by hypothesis. Similarly

$$a_3 x_0 + b_3 y_0 + c_3 z_0 - d_3 = \frac{a_3 d_1 - a_1 d_3}{a_1}.$$

The given equations will therefore be *inconsistent* unless

$$\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & d_1 \\ a_3 & d_3 \end{vmatrix}.$$

If both conditions are satisfied, the solution of the first equation (with y and z arbitrary) will satisfy the second and third also. The solution is *indeterminate* with two 'degrees of choice'; we may say there are ∞^2 solutions.

The results just proved are intuitively evident; for when all cofactors in Δ are zero, the coefficients in the left-hand sides of the given equations are 'proportional'. Clearly the equations will not be consistent unless the right-hand sides are in the *same* proportion; and then the three equations are equivalent to only one.

In 21.62 we shall illustrate the results of this section geometrically.

11.43 Homogeneous linear equations

(1) When the numbers on the right-hand sides are all zero, we obtain the *homogeneous system*

$$a_1 x + b_1 y + c_1 z = 0,$$

$$a_2 x + b_2 y + c_2 z = 0,$$

$$a_3 x + b_3 y + c_3 z = 0.$$

Clearly this is always satisfied by $x = 0$, $y = 0$, $z = 0$.

THEOREM I. *If the homogeneous system is satisfied by values of* x, y, z *which are* NOT ALL ZERO, *then* $\Delta = 0$.

Proof. Proceeding as in 11.41, we find that any set of values (x, y, z) which satisfies the given equations must also satisfy

$$\Delta x = 0, \quad \Delta y = 0, \quad \Delta z = 0.$$

Since x, y, z are *not all* zero, we must have $\Delta = 0$.

COROLLARY I (*a*). *If* $\Delta \neq 0$, *the* ONLY *solution is* $x = 0, y = 0, z = 0$.

In the particular case $z = 1$ we obtain the *non-homogeneous equations*

$$a_1 x + b_1 y + c_1 = 0,$$

$$a_2 x + b_2 y + c_2 = 0,$$

$$a_3 x + b_3 y + c_3 = 0.$$

If these have a common solution, say (x_0, y_0), then the corresponding homogeneous equations have the solution $(x_0, y_0, 1)$ which is certainly not all zero, whatever x_0, y_0 may be. Hence

COROLLARY I (*b*). *If the non-homogeneous equations possess a common solution, then* $\Delta = 0$.

A direct proof of this result is indicated in Ex. 11 (*b*), no. 14.

COROLLARY I (*c*). *If* $\Delta \neq 0$, *the non-homogeneous equations are inconsistent.*

For by Corollary I (*a*), the *only* solution of the homogeneous system is $(0, 0, 0)$, so $(x, y, 1)$ can never be a solution for any choice of x, y.

Δ is sometimes called the *eliminant* of the non-homogeneous system; for $\Delta = 0$ is the necessary and sufficient relation between the coefficients in order that the *three* equations in *two* unknowns x, y can be satisfied simultaneously (cf. 10.41).

Geometrically, the results of Corollaries I (*b*), (*c*) are associated with concurrence of lines in a plane (15.42).

As a converse to Theorem I we have

THEOREM II. *If* $\Delta = 0$, *then the homogeneous equations are satisfied by values of* x, y, z *not all zero.*

First proof. The discussion in 11.42 shows that, when $\Delta = 0$, we have a solution containing at least one arbitrary unknown which can be *chosen* to be non-zero. (The homogeneous equations are automatically consistent because they have the solution $(0, 0, 0)$.)

Second proof. Independently of 11.42, we can prove the theorem by actually writing down a solution of the required type.

(i) *Suppose that at least one cofactor in* Δ *is non-zero*, say $B_3 \neq 0$. Then

$$x = \lambda A_3, \quad y = \lambda B_3, \quad z = \lambda C_3$$

is a solution not all zero, where λ is non-zero but otherwise arbitrary. For

$$a_1 A_3 + b_1 B_3 + c_1 C_3 = 0 \quad \text{(alien cofactors)},$$

$$a_2 A_3 + b_2 B_3 + c_2 C_3 = 0 \quad \text{(alien cofactors)},$$

and

$$a_3 A_3 + b_3 B_3 + c_3 C_3 = 0 \quad (\Delta = 0 \text{ by hypothesis}).$$

(ii) *Suppose all cofactors in* Δ *are zero, but that at least one element of* Δ *is non-zero*, say $c_1 \neq 0$. Let x, y have arbitrary non-zero values

$$x = \lambda c_1, \quad y = \mu c_1 \quad (\text{so } \lambda \neq 0 \text{ and } \mu \neq 0).$$

Choose z to satisfy the first equation:

$$z = -(\lambda a_1 + \mu b_1).$$

With these values, the left-hand side of the second equation is

$$\begin{aligned} a_2 \lambda c_1 + b_2 \mu c_1 - c_2 \lambda a_1 - c_2 \mu b_1 & \\ &= \lambda(a_2 c_1 - a_1 c_2) + \mu(b_2 c_1 - b_1 c_2) \\ &= \lambda B_3 - \mu A_3 \\ &= 0 \end{aligned}$$

by hypothesis. Similarly the third equation is satisfied by the above values.

(2) *Two homogeneous equations in three unknowns: solution in ratios.* If A_3, B_3, C_3 are not all zero, then the equations

$$a_1 x + b_1 y + c_1 z = 0, \quad a_2 x + b_2 y + c_2 z = 0$$

have a non-zero solution $x = \lambda A_3$, $y = \lambda B_3$, $z = \lambda C_3$; this is clear by alien cofactors, or by proceeding as in 11.11. The solution can be written

$$\frac{x}{A_3} = \frac{y}{B_3} = \frac{z}{C_3},$$

i.e.

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

where if a particular denominator is zero, the corresponding numerator must be interpreted as 0 also.

This simple result will be useful later, especially in Ch. 21.

Exercise 11(*b*)

Solve the following systems of equations by use of determinants.

1 $2x-y+5z=2,\quad x+7y-10z=1,\quad x+y+z=2.$

2 $x-y-z=2,\quad x-2y-3z=1,\quad 3x-y+5z=4.$

3 $3x+y-4z=13,\quad 5x-y+3z=5,\quad x+y-z=3.$

*4 If a, b, c are all different, solve

$$x+y+z=1,\quad ax+by+cz=d,\quad a^2x+b^2y+c^2z=d^2.$$

[Use Ex. 11(*a*), no. 21.]

5 Prove that the following system of equations is inconsistent:

$$3x+y-z=0,\quad x-4y+2z=9,\quad 4x-3y+z=7.$$

[Solve the first two for x, y in terms of z, and substitute in left-hand side of the third.]

6 Show that the solution of the system

$$x-y-z=0,\quad 3x+3y-z=6,\quad 2x+y-z=3$$

is indeterminate, and can be written

$$x=1+2\lambda,\quad y=1-\lambda,\quad z=3\lambda,$$

where λ is arbitrary. [Solve the first two equations for x, y in terms of z.]

7 Show that the solution of

$$3x+y-z=0,\quad x-4y+2z=0.\quad 4x-3y+z=0$$

can be written $\quad x=2\lambda,\quad y=7\lambda,\quad z=13\lambda.$

8 Find the values of λ for which the equations

$$\lambda x+y+\sqrt{2}z=0,$$
$$x+\lambda y+\sqrt{2}z=0,$$
$$\sqrt{2}x+\sqrt{2}y+(\lambda-2)z=0$$

have a solution other than $x=y=z=0$. Find also the ratios $x:y:z$ which correspond to each of these values of λ.

9 Find the values of λ for which the equations

$$3x+\lambda y=5,\quad \lambda x-3y=-4,\quad 3x-y=-1$$

are consistent. Solve the equations when λ has these values.

10 Solve (if possible) the equations

$$x+y+kz=4k,$$
$$x+ky+z=-2,$$
$$2x+y+z=-2$$

when (i) $k\neq 0$, $k\neq 1$; (ii) $k=0$; (iii) $k=1$.

11 Eliminate x, y, z from $x+by+cz=0$, $ax+y+cz=0$, $ax+by+z=0$.

12 Solve Ex. 11(*a*), no. 26 by regarding $a=b\cos C+c\cos B$, etc. as homogeneous equations for a, b, c.

13 If the quadratics $ax^2+bx+c=0$, $px^2+qx+r=0$, $lx^2+mx+n=0$ have a common root, show that

$$\begin{vmatrix} a & b & c \\ p & q & r \\ l & m & n \end{vmatrix} = 0.$$

[If α is the common root, then the non-homogeneous system of equations $ax+by+c=0$, $px+qy+r=0$, $lx+my+n=0$ has the solution (α^2, α); use 11.43, Corollary I (*b*).]

14 Find from first principles the condition that the equations

$$a_1x+b_1y+c_1=0, \quad a_2x+b_2y+c_2=0, \quad a_3x+b_3y+c_3=0$$

are satisfied by the same values of x and y. [Assuming $a_2b_3-a_3b_2 \neq 0$, solve the last two for x, y in the form

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}},$$

and substitute these solutions in the first, getting $\Delta = 0$.]

15 If the homogeneous system

$$a_1x+b_1y+c_1z=0, \quad a_2x+b_2y+c_2z=0, \quad a_3x+b_3y+c_3z=0$$

has a solution other than $x=y=z=0$, show by means of no. 14 that $\Delta = 0$. [If $z \neq 0$, the equations can be written $a_1(x/z)+b_1(y/z)+c_1=0$, etc.]

16 *Verify in detail that when* $\Delta \neq 0$, *the solutions given in* 11.41 *actually do satisfy the equations.* [The solution is given by

$$\Delta x = d_1A_1+d_2A_2+d_3A_3, \quad \Delta y = d_1B_1+d_2B_2+d_3B_3,$$
$$\Delta z = d_1C_1+d_2C_2+d_3C_3.$$

Hence

$$\begin{aligned}\Delta(a_1x+b_1y+c_1z) &= d_1(a_1A_1+b_1B_1+c_1C_1)+d_2(a_1A_2+b_1B_2+c_1C_2) \\ &\quad +d_3(a_1A_3+b_1B_3+c_1C_3) \\ &= d_1\Delta\end{aligned}$$

by true and alien cofactors.]

11.5 Factorisation of determinants

In 11.25, ex. (iii) and Ex. 11 (*a*), no. 21 we have examples in which a determinant is expressed as a product of factors *without* expanding it directly. Here we consider various methods of factorising a determinant. Direct expansion should be used only when other methods cannot be further applied.

(i) *Use of the Remainder Theorem.*

Factorise

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

If Δ were expanded, we should obtain a polynomial in a, b, c which is homogeneous of degree 3 (for each term in the expansion consists of a product of factors taken one from each row and one from each column). We may regard this polynomial as a quadratic in a whose coefficients are polynomials in b and c.

When $a = b$, $\Delta = 0$ since it has two columns identical. Hence by applying the Remainder Theorem to Δ regarded as a polynomial in a, we see that $a-b$ is a factor.

Similarly $b-c$, $c-a$ are factors. Hence

$$\Delta \equiv k(b-c)(c-a)(a-b),$$

where k must be numerical because Δ and $(b-c)(c-a)(a-b)$ are both polynomials in a, b, c of total degree 3. To obtain the value of k, *either* compare coefficients of a particular term, say bc^2; *or* substitute particular values for a, b, c, say $a = 0$, $b = 1$, $c = -1$. We find $k = +1$, and so

$$\Delta \equiv (b-c)(c-a)(a-b).$$

(ii) *Considerations of symmetry and skewness.*

Factorise
$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}.$$

As in example (i), the Remainder Theorem shows that $b-c$, $c-a$, $a-b$ are factors. The determinant is a homogeneous polynomial of degree 5 in a, b, c, while the product $(b-c)(c-a)(a-b)$ is homogeneous of degree 3; hence the remaining factor P must be a homogeneous polynomial of degree 2 in a, b, c.

Since Δ and $(b-c)(c-a)(a-b)$ are both skew functions of a, b, c, hence P must by symmetric in a, b, c.

Hence by 10.22, Remark (β), P must be of the form

$$k(a^2+b^2+c^2)+l(bc+ca+ab),$$

where k, l are numerical.

Clearly the expansion of Δ contains no term in a^4, while the factorised form does unless $k = 0$. Hence

$$\Delta \equiv l(bc+ca+ab)(b-c)(c-a)(a-b).$$

By comparing coefficients (e.g. of b^2c^3), or substituting particular values for a, b, c, we find $l = +1$. Hence

$$\Delta \equiv (b-c)(c-a)(a-b)(bc+ca+ab).$$

(iii) *Use of row- and column-operations to make known factors appear explicitly.*

One example has already been given in 11.25, ex. (iii). As another we factorise Δ in ex. (ii) above by this method.

The Remainder Theorem indicates the factors $b-a$ and $c-a$. By $\mathbf{c}_2 \to \mathbf{c}_2 - \mathbf{c}_1$ and removal of $b-a$, followed by $\mathbf{c}_3 \to \mathbf{c}_3 - \mathbf{c}_1$ and removal of $c-a$, we have

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & b+a & c+a \\ a^3 & b^2+ba+a^2 & c^2+ca+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} b+a & c+a \\ b^2+ba+a^2 & c^2+ca+a^2 \end{vmatrix} \quad \text{on expanding by the first row,}$$

$$= (b-a)(c-a)(b-c)\begin{vmatrix} 1 & c+a \\ b+c+a & c^2+ca+a^2 \end{vmatrix} \quad \text{by } \mathbf{c}_1 \to \mathbf{c}_1 - \mathbf{c}_2 \text{ and removal of } b-c,$$

$$= (b-a)(c-a)(b-c)\begin{vmatrix} 1 & c+a \\ b+a & a^2 \end{vmatrix} \quad \text{by } \mathbf{r}_2 \to \mathbf{r}_2 - c\mathbf{r}_1,$$

$$= (b-c)(c-a)(a-b)(bc+ca+ab)$$

on expanding and rearranging cyclically.

Exercise 11(*c*)

Factorise the following determinants.

1 $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix}$.

2 $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$.

3 $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$.

4 $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$.

5 $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$.

6 $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ (b+c)^2 & (c+a)^2 & (a+b)^2 \end{vmatrix}$.

7 $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^4 & b^4 & c^4 \end{vmatrix}$.

*8 $\begin{vmatrix} a^2 & b^2 & c^2 \\ (b+c)^2 & (c+a)^2 & (a+b)^2 \\ bc & ca & ab \end{vmatrix}$.

*9 $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix}$.

Solve the following systems of equations by use of determinants.

10 $x+ay+a^2z = a^3,\quad x+by+b^2z = b^3,\quad x+cy+c^2z = c^3.$

11 $ax+by+cz = 1,\quad a^2x+b^2y+c^2z = 1,\quad a^3x+b^3y+c^3z = 1.$

*12 $ayz+bzx+cxy = 0,\quad yz+zx+xy = xyz,\quad a^2/x+b^2/y+c^2/z = p^2.$

11.6 Derivative of a determinant

If the elements of Δ are functions of x, then $d\Delta/dx$ is equal to the sum of the three determinants obtained from Δ by deriving one row at a time:

$$\frac{d}{dx}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} \frac{da_1}{dx} & \frac{db_1}{dx} & \frac{dc_1}{dx} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ \frac{da_2}{dx} & \frac{db_2}{dx} & \frac{dc_2}{dx} \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \frac{da_3}{dx} & \frac{db_3}{dx} & \frac{dc_3}{dx} \end{vmatrix}.$$

Proof. It is convenient to use the notation (mentioned in 11.22)

$$\Delta = \Sigma \pm a_i b_j c_k$$

for the expansion of Δ. Then

$$\frac{d\Delta}{dx} = \Sigma \pm \frac{da_i}{dx} b_j c_k + \Sigma \pm a_i \frac{db_j}{dx} c_k + \Sigma \pm a_i b_j \frac{dc_k}{dx}.$$

The first sum is the expansion of the determinant obtained from Δ by replacing the top row a_1, b_1, c_1 by da_1/dx, db_1/dx, dc_1/dx; the other two sums are interpreted similarly. This proves the theorem.

A similar result holds with 'rows' replaced by 'columns'.

Example

If three rows of a determinant whose elements are polynomials in x become identical when $x = a$, then $(x-a)^2$ is a factor of Δ.

First solution. $d\Delta/dx$ = sum of three determinants each having two rows the same as Δ. Therefore when $x = a$, $d\Delta/dx = 0$ because each determinant in the sum has two identical rows. Hence $x-a$ is a factor of $d\Delta/dx$. Since also $\Delta = 0$ when $x = a$, $(x-a)^2$ is a factor of Δ by 10.43.

Second solution. Subtract any one of the rows from the remaining two rows. When $x = a$, each element in each of these new rows vanishes; hence by the Remainder Theorem $x-a$ is a factor of each row, so $(x-a)^2$ is a factor of Δ.

11.7 Determinants of order 4

The symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

called a *fourth-order determinant,* is defined to mean

$$a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1 \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

This is a direct generalisation of the definition of 'third-order determinant' in 11.22.†

We mention fourth-order determinants because they arise occasionally in the coordinate geometry of three dimensions. It can be shown that

(i) the properties (1)–(5) in 11.21 continue to hold;
(ii) the determinant can be expanded from any one row or any one column;
(iii) the concept of 'cofactors' is still valid and useful;
(iv) there are expansions by true and alien cofactors.

They can be used to solve systems of four linear equations in four unknowns by the obvious extension of Cramer's rule (11.41), and in elimination problems.

† Determinants of order n can be defined similarly in terms of those of order $n-1$.

The proofs of these statements are not always easy, and for determinants of order four or more it is better to approach the subject from a more advanced point of view than we can consider here.

Exercise 11(*d*)

Write down the derivative of

1 $\begin{vmatrix} 1 & 2 & 3 \\ x & x^2 & x^3 \\ 1 & 2x & 3x^2 \end{vmatrix}$. 2 $\begin{vmatrix} 1 & 1 & x \\ 1 & 2 & x^2 \\ 1 & 3 & x^3 \end{vmatrix}$. 3 $\begin{vmatrix} \log x & x & x^2 \\ 1/x & 1 & 2x \\ 1/x^2 & 0 & -2 \end{vmatrix}$.

4 If $u' = du/dx$, etc., prove

$$\frac{d}{dx}\begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix} = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u''' & v''' & w''' \end{vmatrix}.$$

5 If u, v, w are polynomials in x whose degrees do not exceed 3, prove that

$$\Delta = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix}$$

is also a polynomial in x of degree not greater than 3. [Derive Δ wo x four times, rejecting zero determinants at each stage.]

6 Prove

$$\frac{\partial^2}{\partial x\,\partial y}\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = 2(y-x).$$

7 If

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}, \quad \text{prove} \quad \frac{\partial\Delta}{\partial x}+\frac{\partial\Delta}{\partial y}+\frac{\partial\Delta}{\partial z} = 0.$$

[Derive by rows, regarding $\partial/\partial x + \partial/\partial y + \partial/\partial z$ as operating on the rows.]

8 If

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}, \quad \text{find} \quad \frac{\partial\Delta}{\partial x}+\frac{\partial\Delta}{\partial y}+\frac{\partial\Delta}{\partial z}.$$

Evaluate the following fourth-order determinants.

*9 $\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 2 \\ 4 & 0 & 6 & 1 \end{vmatrix}$. *10 $\begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & 4 & 0 & -1 \\ -3 & 6 & 2 & 4 \\ 0 & 8 & 1 & 2 \end{vmatrix}$. *11 $\begin{vmatrix} 2 & 3 & 7 & 5 \\ 7 & 8 & 2 & 9 \\ 3 & 6 & 7 & 4 \\ 3 & 3 & 9 & 2 \end{vmatrix}$.

*12 Prove

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & p \end{vmatrix} = p\Delta - (Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm),$$

where the notation is that of 11.32, ex. (i). [Expand by the last column, then by the last rows of the third-order determinants so obtained.]

*13 Solve by determinants:

$$\begin{aligned} 3x + y - 5z &= 0, \\ x + 2z + 3t &= 0, \\ y + 3z - 2t &= 0, \\ x + 2y - z - 2t &= 14. \end{aligned}$$

*14 Prove (i) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & r & r^2 & r^3 \\ 1 & r^2 & r^3 & r^4 \\ 1 & r^3 & r^4 & r^5 \end{vmatrix} = 0;$

(ii) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & r & r^2 & r^3 \\ 1 & r^2 & r^4 & r^6 \\ 1 & r^3 & r^6 & r^9 \end{vmatrix} = r^4(r-1)^3(r^2-1)^2(r^3-1).$

*15 Factorise

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & b & c \\ x^2 & a^2 & b^2 & c^2 \\ x^3 & a^3 & b^3 & c^3 \end{vmatrix}.$$

Obtain identities by considering the cofactors of x^3, x^2, x in Δ and the coefficients of x^3, x^2, x in the product.

Miscellaneous Exercise 11(*e*)

1 Without expanding the determinant, explain why

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

is the equation of the line joining the points (x_1, y_1), (x_2, y_2).

Without expanding, prove

2 $\begin{vmatrix} bc & c^2 & b^2 \\ c^2 & ca & a^2 \\ b^2 & a^2 & ab \end{vmatrix} = abc \begin{vmatrix} a & c & b \\ c & b & a \\ b & a & c \end{vmatrix}.$

3 $\begin{vmatrix} a+b & b+c & c+a \\ l+m & m+n & n+l \\ p+q & q+r & r+p \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix}.$

Factorise

4 $\begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix}$.

5 $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ (b+c)^2 & (c+a)^2 & (a+b)^2 \end{vmatrix}$.

6 $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$.

7 $\begin{vmatrix} b^2c^2+a^2d^2 & bc+ad & 1 \\ c^2a^2+b^2d^2 & ca+bd & 1 \\ a^2b^2+c^2d^2 & ab+cd & 1 \end{vmatrix}$.

8 Show that a and $a+b+c$ are factors of

$$\begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ a^2 & (c+a)^2 & c^2 \\ a^2 & b^2 & (a+b)^2 \end{vmatrix},$$

and hence factorise it completely.

9 By factorising the determinants, prove that

$$\begin{vmatrix} 2 & a+b & a^2+b^2 \\ a+b & a^2+b^2 & a^3+b^3 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

10 Prove that $\sin\theta$, $\cos\theta$, $\sin\theta-\cos\theta$ are factors of

$$\begin{vmatrix} \cos\theta & \sin 2\theta & \cos^2\theta \\ \sin\theta & \sin 2\theta & \sin^2\theta \\ \sin\theta & \sin^2\theta & \cos^2\theta \end{vmatrix},$$

and find the remaining factor.

11 Express $\begin{vmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{vmatrix}$

as the sum of four second-order determinants, and hence show that it is equal to

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} p & q \\ r & s \end{vmatrix}.$$

(This gives a rule for multiplying two second-order determinants.)

Solve

12 $\begin{vmatrix} x+1 & 2x & 1 \\ x & 3x-2 & 2x \\ 1 & x & x \end{vmatrix} = 0.$

13 $\begin{vmatrix} x & x^2 & a^3-x^3 \\ b & b^2 & a^3-b^3 \\ c & c^2 & a^3-c^3 \end{vmatrix} = 0$ (assuming that a, b, c are non-zero and distinct).

14 Prove that the vertices of the triangle formed by the lines

$$a_1x+b_1y+c_1 = 0, \quad a_2x+b_2y+c_2 = 0, \quad a_3x+b_3y+c_3 = 0$$

are $\left(\frac{A_1}{C_1}, \frac{B_1}{C_1}\right)$, $\left(\frac{A_2}{C_2}, \frac{B_2}{C_2}\right)$, $\left(\frac{A_3}{C_3}, \frac{B_3}{C_3}\right)$.

15 Show that g, f, c cannot be found so that the equation

$$x^2+y^2+2gx+2fy+c = 0$$

is satisfied by the values (1, 1), (0, −2), (2, 4) of (x, y). Interpret geometrically.

16 Show that the equations

$$2x+3y = 4, \quad 3x+\lambda y = -1, \quad \lambda x-2y = c$$

can be consistent whenever c satisfies $4c^2-156c-439 \geqslant 0$.

17 Discuss the equations

$$x+y+3z = 4, \quad x+2y+4z = 5, \quad x-y+az = b$$

when (i) $a = b = 2$; (ii) $a = 1, b = 2$; (iii) $a = b = 1$.

18 Find the conditions which λ, μ must satisfy for the following system of equations to have (i) a unique solution; (ii) no solution; (iii) an infinity of solutions:

$$x+y+z = 6, \quad x+2y+3z = 10, \quad x+2y+\lambda z = \mu.$$

19 Prove that, for two values of λ, the equations

$$\begin{aligned}(\lambda+2)x+4y+3z &= 6,\\ 2x+(\lambda+9)y+6z &= 12,\\ 3x+12y+(\lambda+10)z &= k,\end{aligned}$$

have no solution unless k is suitably chosen. When k is thus chosen, find the general solution for each of the values of λ.

20 If f, g, h are functions of x which satisfy the differential equations

$$\frac{df}{dx} = f+g+2h, \quad \frac{dg}{dx} = 2g+2h, \quad 8\frac{dh}{dx} = 7f+8g+24h,$$

find all solutions of the form $f = ae^{\lambda x}, g = be^{\lambda x}, h = ce^{\lambda x}$ (where λ is independent of x), giving all the possible values of λ and the corresponding ratios of the constants a, b, c.

21 If

$$W(x,y,z) = \begin{vmatrix} x & y & z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix},$$

where x, y, z are functions of t and $\dot{x} = dx/dt$, etc., prove that

$$W(x,y,z) = x^3 W(1, y/x, z/x).$$

22 Write

$$V_1 = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}, \quad V_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}, \quad V_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix},$$

and

$$f(x) = \begin{vmatrix} x+a & x+b & x+c \\ (x+a)^2 & (x+b)^2 & (x+c)^2 \\ (x+a)^3 & (x+b)^3 & (x+c)^3 \end{vmatrix}.$$

(i) Prove $f(x) = (x+a)(x+b)(x+c)V_3$.

(ii) Calculate $f'(x), f''(x), f'''(x)$ as determinants, and show $f^{(r)}(x) = 0$ for $r \geqslant 4$.

(iii) Using Maclaurin's theorem, prove

$$f(x) = f(0)+V_1x+V_2x^2+V_3x^3,$$

and deduce that V_r $(r = 1, 2, 3)$ is the coefficient of x^r in the expansion of $(x+a)(x+b)(x+c)V_3$. (Cf. Ex. 11(d), no. 15.)

12

SERIES

12.1 The binomial theorem

The reader should already be familiar with the work in this section; we include it for revision and completeness.

12.11 *If n is a positive integer,*

$$(x+a)^n = x^n + {}^nC_1 x^{n-1}a + {}^nC_2 x^{n-2}a^2 + \dots + {}^nC_r x^{n-r}a^r + \dots + a^n,$$

where ${}^nC_r$ denotes the number of selections of r different objects from n, viz.

$$\frac{n(n-1)\dots(n-r+1)}{r!}.$$

Proof. Consider the product $(x+a_1)(x+a_2)\dots(x+a_n)$. To expand it we must multiply each term in each bracket by the terms in the other brackets. This can be done systematically as follows.

If we multiply the x from each bracket, we obtain x^n.

If we multiply the a-term from one bracket and the x-terms from the other brackets, and do this in all possible ways, we obtain

$$(a_1+a_2+\dots+a_n)\,x^{n-1}.$$

Next, multiplying the a-terms from two brackets and the x-terms from the rest, and doing this in all possible ways, we obtain

$$(a_1a_2+a_1a_3+\dots+a_2a_3+\dots)\,x^{n-2},$$

where the coefficient of x^{n-2} consists of the products of different a-terms taken two at a time. The *number* of terms in this coefficient is therefore ${}^nC_2$.

Similarly, multiplying a-terms from three brackets and x-terms from the rest in all possible ways gives

$$(a_1a_2a_3+a_1a_2a_4+\dots)\,x^{n-3},$$

where the *number* of terms in the coefficient of x^{n-3} is ${}^nC_3$.

In general, if we multiply a-terms from r brackets and x-terms from the remaining $n-r$ brackets in all possible ways, we get

$$(a_1a_2\dots a_r+\dots)\,x^{n-r},$$

where the coefficient consists of ${}^nC_r$ terms.

Finally, the product obtained by multiplying a-terms from all brackets is $a_1 a_2 \dots a_n$.

We now put $a_1 = a_2 = \dots = a_n = a$. The original product then becomes $(x+a)^n$. In the expansion, the coefficients of x^{n-1}, x^{n-2}, x^{n-3}, ..., x^{n-r}, ... become

$$na, \quad {}^nC_2 a^2, \quad {}^nC_3 a^3, \quad \dots, \quad {}^nC_r a^r, \quad \dots,$$

and the constant term becomes a^n. This proves the result stated.

12.12 Properties of the binomial expansion

(1) The expansion of $(x+a)^n$ consists of $n+1$ terms.

(2) The $(r+1)$th term is ${}^nC_r a^r x^{n-r}$ and is called the *general term*.

(3) The coefficients of terms at equal distances from each end of the expansion are equal.

For the $(r+1)$th term from the beginning is ${}^nC_r a^r x^{n-r}$, while the $(r+1)$th term from the end is ${}^nC_{n-r} a^{n-r} x^r$; and

$${}^nC_r = \frac{n!}{r!\,(n-r)!} = {}^nC_{n-r}.$$

This fact saves work in numerical calculations.

(4) If n is even, there is a middle term, given by $r = \frac{1}{2}n$.

If n is odd, there are two (equal) middle terms given by $r = \frac{1}{2}(n \pm 1)$.

(5) We can obtain the expansion of $(x+a)^n$ in *ascending* powers of x by interchanging x and a in the result of 12.11:

$$(x+a)^n = a^n + {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + \dots + x^n.$$

This is useful in approximations when x is small compared with a.

(6) If we write $-a$ for a in 12.11, we obtain

$$(x-a)^n = x^n - {}^nC_1 a x^{n-1} + {}^nC_2 a^2 x^{n-2} - \dots + (-1)^r\, {}^nC_r a^r x^{n-r} + \dots + (-1)^n a^n.$$

12.13 Examples

(i) *Find the coefficient of x^9 in the expansion of $(2x - 3/x)^{13}$.*

The general term is

$${}^{13}C_r \left(-\frac{3}{x}\right)^r (2x)^{13-r} = {}^{13}C_r (-3)^r\, 2^{13-r} x^{13-2r}.$$

This will involve x^9 if $13 - 2r = 9$, i.e. if $r = 2$. Hence the required coefficient is

$${}^{13}C_2 (-3)^2\, 2^{11} = 13.3^3.2^{12}.$$

(ii) *Expand $(2+x-3x^2)^5$ in ascending powers of x as far as the term in x^3.*

$$\begin{aligned}(2+x-3x^2)^5 &= \{2+x(1-3x)\}^5\\ &= 2^5 + {}^5C_1 2^4 x(1-3x) + {}^5C_2 2^3 x^2 (1-3x)^2 + {}^5C_3 2^2 x^3 (1-3x)^3 + \dots,\end{aligned}$$

where the unwritten terms all contain no power lower than x^4. On ignoring all such terms, we have

$$32+5.16(x-3x^2)+\frac{5.4}{2.1}8x^2(1-6x)+\frac{5.4.3}{3.2.1}4x^3+\dots$$
$$=32+80x-160x^2-440x^3+\dots$$

after collecting like terms.

(iii) *Find the numerically greatest term in the expansion of* $(1-3x)^7$ *if* $x=\frac{1}{4}$.

The terms in this expansion are alternately positive and negative, but the *numerically* greatest term will be the same as the corresponding term in the expansion of $(1+3x)^7$. If the latter expansion is

$$u_1+u_2x+u_3x^2+\dots+u_8x^7,$$

then
$$\frac{u_{r+1}}{u_r}=\frac{7!}{r!\,(7-r)!}(3x)^r\bigg/\frac{7!}{(r-1)!\,(8-r)!}(3x)^{r-1}$$
$$=\frac{8-r}{r}3x$$
$$=\frac{8-r}{r}.\frac{3}{4}\quad\text{if}\quad x=\tfrac{1}{4}.$$

Now $u_{r+1}\gtreqless u_r$ according as $u_{r+1}/u_r\gtreqless 1$, i.e. according as

$$\frac{3(8-r)}{4r}\gtreqless 1,\quad\text{i.e.}\quad 24-3r\gtreqless 4r,\quad\text{i.e.}\quad 3\tfrac{3}{7}\gtreqless r.$$

Hence if $r\leqslant 3$, $u_{r+1}>u_r$, i.e. $u_4>u_3$, $u_3>u_2$ and $u_2>u_1$. If $r\geqslant 4$, then $u_{r+1}<u_r$. i.e. $u_5<u_4$, $u_6<u_5$, etc. These inequalities show that u_4 is greater than the other terms. Thus the 4th term is the greatest. Its value is

$${}^7C_3(-\tfrac{3}{4})^3=-35\times(\tfrac{3}{4})^3.$$

(iv) *Prove* $\qquad {}^nC_{r-1}+{}^nC_r={}^{n+1}C_r.$

This can be verified directly, or obtained from the identity

$$(1+x)(1+x)^n\equiv(1+x)^{n+1},$$

in which the coefficient of x^r on the right is ${}^{n+1}C_r$. On the left, terms involving x^r will arise from $1\times{}^nC_rx^r$ and $x\times{}^nC_{r-1}x^{r-1}$, so the total coefficient is ${}^nC_r+{}^nC_{r-1}$. The result follows.

When the coefficients ${}^nC_r$ in the expansion of $(1+x)^n$ have been calculated, the above relation shows how to obtain those in $(1+x)^{n+1}$. The following scheme, in which each number is formed by adding the two immediately above it, is called *Pascal's triangle*:

$(1+x)^0$						1					
$(1+x)^1$					1		1				
$(1+x)^2$				1		2		1			
$(1+x)^3$			1		3		3		1		
$(1+x)^4$		1		4		6		4		1	
$(1+x)^5$	1		5		10		10		5		1
...	....										

Exercise 12(*a*)

Write out the expansions of

1 $(3x-2)^4$. 2 $(x+1/x)^5$. 3 $(1-x)(1+x)^4$.

4 Expand $(1+2x-x^2)^6$ in ascending powers as far as the term in x^4.

Write down and simplify the

5 coefficient of x^8 in $(2-x^2)^6$. 6 coefficient of x^{10} in $(1/x^2+x)^{19}$.

7 coefficient of x^{-20} in $(x^3-1/2x^2)^{15}$.

8 term independent of x in $(2x^2-1/x)^{12}$.

9 6th term in $(3x+1/x)^{11}$. 10 coefficient of x^3 in $(2+x-3x^2)^7$.

11 Evaluate correct to 4 places of decimals (i) $(1{\cdot}04)^5$; (ii) $(0{\cdot}98)^5$.

12 Find r if the coefficient of x^r in $(1+x)^{20}$ is twice the coefficient of x^{r-1}.

13 Find the greatest term in the expansion of $(5+4x)^{11}$ when $x=\frac{2}{3}$.

14 Find the greatest *coefficient* in the expansion of $(5+4x)^{11}$.

15 Obtain an identity by equating coefficients of x^r in

$$(1+2x+x^2)(1+x)^n \equiv (1+x)^{n+2}.$$

16 Consider the coefficient of x^r in $(1+x)^m(1+x)^n \equiv (1+x)^{m+n}$ to prove

$${}^mC_r\,{}^nC_0+{}^mC_{r-1}\,{}^nC_1+\ldots+{}^mC_0\,{}^nC_r = {}^{m+n}C_r \quad (r \leqslant m,\ r \leqslant n).$$

17 By considering the coefficients of x^n in $(1+x)^n(1+x)^n \equiv (1+x)^{2n}$, prove

$$c_0^2+c_1^2+c_2^2+\ldots+c_n^2 = \frac{(2n)!}{(n!)^2},$$

where c_r denotes ${}^nC_r$.

18 Prove

$$(1+x)^{n+1}-x^{n+1} \equiv (1+x)^n+x(1+x)^{n-1}+\ldots+x^r(1+x)^{n-r}+\ldots+x^n.$$

Deduce an identity by equating coefficients of x^r.

19 Show that $(2+\sqrt{3})^5+(2-\sqrt{3})^5$ is rational, and find its value.

20 Prove that

$$x+n(x+y)+\frac{n(n-1)}{1.2}(x+2y)+\frac{n(n-1)(n-2)}{3.2.1}(x+3y)+\ldots \text{ to } (n+1) \text{ terms}$$
$$= 2^{n-1}(2x+ny).$$

12.2 Finite series

The reader will already have met arithmetical and geometrical progressions, and perhaps other simple series. We now extend this work.

12.21 Notation and definitions

u_r denotes the rth term of a series.†

$s_n = \sum_{r=1}^{n} u_r$ denotes the sum of the first n terms, and is called the sum of the *finite series* $u_1, u_2, \ldots, u_n$.

† Or *sequence* in the language of 2.71; but when a sequence is considered in relation to its sum-sequence s_n, it is usually called a *series*.

If $u_r = a_r x^r$, then the series is called a *power series* in x.

Later we shall write $s = \lim_{n\to\infty} s_n$ provided this limit exists, and call s the *sum to infinity* of the *infinite series* Σu_r. For example, the sum of the first n terms of the geometrical progression

$$1, \quad x, \quad x^2, \quad x^3, \quad \ldots \quad (x \neq 1)$$

is
$$s_n = \frac{1-x^n}{1-x};$$

and if $|x| < 1$, then $x^n \to 0$ when $n \to \infty$ (2.72), so $s_n \to 1/(1-x)$.

12.22 General methods for summing finite series

Given a series, our problem is to find a formula for s_n, the sum of the first n terms. The methods (1)–(3) listed below will be illustrated in this section.

(1) *Derivation or integration* of a known finite power series and its sum-function (followed perhaps by substitution of some particular value for x).

(2) *The difference method.* If a function $f(r)$ can be found so that

$$u_r \equiv f(r+1) - f(r),$$

then by taking $r = n, n-1, n-2, \ldots, 2, 1$ in turn we have

$$u_n = f(n+1) - f(n),$$
$$u_{n-1} = f(n) - f(n-1),$$
$$u_{n-2} = f(n-1) - f(n-2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$u_2 = f(3) - f(2),$$

and
$$u_1 = f(2) - f(1).$$

By adding,
$$\sum_{r=1}^{n} u_r = f(n+1) - f(1).$$

(3) *Mathematical Induction* (for proving a stated result).

(4) *Use of complex numbers and de Moivre's theorem.* This method, applicable to many trigonometric series, will be considered in 14.5.

12.23 Series involving binomial coefficients

As our standard form of the binomial theorem we take

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + x^n.$$

Used thus, the numbers

$$1 = {}^nC_0, \quad {}^nC_1, \quad {}^nC_2, \quad \dots, \quad 1 = {}^nC_n$$

are called *binomial coefficients*, and are sometimes written

$$\binom{n}{0}, \quad \binom{n}{1}, \quad \binom{n}{2}, \quad \dots, \quad \binom{n}{n};$$

and when the index n is evident from the context and is the same for all, they are abbreviated to

$$c_0, \quad c_1, \quad c_2, \quad \dots, \quad c_n.$$

Thus $$(1+x)^n \equiv c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n. \tag{i}$$

The following methods are useful in dealing with series involving binomial coefficients:

(*a*) Express the given series as a combination of two or more binomial expansions.

(*b*) Obtain the given series by derivation or integration of an identity based on the standard binomial expansion (i).

(*c*) Construct a function in which the given series is the coefficient of a certain power of x, and evaluate this coefficient in another way.

Examples

(i) *Sum* $\qquad c_0 + 2c_1 x + 3c_2 x^2 + \dots + (n+1)\, c_n x^n.$

Method (*a*)

The sum $= (c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n) + (c_1 x + 2c_2 x^2 + \dots + nc_n x^n)$.

The first bracket $= (1+x)^n$. The second is

$$nx + 2\frac{n(n-1)}{2!}x^2 + 3\frac{n(n-1)(n-2)}{3!}x^3 + \dots + nx^n$$

$$= nx\left\{1 + (n-1)x + \frac{(n-1)(n-2)}{2!}x^2 + \dots + x^{n-1}\right\}$$

$$= nx(1+x)^{n-1}.$$

Therefore the given sum $= (1+x)^n + nx(1+x)^{n-1} = (1+x)^{n-1}\{1 + (n+1)\,x\}$.

Method (*b*)

Multiply both sides of the standard identity (i) by x:

$$c_0 x + c_1 x^2 + c_2 x^3 + \dots + c_n x^{n+1} \equiv x(1+x)^n.$$

Derive both sides wo x:

$$c_0+2c_1x+3c_2x^2+\dots+(n+1)c_nx^n \equiv (1+x)^n+nx(1+x)^{n-1},$$

which gives the same result as before.

(ii) *Sum* $\quad c_0+2c_1+3c_2+\dots+(n+1)c_n.$

This is not a power series, but it can be obtained by putting $x = 1$ in ex. (i):

$$c_0+2c_1+3c_2+\dots+(n+1)c_n = 2^{n-1}(n+2).$$

(iii) *Sum* $\quad c_0+\frac{1}{2}c_1+\frac{1}{3}c_2+\dots+\frac{1}{n+1}c_n.$

Method (*b*)

Integrate both sides of identity (i) from 0 to 1:

$$\left[c_0x+\tfrac{1}{2}c_1x^2+\tfrac{1}{3}c_2x^3+\dots+\frac{1}{n+1}c_nx^{n+1}\right]_0^1 = \left[\frac{1}{n+1}(1+x)^{n+1}\right]_0^1,$$

i.e.
$$c_0+\tfrac{1}{2}c_1+\tfrac{1}{3}c_2+\dots+\frac{1}{n+1}c_n = \frac{1}{n+1}(2^{n+1}-1).$$

Method (*a*)

$$\text{The given sum} = 1+\frac{n}{1.2}+\frac{n(n-1)}{1.2.3}+\dots+\frac{n(n-1)\dots 2.1}{1.2\dots n(n+1)}$$

$$= \frac{1}{n+1}\left\{\frac{n+1}{1}+\frac{(n+1)n}{1.2}+\frac{(n+1)n(n-1)}{1.2.3}+\dots+1\right\}$$

$$= \frac{1}{n+1}\{(1+1)^{n+1}-1\}$$

$$= \frac{1}{n+1}(2^{n+1}-1).$$

(iv) *Sum* $\quad c_0c_1+c_1c_2+c_2c_3+\dots+c_{n-1}c_n.$

Method (*c*). Consider

$$(c_0+c_1x+c_2x^2+\dots+c_nx^n)(c_0+c_1x+\dots+c_nx^n) \equiv (1+x)^n(1+x)^n \equiv (1+x)^{2n}.$$

Coefficient of x^{n-1} on the left-hand side is

$$c_0c_{n-1}+c_1c_{n-2}+\dots+c_{n-1}c_0 = c_0c_1+c_1c_2+\dots+c_{n-1}c_n \quad \text{because} \quad c_r = c_{n-r}.$$

Coefficient of x^{n-1} on the right is

$${}^{2n}C_{n-1} = \frac{(2n)!}{(n-1)!\,(n+1)!},$$

which is the value of the required sum.

Ex. 12 (*a*), no. 17 is another illustration of Method (*c*).

Exercise 12(*b*)

1 Prove $c_0+c_1+c_2+\dots+c_n = 2^n$.

2 Prove $c_0+c_2+c_4+\dots = c_1+c_3+c_5+\dots = 2^{n-1}$. [Put $x = -1$ in identity (i), then use no. 1.]

3 Prove $c_1+2c_2x+3c_3x^2+\dots+nc_nx^{n-1} = n(1+x)^{n-1}$, and deduce the sum of

$$c_1+2c_2+3c_3+\dots+nc_n.$$

4 Find $c_1-2c_2+3c_3-\dots+(-1)^{n-1}nc_n$.

5 Prove $c_0+3c_1+5c_2+\dots+(2n+1)c_n = (n+1)2^n$. [Method (*a*); *or* write backwards and add, as for summing on A.P.; *or* derive $x(1+x^2)^n$.]

6 Prove $2+3c_1x+4c_2x^2+\dots+(n+2)c_nx^n = (1+x)^{n-1}\{2+(n+2)x\}$.

7 Prove $c_0c_r+c_1c_{r+1}+c_2c_{r+2}+\dots+c_{n-r}c_n = {}^{2n}C_{n+r}$.

8 Using the identity $(1+x)^n(1-x)^n \equiv (1-x^2)^n$, prove that when n is *even*,

$$c_0^2-c_1^2+c_2^2-\dots+(-1)^n c_n^2 = (-1)^{\frac{1}{2}n}\frac{n!}{\{(\frac{1}{2}n)!\}^2}.$$

What is the value if n is *odd*?

9 Prove
$$c_1^2+2c_2^2+3c_3^2+\dots+nc_n^2 = \frac{(2n-1)!}{\{(n-1)!\}^2}.$$

10 Sum $c_0c_1+2c_1c_2+3c_2c_3+\dots+nc_{n-1}c_n$.

11 Find
$$\frac{x^2}{1.2}c_0+\frac{x^3}{2.3}c_1+\frac{x^4}{3.4}c_2+\dots \text{ to } (n+1) \text{ terms},$$

and deduce that

$$\frac{c_0}{1.2}+\frac{c_1}{2.3}+\frac{c_2}{3.4}+\dots \text{ to } (n+1) \text{ terms} = \frac{2^{n+2}-n-3}{(n+1)(n+2)}.$$

*12 Prove
$$c_1-\tfrac{1}{2}c_2+\tfrac{1}{3}c_3-\dots+(-1)^{n-1}\frac{1}{n}c_n = 1+\tfrac{1}{2}+\tfrac{1}{3}+\dots+\frac{1}{n}.$$

[$c_1+c_2x+\dots+c_nx^{n-1} \equiv \{(1+x)^n-1\}/x$; integrate from -1 to 0.]

13 (i) Find the sum of the coefficients in the expansion of $(1+3x)^6$. [Write $(1+3x)^6 \equiv a_0+a_1x+\dots+a_6x^6$, and put $x = 1$.]

(ii) Calculate $a_0+a_2+a_4+a_6$ and $a_1+a_3+a_5$. [See no. 2.]

*14 Let
$$(1+x+x^2)^n \equiv a_0+a_1x+a_2x^2+\dots+a_{2n}x^{2n}.$$

(i) Deduce other expansions by writing $1/x$, $-x$ in place of x, and prove $a_r = a_{2n-r}$.

(ii) Deduce the expansion of $(1+x^2+x^4)^n$.

(iii) Use the identity $(1+x^2+x^4) \equiv (1+x+x^2)(1-x+x^2)$ to prove that

$$a_0^2-a_1^2+a_2^2-\dots+a_{2n}^2 = a_n.$$

(iv) Calculate $a_0+a_3+\dots+a_{2n-1}$.

(v) Calculate $a_0+2a_1+3a_2+\dots+(2n+1)a_{2n}$.

12.24 Powers of integers

We consider sums of the form $\sum_{r=1}^{n} r^p$ where $p = 1$ or 2 or 3.

(1) $\sum_{r=1}^{n} r$. This is an A.P., and the usual method gives

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$

(2) $\sum_{r=1}^{n} r^2$. *First method.* From the identity

$$(2r+1)^3-(2r-1)^3 \equiv 24r^2+2,$$

we have by putting $r = 1, 2, \ldots, n$ and adding:

$$(2n+1)^3-1^3 = 24\sum_{r=1}^{n} r^2+2n,$$

$$\therefore \quad 24\sum_{r=1}^{n} r^2 = 8n^3+12n^2+4n,$$

and
$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

Second method. Using

$$(r+1)^3-r^3 \equiv 3r^2+3r+1$$

and summing for $r = 1, 2, \ldots, n$, we have

$$(n+1)^3-1^3 = 3\sum_{r=1}^{n} r^2+3\sum_{r=1}^{n} r+n.$$

From (1) this becomes

$$n^3+3n^2+3n = 3\sum_{r=1}^{n} r^2+\tfrac{3}{2}n(n+1)+n,$$

which leads to the result just found.

(3) $\sum_{r=1}^{n} r^3$. We have

$$\{r(r+1)\}^2-\{(r-1)r\}^2 \equiv 4r^3.$$

Summing this for $r = 1, 2, \ldots, n$, we get

$$\{n(n+1)\}^2-0^2 = 4\sum_{r=1}^{n} r^3,$$

so
$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Observe that this result is the square of $\sum_{r=1}^{n} r$.

The sum could also be obtained by using the expansion of $(r+1)^4-r^4$ together with the results of (1), (2).

The preceding series have been summed by using the *difference method*, described in general in 12.22 (2). We next consider some other series which can be treated in this way.

12.25 'Factor' series

(i) *Sum* $1.2+2.3+3.4+\ldots+n(n+1)$.

The general term is $u_r = r(r+1)$. Consider

$$r(r+1)(r+2)-(r-1)r(r+1) = r(r+1)\{(r+2)-(r-1)\}$$
$$= 3r(r+1) = 3u_r.$$

This is a suitable difference relation, and by summing for $r = 1, 2, \ldots, n$,

$$3\sum_{r=1}^{n} r(r+1) = n(n+1)(n+2)-0,$$

i.e. $$\sum_{r=1}^{n} r(r+1) = \tfrac{1}{3}n(n+1)(n+2).$$

(ii) *Sum to n terms* $1.2.3+2.3.4+3.4.5+\ldots$.

The general term is† $u_r = r(r+1)(r+2)$. Consider

$$r(r+1)(r+2)(r+3)-(r-1)r(r+1)(r+2)$$
$$= r(r+1)(r+2)\{(r+3)-(r-1)\}$$
$$= 4r(r+1)(r+2) = 4u_r.$$

Hence $$4\sum_{r=1}^{n} r(r+1)(r+2) = n(n+1)(n+2)(n+3)-0,$$

i.e. $$\sum_{r=1}^{n} r(r+1)(r+2) = \tfrac{1}{4}n(n+1)(n+2)(n+3).$$

The same method can be used for any series in which the terms consist of the same number of factors and the first factors in each term form an A.P. having the same common difference as the successive factors in each term. To obtain a difference function $f(r+1)$, insert an extra factor at the end of the rth term. Observe in exs. (i), (ii) how the sum can be written down from the form of the general term.

Other series may be reducible to this type.

(iii) *Sum* $1.2.3+2.3.5+3.4.7+\ldots$ *to n terms.*

$$u_r = r(r+1)(2r+1)$$
$$= r(r+1)\{2(r+2)-3\}$$
$$= 2r(r+1)(r+2)-3r(r+1).$$

$$\therefore \quad \sum_{r=1}^{n} u_r = 2\sum_{r=1}^{n} r(r+1)(r+2)-3\sum_{r=1}^{n} r(r+1)$$
$$= 2\times\tfrac{1}{4}n(n+1)(n+2)(n+3)-3\times\tfrac{1}{3}n(n+1)(n+2) \quad \text{by exs. (i), (ii),}$$
$$= \tfrac{1}{2}n(n+1)^2(n+2) \quad \text{after factorisation.}$$

Alternatively, $$u_r = 2r^3+3r^2+r,$$

† Here and elsewhere we write down the *simplest* formula which is consistent with the terms given; cf. Remark (α) in 2.71.

$$\therefore \quad \sum_{r=1}^{n} u_r = 2\sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$
$$= 2\times\tfrac{1}{4}n^2(n+1)^2 + 3\times\tfrac{1}{6}n(n+1)(2n+1) + \tfrac{1}{2}n(n+1)$$

by the results in 12.24, and this reduces to the expression just found.

The second method is inconvenient if u_r is of degree higher than 3 in r, unless we know formulae for $\sum_{r=1}^{n} r^4$, etc.

12.26 'Fraction' series

(1) *Examples.*

(i) *Sum*
$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots \textit{ to } n \textit{ terms.}$$
$$u_r = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$
by partial fractions, and this decomposition is of the form $f(r) - f(r+1)$. The difference method is therefore applicable, so that on putting $r = 1, 2, \dots, n$ and adding,
$$\sum_{r=1}^{n} u_r = \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}.$$

(ii) *Sum*
$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\dots \textit{ to } n \textit{ terms.}$$
$$u_r = \frac{1}{r(r+1)(r+2)}.$$

Partial fractions are less suitable here because their use would lead to *three* fractions. We may decompose u_r into *two* fractions by omitting the last factor:
$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2)-r}{r(r+1)(r+2)} = 2u_r.$$

Hence
$$\frac{1}{1.2} - \frac{1}{(n+1)(n+2)} = 2\sum_{r=1}^{n} u_r,$$

and the required sum is
$$\frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

The terms of the series just considered are the reciprocals of those illustrated in 12.25. The same method as in ex. (ii) here can be used to obtain a difference function $f(r)$, viz. write down u_r and omit the last factor in the denominator. This gives u_r in the form $f(r) - f(r+1)$ which, although not exactly the difference considered in 12.22 (2), does enable the method to be used.

Other series may be reducible to the above.

(iii) *Sum*
$$\frac{2}{3.4.5}+\frac{3}{4.5.6}+\frac{4}{5.6.7}+\dots \textit{ to } n \textit{ terms.}$$
$$u_r = \frac{r+1}{(r+2)(r+3)(r+4)} = \frac{(r+2)-1}{(r+2)(r+3)(r+4)}$$
$$= \frac{1}{(r+3)(r+4)} - \frac{1}{(r+2)(r+3)(r+4)}$$
$$= v_r - w_r, \quad \text{say.}$$

Since
$$\frac{1}{r+3}-\frac{1}{r+4}=v_r,\quad \sum_{r=1}^{n} v_r=\frac{1}{4}-\frac{1}{n+4}.$$

Since
$$\frac{1}{(r+2)(r+3)}-\frac{1}{(r+3)(r+4)}=\frac{2}{(r+2)(r+3)(r+4)}=2w_r,$$

hence
$$2\sum_{r=1}^{n} w_r=\frac{1}{3.4}-\frac{1}{(n+3)(n+4)}.$$

After reduction, we find
$$\sum_{r=1}^{n} u_r=\sum_{r=1}^{n} v_r-\sum_{r=1}^{n} w_r=\frac{5}{24}-\frac{2n+5}{2(n+3)(n+4)}.$$

(2) *Direct use of partial fractions.* This method will sum all 'fraction' series summable by the difference method, although often less easily.

Example

(iv) *Sum*
$$\frac{3}{1.2.4}+\frac{5}{2.3.5}+\frac{7}{3.4.6}+\ldots \textit{ to } n \textit{ terms.}$$

$$u_r=\frac{2r+1}{r(r+1)(r+3)}=\frac{A}{r}+\frac{B}{r+1}+\frac{C}{r+3},$$

and by the usual method (4.62) we find $A=\frac{1}{3}$, $B=\frac{1}{2}$, $C=-\frac{5}{6}$.

$$\therefore\quad 6u_r=\frac{2}{r}+\frac{3}{r+1}-\frac{5}{r+3},$$

and
$$6\sum_{r=1}^{n} u_r=2\sum_{r=1}^{n}\frac{1}{r}+3\sum_{r=1}^{n}\frac{1}{r+1}-5\sum_{r=1}^{n}\frac{1}{r+3}$$
$$=2\sum_{r=1}^{n}\frac{1}{r}+3\sum_{r=2}^{n+1}\frac{1}{r}-5\sum_{r=4}^{n+3}\frac{1}{r}$$
$$=2\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\sum_{r=4}^{n}\frac{1}{r}\right)+3\left(\tfrac{1}{2}+\tfrac{1}{3}+\sum_{r=4}^{n}\frac{1}{r}+\frac{1}{n+1}\right)$$
$$-5\left(\sum_{r=4}^{n}\frac{1}{r}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)$$
$$=2(1+\tfrac{1}{2}+\tfrac{1}{3})+3(\tfrac{1}{2}+\tfrac{1}{3})+\frac{3}{n+1}-\frac{5}{n+1}-\frac{5}{n+2}-\frac{5}{n+3}$$
$$=\frac{37}{6}-\frac{2}{n+1}-\frac{5}{n+2}-\frac{5}{n+3}.$$
$$\therefore\quad \sum_{r=1}^{n} u_r=\frac{37}{36}-\frac{1}{3(n+1)}-\frac{5}{6(n+2)}-\frac{5}{6(n+3)}.$$

To sum this series by the method of 12.25 we should write u_r as
$$\frac{(2r+1)(r+2)}{r(r+1)(r+2)(r+3)}$$
in order to make the factors in the denominator successive terms of an A.P., and then express the numerator in the form
$$2r^2+5r+2=2r(r+1)+3r+2.$$

Thus
$$u_r = \frac{2}{(r+2)(r+3)} + \frac{3}{(r+1)(r+2)(r+3)} + \frac{2}{r(r+1)(r+2)(r+3)},$$
and each fraction can be dealt with separately.

Success of the 'partial fraction' method in this example depends on the fact that the sum of the numerators of the three partial fractions is zero, so that the major part $\sum_{r=4}^{n} \frac{1}{r}$ of the sum cancels out. This will always be the case when the degree of the numerator of u_r is at least two lower than that of the denominator.

(3) *Sum to infinity.* The series in 12.24, 12.25 clearly have no limiting sum, but those in exs. (i)–(iv) above do possess such a sum to infinity (12.21), viz.

$$\text{(i) } 1, \quad \text{(ii) } \tfrac{1}{4}, \quad \text{(iii) } \tfrac{5}{24}, \quad \text{(iv) } \tfrac{37}{36}.$$

12.27 Some trigonometric series

(1) *Series of sines or cosines of angles in* A.P.

(i) $C = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha+(n-1)\beta\}$.

Multiply both sides by $2\sin\frac{1}{2}\beta$; then since
$$2\sin\tfrac{1}{2}\beta\, u_r = 2\sin\tfrac{1}{2}\beta\cos\{\alpha+(r-1)\beta\}$$
$$= \sin\{\alpha+(r-\tfrac{1}{2})\beta\} - \sin\{\alpha+(r-\tfrac{3}{2})\beta\}$$
by using one of the formulae for products into sums, and this expression is of the form $f(r+1)-f(r)$, we have by summing for $r = 1, 2, \ldots, n$:
$$2\sin\tfrac{1}{2}\beta\, C = \sin\{\alpha+(n-\tfrac{1}{2})\beta\} - \sin\{\alpha-\tfrac{1}{2}\beta\},$$
$$= 2\cos\{\alpha+\tfrac{1}{2}(n-1)\beta\}\sin\tfrac{1}{2}n\beta$$
by converting the difference into a product.
$$\therefore \quad C = \cos\{\alpha+\tfrac{1}{2}(n-1)\beta\}\frac{\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}.$$

(ii) $S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin\{\alpha+(n-1)\beta\}$.

The result
$$S = \sin\{\alpha+\tfrac{1}{2}(n-1)\beta\}\frac{\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}$$
can be obtained in various ways:

(*a*) Put $\alpha-\frac{1}{2}\pi$ for α in ex. (i). (This shows that the multiplier $2\sin\frac{1}{2}\beta$ for finding C must also suit S.)

(*b*) Multiply both sides by $2\sin\frac{1}{2}\beta$ (the same factor as for C).

(*c*) Derive the result of ex. (i) wo α.

(iii) $C' = \cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) - \ldots + (-1)^{n-1}\cos\{\alpha+(n-1)\beta\}$,

$S' = \sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \ldots + (-1)^{n-1}\sin\{\alpha+(n-1)\beta\}$.

These can be deduced from exs. (i), (ii) respectively by writing $\pi+\beta$ for β; or found directly by using the multiplier $2\sin\frac{1}{2}(\pi+\beta) = 2\cos\frac{1}{2}\beta$.

(2) *Other useful differences.* Results like

$$\tan\theta = \cot\theta - 2\cot 2\theta,$$

$$\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta,$$

$$\tan\theta \sec 2\theta = \tan 2\theta - \tan\theta$$

are easily verified, and can be used to sum suitable trigonometric series.

Example

(iv) *Sum* $\tan\frac{\theta}{2}\sec\theta + \tan\frac{\theta}{4}\sec\frac{\theta}{2} + \tan\frac{\theta}{8}\sec\frac{\theta}{4} + \dots$ *to n terms.*

$$u_r = \tan\frac{\theta}{2^r}\sec\frac{\theta}{2^{r-1}}.$$

If we write $\theta/2^r$ for θ in the last difference just mentioned, we see that

$$u_r = \tan\frac{\theta}{2^{r-1}} - \tan\frac{\theta}{2^r}.$$

$$\therefore \quad \sum_{r=1}^{n} u_r = \tan\theta - \tan\frac{\theta}{2^n}.$$

The sum to infinity is $\tan\theta$.

Exercise 12(*c*)

Find a formula for each of the following sums.

1 $1^3 + 2^3 + \dots + (2n+1)^3$.

2 $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2$.

3 $1^2 - 2^2 + 3^2 - 4^2 + \dots$ to n terms if n is (i) even; (ii) odd. Give a formula which includes both cases.

4 Prove that the sum of the products in pairs (without repetitions) of the first n integers is $\frac{1}{24}n(n^2-1)(3n+2)$. [If s_n is the required sum, then

$$\left(\sum_{r=1}^{n} r\right)^2 = 2s_n + \sum_{r=1}^{n} r^2.]$$

5 Expand $(r+1)^4 - r^4$. Using the formulae for Σr, Σr^2, deduce $\sum_{r=1}^{n} r^3$.

6 (i) By writing $r^2 \equiv r(r+1) - r$, obtain $\sum_{r=1}^{n} r^2$ by using 12.25.

(ii) Writing $r^3 \equiv r(r^2-1) + r \equiv (r-1)r(r+1) + r$, obtain $\sum_{r=1}^{n} r^3$.

7 Calculate $\sum_{r=n+1}^{2n} r(r+1)$.

8 Express $r(r+2)(2r-1)$ in the form $ar(r+1)(r+2) + br(r+1) + cr + d$, and hence calculate $\sum_{r=1}^{n} r(r+2)(2r-1)$.

9 Calculate $\sum_{r=1}^{n} (n-r)(r+1)$. [$u_r = n(r+1) - r(r+1)$.]

Sum the following series to n terms, and find the sum to infinity when it exists.

10 $1^2.2+2^2.3+3^2.4+\dots$

11 $3.5.7+5.7.9+7.9.11+\dots$ [Method of 12.25.]

12 $1.4.7+2.5.8+3.6.9+\dots$

13 $\frac{1^3}{1}+\frac{1^3+2^3}{2}+\frac{1^3+2^3+3^3}{3}+\dots$

14 $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots$

15 $\frac{1}{3.5.7}+\frac{1}{5.7.9}+\frac{1}{7.9.11}+\dots$

16 $\Sigma\frac{2r+1}{r(r+1)(r+2)}$.

17 $\frac{1}{3.5.7}+\frac{4}{5.7.9}+\frac{7}{7.9.11}+\dots$

Use partial fractions to find the following sums, and also the sum to infinity.

18 $\sum_{r=1}^{n}\frac{1}{r(r+2)}$.

19 $\sum_{r=1}^{n}\frac{r+1}{r(r+2)(r+3)}$.

Sum the following to n terms.

20 $\cos\theta\sin 2\theta+\cos 2\theta\sin 3\theta+\cos 3\theta\sin 4\theta+\dots$

[$\cos r\theta\sin(r+1)\theta = \frac{1}{2}\{\sin(2r+1)\theta+\sin\theta\}$.]

21 $\cos^2\theta+\cos^2 2\theta+\cos^2 3\theta+\dots$ [$\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$.]

22 $\operatorname{cosec} 2\theta+\operatorname{cosec} 4\theta+\operatorname{cosec} 8\theta+\dots$

23 $\tan\theta+2\tan 2\theta+4\tan 4\theta+8\tan 8\theta+\dots$

24 $\sec^2\theta+4\sec^2 2\theta+16\sec^2 4\theta+64\sec^2 8\theta+\dots$ [Use no. 23.]

25 $\sec\theta\sec 2\theta+\sec 2\theta\sec 3\theta+\sec 3\theta\sec 4\theta+\dots$ [$\tan(r+1)\theta-\tan r\theta = \dots$]

26 Prove that $\tan^{-1}(r+1)-\tan^{-1}r = \cot^{-1}(1+r+r^2)$, and hence sum

$$\cot^{-1}3+\cot^{-1}7+\cot^{-1}13+\dots+\cot^{-1}(1+n+n^2).$$

What is the sum to infinity?

12.28 Mathematical Induction

This is a general principle for proving a given statement which involves a positive integer n.

Let ϕ_n be such a statement. For example, ϕ_n might be '$n(n+1)$ is always divisible by 2', or 'the sum to n terms of $1.1!+2.2!+3.3!+\dots$ is $(n+1)!-1$'.

If by assuming the truth of ϕ_k we can prove the truth of ϕ_{k+1} (i.e. if the statement for any particular integer always implies the corresponding statement for the next integer), and if ϕ_1 is known to be true, then ϕ_n is true for all positive integers n. For by taking $k = 1$, we have that ϕ_1 implies ϕ_2; taking $k = 2$, ϕ_2 implies ϕ_3; and we can continue thus until any positive integer n is reached.

The principle holds for negative integers also, since a statement about $-n$ is equivalent to a statement about the positive integer n.

Examples

(i) *Prove* $$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

Suppose that, for some integer k,

$$\sum_{r=1}^{k} r^2 = \tfrac{1}{6}k(k+1)(2k+1).$$

This is the *induction hypothesis.*

Then $$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$$
$$= \tfrac{1}{6}k(k+1)(2k+1)+(k+1)^2 \quad \text{by the induction hypothesis,}$$
$$= \tfrac{1}{6}(k+1)\{k(2k+1)+6(k+1)\}$$
$$= \tfrac{1}{6}(k+1)\{2k^2+7k+6\}$$
$$= \tfrac{1}{6}(k+1)(k+2)(2k+3),$$

which is similar to the induction hypothesis, *but with* $k+1$ *instead of* k. Hence *if* the result holds for $n = k$, *then* it also holds for $n = k+1$.

When $n = 1$, the result is true because the left-hand side is $1^2 = 1$ and the right-hand side is $\tfrac{1}{6}.1.2.3 = 1$.

Hence by Mathematical Induction the statement is true for all positive integers n.

(ii) *Prove that* $3^{2n}+7$ *is always divisible by* 8.

Write $f(n) = 3^{2n}+7$, and consider

$$f(k+1)-f(k) = (3^{2k+2}+7)-(3^{2k}+7)$$
$$= 3^{2k+2}-3^{2k}$$
$$= 3^{2k}(3^2-1) = 8.3^{2k}.$$

If $f(k)$ is divisible by 8, this relation shows that $f(k+1)$ is also divisible by 8. Also $f(1) = 3^2+7 = 16$, which is in fact divisible by 8. Hence the result follows by Induction.

(iii) *Prove the binomial theorem by Induction.*

Suppose the theorem holds for some particular value $n = k$:

$$(x+a)^k = x^k + {}^kC_1x^{k-1}a + \ldots + {}^kC_rx^{k-r}a^r + \ldots + a^k.$$

Then in the expansion of $(x+a)^{k+1} = (x+a)(x+a)^k$, the coefficient of x^{k-r+1} is (for $r = 1, 2, \ldots, k$)

$${}^kC_ra^r + {}^kC_{r-1}a^r = {}^{k+1}C_ra^r$$

by 12.13, ex. (iv). This expression is also correct when $r = 0$ or $k+1$. Hence

$$(x+a)^{k+1} = x^{k+1} + {}^{k+1}C_1x^ka + \ldots + {}^{k+1}C_rx^{k-r+1}a^r + \ldots + a^{k+1},$$

which is similar to the induction hypothesis, but with $k+1$ for k.

The result holds (trivially) when $n = 1$. The binomial theorem follows by Induction.

(iv) *Leibniz's theorem on* $d^n(uv)/dx^n$ *was proved by Induction in* 6.62.

Remark. Mathematical Induction applied to the summation of a series is equivalent to the difference method. For to prove that, for all positive integers n,

$$u_1 + u_2 + \ldots + u_n = f(n),$$

we assume that, for some k,

$$u_1 + u_2 + \ldots + u_k = f(k),$$

and then show that

$$u_1 + u_2 + \ldots + u_{k+1} = f(k+1).$$

Since $\quad u_1 + u_2 + \ldots + u_{k+1} = (u_1 + u_2 + \ldots + u_k) + u_{k+1} = f(k) + u_{k+1}$,

this is equivalent to showing that

$$u_{k+1} = f(k+1) - f(k),$$

so that $f(r)$ is a suitable difference function.

Exercise 12(*d*)

Prove the following results by Mathematical Induction.

1 $1 + 3 + 5 + \ldots + (2n-1) = n^2$.

2 $1.1! + 2.2! + 3.3! + \ldots$ to n terms $= (n+1)! - 1$.

3 $1^2 + 4^2 + 7^2 + \ldots + (3n-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$.

4 $\dfrac{8}{3.5} - \dfrac{12}{5.7} + \dfrac{16}{7.9} - \ldots$ to n terms $= \frac{1}{3} + (-1)^{n-1}\dfrac{1}{2n+3}$.

5 (i) $n(n+1)(n+2)$ is divisible by 6; (ii) $n^3 + 2n$ is divisible by 3.

6 (i) $9^n - 1$ is divisible by 8; (ii) $9^n - 8n - 1$ is divisible by 64.

7 $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$.

[$x^{2n+1} + y^{2n+1} \equiv x^2(x^{2n-1} + y^{2n-1}) - y^{2n-1}(x^2 - y^2)$.]

8 If $2u_1 = a + b$, $2u_2 = b + u_1$, and $2u_{n+2} = u_n + u_{n+1}$ $(n \geqslant 1)$, prove

$$3u_n = a\{1 - (-\tfrac{1}{2})^n\} + b\{2 + (-\tfrac{1}{2})^n\},$$

and find $\lim_{n\to\infty} u_n$.

9 Prove

$$\cos\theta \sec\theta + \cos 2\theta \sec^2\theta + \ldots + \cos n\theta \sec^n\theta = \frac{\sin(n+1)\theta}{\sin\theta \cos^n\theta} - 1.$$

[Use the answer to construct a difference function: see Remark in 12.28.]

10 Prove

$$\operatorname{cosec}\alpha \operatorname{cosec}(\alpha+\beta) + \operatorname{cosec}(\alpha+\beta)\operatorname{cosec}(\alpha+2\beta) + \operatorname{cosec}(\alpha+2\beta)\operatorname{cosec}(\alpha+3\beta) + \ldots$$

to n terms is $\quad \operatorname{cosec}\beta\{\cot\alpha - \cot(\alpha+n\beta)\}$.

12.3 Infinite series

12.31 Behaviour of an infinite series

(1) *Convergence.* If $s_n = \sum_{r=1}^{n} u_r$ and $\lim_{n\to\infty} s_n$ exists and has the value s, we defined s to be the *sum to infinity* of the *infinite series* Σu_r (12.21).

We also say that the series Σu_r *converges* (or *is convergent*) to s, and write

$$s = \sum_{r=1}^{\infty} u_r$$

or

$$s = u_1 + u_2 + u_3 + \dots.$$

It will be clear from the definition that s is not a 'sum' in the ordinary sense of 'the result of adding a number of terms', but is the *limit* of such a sum. If we attempt to treat infinite series like finite ones (i.e. without consideration of convergence), then paradoxical results may arise.

For example, suppose $x > 0$ and consider the series

$$1 + x + \frac{1}{x} + x^2 + \frac{1}{x^2} + x^3 + \frac{1}{x^3} + \dots.$$

One is tempted to regard this as the sum of two infinite G.P.'s

$$1 + x + x^2 + x^3 + \dots \quad \text{and} \quad \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \dots,$$

whose sums to infinity are (see end of 12.21)

$$\frac{1}{1-x} \quad \text{and} \quad \frac{1/x}{1-1/x} = \frac{1}{x-1},$$

which add up to zero. We thus appear to have a series of positive terms whose sum to infinity is zero. The explanation is simply that the first G.P. converges (i.e. possesses a sum to infinity) only when $|x| < 1$, while the second converges only when $|1/x| < 1$, i.e. $|x| > 1$; there is no value of x for which both converge, and hence the original series has no sum to infinity.

Another example is

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \dots = (1 + \tfrac{1}{3} + \tfrac{1}{5} + \dots) + \tfrac{1}{2}(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots),$$

from which

$$\tfrac{1}{2}(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots) = 1 + \tfrac{1}{3} + \tfrac{1}{5} + \dots,$$

i.e.

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \dots = 1 + \tfrac{1}{3} + \tfrac{1}{5} + \dots,$$

a result clearly wrong since each term on the left is less than the corresponding one on the right.

When s_n does not tend to a limit when $n \to \infty$, there are various possibilities, illustrated by the following example.

(2) *The infinite* G.P. $1+x+x^2+x^3+\ldots$.

If $x \neq 1$, then $s_n = (1-x^n)/(1-x)$. Hence by 2.72

(i) if $-1 < x < 1$, then $x^n \to 0$ when $n \to \infty$, and so $s_n \to 1/(1-x)$;

(ii) if $x > 1$, then $x^n \to \infty$ and so $s_n \to +\infty$; we say s_n is *properly divergent to* $+\infty$;

(iii) if $x < -1$, then $x^n \to +\infty$ when n is even and $x^n \to -\infty$ when n is odd: s_n *oscillates infinitely*;

(iv) if $x = -1$, the series is $1-1+1-1+\ldots$, so that $s_n = 0$ if n is even and $s_n = 1$ if n is odd: s_n *oscillates finitely* (between 0 and 1).

If $x = 1$, the formula for s_n is meaningless, but the series becomes $1+1+1+\ldots$, so that $s_n = n$ and $s_n \to +\infty$ when $n \to \infty$: s_n is properly divergent to $+\infty$.

(3) *Divergence.* In this book we shall not give a *detailed* study of the behaviour of infinite series, and it will usually be sufficient if we employ the term '*divergent*' to mean 'not convergent'. Thus our 'divergence' will cover proper divergence to $+\infty$, proper divergence to $-\infty$, finite oscillation and infinite oscillation.

We abbreviate 'Σu_r is convergent' to 'Σu_r C'; similarly, 'Σu_r D' means 'Σu_r is divergent (i.e. not convergent)'.

12.32 General properties

(1) *If* Σu_r *converges, then* $\lim_{n\to\infty} u_n = 0$.

For by hypothesis $s_n \to s$ when $n \to \infty$; and since $u_n = s_n - s_{n-1}$, therefore $u_n \to s - s = 0$ when $n \to \infty$.

(2) *The converse of this result is false:* if $u_n \to 0$, the series Σu_r may not converge.

Example

$\Sigma(1/r)$ *is divergent.*

We have $$\tfrac{1}{3}+\tfrac{1}{4} > \tfrac{1}{4}+\tfrac{1}{4} = \tfrac{1}{2},$$
$$\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8} > \tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8} = \tfrac{1}{2},$$
$$\tfrac{1}{9}+\tfrac{1}{10}+\ldots+\tfrac{1}{16} > \tfrac{8}{16} = \tfrac{1}{2}, \quad \text{etc.},$$
so that $$s_4 > 1+\tfrac{1}{2}+\tfrac{1}{2} = 1+2\times\tfrac{1}{2},$$
$$s_8 > 1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2} = 1+3\times\tfrac{1}{2},$$
$$s_{16} > 1+4\times\tfrac{1}{2}, \quad \text{etc.},$$
and in general $$s_{2^p} > 1+p\times\tfrac{1}{2}.$$

Hence when $p \to \infty$, $s_{2^p} \to \infty$.

Given a positive integer $n > 1$, a positive integer p can be found so that $2^{p-1} < n \leqslant 2^p$, and therefore (since all terms of the series are *positive*)

$$s_{2^{p-1}} < s_n \leqslant s_{2^p}.$$

When $n \to \infty$, also $2^p \to \infty$, i.e. $p \to \infty$; hence $s_n \to \infty$ because $s_{2^{p-1}} \to \infty$. This shows that the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \dots,$$

sometimes called the *harmonic series*, is (properly) divergent (to $+\infty$), although $u_n = 1/n \to 0$ when $n \to \infty$.

Remark. We have shown that $u_n \to 0$ does not imply convergence of Σu_r; but *if* $u_n \nrightarrow 0$ *when* $n \to \infty$, Σu_r *must be divergent* (otherwise (1) would be contradicted).

Thus $\Sigma r/(r+1)$ D because $u_n = n/(n+1) \to 1$ when $n \to \infty$.

(3) *Convergence or divergence of a series is unaltered by*

(*a*) *removing a* FINITE *number of terms from the series;*

(*b*) *multiplying every term by a* NON-ZERO *constant.*

These facts are clear from the properties of limits (2.3). Property (*a*) is useful when the first few terms of a series behave irregularly, and also shows that any test for convergence which we shall give need apply only 'for all $n \geqslant m$', where m is a fixed number; i.e. from some definite term of the series onwards.

(4) *If* Σu_r, Σv_r *both converge, to sums* s, t *respectively, then* $\Sigma(au_r + bv_r)$ *converges to* $as + bt$, a *and* b *being constants.*

For if
$$s_n = \sum_{r=1}^{n} u_r, \quad t_n = \sum_{r=1}^{n} v_r,$$
then
$$\sum_{r=1}^{n} (au_r + bv_r) = as_n + bt_n$$
$$\to as + bt \quad \text{when} \quad n \to \infty, \quad \text{by 2.3, (ii)}.$$

Similarly, if one of Σu_r, Σv_r converges and the other diverges, then $\Sigma(au_r + bv_r)$ diverges. However, if both diverge, $\Sigma(au_r + bv_r)$ may possibly converge, e.g. if $u_r = r + 2^{-r}$, $v_r = r$, $a = 1$, $b = -1$; roughly speaking, the divergent parts may cancel out.

(5) *If* s_n *steadily increases when* n *increases, then* Σu_r *either converges or properly diverges to* $+\infty$. *If also* s_n *is bounded, then* Σu_r *is convergent.*

This follows from 2.77. In particular, if Σu_r consists of positive terms, then s_n is certainly increasing since $s_{n+1} = s_n + u_{n+1} > s_n$. Hence

a series of positive terms is either convergent or properly divergent to $+\infty$.

The behaviour of the series considered in 12.2 can be determined directly because a formula for s_n can be found; cf. 12.26 (3). When no such formula is known, we resort to *tests* for convergence or divergence. Although these may show that the series converges, they do not help us to evaluate its sum to infinity.

We recall that (2.71) the statement '$\lim_{n\to\infty} s_n = s$' means that if any positive number ϵ however small is given, then a number m (in general depending on ϵ) can be found such that, for *all* $n \geqslant m$,

$$s-\epsilon < s_n < s+\epsilon.$$

12.4 Series of positive terms

We suppose all terms of our series are positive in this section. Series whose terms are all negative can be included by first removing the factor -1 (see 12.32 (3) (*b*)).

12.41 Comparison tests

In these we compare the given series Σu_r with a series Σv_r whose behaviour (C or D) is known. Roughly, a series which is less term by term than a convergent series is also convergent, and one greater than a divergent series term by term is divergent.

(1) *Test for convergence.*
If Σv_r C, *and*

EITHER (*a*) $u_n \leqslant cv_n$ *for all* $n \geqslant m$, *where* c *is a positive constant,*

OR (*b*) $\lim_{n\to\infty} (u_n/v_n) = l \geqslant 0,$

then also Σu_r C.

Proof of (*a*). If $\sum_{r=1}^{\infty} v_r = t$, then since all the terms v_r are positive,

$$v_1+v_2+\ldots+v_n < t \quad \text{for all } n.$$

By hypothesis,

$$\begin{aligned} u_m+u_{m+1}+\ldots+u_n &\leqslant c(v_m+v_{m+1}+\ldots+v_n) \\ &< c(v_1+\ldots+v_m+\ldots+v_n) \\ &< ct. \end{aligned}$$

Hence for all $n \geqslant m$,

$$s_n = u_1+\ldots+u_m+\ldots+u_n < u_1+\ldots+u_{m-1}+ct = K, \text{ say.}$$

Therefore by 12.32 (5), Σu_r C.

Proof of (*b*). Given $\epsilon > 0$ however small, there is a number m such that

$$l-\epsilon < \frac{u_n}{v_n} < l+\epsilon \quad \text{for } all\ n \geqslant m.$$

The right-hand inequality shows that

$$u_n < (l+\epsilon)\,v_n \quad \text{for all } n \geqslant m.$$

Taking $c = l+\epsilon$ in (*a*), the result follows.

(2) *Test for* (*proper*) *divergence.*

If Σv_r D, *and*

EITHER (*a*) $u_n \geqslant cv_n$ *for all* $n \geqslant m$, *where* c *is a positive constant,*

OR (*b*) $\lim\limits_{n\to\infty}(u_n/v_n) = l > 0$ (*but* NOT $l = 0$),†

then Σu_r D.

Proof of (*a*). Since Σv_r D, then (see 12.32 (5))

$$v_m+v_{m+1}+\ldots+v_n \to \infty \quad \text{when} \quad n\to\infty.$$

By hypothesis,

$$u_m+u_{m+1}+\ldots+u_n \geqslant c(v_m+v_{m+1}+\ldots+v_n)$$
$$\to \infty \quad \text{when} \quad n\to\infty.$$

$$\therefore \quad s_n = u_1+\ldots+u_m+\ldots+u_n \to \infty \quad \text{when} \quad n\to\infty,$$

i.e. Σu_r D.

Proof of (*b*). From the left-hand inequality in the proof of (1) (*b*),

$$u_n > (l-\epsilon)\,v_n \quad \text{for all } n \geqslant m.$$

If $l > 0$, then $l-\epsilon > 0$ for all positive ϵ sufficiently small; we may take $c = l-\epsilon$ in (*a*), and the result follows.

(3) *Standard comparison series* are

(i) the G.P. Σx^r, which C if $0 \leqslant x < 1$ and D if $x \geqslant 1$ (12.31 (2));

(ii) the harmonic series $\Sigma(1/r)$ which D (12.32 (2)).

Another is $\Sigma(1/r^p)$; for various values of p this gives a whole family of series, which we now consider.

(iii) $\Sigma(1/r^p)$ C *if* $p > 1$ *and* D *if* $p \leqslant 1$.

(*a*) $$s_n = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}, \qquad \text{there being } n \text{ } terms,$$

$$< \left(\frac{1}{1^p}\right)+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\ldots+\frac{1}{7^p}\right)+\ldots,$$

there being n *brackets*;

† The result may be false if $l=0$; e.g. $\Sigma(1/r)$ D (12.32 (2)) and $(1/n^2)/(1/n)\to 0$, yet $\Sigma(1/r^2$ C (see (3) below).

the inequality is clear since the first, second, third, ... terms in s_n have been replaced by brackets containing 1, 2, 4, ... positive terms. Making an upper estimate for the contents of each bracket, we have when $p > 0$:

$$s_n < \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \ldots + \frac{1}{4^p}\right) + \ldots \quad (n \text{ brackets})$$

$$= \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \ldots \quad (n \text{ terms})$$

$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \ldots \quad (n \text{ terms})$$

$$= \left\{1 - \left(\frac{1}{2^{p-1}}\right)^n\right\} \Big/ \left\{1 - \frac{1}{2^{p-1}}\right\} \quad \text{on summing the G.P.,}$$

$$< \frac{1}{1 - (\frac{1}{2})^{p-1}} \quad \text{for all } n \text{ if } p > 1,$$

since then the common ratio $(\frac{1}{2})^{p-1}$ is less than 1.

Hence when $p > 1$, s_n is bounded; so by 12.32 (5) the series C.

(*b*) The case $p = 1$ gives the harmonic series already investigated in 12.32 (2).

(*c*) If $p < 1$, then $n^p < n$ when $n > 1$, so that $1/n^p > 1/n$. Hence by comparison with $\Sigma(1/r)$, $\Sigma(1/r^p)$ D when $p < 1$.

Alternatively, if $0 < p < 1$ then (without referring to the harmonic series)

$$s_n = \frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p} > n \times \frac{1}{n^p} = n^{1-p} \to \infty \quad \text{when} \quad n \to \infty.$$

If $p \leqslant 0$, then clearly $s_n \to \infty$. Hence $\Sigma(1/r^p)$ D when $p < 1$.

Examples

(i) *Examine the series*

$$\frac{5}{1.3.4} + \frac{7}{2.4.5} + \frac{9}{3.5.6} + \ldots.$$

Here
$$u_n = \frac{2n+3}{n(n+2)(n+3)}.$$

[When n is large, $u_n \doteqdot 2n/(n.n.n) = 2/n^2$; so for large n, u_n behaves like $2/n^2$. It is therefore likely that Σu_n C, and we try the comparison test for convergence.]

Using the 'inequality' form of the test,

$$u_n = \frac{2n+3}{n(n+2)(n+3)} < \frac{3n}{n.n.n} \quad \text{if} \quad n > 3,$$

i.e. $u_n < 3/n^2$ if $n > 3$. Taking $v_n = 1/n^2$, then $u_n < 3v_n$ for $n > 3$, and Σv_n C (being the series $\Sigma(1/r^p)$ when $p = 2$). Hence Σu_n C also.

Alternatively, using the 'limit' form of the test, and taking $v_n = 1/n^2$ as before,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{2n+3}{n(n+2)(n+3)}\bigg/\frac{1}{n^2}\right) = 2.$$

Hence Σu_n c because Σv_n c.

(ii) *Examine* $$\frac{1}{1^3}+\frac{1+2}{2^3}+\frac{1+2+3}{3^3}+\ldots.$$

$$u_n = \frac{1+2+\ldots+n}{n^3} = \frac{\frac{1}{2}n(n+1)}{n^3} = \frac{n+1}{2n^2}.$$

[When n is large, $u_n \doteqdot n/2n^2 = 1/2n$, so that probably Σu_n D.]

We have

$$u_n > \frac{n}{2n^2} = \frac{1}{2n} \quad \text{for all } n,$$

so if $v_n = 1/n$, then $u_n > \frac{1}{2}v_n$. Therefore Σu_n diverges with Σv_n.

Alternatively,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{n+1}{2n^2}\bigg/\frac{1}{n}\right) = \tfrac{1}{2},$$

and the same result follows.

Exercise 12(*e*)

Show that the following series do not converge because $u_n \not\to 0$.

1 $\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\ldots.$

2 $\frac{1+x^2}{x}-\frac{1+x^4}{x^2}+\frac{1+x^6}{x^3}-\ldots.$

3 $\Sigma\frac{1}{\sqrt{(r+1)}-\sqrt{r}}.$

Using the result of 12.41 (3) (iii), *state which of the following series converge.*

4 $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots.$

5 $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\ldots.$

6 $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots.$

7 $\frac{1}{\sqrt{8}}+\frac{1}{\sqrt{27}}+\frac{1}{\sqrt{64}}+\ldots.$

Use the comparison test to ascertain the behaviour of the following series.

8 $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\ldots.$

9 $\frac{1}{1^2+1}+\frac{1}{2^2+2}+\frac{1}{3^2+3}+\ldots.$

10 $\frac{1}{2^2}+\frac{2}{3^2}+\frac{3}{4^2}+\ldots.$

11 $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+\ldots.$

12 $\Sigma\frac{\sqrt{r}}{r+1}.$

13 $\Sigma\frac{1}{\sqrt{(r^2-2r)}}.$

14 $\Sigma\frac{2^r+1}{3^r+1}.$

15 $\Sigma\frac{5^r}{r!}.$ $\left[\text{If } n > 5,\ u_n = \frac{5}{1}\cdot\frac{5}{2}\cdots\frac{5}{n} = \left(\frac{5}{1}\cdot\frac{5}{2}\cdots\frac{5}{5}\right)\frac{5}{6}\cdots\frac{5}{n} < \frac{5^5}{5!}\left(\frac{5}{6}\right)^{n-5}.\right]$

16 Prove that $\frac{1}{1+x}+\frac{1}{1+x^2}+\frac{1}{1+x^3}+\ldots$ c if $x > 1$, D if $0 \leqslant x \leqslant 1$.

17 Use the argument in no. 15 to prove that $\Sigma(x^r/r!)$ c for all $x \geqslant 0$.

18 If Σu_r c, prove that Σu_r^2 and $\Sigma\{u_r/(1+u_r)\}$ also c. [Use the comparison test.]

19 If Σu_r^2 c, prove $\Sigma(u_r/r)$ also c. $\left[\left(u_n - \frac{1}{n}\right)^2 \geqslant 0\right.$, therefore $\left.2\frac{u_n}{n} \leqslant u_n^2 + \frac{1}{n^2}.\right]$

20 (i) Verify that $$\frac{1}{r^2} < \frac{1}{(r-1)r} \quad \text{when} \quad r > 2.$$

Deduce that $$s_n = \sum_{r=1}^{n} \frac{1}{r^2} < 1 + \sum_{r=2}^{n} \frac{1}{(r-1)r} = 2 - \frac{1}{n} < 2.$$

(This shows that $\Sigma(1/r^2)$ converges and has sum to infinity $s \leqslant 2$.)

*(ii) Prove $$\frac{1}{(r+1)^2} < \frac{1}{r} - \frac{1}{r+1} < \frac{1}{r^2} \quad \text{when} \quad r \geqslant 1,$$
and deduce that
$$1 + \frac{1}{2^2} + \dots + \frac{1}{(n-1)^2} + \frac{1}{n} < \sum_{r=1}^{\infty} \frac{1}{r^2} < 1 + \frac{1}{2^2} + \dots + \frac{1}{n^2} + \frac{1}{n}.$$

(This shows that $1/(n+1) < s - s_n < 1/n$. Taking $n = 100$, we see that the first hundred terms give s correct to only two decimal places. It can be proved that $s = \frac{1}{6}\pi^2$: see 14.34, ex. (ii).)

*21 If Σu_r c, prove $u_{n+1} + u_{n+2} + \dots + u_{n+p} \to 0$ when $n \to \infty$, where p is any *fixed* positive integer.

12.42 d'Alembert's ratio test

The tests in 12.41 required an auxiliary series Σv_r; that now to be stated involves only the given series itself.

Σu_r is convergent if

EITHER (a) $u_{n+1}/u_n \leqslant k < 1$ *for all* $n \geqslant m$, *where* k *is constant,*

OR (b) $\lim_{n\to\infty} (u_{n+1}/u_n) = l < 1$.

Σu_r is (properly) divergent if

EITHER (a′) $u_{n+1}/u_n \geqslant 1$ *for all* $n \geqslant m$,

OR (b′) $\lim_{n\to\infty} (u_{n+1}/u_n) = l > 1$.

Proofs

(a) By hypothesis,

$$u_{m+1} \leqslant ku_m;$$
$$u_{m+2} \leqslant ku_{m+1} \leqslant k^2u_m \quad \text{from the previous line;}$$
$$u_{m+3} \leqslant ku_{m+2} \leqslant k^3u_m \quad \text{from the previous line;}$$

and so on. Hence each term of the series

$$u_{m+1} + u_{m+2} + u_{m+3} + \dots$$

is less than or equal to the corresponding term of the G.P.

$$ku_m + k^2u_m + k^3u_m + \dots,$$

which converges since $0 < k < 1$ by hypothesis. Therefore Σu_r converges, since the first m terms $u_1+u_2+\ldots+u_m$ do not affect its behaviour.

(*b*) Given $\epsilon > 0$, there is a number m such that

$$l-\epsilon < \frac{u_{n+1}}{u_n} < l+\epsilon \quad \text{for all } n \geqslant m.$$

Since $l < 1$, we have $l+\epsilon < 1$ for all ϵ sufficiently small. Taking $k = l+\epsilon$ in the right-hand part of the inequality, the result follows from (*a*).

(*a′*) By hypothesis,

$$u_{m+1} \geqslant u_m, \quad u_{m+2} \geqslant u_{m+1} \geqslant u_m, \quad \text{etc.}$$

Hence for $n \geqslant m$ we have $u_n \geqslant u_m$, so that $u_n \not\to 0$ when $n \to \infty$, proving non-convergence of Σu_r. The series must actually be properly divergent since it consists of positive terms (12.32 (5)).

(*b′*) Since $l > 1$, we have $l-\epsilon > 1$ for all ϵ sufficiently small. The left-hand part of the inequality in the proof of (*b*) then shows

$$\frac{u_{n+1}}{u_n} > 1 \quad \text{for all } n \geqslant m,$$

and the result follows by (*a′*).

Remarks

(α) *The* COMPLETE *condition* (*a*) *is essential:* the result may be false if we merely have

$$\frac{u_{n+1}}{u_n} < 1 \quad \text{for all } n \geqslant m.$$

Thus although $\Sigma(1/r)$ D,

$$\frac{u_{n+1}}{u_n} = \frac{1}{n+1}\bigg/\frac{1}{n} = \frac{n}{n+1} < 1 \quad \text{for all } n.$$

No fixed k independent of n can be found for which

$$\frac{u_{n+1}}{u_n} \leqslant k \quad \textit{and} \quad k < 1$$

because $n/(n+1)$ can be made as close to 1 as we please by taking n sufficiently large.

(β) *The tests give no information if* $\lim_{n\to\infty} (u_{n+1}/u_n) = 1$. For example, $\Sigma(1/r)$ D and $\Sigma\,(1/r^2)$ C, but in both cases $u_{n+1}/u_n \to 1$ when $n \to \infty$.

If use of the 'limit' form of the ratio test leads to the limit 1, then we must revert to the 'inequality' form since this can give a decision when the 'limit' form is inconclusive. For a power series it is usually best to treat this exceptional case from first principles.

(γ) *No conclusion can be drawn from non-existence of the limit of* u_{n+1}/u_n. Thus for the series†

$$\frac{1}{2}+\frac{1}{4^2}+\frac{1}{2^3}+\frac{1}{4^4}+\frac{1}{2^5}+\frac{1}{4^6}+\dots,$$

successive values of u_{n+1}/u_n are

$$\frac{1}{2^3},\quad 2,\quad \frac{1}{2^5},\quad 2^3,\quad \frac{1}{2^7},\quad 2^5,\quad \frac{1}{2^9},\quad 2^7,\quad \dots,$$

so that u_{n+1}/u_n oscillates infinitely. The series is nevertheless convergent (because, roughly, it is the sum of two convergent G.P.'s):

$$\begin{aligned} s_{2n} &= \frac{1}{2}+\frac{1}{4^2}+\frac{1}{2^3}+\frac{1}{4^4}+\dots+\frac{1}{2^{2n-1}}+\frac{1}{4^{2n}} \\ &= \frac{1}{2}\left\{1+\frac{1}{2^2}+\dots+\left(\frac{1}{2^2}\right)^{n-1}\right\}+\frac{1}{4^2}\left\{1+\frac{1}{4^2}+\dots+\left(\frac{1}{4^2}\right)^{n-1}\right\} \\ &= \frac{\frac{1}{2}\left\{1-\left(\frac{1}{2^2}\right)^n\right\}}{1-\frac{1}{2^2}}+\frac{\frac{1}{4^2}\left\{1-\left(\frac{1}{4^2}\right)^n\right\}}{1-\frac{1}{4^2}} \\ &\to \tfrac{2}{3}+\tfrac{1}{15} \quad \text{when} \quad n\to\infty; \end{aligned}$$

and‡
$$s_{2n+1} = s_{2n}+\frac{1}{2^{2n+1}}\to \tfrac{11}{15}+0 \quad \text{when} \quad n\to\infty.$$

Hence the series converges, and has sum to infinity $\frac{11}{15}$. Observe also that for this series u_n does not *steadily* decrease to zero when $n \to \infty$

Example

Examine the series $1+2x+3x^2+4x^3+\dots$ *for all positive* x.

If $x = 0$, the series is $1+0+0+\dots$, which converges to the sum 1.

If $x \neq 0$, then since $u_n = nx^{n-1}$,

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{n}x \to x \quad \text{when} \quad n\to\infty.$$

The ratio test shows that if $0 < x < 1$ the series C, and if $x > 1$ the series D.

† The reader may feel that an 'irregular' series like this is not a fair example; but in fact the terms are constructed according to a quite definite law: 'the rth term is the rth power of 2 or 4, according as r is odd or even'. In this example it is easy to write down a formula for u_r, viz. $u_r = \{3+(-1)^r\}^{-r}$.

‡ See the footnote on p. 451.

When $x = 1$, the test gives no information; but then the series is $1+2+3+\ldots$, which clearly D.

Hence the series C when $0 \leqslant x < 1$, D when $x \geqslant 1$. (See Ex. 12 (*f*), no. 12 for a method of finding the sum to infinity.)

Exercise 12(*f*)

Test the following series.

1 $1+\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\ldots.$

2 $\frac{2}{2^7}+\frac{2^2}{3^7}+\frac{2^3}{4^7}+\ldots.$

3 If a and b are positive, prove that

$$\frac{a+1}{b+1}+\frac{(a+1)(2a+1)}{(b+1)(2b+1)}+\frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}+\ldots$$

converges if $a < b$ and diverges if $a \geqslant b$.

Test the following, assuming x to be positive.

4 $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots.$

5 $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\ldots.$

6 $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots.$

7 $\frac{1}{x}+\frac{2!}{x^2}+\frac{3!}{x^3}+\ldots.$

8 $x+2^3x^2+3^3x^3+4^3x^4+\ldots.$

9 $1+(1+3)x+(1+3^2)x^2+(1+3^3)x^3+\ldots.$

10 $\frac{1}{x+1}+\frac{x}{x+2}+\frac{x^2}{x+3}+\ldots.$

11 (i) $\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\frac{1.3.5.7}{2.4.6.8}x^4+\ldots \quad (x \neq 1).$

*(ii) By showing that

$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} > \frac{1.2.4\ldots(2n-2)}{2.4.6\ldots(2n)} = \frac{1}{2n},$$

prove that the series in (i) diverges when $x = 1$.

*12 By deriving the identity

$$1+x+x^2+\ldots+x^n \equiv \frac{1}{1-x}-\frac{x^{n+1}}{1-x},$$

obtain a formula for $1+2x+3x^2+\ldots+nx^{n-1}$, and hence verify from first principles the conclusions of the worked example in 12.42.

12.43 Speed of convergence of a series

A series is said to be 'rapidly convergent' if s_n gives a good approximation to the sum to infinity s for fairly small values of n. In this sense the G.P.

$$1+x+x^2+x^3+\ldots \quad (|x| < 1)$$

is rapidly convergent when $|x|$ is small, since

$$s_n = \frac{1-x^n}{1-x} = s - \frac{x^n}{1-x},$$

$$|s-s_n| = \frac{|x|^n}{1-x},$$

and $|x|^n$ is small for small n. It converges less rapidly when x is just less than 1. On the other hand, Ex. 12 (*e*), no. 20 (ii) shows that $\Sigma(1/r^2)$ converges slowly.

These notions are necessarily vague since the terms 'a good approximation', 'fairly small n' have not been specified; they are *relative* and not absolute notions, and can become precise only after we have selected some definite series by which to fix our 'standard of rapidity'. However, they are useful in a descriptive way.

The essence of the comparison test for convergence (12.41 (1)) is that if Σu_r is term by term less than a convergent series $c\Sigma v_r$, then Σu_r will converge. We may say that 'Σu_r converges at least as rapidly as Σv_r;' and if $\lim_{n\to\infty} (u_n/v_n) = 0$, that '$\Sigma u_n$ converges faster than Σv_n'. Similar language would describe relative rates of divergence.

d'Alembert's test for convergence consists fundamentally of comparing the given series Σu_r with a G.P. If Σu_r happens to converge less rapidly than *any* G.P., the test is ineffective; this is the case with $\Sigma(1/r^2)$ (and indeed with any of the series $\Sigma(1/r^p)$ for $p > 1$), and is usually so when $\lim (u_{n+1}/u_n) = 1$.

Tests of greater delicacy than d'Alembert's can be formulated, but no finality can be attained since it can be shown that, however delicate the test, a series can be constructed for which that test is ineffective. In 12.44 we give a test which is of no fixed standard of delicacy, and which deals particularly easily with series like $\Sigma(1/r^p)$, $\Sigma\{1/r(\log r)^p\}$ for which the ratio test is indecisive.

12.44 Infinite series and infinite integrals

The definitions of 'sum to infinity of a series' (as the limit of a certain 'finite' sum, 12.21) and 'value of an infinite integral' (as the limit of the corresponding 'finite' integral, 4.92) are somewhat similar. For a certain class of functions $f(x)$ we now give a theorem relating

$$\sum_{r=1}^{\infty} f(r) \quad \text{and} \quad \int_1^{\infty} f(t)\,dt,$$

in which convergence of one implies that of the other.

(1) *The Maclaurin–Cauchy integral test.*

If $f(x)$ is continuous and steadily decreases to zero for $x \geqslant 1$, then $\sum_{r=1}^{\infty} f(r)$ and $\int_1^\infty f(t)\,dt$ either both converge or both diverge. When they converge,

$$\int_1^\infty f(t)\,dt \leqslant \sum_{r=1}^{\infty} f(r) \leqslant \int_1^\infty f(t)\,dt + f(1).$$

Proof. Since $f(x)$ steadily decreases to zero, we have $f(x) > 0$ for all $x \geqslant 1$, so that $\sum_{r=1}^{n} f(r)$, $\int_1^x f(t)\,dt$ *are increasing functions of* n, x *respectively.*

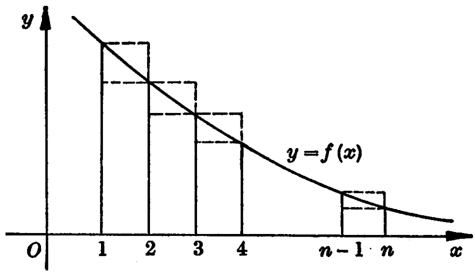

Fig. 124

If $r < x < r+1$, then

$$f(r+1) < f(x) < f(r).$$

Integrating this form r to $r+1$,

$$f(r+1) < \int^{r+1} f(x)\,dx < f(r). \tag{i}$$

Summing for $r = 1, 2, \ldots, n-1$,

$$\sum_{r=2}^{n} f(r) < \int_1^n f(t)\,dt < \sum_{r=1}^{n-1} f(r). \tag{ii}$$

The argument is illustrated geometrically by considering the sums of the inner and outer rectangles in fig. 124.

(a) Suppose $\int_1^\infty f(t)\,dt$ exists. Since $\int_1^n f(t)\,dt$ is an increasing function of n, it cannot attain its limit; therefore for each n,

$$\int_1^n f(t)\,dt < \int_1^\infty f(t)\,dt.$$

From the left of (ii), $\displaystyle \sum_{r=2}^{n} f(r) < \int_1^\infty f(t)\,dt,$

so that $\sum_{r=2}^{n} f(r)$ is bounded. Hence (2.77) $\Sigma f(r)$ converges, and

$$\sum_{r=1}^{\infty} f(r) \leqslant \int_1^{\infty} f(t)\,dt + f(1).$$

It now follows from the right of (ii) by letting $n \to \infty$ that

$$\int_1^{\infty} f(t)\,dt \leqslant \sum_{r=1}^{\infty} f(r).$$

(*b*) If $\Sigma f(r)$ converges, then

$$\sum_{r=1}^{n-1} f(r) < \sum_{r=1}^{\infty} f(r).$$

By the right of (ii), $$\int_1^n f(t)\,dt < \sum_{r=1}^{\infty} f(r),$$

so that $\int_1^n f(t)\,dt$ is bounded and therefore tends to a limit, say l, when $n \to \infty$.

Since† $\int_1^x f(t)\,dt$ is an increasing function of x, then if $n-1 < X < n$,

$$\int_1^{n-1} f(t)\,dt < \int_1^X f(t)\,dt < \int_1^n f(t)\,dt.$$

When $X \to \infty$, also $n \to \infty$, and so $\int_1^X f(t)\,dt \to l$, i.e. $\int_1^{\infty} f(t)\,dt$ exists.

(*c*) Since the series converges when and only when the integral exists, it follows that divergence of one implies divergence of the other.

(2) *The function* $$\phi(n) \equiv \sum_{r=1}^{n} f(r) - \int_1^n f(t)\,dt$$

steadily decreases as n increases, and lies between 0 *and* $f(1)$ *for all n.*

Proof. By (ii), $$0 < \sum_{r=1}^{n} f(r) - \int_1^n f(t)\,dt < f(1);$$

† This step is necessary because the fact that $\int_1^n f(t)\,dt \to l$ when $n \to \infty$ (n being a positive integer) does not by itself imply the existence of $\int_1^{\infty} f(t)\,dt$ (the limit of $\int_1^X f(t)\,dt$ when $X \to \infty$, X being a continuous variable): cf. Remark (β) in 2.71. Thus $\int_1^n \cos \pi t\,dt = 0$ for all integers n, and hence its limit when $n \to \infty$ is 0; but $\int_1^{\infty} \cos \pi t\,dt$ does not exist since $\int_1^X \cos \pi t\,dt = \frac{1}{\pi} \sin \pi X$, which oscillates when $X \to \infty$.

also
$$\phi(n+1)-\phi(n) = f(n+1)-\int_n^{n+1} f(t)\,dt < 0$$

by using (i). Hence $\phi(n)$ decreases as n increases, but $0 < \phi(n) < f(1)$.

COROLLARY. *It follows that* $\lim_{n\to\infty}\phi(n)$ *exists and is non-negative.*

The corollary is particularly interesting when both series and integral diverge, as in example (i) following.

Examples

(i) Taking $f(x) = 1/x$ in the corollary,
$$\phi(n) = 1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}-\log n.$$

Hence $\phi(n)$ tends to some limit γ (*Euler's constant*: cf. 4.43 (8)) when $n\to\infty$, and $0 \leqslant \gamma < 1$.

(ii) *Discuss the series* $\Sigma(1/r^p)$ *by means of the integral test.*

With $f(x) = 1/x^p$, then if $p \neq 1$,
$$\int_1^X f(t)\,dt = \frac{1}{p-1}(1-X^{-p+1}),$$
while if $p = 1$,
$$\int_1^X f(t)\,dt = \log X.$$

If $p > 1$, the integral tends to $1/(p-1)$, and if $0 \leqslant p \leqslant 1$ the integral tends to $+\infty$, when $X\to\infty$. Hence $\Sigma(1/r^p)$ C if $p>1$, D if $p \leqslant 1$ (cf. 12.41 (3)).

The general theorem also shows that when $p > 1$,
$$\frac{1}{p-1} \leqslant \sum_{r=1}^{\infty}\frac{1}{r^p} \leqslant \frac{p}{p-1}.$$

In fact $\sum_{r=1}^{\infty}\frac{1}{r^p} > 1$, which is a better lower estimate when $p > 2$.

Exercise 12(g)

1 Prove $\Sigma\frac{1}{r\log(2r)}$ diverges.

2 Discuss $\Sigma\frac{1}{r(\log r)^p}$.

3 If $a > 0$, prove that the sum to infinity of
$$\frac{1}{a^2}+\frac{1}{(a+1)^2}+\frac{1}{(a+2)^2}+\ldots$$
lies between $1/a$ and $1/a+1/a^2$.

4 If $a > 0$, find numbers between which the sum to infinity of
$$\frac{1}{a^2}+\frac{1}{a^2+1^2}+\frac{1}{a^2+2^2}+\ldots$$
must lie.

5 Sketch the graph of $y = \log x$, and show that

$$\int_1^n \log x\,dx < \sum_{r=2}^{n} \log r < \int_1^n \log x\,dx + \log n.$$

Deduce the value of $\lim_{n\to\infty} \{(n!)^{1/n}/n\}$.

12.5 Series of positive and negative terms

The special difficulty with such series is that s_n is not necessarily a steadily increasing function of n, so that the principle in 12.32 (5) does not apply directly.

12.51 Alternating signs (theorem of Leibniz)

The simplest kind of series with terms of mixed sign is that in which the signs are alternately $+$, $-$.

If (i) *the terms are alternately* $+$, $-$, *say*

$$u_1 - u_2 + u_3 - u_4 + \ldots$$

where each u *is positive,*

$$\left.\begin{array}{ll}\text{(ii)} & u_{n+1} < u_n \\ \text{(iii)} & \lim_{n\to\infty} u_n = 0\end{array}\right\} \text{ i.e. } u_n \text{ STEADILY } \textit{decreases to zero,}$$

then the series is convergent.

Proof

$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n}).$$

By (ii), each bracket is positive. Hence s_{2n} is a steadily increasing positive function of n. Also

$$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots - (u_{2n-2} - u_{2n-1}) - u_{2n};$$

each bracket is positive by (ii), and $u_{2n} > 0$ by (i). Hence $s_{2n} < u_1$. Therefore by 2.77 s_{2n} tends to a limit, say l, when $n \to \infty$.

Further,†

$$s_{2n+1} = s_{2n} + u_{2n+1},$$

and since $\lim_{n\to\infty} u_{2n+1} = 0$ by (iii), therefore

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} = l.$$

Hence, whether n is even or odd, $s_n \to l$ when $n \to \infty$, i.e. the series converges.

† The fact that $s_{2n} \to l$ does not by itself show that the series converges. Thus for the series $1 - 1 + 1 - 1 + \ldots$, $s_{2n} = 0$ and so $\lim s_{2n} = 0$; but the series does not converge since $s_{2n+1} = 1$: it oscillates finitely.

Example

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ *converges* because the terms satisfy all three conditions of the theorem. The argument used in the proof shows that the sum to infinity lies between $\frac{1}{2}$ and 1; for $s_{2n} < u_1 = 1$, and

$$s_{2n} = (1-\tfrac{1}{2})+(\tfrac{1}{3}-\tfrac{1}{4})+\ldots+\left(\frac{1}{2n-1}-\frac{1}{2n}\right) > (1-\tfrac{1}{2}) = \tfrac{1}{2}.$$

Also see Ex. 12 (*h*), no. 8.

If any one of the conditions (i), (ii), (iii) is omitted, the series may not converge. Thus since the series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots, \quad \tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\ldots$$

D, C respectively, the 'difference series'

$$1-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{8}+\tfrac{1}{4}-\tfrac{1}{16}+\ldots$$

diverges (12.32 (4)); conditions (i), (iii) are satisfied by it, but not (ii).

12.52 Absolute convergence

Consider the pairs of series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots, \qquad 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots;$$

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots, \quad 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.$$

The first of each pair converges, by 12.51; but in the corresponding series where all the terms are made positive, the first diverges (being the harmonic series) while the second still converges (12.41 (3)).

It is convenient to distinguish between series which remain convergent when all their terms are made positive, and those which do not.

Definition. If the series Σu_r is such that† $\Sigma\,|u_r|$ converges, then Σu_r is said to be *absolutely convergent* (A.C.).

Observe that nothing has been asserted in this definition about the convergence of Σu_r itself; it is a *different* series $\Sigma\,|u_r|$ (i.e. Σu_r with all its terms made positive) which converges.

Examples

(i) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ is not A.C. because $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ diverges.

(ii) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$ is A.C. because $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ converges.

(iii) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ is A.C. because $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ converges.

† See 1.14 for the meaning of $|x|$.

The importance of absolutely convergent series is (roughly) that they behave like series of positive terms. For example, compare Theorems II, III in 12.54.

Although the definition states nothing about the convergence of Σu_r, we can in fact prove that *an absolutely convergent series is also convergent* in the ordinary sense. More explicitly—

If $\Sigma\,|u_r|$ *converges, then* Σu_r *also converges.*

Before giving the proof of this, we illustrate the underlying idea. Suppose that the series Σu_r in question is

$$u_1+u_2-u_3+u_4-u_5-u_6+0-u_8+u_9-u_{10}+u_{11}+u_{12}+\dots.$$

Consider the following two series:

$$u_1+u_2+0\;+u_4+0\;+0\;+0+0\;+u_9+0\;\;+u_{11}+u_{12}+\dots,$$

$$0\;+0\;+u_3+0\;+u_5+u_6+0+u_8+0\;+u_{10}+0\;\;+0\;\;+\dots.$$

The first is formed by replacing all negative terms in the given series by 0's; the second is obtained by replacing all positive terms by 0 and changing the sign of the negative ones.

Compare each of these series of positive terms with the series

$$u_1+u_2+u_3+u_4+u_5+u_6+0+u_8+u_9+u_{10}+u_{11}+u_{12}+\dots,$$

which is convergent by hypothesis. The comparison test (12.41 (1)) with $c = 1$ shows that each series converges. Hence by 12.32 (4) their difference (which is the given series) also converges.

General proof. Consider the series

$$v_1+v_2+v_3+\dots,\qquad w_1+w_2+w_3+\dots,$$

where we define

$$v_r=\begin{cases}u_r & \text{if}\quad u_r>0,\\ 0 & \text{if}\quad u_r\leqslant 0,\end{cases}\qquad\text{and}\qquad w_r=\begin{cases}-u_r & \text{if}\quad u_r<0,\\ 0 & \text{if}\quad u_r\geqslant 0.\end{cases}$$

(Thus relative to the given series Σu_r, Σv_r is the non-negative part, and Σw_r is the non-positive part with its sign changed.)

Clearly $v_r \leqslant |u_r|$ and $w_r \leqslant |u_r|$. Comparing each of Σv_r, Σw_r (which are series of positive terms) with $\Sigma\,|u_r|$ which is *given* to be convergent, the comparison test with $c = 1$ shows that they converge. Hence $\Sigma(v_r-w_r)$ converges; i.e., since $u_r = v_r-w_r$ for all r, Σu_r converges.

The property just proved gives a possible test for convergence of a series of terms with mixed signs, viz. *prove it is* A.C. For example,

$$1-\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}-\frac{1}{6^2}+\frac{1}{7^2}-\dots$$

converges because it is A.C. However, the test is 'one-sided' because a series which is not A.C. may yet be C; e.g. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ is not A.C., but is in fact C. Such a series may be called *conditionally convergent* (C.C.) or *semi-convergent.*

Examples

*(iv) *The 'triangle inequality'*† *for infinite series.*

If Σu_r *is* A.C., *its sum to infinity is numerically less than or equal to the sum of* $\Sigma |u_r|$:

$$\left|\sum_{r=1}^{\infty} u_r\right| \leqslant \sum_{r=1}^{\infty} |u_r|.$$

With the above notation we have $u_r = v_r - w_r$, $|u_r| = v_r + w_r$. Also, since Σv_r, Σw_r converge,

$$\left|\sum_{r=1}^{\infty} u_r\right| = \left|\sum_{r=1}^{\infty} v_r - \sum_{r=1}^{\infty} w_r\right|$$

$$\leqslant \sum_{r=1}^{\infty} v_r + \sum_{r=1}^{\infty} w_r \quad \text{since† both these sums are } \textit{positive},$$

$$= \sum_{r=1}^{\infty} |u_r|.$$

The device used in this example can also be employed to prove that

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx,$$

where $b > a$ and $f(x)$ is continuous; cf. Ex. 7 (a), no. 5. Define

$$f_1(x) = \begin{cases} f(x) & \text{if } f(x) > 0, \\ 0 & \text{if } f(x) \leqslant 0, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} -f(x) & \text{if } f(x) < 0, \\ 0 & \text{if } f(x) \geqslant 0. \end{cases}$$

Then $f_1(x), f_2(x)$ are non-negative functions, and

$$|f(x)| = f_1(x) + f_2(x), \quad f(x) = f_1(x) - f_2(x).$$

Also

$$\left|\int_a^b f(x)\,dx\right| = \left|\int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx\right|$$

$$\leqslant \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx, \quad \text{both integrals being non-negative,‡}$$

$$= \int_a^b \{f_1(x) + f_2(x)\}\,dx = \int_a^b |f(x)|\,dx.$$

(v) *Discuss the series*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

for all values of x.

If $x = 0$, the series is $1+0+0+\dots$, which converges to the sum 1.

† See 1.14, (i). ‡ By 4.15(9).

If $x > 0$, all the terms are positive and we can use d'Alembert's ratio test:

$$\frac{u_{n+1}}{u_n} = \frac{x^n}{n!}\bigg/\frac{x^{n-1}}{(n-1)!} = \frac{x}{n};$$

thus since $\qquad \dfrac{u_{n+1}}{u_n} < \frac{1}{2}$ for all $n > 2x$,

or alternatively, since $\qquad \lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = 0,$

the series converges for all positive x.

If $x < 0$, write $x = -y$ so that $y > 0$. The series becomes†

$$1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \ldots.$$

By the case just considered, this is A.C. for all positive y, and hence it is also C. The given series therefore converges for all negative x.

Consequently, *the series converges for all x.*

N.B.—From 12.32 (1) it follows that (cf. 2.74)

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \textit{for all } x \textit{ independent of } n.$$

(vi) *Discuss the series* $\qquad x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

for all values of x.

First consider the series $\qquad y + \dfrac{y^2}{2} + \dfrac{y^3}{3} + \dfrac{y^4}{4} + \ldots,$

where $y \geqslant 0$. If $y = 0$, the series is $0+0+0+\ldots$ which converges to sum 0. If $y \neq 0$, the ratio test gives

$$\frac{u_{n+1}}{u_n} = \frac{y^{n+1}}{n+1}\bigg/\frac{y^n}{n} = \frac{n}{n+1} y \to y \quad \text{when} \quad n \to \infty;$$

so if $0 < y < 1$ the series converges.

The y-series therefore converges for $0 \leqslant y < 1$; hence the given series in x is A.C. for $-1 < x < 1$, and thus also converges in this range.

If $x = 1$, the given series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, which C (12.51).

If $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots$, which is properly divergent to $-\infty$ (it is -1 times the harmonic series).

If $x < -1$, each term of the series is negative, but its *numerical* value is greater than that of the corresponding term of $1 + \frac{1}{2} + \frac{1}{3} + \ldots$. Hence the series properly diverges to $-\infty$.

If $x > 1$, then $x = 1 + c$ where $c > 0$, and

$$x^n = (1+c)^n = 1 + nc + \tfrac{1}{2}n(n-1)c^2 + \ldots > \tfrac{1}{2}n(n-1)c^2,$$

since all the terms of this binomial expansion are positive. Therefore

$$\frac{x^n}{n} > \tfrac{1}{2}(n-1)c^2 \quad \text{and so‡} \quad \frac{x^n}{n} \not\to 0 \quad \text{when} \quad n \to \infty.$$

The series cannot converge in this case.

† In fact we are considering $1 - |x| + \dfrac{|x|^2}{2!} - \dfrac{|x|^3}{3!} + \ldots.$

‡ This also follows at once from 2.73.

Summarising, *the series* C *for* $-1 < x \leqslant 1$, *and* D *for* $x \leqslant -1$ *or* $x > 1$. The range $-1 < x \leqslant 1$ is called the *interval of convergence* of this series.

12.53 The modified ratio test

We introduce here a modification of d'Alembert's ratio test which makes it directly applicable to series whose terms are not all positive.

If EITHER (*a*) $\left|\dfrac{u_{n+1}}{u_n}\right| \leqslant k < 1$ *for all* $n \geqslant m$, *where k is constant,*

$$\text{OR } (b) \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = l < 1,$$

then Σu_r *converges.*

For either condition ensures that $\Sigma\,|u_r|$ C, being what is obtained by applying the original ratio test to this series. Hence Σu_r is A.C., and therefore C.

If EITHER (*a'*) $\left|\dfrac{u_{n+1}}{u_n}\right| \geqslant 1$ *for all* $n \geqslant m$,

$$\text{OR } (b') \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = l > 1,$$

then Σu_r *is divergent* (*i.e. not convergent*).

(At first sight all we can say is that $\Sigma\,|u_r|$ D, i.e. that Σu_r is not A.C.; this does not prevent Σu_r from converging.)

Condition (*a'*) implies that $|u_n| \geqslant |u_m|$ for all $n \geqslant m$, so that $u_n \not\to 0$ when $n \to \infty$. Given $\epsilon > 0$, condition (*b'*) implies that there is a number m such that $|u_{n+1}| > (l-\epsilon)\,|u_n|$ for all $n \geqslant m$, and the result follows from (*a'*) as in the proofs in 12.42.

Examples

(i) *Discuss the series of* 12.52, *ex.* (vi) *in this way.*

If $x = 0$ the series converges to sum 0. Since $u_n = (-1)^{n-1}x^n/n$, we have when $x \neq 0$ that

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|-\frac{n}{n+1}x\right| \to |x| \quad \text{when} \quad n \to \infty.$$

Hence if $|x| < 1$ the series C; if $|x| > 1$ the series D. Only the cases $|x| = 1$ remain to be investigated, and these are treated separately as before.

(ii) *Discuss the series*

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots.$$

If $x = 0$, the series is $1+0+0+\dots$, which converges to 1.

If $x \neq 0$, then from

$$u_n = \frac{m(m-1)\dots(m-n+1)}{n!}x^n$$

we find
$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{m-n}{n+1}x\right| \to |-x| = |x| \quad \text{when} \quad n \to \infty.$$

Hence *if* $|x| < 1$ *the series* C; *if* $|x| > 1$ *the series* D.

The cases $|x| = 1$ cannot be decided by means of the tests at our disposal. However, it can be proved that

if $x = 1$, the series C if $m > -1$ and D if $m \leqslant -1$;

and if $x = -1$, the series C if $m \geqslant 0$ and D if $m < 0$.

12.54 Regrouping and rearrangement of terms of an infinite series

When dealing with finite sums it is permissible to group the terms in any manner by means of brackets, or to rearrange the order of the terms, without thereby altering the value of the sum. These properties may not extend to infinite series.†

Example

(i) The series
$$(1-1)+(1-1)+(1-1)+\dots,$$
i.e.
$$0+0+0+\dots,$$
converges to zero. If we remove the brackets we obtain the series
$$1-1+1-1+1-1+\dots,$$
which diverges since s_n oscillates between 0 and 1. If we insert brackets as
$$1+(-1+1)+(-1+1)+\dots,$$
we obtain
$$1+0+0+\dots$$
which converges to 1.

This example shows that, in general, brackets cannot be removed or inserted in an infinite series without altering its behaviour or its sum. However, the following result is easily proved by using the definition of 'sum to infinity'.

THEOREM I. *If Σu_r converges to s, then it will still converge to s when brackets are inserted in any way.*

Proof. Let $n_1 < n_2 < n_3 < \dots$ be any infinite sequence of positive integers, and write
$$t_1 = \sum_{r=1}^{n_1} u_r, \quad t_2 = \sum_{r=n_1+1}^{n_2} u_r, \quad \dots.$$

Then we wish to prove that Σt_i converges to s.

Given $\epsilon > 0$, there corresponds a number N such that, for all $n \geqslant N$,
$$\left|\sum_{r=1}^{n} u_r - s\right| < \epsilon. \tag{i}$$

† They will hold if we bracket or rearrange only a *finite* number of the terms; but this is relatively trivial.

Let n_p be the first integer of the above sequence which is greater than or equal to N. Then (i) holds for all $n \geqslant n_p$, and so

$$\left|\sum_{r=1}^{n_m} u_r - s\right| < \epsilon \qquad \text{(ii)}$$

for all $m \geqslant p$ (since the integers $n_{p+1}, n_{p+2}, \ldots$ are a selection from the set of *all* integers greater than n_p).

Since terms of a 'finite' sum can be grouped in any way,

$$\sum_{r=1}^{n_m} u_r = \sum_{i=1}^{m} t_i.$$

Hence result (ii) becomes

$$\left|\sum_{i=1}^{m} t_i - s\right| < \epsilon \quad \text{for all } m \geqslant p;$$

i.e. Σt_i converges to s.

The converse of this theorem is false: if a series with the terms grouped is convergent, the series obtained by *removing* the brackets may be divergent. This is illustrated by ex. (i), which also shows that there is no general theorem for divergent series like Theorem I; but see Ex. 12 (*h*), no. 14.

Example

(ii) *If s_n denotes the sum of n terms of the series*

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \ldots,$$

and t_n denotes the sum of n terms of

$$1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{5} - \tfrac{1}{10} - \tfrac{1}{12} + \tfrac{1}{7} - \ldots$$

(formed from the first by taking one positive term followed by two negative ones) then the sum to infinity of the second is HALF *that of the first.*

Consider

$$\begin{aligned} t_{3n} &= (1 - \tfrac{1}{2}) - \tfrac{1}{4} + (\tfrac{1}{3} - \tfrac{1}{6}) - \tfrac{1}{8} + (\tfrac{1}{5} - \tfrac{1}{10}) - \tfrac{1}{12} + \ldots + \left(\frac{1}{2n-1} - \frac{1}{4n-2}\right) - \frac{1}{4n} \\ &= \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \ldots + \frac{1}{4n-2} - \frac{1}{4n} \\ &= \tfrac{1}{2} s_{2n}. \end{aligned}$$

The first series is known to converge, say to s; hence when $n \to \infty$, $s_{2n} \to s$ and therefore $t_{3n} \to \tfrac{1}{2}s$. Since also

$$t_{3n+1} = t_{3n} + \frac{1}{2n+1} \to \tfrac{1}{2}s$$

and

$$t_{3n+2} = t_{3n} + \frac{1}{2n+1} - \frac{1}{4n+2} \to \tfrac{1}{2}s,$$

therefore $t_n \to \tfrac{1}{2}s$ when $n \to \infty$; this is the result stated.

Example (ii) illustrates that, if a convergent series is rearranged, the new series may converge to a sum different from the original. It can be proved that *a given non-absolutely convergent series can be rearranged so that the new series will converge to any pre-assigned limit; and it can be arranged to diverge* (Riemann's rearrangement theorem). For series of positive terms the situation is simpler; and we shall prove that the same is true of A.C. series.

THEOREM II. *A convergent series of positive terms can be rearranged in any manner without altering the convergence or the sum to infinity.*

Proof. Let Σu_r be a convergent series of positive terms, with sum to infinity s. Let $\Sigma u'_r$ be a rearrangement of Σu_r, and write $s'_n = \sum_{r=1}^{n} u'_r$.

Each term of $\Sigma u'_r$ appears somewhere in Σu_r. Hence for a given n, there is a number m such that $s'_n \leqslant s_m$. Since $\sum_{r=1}^{n} u_r$ is a steadily increasing function of n (all $u_r \geqslant 0$), hence $s_m \leqslant s$ and so $s'_n \leqslant s$. Since s'_n is also a steadily increasing function of n (because all $u'_r \geqslant 0$), therefore by 2.77 s'_n tends to a limit s' where $s' \leqslant s$. Thus $\Sigma u'_r$ *certainly converges*, and its sum is s'.

Similarly, since each term of Σu_r appears somewhere in $\Sigma u'_r$, the preceding argument will prove that $s \leqslant s'$. Consequently $s' = s$.

N.B.—Observe that we could not prove $s \leqslant s'$ first because we do not know whether $\Sigma u'_r$ converges or not until we have proved $s'_n \leqslant s$.

THEOREM III. *An* A.C. *series can be rearranged in any manner without altering the absolute convergence or the sum to infinity.*

Proof. If $\Sigma u'_r$ is a rearrangement of Σu_r, then certainly $\Sigma u'_r$ is A.C. since $\Sigma |u'_r|$ is a rearrangement of $\Sigma |u_r|$, and Theorem II applies.

We use the notation v_r, w_r of 12.52, and its extension v'_r, w'_r for the rearranged series $\Sigma u'_r$. Then clearly $\Sigma v'_r$, $\Sigma w'_r$ are respectively the rearrangements of Σv_r, Σw_r entailed by the rearrangement $\Sigma u'_r$ of Σu_r. Since Σv_r, Σw_r are convergent series† of positive terms, Theorem II gives

$$\sum_{r=1}^{\infty} v'_r = \sum_{r=1}^{\infty} v_r \quad \text{and} \quad \sum_{r=1}^{\infty} w'_r = \sum_{r=1}^{\infty} w_r.$$

$$\therefore \quad \sum_{r=1}^{\infty} u'_r = \sum_{r=1}^{\infty} (v'_r - w'_r) = \sum_{r=1}^{\infty} v'_r - \sum_{r=1}^{\infty} w'_r = \sum_{r=1}^{\infty} v_r - \sum_{r=1}^{\infty} w_r = \sum_{r=1}^{\infty} u_r.$$

We shall not discuss the rearrangement of infinite series any further; but enough has been said to caution the reader against treating infinite series analogously to finite ones.

Exercise 12(*h*)

Determine the behaviour of the following series.

1 $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$

2 $\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}-\frac{1}{4.5}+\dots$

3 $1-\frac{1}{2}\sqrt{2}+\frac{1}{3}\sqrt{3}-\frac{1}{4}\sqrt{4}+\dots$

4 $\frac{1}{2}-\frac{4}{3}+\frac{5}{6}-\frac{8}{7}+\frac{9}{10}-\frac{12}{11}+\dots$

5 $\log\frac{1}{2}+\log\frac{4}{3}+\log\frac{5}{6}+\log\frac{8}{7}+\log\frac{9}{10}+\log\frac{12}{11}+\dots$

6 State which of the series in nos. 1–3 are A.C.

7 Show that $\Sigma\{(\cos rx)/r^2\}$ converges for all values of x.

8 Under the conditions of Leibniz's rule (12.51), prove $u_1 - u_2 < s < u_1$.

Discuss the behaviour of the following series for all values of x.

9 $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots$

10 $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$

11 $\frac{x}{2^2}+\frac{x^2}{3^2}+\frac{x^3}{4^2}+\dots$

† See the General Proof in 12.52.

*12 $\Sigma\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{r}\right)x^r$.

$$\left[\frac{u_{n+1}}{u_n}=\frac{a_{n+1}}{a_n}x=\left\{1+\frac{1}{(n+1)\,a_n}\right\}x\to x,\text{ where } a_n=\sum_{r=1}^{n}\frac{1}{r}\right].$$

*13 $\Sigma\dfrac{n!}{1.3.5\ldots(2n+1)}x^n$. [The cases $|x|=2$ may be omitted.]

*14 If $u_r \geqslant 0$ for all r, prove that (with the notation of 12.54, Theorem I)

$$\text{if}\quad n_m \leqslant n < n_{m+1},\quad \text{then}\quad \sum_{i=1}^{m} t_i \leqslant \sum_{r=1}^{n} u_r \leqslant \sum_{i=1}^{m+1} t_i,$$

and deduce that *the converse of Theorem I holds for a series of positive terms.*

*15 If Σu_r is conditionally convergent, prove that (with the notation of 12.52) Σv_r and Σw_r diverge to $+\infty$. $\left[\text{If } s_n=\sum_{r=1}^{n}u_r,\ t_n=\sum_{r=1}^{n}|u_r|,\text{ then } s_n\to s \text{ and } t_n\to+\infty;\right.$ also $\left.\sum_{r=1}^{n}v_r=\frac{1}{2}(t_n+s_n),\ \sum_{r=1}^{n}w_r=\frac{1}{2}(t_n-s_n).\right]$

*16 If the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$ is rearranged as

$$1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{4}+\ldots$$

(taking two positive terms followed by one negative term), prove that the sum of the new series is $\frac{3}{2}$ times that of the original. [Prove $t_{3n}=s_{4n}+\frac{1}{2}s_{2n}$.]

12.6 Maclaurin's series

12.61 Expansion of a function as a power series

We proved in 6.52 that if $f(t)$ satisfies certain conditions of continuity and derivability for $0 \leqslant t \leqslant x$, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+R_n(x),$$

where (see 6.54 (1))

$$R_n(x)=\begin{cases}\dfrac{x^n}{n!}f^{(n)}(\theta x) & \text{(Lagrange's remainder)},\\[2ex] \dfrac{x^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta x) & \text{(Cauchy's remainder)}\end{cases}$$

and θ is some function of x and n (in general not the *same* function in the two forms of remainder) which satisfies $0<\theta<1$.

In 6.53 we showed that (under suitable conditions) this result gives a polynomial approximation to $f(x)$. We now use it to represent $f(x)$ by an *infinite series.*

Suppose that for each positive integer n, however large, $f^{(n)}(t)$ is continuous for $0 \leqslant t \leqslant x$; then the conditions for validity of Maclaurin's theorem hold for all orders. Consider the infinite series

$$f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\ldots. \qquad \text{(i)}$$

We have
$$s_n(x)=f(0)+xf'(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0),$$
$$=f(x)-R_n(x)$$

by Maclaurin's theorem. *If* $R_n(x)\to 0$ *when* $n\to\infty$ (perhaps only for some range of values of x, say $|x|<c$), then

$$s_n(x)\to f(x) \quad \text{when} \quad n\to\infty, \quad \text{for} \quad |x|<c;$$

hence the infinite series (i) converges and has sum-function $f(x)$ when $|x|<c$. We write

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^r}{r!}f^{(r)}(0)+\ldots \quad (|x|<c), \qquad \text{(ii)}$$

and call this series the *expansion* of $f(x)$ as a power series in x over the interval $|x|<c$.

This idea will now be applied to obtain series for some of the elementary functions; the results are important, and will be discussed in detail in 12.7. We use the results on nth derivatives obtained in 6.6; only when such convenient formulae are known is direct discussion of $R_n(x)$ possible.

12.62 Expansion of some elementary functions

(1) *The exponential series.*

If $f(x)=e^x$, then $f^{(n)}(x)=e^x$ for every positive integer n, and Lagrange's remainder is

$$R_n(x)=\frac{x^n}{n!}e^{\theta x} \quad (0<\theta<1).$$

Since $x^n/n!\to 0$ when $n\to\infty$ for any fixed value of x (2.74), therefore $R_n(x)\to 0$ when $n\to\infty$, for all x. As $f^{(r)}(0)=1$ for each r,

$$\boldsymbol{e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^r}{r!}+\ldots \quad (\textit{all } x).}$$

(2) *The sine and cosine series.*

If $f(x)=\sin x$, then $f^{(n)}(x)=\sin(x+\tfrac{1}{2}n\pi)$, and Lagrange's remainder is

$$R_n(x)=\frac{x^n}{n!}\sin(\theta x+\tfrac{1}{2}n\pi),$$

$$\therefore \quad |R_n(x)|\leqslant\frac{|x|^n}{n!}\to 0 \quad \text{when} \quad n\to\infty, \quad \text{for all } x.$$

Also $f^{(2r)}(0) = 0$ and $f^{(2r-1)}(0) = (-1)^{r-1}$ for each r, so

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{r-1}\frac{x^{2r-1}}{(2r-1)!} + \dots \quad (\textit{all } x).$$

Similarly

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^{r-1}\frac{x^{2r-2}}{(2r-2)!} + \dots \quad (\textit{all } x).$$

(3) *The logarithmic series.*

If $f(x) = \log(1+x)$, then $f^{(n)}(x) = (-1)^{n-1}\{(n-1)!/(1+x)^n\}$ and so $f^{(n)}(0) = (-1)^{n-1}(n-1)!$. We are supposing $x > -1$, for otherwise $\log(1+t)$ would not be defined in part of the range $x \leqslant t \leqslant 0$.

When $0 \leqslant x \leqslant 1$, Lagrange's remainder

$$R_n(x) = (-1)^{n-1}\frac{x^n}{n}\frac{1}{(1+\theta x)^n}$$

shows that $\qquad |R_n(x)| < \dfrac{1}{n} \to 0 \quad \text{when} \quad n \to \infty.$

When $x < 0$, $-x/(1+\theta x)$ may be greater than 1, so that Lagrange's remainder

$$-\frac{1}{n}\left(\frac{-x}{1+\theta x}\right)^n$$

might tend to $-\infty$ when $n \to \infty$ (2.73); insufficient is known about the behaviour of θ as a function of n.

Cauchy's remainder can be written

$$R_n(x) = (-1)^{n-1}\frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1},$$

so that if $-1 < x < 0$, $\qquad |R_n(x)| < \dfrac{|x|^n}{1-|x|}$

because $(1-\theta)/(1+\theta x)$ is a positive proper fraction and (see 1.14)

$$|1+\theta x| \geqslant 1 - \theta|x| > 1 - |x|.$$

Hence $R_n(x) \to 0$ when $n \to \infty$. (Cauchy's remainder also covers the case $0 \leqslant x \leqslant 1$ already treated.) Thus

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r-1}\frac{x^r}{r!} + \dots \quad (-1 < x \leqslant 1).$$

Since the series converges only for $-1 < x \leqslant 1$ (see 12.52, ex. (vi)), this is the most that can be proved concerning its sum to infinity.

(4) *The binomial series.*
If $f(x) = (1+x)^m$, then

$$f^{(n)}(x) = m(m-1)\dots(m-n+1)(1+x)^{m-n}.$$

If m is a positive integer, Maclaurin's formula terminates when $n = m+1$, and gives the usual binomial expansion containing $m+1$ terms. If m is not a positive integer, then $f^{(n)}(x)$ will be continuous when $n > m$ only if $x > -1$.

Cauchy's remainder can be written

$$R_n(x) = \frac{m(m-1)\dots(m-n+1)}{(n-1)!} x^n (1+\theta x)^{m-1} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}.$$

When $|x| < 1$,

$$a_n = \frac{(m-1)\dots(m-n+1)}{(n-1)!} x^n \to 0 \quad \text{when} \quad n \to \infty,$$

by 2.75. Also $(1-\theta)/(1+\theta x)$ is a positive proper fraction, and

$$(1+\theta x)^{m-1} < \begin{cases} (1+|x|)^{m-1} & \text{if} \quad m > 1, \\ (1-|x|)^{m-1} & \text{if} \quad m < 1. \end{cases}$$

$$\therefore \quad |R_n(x)| < |m|\,(1 \pm |x|)^{m-1}\,|a_n| \to 0 \quad \text{when} \quad n \to \infty.$$

Hence

$$\mathbf{(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots}$$
$$\mathbf{+ \frac{m(m-1)\dots(m-r+1)}{r!}x^r + \dots \quad (|x| < 1).}$$

If m is a fraction with even denominator, $(1+x)^m$ has two values which are equal and opposite. Here we intend the *positive* value, since the sum of the series when $x = 0$ is clearly $+1$.

As in (3), Lagrange's remainder would cover only the range $0 \leqslant x < 1$. We omit consideration of the cases $x = \pm 1$.

(5) *Gregory's series for* $\tan^{-1} x$.
If $f(x) = \tan^{-1} x$, then

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x^2)^{-\frac{1}{2}n} \sin(n \cot^{-1} x),$$

so
$$f^{(2r)}(0) = 0 \quad \text{and} \quad f^{(2r-1)}(0) = (-1)^{r-1}(2r-2)!.$$

Lagrange's remainder is

$$R_n(x) = \frac{x^n}{n}(-1)^{n-1}(1+\theta^2 x^2)^{-\frac{1}{2}n} \sin(n \cot^{-1}(\theta x)),$$

so by 2.73
$$|R_n(x)| \leqslant \frac{1}{n}|x|^n \to 0 \quad \text{when} \quad n \to \infty \quad \text{if} \quad |x| \leqslant 1.$$

Hence $$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\dots+(-1)^{r-1}\frac{x^{2r-1}}{2r-1}+\dots \quad (|x|\leqslant 1).$$

The value of $\tan^{-1}x$ in the range $-\frac{1}{4}\pi$ to $+\frac{1}{4}\pi$ is intended (i.e. the *principal* value when $-1\leqslant x\leqslant +1$), since when $x=0$ the sum of the series is clearly zero. No more than the above can be proved about the sum because the series converges only for $|x|\leqslant 1$: see Ex. 12 (*h*), no. 9.

12.63 Note on formal expansions

In Ex. 6 (*b*), no. 15 we considered the function $y=\tan^{-1}x$. Having proved that it satisfies the differential equation

$$(1+x^2)\,y_{n+2}+2(n+1)\,xy_{n+1}+n(n+1)\,y_n=0,$$

we put $x=0$ and showed that the Maclaurin coefficients $a_r=(y_r)_{x=0}$ are given by

$$a_{n+2}=-n(n+1)\,a_n.$$

Since $a_0=0$ and $a_1=1$, all coefficients with even suffix are zero; while by successive application of the above recurrence formula to those with odd suffix, we find that

$$a_{2n-1}=(-1)^{n-1}(2n-2)! \quad (n=1,2,\dots).$$

Assuming that y possesses a Maclaurin expansion, we thus obtain for it the series

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\dots+(-1)^{r-1}\frac{x^{2r-1}}{2r-1}+\dots. \quad \text{(i)}$$

Since this converges for $|x|\leqslant 1$ (see Ex. 12 (*h*), no. 9), *it is reasonable to expect* this series to have $y=\tan^{-1}x$ for its sum-function when $|x|\leqslant 1$. However, as we have nowhere considered $R_n(x)$, we have not *proved* by this argument that the series has sum-function $\tan^{-1}x$, nor even that $\tan^{-1}x$ can be expanded at all as a power series.

It can be proved that, within its interval of convergence, every power series is the Maclaurin series of its sum-function. In other words, if $\Sigma a_r x^r$ converges to $s(x)$ (for $|x|<c$, say), then $a_n=s^{(n)}(0)/n!$ $(n=0,1,2,\dots)$. Assuming this, what we have shown by the above argument is that, if the series (i) converges to $s(x)$ when $|x|\leqslant 1$, and $f(x)=\tan^{-1}x$, then

$$s(0)=f(0),\quad s'(0)=f'(0),\quad \dots,\quad s^{(r)}(0)=f^{(r)}(0),\quad \dots. \quad \text{(ii)}$$

These equations do *not* prove that $s(x)$ is identical with $f(x)$ for $|x|\leqslant 1$. For example, if

$$C(x)=e^{-1/x^2}\quad (x\neq 0),\quad C(0)=0,$$

it can be shown that this continuous function $C(x)$ has continuous derivatives of all orders for $|x|<1$ and that $C^{(n)}(0)=0$ $(n=1,2,\dots)$. Hence $s(x)=f(x)+C(x)$ satisfies equations (ii), although $C(x)\not\equiv 0$. In short, *the Maclaurin series of $f(x)$, even if convergent, may not have $f(x)$ for its sum-function.* Further, it can be shown by examples that even if $f(x)$ and its derivatives of all orders are continuous in a certain interval, the Maclaurin series of $f(x)$ may not converge at all for that interval.

Thus, to ensure the existence and validity of the Maclaurin expansion of $f(x)$ in 12.61, it is *essential* to prove somehow that $R_n(x) \to 0$ when $n \to \infty$, for all x in the interval concerned. However, despite the cautionary nature of these remarks, it is true that for 'ordinary' functions the formal expansions obtained as above are valid whenever they converge; and with this assurance the reader may accept them with confidence.

Exercise 12(*i*)

State the sum to infinity of

1 $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$. *2 $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$.

3 Assuming the exponential series, state series for $\operatorname{ch} x$ and $\operatorname{sh} x$.

Obtain Maclaurin series, stating the range of validity, for the following functions.

4 $e^x \sin x$. [See 6.61, (vi).] 5 $e^{-x}\cos x$. 6 $\cos x \operatorname{ch} x$.

7 Prove $\dfrac{d^n}{dx^n}\{e^{x\cos\alpha}\cos(x\sin\alpha)\} = e^{x\cos\alpha}\cos(x\sin\alpha+n\alpha)$,

and deduce that for all x,

$$e^{x\cos\alpha}\cos(x\sin\alpha) = 1+x\cos\alpha+\frac{x^2}{2!}\cos 2\alpha+\frac{x^3}{3!}\cos 3\alpha+\ldots.$$

*8 By showing that $R_n(x,h) \to 0$ in Taylor's theorem, prove that

$$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{2!}\cos x + \frac{h^3}{3!}\sin x - \ldots$$

for all x and h. [See Ex. 6(*b*), no. 24.]

*9 Obtain a Taylor series for $\sin(x+h)$.

10 If $y = \operatorname{ch}(\sin^{-1}x)$, prove

$$(1-x^2)y_2 - xy_1 - y = 0 \quad \text{and} \quad (1-x^2)y_{n+2}-(2n+1)xy_{n+1} = (n^2+1)y_n.$$

If $$\operatorname{ch}(\sin^{-1}x) = a_0+a_1x+a_2x^2+\ldots,$$

obtain an expression for a_{2n}, and prove $a_{2n+1} = 0$.

11 If $x = \cos\theta$ and $y = \cos n\theta$ where $n > 1$, prove $(1-x^2)y_2 - xy_1 + n^2y = 0$. Assuming that $y = a_0+a_1x+a_2x^2+\ldots$, prove

$$(k+1)(k+2)a_{k+2}+(n^2-k^2)a_k = 0 \quad (k = 0, 1, 2, \ldots).$$

If n is an *integer*, show that the assumed expansion is a *polynomial* of degree n; and if n is *even* and greater than 2, prove $a_4 = \frac{1}{24}(-1)^{\frac{1}{2}n}n^2(n^2-4)$.

12 If $y = \cos\log(1+x)$, prove

$$(1+x)^2y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2+1)y_n = 0.$$

If y can be expanded as $\sum_{r=0}^{\infty} a_r x^r$, prove

$$(n+1)(n+2)a_{n+2}+(2n+1)(n+1)a_{n+1}+(n^2+1)a_n = 0,$$

and hence determine the expansion up to the term in x^6.

12.7 Applications of the series in 12.62

12.71 Binomial series

(1) The expression $\dfrac{m(m-1)\ldots(m-r+1)}{r!}$, often abbreviated† to $\dbinom{m}{r}$, is called a *binomial coefficient*. The expansion of $(a+x)^m$ can be reduced to the standard one in 12.62 (4) as follows:

(*a*) if $\left|\dfrac{x}{a}\right| < 1$ and a^m exists, $\quad (a+x)^m = a^m\left(1+\dfrac{x}{a}\right)^m = $ etc.;

(*b*) if $\left|\dfrac{x}{a}\right| > 1$ and x^m exists, $\quad (a+x)^m = x^m\left(1+\dfrac{a}{x}\right)^m = $ etc.

Examples

(i) *Write down the general term and the first four terms of the expansion of* $(1+2x)^{-3}$ *in ascending powers of* x, *stating when it is valid.*

There will be an expansion in ascending powers of x if $|2x| < 1$, i.e. if $|x| < \frac{1}{2}$, and then the $(r+1)$th term is

$$\frac{(-3)(-4)\ldots(-3-r+1)}{r!}(2x)^r = \frac{(-1)^r 3.4\ldots(r+2)}{r!}(2x)^r$$

$$= (-1)^r\frac{(r+1)(r+2)}{1.2}2^r x^r = (-1)^r 2^{r-1}(r+1)(r+2)x^r.$$

We can obtain the first four terms by writing $r = 0, 1, 2, 3$ successively in this formula:

$$(1+2x)^{-3} = 1-6x+24x^2-80x^3+\ldots \quad (|x| < \tfrac{1}{2}).$$

(ii) *Find the* 9*th term in the expansion of* $(1-4x)^{-2}$ *if* $|x| > \frac{1}{4}$.

Since $|4x| > 1$, the expansion can be made only if we expand in powers of $1/x$, i.e. in *descending* powers of x.

$$(1-4x)^{-2} = \left(\frac{1}{4x}-1\right)^{-2}(4x)^{-2} = \frac{1}{16x^2}\left(1-\frac{1}{4x}\right)^{-2},$$

and the 9th term in this is (putting $r = 8$ in the general term)

$$\frac{1}{16x^2}\frac{(-2)(-3)\ldots(-9)}{8!}\left(\frac{1}{4x}\right)^8 = (-1)^8\frac{2.3\ldots9}{8!}\left(\frac{1}{4x}\right)^{10} = \frac{9}{(4x)^{10}}.$$

(iii) *Calculate* $\sqrt{102}$ *correct to four places of decimals by using the binomial series.*

$$\sqrt{102} = \sqrt{\{100(1+0{\cdot}02)\}} = 10(1+0{\cdot}02)^{\frac{1}{2}},$$

and $\quad (1+0{\cdot}02)^{\frac{1}{2}} = 1+\frac{1}{2}(0{\cdot}02)+\dfrac{\frac{1}{2}.-\frac{1}{2}}{2!}(0{\cdot}02)^2+\dfrac{\frac{1}{2}.-\frac{1}{2}.-\frac{3}{2}}{3!}(0{\cdot}02)^3+\ldots.$

† Cf. 12.23; the symbol ${}^mC_r$ could not be used unless m is a *positive integer*.

Since we require the result to be correct to 4 places, we must work with 5 places; and owing to the factor 10, $(1{\cdot}02)^{\frac{1}{2}}$ must be found to 6 places. Calculating the terms shown, we find

$$\begin{aligned}(1{\cdot}02)^{\frac{1}{2}} &= 1+0{\cdot}01-0{\cdot}000{,}05+0{\cdot}000{,}000{,}5+\dots\\ &= 1{\cdot}009{,}950{,}5\dots,\end{aligned}$$

the next term clearly† being too small to influence the sixth place. Hence

$$\sqrt{102} \doteqdot 10{\cdot}0995.$$

(iv) *Expand* $(1+x+x^2+x^3)^{-5}$ *as far as the term in* x^3.

Since $1+x+x^2+x^3 = (1-x^4)/(1-x)$, the given expression is

$$\begin{aligned}(1-x)^5(1-x^4)^{-5} &= (1-5x+10x^2-10x^3+\dots)(1+5x^4+\dots)\\ &= 1-5x+10x^2-10x^3+\dots\end{aligned}$$

on neglecting all terms in x^4 and higher powers.

(v) *Find the coefficient of* x^r *in the expansion of* $1/\{(2-x)(1+3x)\}$ *in ascending powers of* x, *stating the condition of validity.*

By resolving into partial fractions we find

$$\begin{aligned}\frac{1}{(2-x)(1+3x)} &= \frac{1}{7}\frac{1}{2-x}+\frac{3}{7}\frac{1}{1+3x}\\ &= \tfrac{1}{14}(1-\tfrac{1}{2}x)^{-1}+\tfrac{3}{7}(1+3x)^{-1}.\end{aligned}$$

The first bracket can be expanded in ascending powers of x if $|\tfrac{1}{2}x| < 1$, i.e. $|x| < 2$; the second can be so expanded if $|3x| < 1$, i.e. $|x| < \tfrac{1}{3}$. Hence both can be expanded if $|x| < \tfrac{1}{3}$, and then the coefficient of x^r is

$$\begin{aligned}&\frac{1}{14}\frac{(-1)(-2)\dots(-r)}{r!}\left(-\frac{1}{2}\right)^r+\frac{3}{7}\frac{(-1)(-2)\dots(-r)}{r!}3^r\\ &= \tfrac{1}{14}(\tfrac{1}{2})^r+\tfrac{3}{7}(-3)^r = \frac{1}{7}\left\{\frac{1}{2^{r+1}}-(-3)^{r+1}\right\}.\end{aligned}$$

(2) *Summation of series reducible to binomial expansions.*

In the result

$$\begin{aligned}(1-x)^{-3} &= 1+(-3)(-x)+\frac{-3\,.\,-4}{1\,.\,2}(-x)^2+\frac{-3\,.\,-4\,.\,-5}{1\,.\,2\,.\,3}(-x)^3+\dots\\ &= 1+3x+\frac{3\,.\,4}{1\,.\,2}x^2+\frac{3\,.\,4\,.\,5}{1\,.\,2\,.\,3}x^3+\dots,\end{aligned}$$

which is valid when $|x| < 1$, let us put $x = \tfrac{1}{2}$. The series becomes

$$1+\tfrac{3}{2}+\frac{3\,.\,4}{1\,.\,2}(\tfrac{1}{2})^2+\frac{3\,.\,4\,.\,5}{1\,.\,2\,.\,3}(\tfrac{1}{2})^3+\dots,$$

† See 12.81, ex. (iii) for an estimate showing that the error in stopping here does not affect the sixth place even when we allow for the succeeding *infinite series* of terms.

which can be written
$$1+\tfrac{3}{2}+\frac{3.4}{2.4}+\frac{3.4.5}{2.4.6}+\ldots.$$

Our result above shows that the sum to infinity of this series is $(1-\tfrac{1}{2})^{-3}=8$.

Conversely, many series of this form (in which the factors in the numerator and denominator of each term are in A.P., while the *number* of such factors is the same but increases as we proceed along the series) can be reduced to binomial expansions, and hence their sum to infinity written down.

Examples

(vi) *Find the sum to infinity of*
$$1-\frac{1}{4}+\frac{1.3}{4.8}-\frac{1.3.5}{4.8.12}+\ldots.$$

The factors in the numerators form an A.P. with common difference 2, and those in the denominators have common difference 4. We begin by dividing each factor in the numerator of each term by 2, and each factor in the denominator by 4:
$$1-\frac{\frac{1}{2}}{1}(\tfrac{2}{4})+\frac{\frac{1}{2}.\frac{3}{2}}{1.2}(\tfrac{2}{4})^2-\frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{1.2.3}(\tfrac{2}{4})^3+\ldots.$$

The factors in the numerators now ascend by the common difference 1, whereas for a binomial series they *descend* by 1. This can be adjusted here by introducing signs as follows:
$$1+\frac{-\frac{1}{2}}{1}(\tfrac{1}{2})+\frac{-\frac{1}{2}.-\frac{3}{2}}{1.2}(\tfrac{1}{2})^2+\frac{-\frac{1}{2}.-\frac{3}{2}.-\frac{5}{2}}{1.2.3}(\tfrac{1}{2})^3+\ldots,$$
and this series is clearly the expansion of $(1+x)^m$ with $x=\tfrac{1}{2}$ and $m=-\tfrac{1}{2}$. Hence the required sum to infinity is $(1+\tfrac{1}{2})^{-\frac{1}{2}}=\sqrt{\tfrac{2}{3}}=\tfrac{1}{3}\sqrt{6}$.

(vii) $$\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots$$
$$=\frac{\frac{1}{2}}{1}(\tfrac{2}{3})+\frac{\frac{1}{2}.\frac{3}{2}}{1.2}(\tfrac{2}{3})^2+\frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{1.2.3}(\tfrac{2}{3})^3+\frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}}{1.2.3.4}(\tfrac{2}{3})^4+\ldots$$
$$=\left\{1+\frac{-\frac{1}{2}}{1}(-\tfrac{2}{3})+\frac{-\frac{1}{2}.-\frac{3}{2}}{1.2}(-\tfrac{2}{3})^2+\frac{-\frac{1}{2}.-\frac{3}{2}.-\frac{5}{2}}{1.2.3}(-\tfrac{2}{3})^3+\ldots\right\}-1$$
$$=(1-\tfrac{2}{3})^{-\frac{1}{2}}-1=\sqrt{3}-1.$$

Exercise 12(*j*)

Write down and simplify the general term and the first four terms of the expansions of the following functions in ascending powers of x, stating when these expansions are valid.

1 $(1+x)^{-3}$. 2 $(1-x)^{\frac{1}{2}}$. 3 $(1+3x)^{-\frac{1}{3}}$. 4 $(4+x^2)^{\frac{1}{2}}$.

5 $\dfrac{x}{(1-x)^2}$. 6 $\dfrac{1-x}{2+x}$. 7 $\dfrac{1+x^2}{(1-x)^3}$.

Find the named terms in the following expansions.

8 8th term in $(1-2x)^{-3}$ if $|x| < \frac{1}{2}$. 9 10th term in $(1-3x)^{-2}$ if $|x| > \frac{1}{3}$.

10 What is the first negative coefficient in the expansion of $(3+2x)^{\frac{7}{3}}$ if $|x| < \frac{3}{2}$? [Consider the sign of a_{r+1}/a_r.]

*11 If u_r is the $(r+1)$th term in the expansion of $(1+x)^m$, $|x| < 1$, show that

$$\frac{u_{r+1}}{u_r} = -\left(1-\frac{m+1}{r}\right)x.$$

Deduce that when r becomes greater than $m+1$, then $u_{r+1}/u_r \lesseqgtr 0$ according as $x \gtrless 0$; and hence that, *after a certain stage in the expansion, the terms are alternately positive and negative if $x > 0$, but all have the same sign if $x < 0$.*

Find the numerically greatest term or terms in the expansion of

12 $(1+x)^{21/2}$ when $x = \frac{3}{4}$. [Use the argument in 12.13, ex. (iii).]

13 $(1-x)^{-10}$ when $x = \frac{3}{4}$. 14 $(1+x)^{-\frac{1}{3}}$ when $x = \frac{1}{2}$.

Use the binomial series to calculate correct to four places of decimals

15 $(4{\cdot}08)^{\frac{3}{2}}$. 16 $\sqrt{98}$.

17 Expand $(1+x+2x^2)^{-\frac{1}{2}}$ as far as the term in x^3.

State the condition that the following can be expanded in ascending powers of x, and find the coefficient of x^r in the expansion.

18 $\dfrac{2+3x}{(1-x)(1+2x)}$. 19 $\dfrac{x}{6-x-x^2}$.

20 $\dfrac{x^2}{(1-x)^2(2-x)}$. 21 $\dfrac{1}{1+x+x^2}\left[=\dfrac{1-x}{1-x^3}\right]$.

*22 $(1+x+x^2+\ldots+x^{m-1})^n$ if $r < m$.

23 Find the coefficient of x^{2r} and of x^{2r+1} in the expansion of

$$\sqrt{\frac{1+x}{1-x}}, \quad |x| < 1. \quad \left[\text{Write as } \frac{1+x}{\sqrt{(1-x^2)}}.\right]$$

Find the sum to infinity of

24 $1+\dfrac{1}{8}+\dfrac{1.3}{8.16}+\dfrac{1.3.5}{8.16.24}+\ldots$. 25 $1-\dfrac{1}{4}+\dfrac{1.4}{4.8}-\dfrac{1.4.7}{4.8.12}+\ldots$.

26 $\dfrac{4}{20}+\dfrac{4.7}{20.30}+\dfrac{4.7.10}{20.30.40}+\ldots$. 27 $\dfrac{3}{2.4}+\dfrac{3.4}{2.4.6}+\dfrac{3.4.5}{2.4.6.8}+\ldots$.

*28 Verify that when $|x|<1$,

$$\sum_{r=1}^{\infty} x^{r-1} = \frac{1}{1-x}, \quad \sum_{r=1}^{\infty} rx^{r-1} = \frac{1}{(1-x)^2} \quad \text{and} \quad \sum_{r=1}^{\infty} \tfrac{1}{2}r(r+1)\,x^{r-1} = \frac{1}{(1-x)^3}.$$

Hence calculate $\displaystyle\sum_{r=1}^{\infty} (r^2+3r-3)\,x^{r-1}$ when $|x| < 1$.

$[r^2+3r-3 \equiv -3+2r+r(r+1).]$

*29 By writing Pascal's triangle (12.13, ex. (iv)) as shown here, we observe that the nth vertical column gives the coefficients in the binomial series for $(1-x)^{-n}$. Justify this rule by proving

$$\binom{-n-1}{r}+\binom{-n-1}{r-1}=\binom{-n}{r}.$$

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1
...	...	...	...	...	...

12.72 Exponential series

(1) *Irrationality of e.* Taking $x = 1$ in the series of 12.62 (1),

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{r!}+\ldots.$$

From this it is possible to calculate e to any specified degree of accuracy (see the example in 6.53).

Since this series consists entirely of positive terms,

$$e > 1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}.$$

The error in this estimate is

$$\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\ldots$$

$$=\frac{1}{(n+1)!}\left\{1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\ldots\right\}$$

$$<\frac{1}{(n+1)!}\left\{1+\frac{1}{n+1}+\frac{1}{(n+1)^2}+\ldots\right\}$$

$$=\frac{1}{(n+1)!}\,\frac{1}{1-1/(n+1)} \quad \text{by summing the G.P.,}$$

$$=\frac{1}{(n+1)!}\,\frac{n+1}{n}=\frac{1}{n!\,n}.$$

These results can be used to prove that e *is an irrational number*, i.e. that e cannot be expressed in the form p/q where p, q are integers. For suppose if possible that $e = p/q$; then from the above inequalities with $n = q$,

$$1+\frac{1}{1!}+\ldots+\frac{1}{q!}<\frac{p}{q}<1+\frac{1}{1!}+\ldots+\frac{1}{q!}+\frac{1}{q!\,q}.$$

Multiplying by $q!$, we have

$$I < p.(q-1)! < I+\frac{1}{q},$$

where
$$I = q! + \frac{q!}{1!} + \ldots + \frac{q!}{q!}$$

is an *integer*, and so is $p.(q-1)!$. Our conclusion is that the integer $p.(q-1)!$ lies between I and $I+1/q$, which is impossible since $1/q$ is a proper fraction. Hence e cannot be written as p/q for any pair of integers p, q; i.e. e is irrational.

(2) a^x, $\operatorname{ch} x$, $\operatorname{sh} x$.

If $a > 0$ and $m = \log a$, then $a = e^m$ and

$$a^x = e^{mx} = 1 + mx + \frac{m^2x^2}{2!} + \frac{m^3x^3}{3!} + \ldots \quad \text{for all } x,$$

i.e.
$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots \quad (\text{all } x).$$

From the definitions $\operatorname{ch} x = \frac{1}{2}(e^x + e^{-x})$, $\operatorname{sh} x = \frac{1}{2}(e^x - e^{-x})$ and 12.32 (4),

$$\left.\begin{aligned} \operatorname{ch} x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r-2}}{(2r-2)!} + \ldots, \\ \operatorname{sh} x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r-1}}{(2r-1)!} + \ldots, \end{aligned}\right\} \quad (\textit{all } x).$$

(3) *Series reducible to exponential series.* The first step is always to write down the general term of the given series, and then put it into a suitable form.

Examples

(i) *Sum to infinity* $\quad \frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \ldots.$

Here
$$u_r = \frac{r^2}{(r+1)!}.$$

The numerator $\quad r^2 = (r+1)r - r = (r+1)r - (r+1) + 1,$

and so
$$u_r = \frac{1}{(r-1)!} - \frac{1}{r!} + \frac{1}{(r+1)!} \quad \text{if} \quad r \geqslant 1.$$

$$\begin{aligned} \therefore \quad \sum_{r=1}^{\infty} u_r &= \sum_{r=1}^{\infty} \frac{1}{(r-1)!} - \sum_{r=1}^{\infty} \frac{1}{r!} + \sum_{r=1}^{\infty} \frac{1}{(r+1)!} \\ &= e - (e-1) + \left(e - 1 - \frac{1}{1!}\right) \\ &= e - 1. \end{aligned}$$

Sometimes the decomposition does not hold for a few terms at the beginning.

(ii) *Calculate*
$$\sum_{r=1}^{\infty} \frac{(r+2)^2}{r!}.$$

Since
$$(r+2)^2 = r^2+4r+4 = r(r-1)+5r+4,$$
$$u_r = \frac{1}{(r-2)!}+\frac{5}{(r-1)!}+\frac{4}{r!} \quad \text{if} \quad r \geqslant 2.$$
$$\therefore \quad \sum_{r=1}^{\infty} u_r = \frac{3^2}{1!}+\sum_{r=2}^{\infty} u_r$$
$$= 9+e+5(e-1)+4\left(e-1-\frac{1}{1!}\right)$$
$$= 10e-4.$$

Exercise 12(*k*)

Give the coefficient of x^r in the expansion of

1 e^{3x+2}. 2 $\dfrac{1+x}{e^x}$. 3 $\dfrac{x^2+3x+1}{e^x}$. 4 $\dfrac{e^{3x}+e^x}{e^{2x}}$.

5 Write down the series whose sum to infinity is (i) $1/e$; (ii) e^2; (iii) $(e+1/e)^2$.

Find the sum to infinity of each of the following series.

6 $3-\dfrac{3^2}{2!}+\dfrac{3^3}{3!}-\dfrac{3^4}{4!}+\dots$

7 $1+\dfrac{4^2}{3!}+\dfrac{4^4}{5!}+\dfrac{4^6}{7!}+\dots$

8 $1+\dfrac{2}{1!}+\dfrac{3}{2!}+\dfrac{4}{3!}+\dots$

9 $\dfrac{1}{1!}+\dfrac{1+2}{2!}+\dfrac{1+2+3}{3!}+\dots$

10 $1+\dfrac{3}{2}+\dfrac{5}{2.4}+\dfrac{7}{2.4.6}+\dots$

11 $\displaystyle\sum_{r=1}^{\infty} \frac{x^r}{(r+1)!}$.

12 $\displaystyle\sum_{r=1}^{\infty} \frac{x^r}{(r+2)\,r!}$.

12.73 Logarithmic series

(1) There are several important forms of the result
$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots \quad (-1 < x \leqslant 1).$$

(*a*) Replacing x by $-x$,
$$\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\dots$$
provided $-1 < -x \leqslant 1$, i.e. *provided* $-1 \leqslant x < 1$.

(*b*) Subtracting the above two results, which are simultaneously valid if $-1 < x < 1$,
$$\log(1+x)-\log(1-x) = 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right),$$

i.e. $$\tfrac{1}{2}\log\frac{1+x}{1-x} = x+\frac{x^3}{3}+\frac{x^5}{5}+\dots \quad (-1 < x < 1).$$

(c) In (b) write
$$\frac{1+x}{1-x} = y, \quad \text{i.e.} \quad x = \frac{y-1}{y+1}.$$

The condition $-1 < x < 1$ becomes $y > 0$, and

$$\tfrac{1}{2}\log y = \left(\frac{y-1}{y+1}\right)+\frac{1}{3}\left(\frac{y-1}{y+1}\right)^3+\frac{1}{5}\left(\frac{y-1}{y+1}\right)^5+\dots \quad (y > 0).$$

(d) In (b) put
$$\frac{1+x}{1-x} = \frac{p+1}{p}, \quad \text{i.e.} \quad x = \frac{1}{2p+1}.$$

Then
$$\log(p+1)-\log p = 2\left\{\frac{1}{2p+1}+\frac{1}{3}\frac{1}{(2p+1)^3}+\frac{1}{5}\frac{1}{(2p+1)^5}+\dots\right\},$$

and the left-hand side is defined only if $p > 0$, which corresponds to the condition $0 < x < 1$ under which (b) certainly holds.

The series (b), (c), (d) are more rapidly convergent than either the standard series or (a) separately; (c) is used for calculating natural logarithms, and (d) for calculating logarithms of consecutive numbers.

Example

(i) *Find the coefficient of* x^r *in the expansion of* $\log(1-x+x^2)$ if $|x| < 1$.

$$\log(1-x+x^2) = \log\frac{1+x^3}{1+x} = \log(1+x^3)-\log(1+x).$$

We have $$\log(1+x^3) = x^3-\frac{x^6}{2}+\frac{x^9}{3}-\dots+(-1)^{r-1}\frac{x^{3r}}{r}+\dots$$

and $$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\dots+(-1)^{r-1}\frac{x^r}{r}+\dots.$$

If r is not a multiple of 3, the coefficient of x^r is $(-1)^{r-1}/r$.
If r is a multiple of 3, the coefficient of x^r is

$$(-1)^{\frac{1}{3}r-1}\frac{1}{\frac{1}{3}r}-(-1)^{r-1}\frac{1}{r} = (-1)^{r-1}\frac{2}{r}.$$

(2) *Series reducible to logarithmic expansions.*

Examples

(ii) $$\frac{1}{1.2}+\frac{1}{2.2^2}+\frac{1}{3.2^3}+\dots = \frac{\frac{1}{2}}{1}+\frac{(\frac{1}{2})^2}{2}+\frac{(\frac{1}{2})^3}{3}+\dots$$
$$= -\log(1-\tfrac{1}{2}) = \log 2.$$

If the series is not immediately recognisable, we begin by writing down the general term.

(iii) *Sum to infinity* $\dfrac{1}{1.3}+\dfrac{1}{2.5}+\dfrac{1}{3.7}+\dots.$

$$u_r = \frac{1}{r(2r+1)}$$
$$= \frac{1}{r}-\frac{2}{2r+1} \quad \text{on using partial fractions,}$$
$$= \frac{2}{2r}-\frac{2}{2r+1},$$

where the denominators are now consecutive numbers. Then

$$s_n = 2\left\{(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{4}-\tfrac{1}{5})+\dots+\left(\frac{1}{2n}-\frac{1}{2n+1}\right)\right\}$$
$$\to 2\{1-\log 2\} \quad \text{when} \quad n\to\infty,$$

since $$\log 2 = 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots.$$

The sum of the series is therefore $2-\log 4$.

(iv) *Find the sum to infinity of*

$$\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\dots$$

whenever it converges.

$$u_r = \frac{x^r}{r(r+1)} = x^r\left(\frac{1}{r}-\frac{1}{r+1}\right).$$
$$\therefore \quad \sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} x^r\left(\frac{1}{r}-\frac{1}{r+1}\right)$$
$$= \sum_{r=1}^{\infty}\frac{x^r}{r}-\frac{1}{x}\sum_{r=1}^{\infty}\frac{x^{r+1}}{r+1}$$

provided that each series converges (12.32(4)), and when $-1 \leqslant x < 1$ this is the case because the first is $-\log(1-x)$ and the second is $-(1/x)\{\log(1-x)+x\}$. Hence

$$\sum_{r=1}^{\infty} u_r = \left(\frac{1}{x}-1\right)\log(1-x)+1 \quad \text{when} \quad -1\leqslant x<1.$$

When $x = 1$,

$$u_r = \frac{1}{r(r+1)} = \frac{1}{r}-\frac{1}{r+1}, \quad s_n = 1-\frac{1}{n+1}\to 1 \text{ when } n\to\infty,$$

and the sum is 1.

It is easily verified that the series does not converge unless $-1 \leqslant x \leqslant 1$, so the problem has been solved completely.

(v) *Sum to infinity* $\dfrac{1}{2.3.4}+\dfrac{1}{4.5.6}+\dfrac{1}{6.7.8}+\dots.$

$$u_r = \frac{1}{2r(2r+1)(2r+2)}$$
$$= \frac{A}{2r}+\frac{B}{2r+1}+\frac{C}{2r+2},$$

and we find by the usual method that $A = \frac{1}{2}$, $B = -1$, $C = \frac{1}{2}$. Hence

$$u_r = \frac{\frac{1}{2}}{2r} - \frac{1}{2r+1} + \frac{\frac{1}{2}}{2r+2}$$

$$= \frac{1}{2}\left(\frac{1}{2r} - \frac{1}{2r+1}\right) - \frac{1}{2}\left(\frac{1}{2r+1} - \frac{1}{2r+2}\right).$$

$$\therefore \quad s_n = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r} - \frac{1}{2r+1}\right) - \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r+1} - \frac{1}{2r+2}\right)$$

$$\to -\tfrac{1}{2}(\log 2 - 1) - \tfrac{1}{2}(\log 2 - 1 + \tfrac{1}{2}) \quad \text{when} \quad n \to \infty,$$

$$= \tfrac{3}{4} - \log 2.$$

Exercise 12(*l*)

Find the coefficient of x^r and give the first four terms in the expansions of the following, together with the conditions of validity.

1 $\log(1-4x)$. 2 $\log(2+x)$. 3 $\log(1+x)^2$.

4 $\log(1+5x+6x^2)$. 5 $(1-x)\log(1+x)$. 6 $\log\dfrac{1+x+x^2}{1-x}$.

7 $\log(1-x+x^2-x^3)$.

8 Expand $\log(1+2x+3x^2)$ as far as the term in x^4.

9 If $|x| > 1$, expand $\log(1+x) - \log x$ in powers of $1/x$, giving the general term.

10 Taking $x = \frac{1}{3}$ in 12.73(*b*), prove $\log 2 \doteqdot 0{\cdot}6931472$ correct to 7 places of decimals.

11 Taking $x = \frac{1}{9}$ in series (*b*) and using no. 10, prove $\log 10 \doteqdot 2{\cdot}3025851$.

12 Use series (*d*) and no 10. to prove $\log 3 \doteqdot 1{\cdot}098612$.

13 Prove $2\log 7 + \log 2 = 2\log 10 - \sigma$, where

$$\sigma = 0{\cdot}02 + \tfrac{1}{2}(0{\cdot}02)^2 + \tfrac{1}{3}(0{\cdot}02)^3 + \ldots,$$

and deduce that $\log 7 \doteqdot 1{\cdot}945910$.

Find the sum to infinity of the following series.

14 $\dfrac{1}{1.2} - \dfrac{1}{2.2^2} + \dfrac{1}{3.2^3} - \ldots.$ 15 $\dfrac{1}{1.3} + \dfrac{1}{3.3^3} + \dfrac{1}{5.3^5} + \ldots.$

16 $\dfrac{x^2}{2} + \dfrac{x^4}{4} + \dfrac{x^6}{6} + \ldots$, $|x| < 1$. 17 $\dfrac{1}{1.2} + \dfrac{1}{3.4} + \dfrac{1}{5.6} + \ldots.$

18 $\dfrac{1}{1.2} - \dfrac{1}{2.3} + \dfrac{1}{3.4} - \ldots.$ 19 $\dfrac{1}{1.2.3} + \dfrac{1}{3.4.5} + \dfrac{1}{5.6.7} + \ldots.$

20 $\dfrac{1}{1.2.3} + \dfrac{1}{5.6.7} + \dfrac{1}{9.10.11} + \ldots.$ [$u_r = \dfrac{\frac{1}{2}}{4r-3} - \dfrac{1}{4r-2} + \dfrac{\frac{1}{2}}{4r-1}$, which can be written $\dfrac{1}{2}\left(\dfrac{1}{4r-3} - \dfrac{1}{4r-2} + \dfrac{1}{4r-1} - \dfrac{1}{4r}\right) - \dfrac{1}{4}\left(\dfrac{1}{2r-1} - \dfrac{1}{2r}\right)$; consider $\lim_{n\to\infty} s_n$.]

21 $\dfrac{x^2}{1.3} + \dfrac{x^4}{3.5} + \dfrac{x^6}{5.7} + \ldots$, $|x| < 1$. 22 $\dfrac{4x}{1.3} + \dfrac{6x^2}{2.4} + \dfrac{8x^3}{3.5} + \ldots$, $|x| < 1$.

23 $(1+\frac{1}{2})-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2^3}\right)+\frac{1}{3}\left(\frac{1}{2^2}+\frac{1}{2^5}\right)-\ldots$.

24 If $y = 2x^2-1$, prove (under conditions to be stated) that

$$\frac{1}{x^2}+\frac{1}{2x^4}+\frac{1}{3x^6}+\ldots = \frac{2}{y}+\frac{2}{3y^3}+\frac{2}{5y^5}+\ldots.$$

25 If $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$, prove

$$\sin^2\alpha+\tfrac{1}{2}\sin^4\alpha+\ldots+\frac{1}{n}\sin^{2n}\alpha+\ldots$$
$$= 2^2\sin^2\tfrac{1}{2}\alpha+\frac{2^3\sin^4\frac{1}{2}\alpha}{2}+\ldots+\frac{2^{n+1}\sin^{2n}\frac{1}{2}\alpha}{n}+\ldots.$$

26 If $f(x) = x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots$ and $|x| < 1$, prove $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

12.74 Gregory's series and the calculation of π

If we put $x = 1$ in the result of 12.62 (5), we obtain

$$\tfrac{1}{4}\pi = 1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots,$$

a relation discovered by Leibniz. This series converges very slowly (see Ex. 12(*m*) no. 4) and is impracticable for calculating π.

A better method, given by Machin, depends on the result

$$4\tan^{-1}\tfrac{1}{5}-\tan^{-1}\tfrac{1}{239} = \tfrac{1}{4}\pi,$$

which we now prove. Let $a = \tan^{-1}\frac{1}{5}$, so that $\tan a = \frac{1}{5}$ and

$$\tan 4a = \frac{2\tan 2a}{1-\tan^2 2a} = \frac{4\tan a}{1-\tan^2 a}\Big/\left[1-\left\{\frac{2\tan a}{1-\tan^2 a}\right\}^2\right]$$
$$= \left[\frac{\frac{4}{5}}{1-\frac{1}{25}}\right]\Big/\left[1-\left\{\frac{\frac{2}{5}}{1-\frac{1}{25}}\right\}^2\right]$$
$$= \tfrac{120}{119},$$

which is a number just greater than 1. Hence $4a$ is just greater than $\frac{1}{4}\pi$; if we write $4a = \frac{1}{4}\pi+\tan^{-1}x$, then

$$\tfrac{120}{119} = \tan(\tfrac{1}{4}\pi+\tan^{-1}x) = \frac{1+x}{1-x},$$

and $x = \frac{1}{239}$.

Thus on putting $x = \frac{1}{5}, \frac{1}{239}$ in Gregory's series we have

$$\tfrac{1}{4}\pi = 4\left\{\frac{1}{5}-\frac{1}{3.5^3}+\frac{1}{5.5^5}-\ldots\right\}-\left\{\frac{1}{239}-\frac{1}{3.239^3}+\ldots\right\}.$$

It can be shown that 6 terms from the first bracket and 2 from the second give π to eight decimal places; while 21 and 3 terms from the respective brackets give π to 16 places.

12.8 Series and approximations

12.81 Estimation of the error in $s \doteqdot s_n$

Given a convergent infinite series, we can approximate to its sum to infinity by taking the first n terms. If the series has been obtained by an application of Maclaurin's or Taylor's theorem as in 12.61, the remainder term R gives the error in the approximation (see 6.53); but owing to the indeterminate nature of the number θ, an estimate of R has to be made in practice. When R cannot be obtained explicitly because of difficulty in calculating $f^{(n)}(x)$, or when the sum-function of the given series is not known, the method in 12.72 (1) can be used.

Examples

(i) If we take $\log(1+x) \doteqdot x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^n}{n}$,

then the error is (see 12.62 (3))

$$R_{n+1}(x) = \frac{(-1)^n x^{n+1}}{(n+1)(1+\theta x)^{n+1}}.$$

When $x > 0$, $\quad 1+\theta x > 1$ and so $|R_{n+1}(x)| < \dfrac{x^{n+1}}{n+1}$.

When $x < 0$, $\quad |1+\theta x| \geqslant 1-\theta|x| > 1-|x| = 1+x$,

and
$$|R_{n+1}(x)| < \frac{|x|^{n+1}}{(n+1)(1+x)^{n+1}}.$$

(ii) If we take $\quad e^x \doteqdot 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!}$,

the error is (see 12.62 (1)) $R_n(x) = (x^n/n!)\,e^{\theta x}$. When $x < 0$, $e^{\theta x} < 1$ and $|R_n(x)| < |x|^n/n!$; but if $x > 0$, $e^{\theta x} < e^x$ and all we can assert is that

$$|R_n(x)| < \frac{x^n}{n!} e^x,$$

which is of no help since we are trying to approximate to e^x.

The error is
$$\frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \ldots$$
$$= \frac{x^n}{n!}\left(1 + \frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \ldots\right),$$

and when $x > 0$ this is less than

$$\frac{x^n}{n!}\left(1 + \frac{x}{n+1} + \frac{x^2}{(n+1)^2} + \ldots\right) = \frac{x^n}{n!} \frac{1}{1 - x/(n+1)}$$

if $x < n+1$, on summing the G.P. Thus if $x > 0$ and $n > x-1$, the error in the approximation is less than

$$\frac{x^n}{n!} \frac{n+1}{n+1-x}.$$

The argument just used is essentially that in Proof (*a*) of d'Alembert's ratio test (12.42).

If $|u_{r+1}/u_r| < k < 1$ *for all* $r > m$, *the error in taking* $s \doteqdot s_n$ $(n \geqslant m)$ *is numerically less than* $|u_{n+1}|/(1-k)$.

For when $n \geqslant m$, $$|u_{n+2}| < k\,|u_{n+1}|,$$
$$|u_{n+3}| < k\,|u_{n+2}| < k^2\,|u_{n+1}|, \quad \text{etc.}$$

The error after n terms is
$$u_{n+1}+u_{n+2}+u_{n+3}+\ldots,$$
which is numerically less than or equal to (cf. 12.52, ex. (iv))
$$\begin{aligned}|u_{n+1}|+|u_{n+2}|+|u_{n+3}|+\ldots\\ &< |u_{n+1}|\,(1+k+k^2+\ldots)\\ &= |u_{n+1}|\times\frac{1}{1-k}\end{aligned}$$
on summing the G.P., since $0 < k < 1$.

Another useful result is based on Leibniz's rule (12.51).

If for all $n > m$ the terms of a series alternate in sign and steadily decrease to zero in numerical value, the error in taking $s \doteqdot s_n$ $(n > m)$ cannot exceed the modulus of the first term neglected.

Let the series be
$$u_1-u_2+u_3-\ldots+(-1)^{m-1}u_m+(-1)^m u_{m+1}+\ldots,$$
where for all $n > m$, $u_n > 0$ and $u_{n+1} < u_n$. The error after n terms $(n > m)$ is
$$(-1)^n\{u_{n+1}-u_{n+2}+u_{n+3}-\ldots\}.$$

The argument in the proof of Leibniz's rule shows that, for n fixed,
$$u_{n+1}-u_{n+2}+u_{n+3}-\ldots$$
converges to a limit E_n, where $0 < E_n < u_{n+1}$. Hence the numerical value of the error does not exceed u_{n+1}.

Example

(iii) Consider ex. (iii) in 12.71: we wish to show that all terms after the fourth in the expansion of $(1+0{\cdot}02)^{\frac{1}{2}}$ cannot influence the 6th place of decimals. The general term is
$$u_{r+1}=\frac{\frac{1}{2}\,.-\frac{1}{2}\,.-\frac{3}{2}\ldots(\frac{1}{2}-r+1)}{r!}(0{\cdot}02)^r \quad (r \geqslant 1).$$
$$\therefore \quad \frac{u_{r+2}}{u_{r+1}}=\frac{\frac{1}{2}-r}{r+1}(0{\cdot}02)=-\frac{2r-1}{r+1}(0{\cdot}01) \quad (r \geqslant 1).$$

Hence for $r \geqslant 1$ (i.e. from the second term onwards) the signs alternate; and since
$$\frac{2r-1}{r+1}(0{\cdot}01)=\left(2-\frac{3}{r+1}\right)0{\cdot}01 < 0{\cdot}02,$$

the terms steadily decrease numerically. Hence the error is numerically not greater than

$$\left|\frac{\frac{1}{2}.-\frac{1}{2}.-\frac{3}{2}.-\frac{5}{2}}{4!}(0{\cdot}02)^4\right| = \frac{3.5}{2^4 4!}(0{\cdot}02)^4 = 6{\cdot}25\times 10^{-9}.$$

12.82 Formal approximations

The following examples illustrate methods of obtaining approximations, but no error estimate is made.

(i) *If x is small, find the best approximation for* $\log(1+x)$ *which has the form*

$$(a)\ \frac{ax}{1+bx},\quad (b)\ \frac{x(1+ax)}{1+bx+cx^2}.$$

(*a*) By expanding both sides of

$$\log(1+x) = \frac{ax}{1+bx}$$

when x is small, we have

$$x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\ldots = ax(1-bx+b^2x^2-\ldots)$$
$$= ax-abx^2+ab^2x^3-\ldots.$$

This will hold approximately for all small x if $a = 1$ and $ab = \frac{1}{2}$, i.e. $b = \frac{1}{2}$. The expansions differ in their x^3-terms since $ab^2 \neq \frac{1}{3}$. Hence

$$\log(1+x) \doteqdot \frac{2x}{2+x}\quad \textit{correct to order } x^2.$$

(*b*) A similar method could be used, but the following is more convenient. From

$$x(1+ax) = (1+bx+cx^2)\log(1+x),$$
$$x+ax^2 = (1+bx+cx^2)(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots)$$
$$= x+(b-\tfrac{1}{2})x^2+(c-\tfrac{1}{2}b+\tfrac{1}{3})x^3+(-\tfrac{1}{2}c+\tfrac{1}{3}b-\tfrac{1}{4})x^4+\ldots.$$

We can obtain three equations for a, b, c by equating coefficients of x^2, x^3, x^4:

$$a = b-\tfrac{1}{2},\quad 0 = c-\tfrac{1}{2}b+\tfrac{1}{3},\quad 0 = -\tfrac{1}{2}c+\tfrac{1}{3}b-\tfrac{1}{4},$$

from which $a = \frac{1}{2}$, $b = 1$, $c = \frac{1}{6}$. Hence

$$\log(1+x) \doteqdot \frac{x(1+\frac{1}{2}x)}{1+x+\frac{1}{6}x^2};$$

and since it can be verified that, for these values, the coefficient of x^5 on the right is non-zero, this approximation is *correct to order x^4*.

(ii) *Prove* $$\left(1+\frac{1}{n}\right)^n = e\left(1-\frac{1}{2n}+\frac{11}{24n^2}+O\left(\frac{1}{n^3}\right)\right).$$

If $y = (1+1/n)^n$, then

$$\log y = n\log\left(1+\frac{1}{n}\right)$$
$$= n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+O\left(\frac{1}{n^4}\right)\right)$$
$$= 1-\frac{1}{2n}+\frac{1}{3n^2}+O\left(\frac{1}{n^3}\right).$$

Writing $\exp(x) = e^x$ for ease of printing, we have

$$y = \exp\left(1-\frac{1}{2n}+\frac{1}{3n^2}+O\left(\frac{1}{n^3}\right)\right)$$
$$= e\left[1+\left\{-\frac{1}{2n}+\frac{1}{3n^2}+O\left(\frac{1}{n^3}\right)\right\}+\frac{1}{2!}\left\{-\frac{1}{2n}+\frac{1}{3n^2}+O\left(\frac{1}{n^3}\right)\right\}^2+O\left(\frac{1}{n^3}\right)\right]$$
$$= e\left[1-\frac{1}{2n}+\frac{11}{24n^2}+O\left(\frac{1}{n^3}\right)\right].$$

(iii) *If p is small, prove that a root of the equation $e^x+x = 1+p$ is approximately $\frac{1}{2}p-\frac{1}{16}p^2$.*

For a first approximation we expand the left-hand side as far as terms in x:

$$1+2x \doteqdot 1+p, \qquad \therefore \quad x = \tfrac{1}{2}p.$$

For the second approximation, expand as far as x^2:

$$1+2x+\tfrac{1}{2}x^2 \doteqdot 1+p, \quad \text{i.e.} \quad 2x(1+\tfrac{1}{4}x) \doteqdot p.$$

Using the first approximation† $x \doteqdot \frac{1}{2}p$, this becomes

$$2x(1+\tfrac{1}{8}p) \doteqdot p,$$

so
$$x \doteqdot \tfrac{1}{2}p(1+\tfrac{1}{8}p)^{-1} \doteqdot \tfrac{1}{2}p-\tfrac{1}{16}p^2.$$

12.83 Calculation of certain limits

Example

Find
$$\lim_{x\to 0}\frac{e^x-1-x}{x-\log(1+x)}.$$

We have
$$e^x = 1+x+\frac{x^2}{2!}+ax^3,$$

where
$$a = \frac{1}{3!}+\frac{x}{4!}+\frac{x^2}{5!}+\ldots,$$

$$\therefore \quad |a| \leqslant \frac{1}{3!}+\frac{|x|}{4!}+\frac{|x|^2}{5!}+\ldots$$
$$< \frac{1}{3!}\left(1+\frac{|x|}{4}+\frac{|x|^2}{4^2}+\ldots\right)$$
$$= \frac{1}{3!}\frac{1}{1-\frac{1}{4}|x|} \quad \text{if} \quad |x| < 4.$$

Similarly
$$\log(1+x) = x-\frac{x^2}{2}+bx^3,$$

where
$$|b| \leqslant \frac{1}{3}+\frac{|x|}{4}+\frac{|x|^2}{5}+\ldots$$
$$< \tfrac{1}{3}(1+|x|+|x|^2+\ldots)$$
$$= \frac{1}{3}\frac{1}{1-|x|} \quad \text{if} \quad |x| < 1.$$

† To keep the working *linear* in x.

Hence
$$\frac{e^x-1-x}{x-\log(1+x)}=\left\{\frac{x^2}{2!}+ax^3\right\}\Big/\left\{\frac{x^2}{2}-bx^3\right\}$$
$$=\frac{\frac{1}{2}+ax}{\frac{1}{2}-bx}\to 1 \quad\text{when}\quad x\to 0.$$

The above argument is often given incompletely as follows:

$$\lim_{x\to 0}\frac{e^x-1-x}{x-\log(1+x)}=\lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\dots\right)-1-x}{x-\left(x-\frac{x^2}{2}+\dots\right)}$$
$$=\lim_{x\to 0}\frac{\frac{1}{2!}+\dots}{\frac{1}{2}+\dots}=1.$$

The result is easily obtained by two applications of l'Hospital's rules (6.9).

Exercise 12(*m*)

Prove the statements in nos. 1–4.

1 The error in taking $\log\{(n+1)/n\} \doteqdot 2/(2n+1)$ is less than
$$\frac{1}{6n(n+1)(2n+1)}.$$

2 The remainder after n terms of
$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$$
lies between
$$\left(1+\frac{1}{n+1}\right)\frac{1}{n!}\quad\text{and}\quad\left(1+\frac{1}{n}\right)\frac{1}{n!}.$$

3 If $|x|<1$, the error in taking
$$\tfrac{1}{2}\log\frac{1+x}{1-x}\doteqdot x+\frac{x^3}{3}+\dots+\frac{x^{2n-1}}{2n-1}$$
is less than the numerical value of $x^{2n+1}/\{(2n+1)(1-x^2)\}$. Putting $n=2$, prove that $\log 11 \doteqdot 2\log 3+0{\cdot}20067$ *correct to 5 places of decimals.*

*4 The error in taking
$$\tfrac{1}{4}\pi \doteqdot 1-\tfrac{1}{3}+\tfrac{1}{5}-\dots+(-1)^n\frac{1}{2n-1}$$
is numerically less than $1/(2n+1)$. Deduce that the first 50 terms of the series give $\frac{1}{4}\pi$ correct to only two places of decimals.

5 Use the expansion of $(1-\frac{1}{50})^{-\frac{1}{2}}$ to find $\sqrt{2}$ *correct to 7 places of decimals.*

6 By considering $(1+\frac{3}{125})^{\frac{1}{3}}$, show $\sqrt[3]{2}\doteqdot 1{\cdot}2599211$ *correct to 7 places of decimals.*

Verify the following formal approximations for small x.

7 $\sqrt[n]{(1+x)}-\dfrac{2n+(n+1)x}{2n+(n-1)x}\doteqdot\dfrac{n^2-1}{12n^3}x^3.$

8 $e^x/\sqrt{(1+2x)} \doteqdot 1+x^2-\frac{4}{3}x^3$. Write down the corresponding approximation for $1/\{e^x\sqrt{(1-2x)}\}$.

9 $\log\{\frac{1}{2}(1+e^x)\} \doteqdot \frac{1}{2}x+\frac{1}{8}x^2$ *correct to order* x^3.

10 $(1+x)^{1+x} \doteqdot 1+x+x^2+\frac{1}{2}x^3$. [Take logarithms.]

11 $\dfrac{x}{e^x-1} \doteqdot 1-\dfrac{x}{2}+\dfrac{x^2}{12}-\dfrac{x^4}{720}$. [Method of 12.82, ex. (i) (*b*).]

12 Find an approximation of the form $a+bx$ for $(8+3x)^{\frac{2}{3}}/\{1+(1-x)^6(4+x)^{\frac{3}{2}}\}$, x being small.

13 Find an approximation of the form $a+bx+cx^2$ for $1/\{(1-x)\sqrt{(1+x)}\}$, x being small.

14 Find approximately the small positive value of x which satisfies

$$(1+x)^5 = 1{\cdot}03(1-x)^3.$$

15 Neglecting x^4, prove that $e^x\log(1+x) \doteqdot -\log(1-x)$. Hence find an approximation to the root of $e^x\log(1+x) = \frac{1}{2}$ which lies between 0 and 1.

16 If p is small, prove that $x = p-p^2$ is an approximate solution of $x e^x = p$.

17 If p is small, prove that a root of $x\log x+x = 1+p$ is approximately $1+\frac{1}{2}p$, and find a closer approximation. [Put $x = 1+h$, where h is small.]

Calculate the following limits.

18 $\lim\limits_{x\to 0}\dfrac{x-\log(1+x)}{x^2}$.

19 $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}$ $(a>0)$.

Miscellaneous Exercise 12(n)

1 Prove $\sum\limits_{r=0}^{n}\dfrac{1}{r!\,(n-r)!} = \dfrac{2^n}{n!}$.

2 Prove $(c_0+c_1+\ldots+c_n)^2 = 1+{}^{2n}C_1+{}^{2n}C_2+\ldots+{}^{2n}C_{2n}$.

3 Calculate $c_0^2+c_1^2+\ldots+c_n^2$, and prove

$$c_1^2+2c_2^2+\ldots+nc_n^2 = \tfrac{1}{2}n(c_0^2+c_1^2+\ldots+c_n^2).$$

4 (i) Verify the identity

$$(1+x)^n \equiv 1+x\{(1+x)^{n-1}+(1+x)^{n-2}+\ldots+(1+x)+1\},$$

and use it to prove ${}^nC_{r+1} = {}^{n-1}C_r+{}^{n-2}C_r+{}^{n-3}C_r+\ldots+{}^rC_r$ for $0<r<n$.

*(ii) Conversely, assuming this relation, prove the binomial theorem.

5 For the series $(1)+(3+5)+(7+9+11)+(13+15+17+19)+\ldots$, find
(i) the number of terms in the first $r-1$ brackets;
(ii) the first and last terms of the rth bracket;
(iii) the sum of the terms in the rth bracket.
(iv) Write down the sum to m terms of $1+3+5+\ldots$, and show that the sum of the terms in the first n brackets is t_n^2, where $t_n = \sum\limits_{r=1}^{n} r$.

Deduce from (iii) and (iv) that $\sum\limits_{r=1}^{n} r^3 = \left(\sum\limits_{r=1}^{n} r\right)^2$.

Find the sum to n terms and to infinity of

6 $1+\dfrac{1}{1+2}+\dfrac{1}{1+2+3}+\dots$ 7 $\dfrac{1}{2.3.4}+\dfrac{3}{3.4.5}+\dfrac{5}{4.5.6}+\dots$

8 If $r > 0$, prove $\tan^{-1}(1/2r^2) = \tan^{-1}(2r+1)-\tan^{-1}(2r-1)$. Hence find

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{2r^2}\right).$$

9 If n straight lines are drawn in a plane so that no two of them are parallel and no three concurrent, the number of regions into which the plane is divided is denoted by $f(n)$. Prove $f(n+1)-f(n) = n+1$, and deduce the expression for $f(n)$.

10 Prove that $(1+2\cos x+2\cos 2x+\dots+2\cos nx)\sin\frac{1}{2}x = \sin(n+\frac{1}{2})x$.

Deduce that† $$\int_0^{\pi} \frac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}\,dx = \pi$$

when n is a positive integer or zero. What is the value when n is a negative integer?

11 (i) Prove

$$\frac{\sin nx}{\sin x} = 2\cos(n-1)x+2\cos(n-3)x+\dots+\begin{cases}1\\ 2\cos x\end{cases}$$

according as n is odd or even. Deduce that

$$\int_0^{\pi}\frac{\sin nx}{\sin x}\,dx = \pi \quad\text{or}\quad 0$$

according as n is an odd or even positive integer.

(ii) Prove $$\int_0^{\pi}\left(\frac{\sin nx}{\sin x}\right)^2 dx = n\pi$$

for all positive integers n. [Square the relation in (i), and use Ex. 4 (l), no. 30 (i).]

12 Prove that $$\frac{\sin\theta}{\sin(\theta/2^n)} = 2^n\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\dots\cos\frac{\theta}{2^n}.$$

Deduce the values of

$$\text{(i)}\ \sum_{r=1}^{n}\log\cos\frac{\theta}{2^r};\quad \text{(ii)}\ \sum_{r=1}^{n}\frac{1}{2^r}\tan\frac{\theta}{2^r};\quad \text{(iii)}\ \sum_{r=1}^{\infty}\frac{1}{2^r}\tan\frac{\theta}{2^r}.$$

Use Mathematical Induction to prove the following (*nos.* 13–16).

13 $1^4-2^4+3^4-\dots$ to n terms $= (-1)^{n-1}\frac{1}{2}n(n^3+2n^2-1)$.

14 $\dfrac{d^n}{dx^n}(x^{n-1}e^{1/x}) = (-1)^n e^{1/x}/x^{n+1}$.

15 If $$u_n = \frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n},$$

then $$u_n = 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots+\frac{1}{2n-1}-\frac{1}{2n}.$$

Deduce that $\lim\limits_{n\to\infty} u_n = \log 2$.

† In nos. 10, 11 the expression $(\sin mx)/(\sin x)$ is defined to be m when $x = 0$; i.e. the definition is 'completed by continuity' (see 2.64).

16 If $u_{n+2}-(\alpha+\beta)u_{n+1}+\alpha\beta u_n = 0$ for $n \geqslant 1$, then

$$(\alpha-\beta)u_n = (\alpha^{n-1}-\beta^{n-1})u_2-\alpha\beta(\alpha^{n-2}-\beta^{n-2})u_1.$$

17 If s_n denotes the sum of n terms of

$$\frac{x}{1+x}+\frac{2x^2}{1+x^2}+\frac{4x^4}{1+x^4}+\frac{8x^8}{1+x^8}+\frac{16x^{16}}{1+x^{16}}+\ldots,$$

prove that $s_n = x/(1-x)-mx^m/(1-x^m)$ where $m = 2^n$ and $x \neq 1$. Deduce that the series converges when $|x| < 1$. [Either use Induction, or take logarithms and derive the identity obtained from

$$\frac{1}{1-x} = \frac{1+x}{1-x^2} = \frac{(1+x)(1+x^2)}{1-x^4} = \ldots.]$$

18 Assuming that $$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{6},$$

prove that

$$\text{(i)}\ 1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots = \frac{\pi^2}{8};\quad \text{(ii)}\ 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots = \frac{\pi^2}{12}.$$

19 (i) Find the coefficient of x^4 and of x^7 in the expansion of

$$(1+x+x^2+x^3+x^4)^3.$$

(ii) How many solutions in non-negative integers has the equation

$$(a)\ a+b+c = 4;\quad (b)\ a+b+c = 7?$$

20 In how many ways can a total of 10 be thrown with three dice of different colours, the faces being numbered 1, 2, 3, 4, 5, 6? [We require the number of positive integral solutions not greater than 6 of the equation $a+b+c = 10$. This number is the coefficient of x^{10} in the expansion of

$$(x+x^2+x^3+x^4+x^5+x^6)^3.]$$

21 If $x^2+px+q = 0$ has roots α, β, prove that (for a range of x to be stated),

$$-\log(1+px+qx^2) = (\alpha+\beta)x+\tfrac{1}{2}(\alpha^2+\beta^2)x^2+\tfrac{1}{3}(\alpha^3+\beta^3)x^3+\ldots.$$

22 A circular arc subtends an angle θ at the centre of the circle, and a, b, c are the lengths of the chords of the arc, of $\frac{2}{3}$ of the arc, and of $\frac{1}{3}$ of the arc, respectively. Find numbers x, y, z independent of θ, a, b, c such that the length of the arc is approximately $ax+by+cz$, *correct to order* θ^6. [Use the series for $\sin x$.]

Find the sum to infinity of

23 $1+\frac{3}{8}+\frac{3.9}{8.16}+\frac{3.9.15}{8.16.24}+\ldots.$

24 $\Sigma\frac{r+2}{r(r+1)3^r}.$

25 $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\frac{8}{9!}+\ldots.$

26 $1-\frac{2}{1!}+\frac{3}{2!}-\frac{4}{3!}+\ldots.$

27 $\Sigma\frac{r+3}{r+1}\frac{1}{(r-1)!}.$

28 $\frac{9}{16}+\frac{9.15}{16.24}+\frac{9.15.21}{16.24.32}+\ldots.$

29 $\frac{1}{1.2.3}+\frac{1}{2.3.5}+\frac{1}{3.4.7}+\frac{1}{4.5.9}+\ldots.$

30 $\Sigma\frac{r-2}{r(r+1)(r+3)}.$

31 $1+\frac{2^2x}{1!}+\frac{3^2x^2}{2!}+\frac{4^2x^3}{3!}+\ldots.$

32 $\Sigma \dfrac{x^r}{r(r+1)(r+2)} \quad (-1 \leqslant x \leqslant 1)$.

33 (i) $1+\dfrac{x^4}{4!}+\dfrac{x^8}{8!}+\dfrac{x^{12}}{12!}+\ldots$; (ii) $\dfrac{x^3}{3!}+\dfrac{x^7}{7!}+\dfrac{x^{11}}{11!}+\ldots$.

34 If $p > 1$, prove

$$\frac{2}{p}+\frac{2}{3p^3} < \log\frac{p+1}{p-1} < \frac{2}{p}+\frac{2}{3p^3}+\frac{2}{5p^3(p^2-1)}.$$

Taking $p = 26, 31, 49$, calculate $\log 5$ to 3 *places of decimals*. [Choose a, b, c so that $a\log\frac{27}{25}+b\log\frac{32}{30}+c\log\frac{50}{48} = \log 5$.]

35 Denoting by ${}^nP_r$ the number of arrangements of n unlike things taken r at a time, prove $1+\sum_{r=1}^{n} {}^nP_r =$ the largest integer less than $e\,(n!)$. [Use Ex. 12 (m), no. 2.]

13

COMPLEX ALGEBRA AND GENERAL THEORY OF EQUATIONS

13.1 Complex numbers

13.11 Extension of the real number system; $\sqrt{(-1)}$

Consider the six equations

$$2x = 6, \quad \text{(i)} \qquad x^2 = 9, \quad \text{(iv)}$$
$$2x = 5, \quad \text{(ii)} \qquad x^2 = 2, \quad \text{(v)}$$
$$2x+3 = 0, \quad \text{(iii)} \qquad x^2+1 = 0. \quad \text{(vi)}$$

If we knew nothing about fractions or negative numbers, we should only be able to assert that (i) and (iv) are both satisfied by $x = 3$, and that the others have *no solution* (i.e. *that we could find no natural numbers which satisfy these*).

If we are subsequently acquainted with fractions (i.e. if our list of numbers is extended by admitting *rational numbers*), we can say that (i) and (iv) are satisfied by $\frac{3}{1}$, while (ii) is satisfied by $\frac{5}{2}$. The others would remain insoluble.

Not until *signed numbers* are introduced can we solve (iii) by $x = -\frac{3}{2}$; and then (iv) is found to have *two* solutions $\pm\frac{3}{1}$. We still have no solutions for (v), (vi).

To enable us to say that (v) has a solution (the necessity for doing this was indicated in 1.11), we introduce the number $\sqrt{2}$ which can be proved (see 1.11) to be *non-rational*. Even then, (vi) *has no solution*.

It will now be clear that what can be said about solutions of equations (i)–(vi) depends entirely on what we are prepared to call a 'number'. In other words, *the possibility of solving a given equation depends on how far the concept of 'number' has been extended.* Fractional and negative numbers enable all *linear* equations with integral coefficients to be solved, but not all quadratics.

If we attempt to solve (vi), written in the form $x^2 = -1$, in the same manner as for (v), we should obtain the formal solution $x = \sqrt{(-1)}$; that is, $\sqrt{(-1)}$ is what we should write for the solution of (vi) *if such a solution existed*. Whether or not (vi) can have a solution depends on the meaning of 'number', so that a question like 'Does $\sqrt{(-1)}$ exist?' is futile until we specify what kind of 'numbers' we are going to work with.

Hence we have two alternatives: either (*a*) we leave $\sqrt{(-1)}$ as a meaningless symbol; or (*b*) we generalise still further the concept of 'number' to include such expressions as this. It is clear that, unless we choose (*b*), we can never have any complete theory even of quadratic equations. To arrive at our generalisation we follow the historical sequence of development; we use the symbol i as an abbreviation (introduced by Euler) for $\sqrt{(-1)}$ in 13.12, 13.13.

13.12 First stage: formal development

At first i was regarded as something mathematically disreputable† whose use, nevertheless, gave results quickly and simply to problems which could be solved more laboriously by other (respectable) means. The procedure was formal: i was treated like an ordinary algebraic symbol (with the peculiarity of satisfying $i^2 = -1$), and was used in the spirit of 'press on and see where we get'. Cardan (1501–76) was probably the first to do this.

Accordingly we should have (a, b, c, d being 'ordinary numbers'):

$$(a+bi)+(c+di) = (a+c)+(b+d)\,i \tag{i}$$

by regrouping and taking out the factor i;

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2\\ &= (ac-bd)+(bc+ad)\,i\end{aligned} \tag{ii}$$

by first multiplying out the brackets, and then using $i^2 = -1$ and grouping.

Also, if $a+bi = c+di$, then $a-c = (d-b)\,i$. Squaring both sides gives $(a-c)^2 = -(d-b)^2$, which would assert that a positive number is equal to a negative number and is a contradiction unless both are zero, i.e. $a-c = 0$ and $d-b = 0$. Hence

$$\textit{if}\quad a+bi = c+di, \quad \textit{then}\quad a = c \quad \textit{and}\quad b = d. \tag{iii}$$

These examples show how expressions involving i would behave *if the rules of ordinary algebra are assumed to apply to them.*

Owing to the mystery associated with it, i was named an *imaginary number*, and expressions involving it *imaginary expressions*; while

† Earlier, negative numbers carried a similar stigma.

ordinary numbers and expressions were† called *real* when contrast was desirable.

This intuitive approach, however unsystematic it may appear, did in fact lead to many advances because it produced results which could be verified subsequently by orthodox means.

13.13 Second stage: geometrical representation

In the spirit of 13.12 we see that if any real number a is multiplied by i, we obtain an imaginary number ia; multiplying again by i, we get $iia = i^2a = -1a = -a$, which is again a real number. Thus *two successive multiplications by i convert a real number into its negative.*

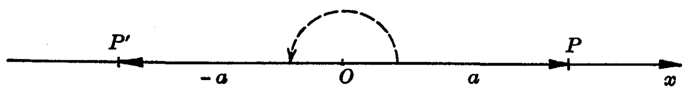

Fig. 125

Consider now the geometrical representation of this result. Suppose for definiteness that a is positive, and let OP be the corresponding segment of the x-axis. Then *two* multiplications by i have the effect of rotating OP, originally along the positive x-axis, through *two* right-angles (say in the *counterclockwise* sense, to agree with trigonometrical conventions) to position OP'. It is therefore reasonable to interpret *one* multiplication by i as turning OP counterclockwise through *one* right-angle. Hence the imaginary number ia is represented by a point P'' at distance a, *measured along Oy* (fig. 126). Similar considerations apply when a is negative.

This geometrical representation was published in 1806 by Argand, although others had the idea a little earlier. In it we are regarding i not as a number at all, but as an *operator* which rotates lines OP counterclockwise about the origin O through a right-angle. (Observe that ordinary numbers can also be interpreted as operators in the process of multiplication: the 2 in $2a$ as the operator which doubles the length of OP and preserves the sense; the -2 in $-2a$ as the operator which doubles the length of OP and reverses the sense; and so on.)

We can now represent real numbers by points on the (positive or negative) x-axis and imaginary numbers of the type ia by points on the (positive or negative) y-axis. Can expressions of the form $a+bi$ be represented? If we interpret this as corresponding to a step of

† And still are (see 1.11).

length a along Ox, followed by a turn counterclockwise through one right-angle and a step of length b parallel to Oy, we end up at what we should refer to in graphical work as *the point* (a, b), say Q (fig. 127).

Although the interpretation of i as an operator and Argand's representation of $a+bi$ both have important applications, the significant thing at this stage is that we are led away from the vague 'imaginary expression' $a+bi$ towards the familiar and precise idea

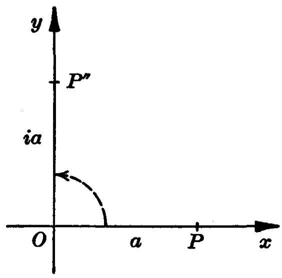

Fig. 126

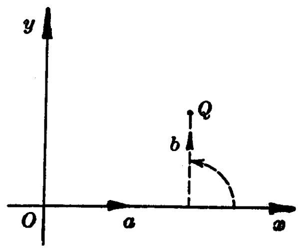

Fig. 127

of the coordinates of a point (a, b) in a plane, i.e. *a pair of numbers* a, b *taken in a definite order*. It is from this notion that we can reconstruct the theory of 'imaginary' numbers on a logical basis.

13.14 Third stage: logical development

(1) Our aim now is to make a fresh start by setting up and developing an algebra of *ordered number-pairs*†. That is, we begin with the definite idea of a pair of numbers, and as far as possible try to retrace our steps to get at i indirectly. This approach is contained in the work of Gauss (1777–1855) and others.

We write our number-pairs, usually called *complex numbers*, as $[a, b]$ where a, b are ordinary (real) numbers. As we are about to invent a new kind of algebra, we can make the rules as we please provided they do not contradict each other or lead to contradiction. However, our purpose is to make the new algebra as much like ordinary algebra as possible; and so, with a view to preserving the *structure* of the formal results (i)–(iii) of 13.12, we define the fundamental operations with number-pairs as follows:

(*a*) *Equal number-pairs.*

$$[a, b] = [c, d] \quad \text{if and only if} \quad a = b \quad \text{and} \quad c = d.$$

† The reader is in fact already skilled in dealing with ordered number-pairs of another kind, viz. ordinary fractions: see Ex. 13(*a*), no. 26.

(*b*) *Addition of number-pairs.*

$$[a, b]+[c, d] \quad \text{is to mean} \quad [a+c, b+d].$$

(*c*) *Multiplication of number-pairs.*

$$[a, b]\times[c, d] \quad \text{is to mean} \quad [ac-bd, bc+ad].$$

Since subtraction is the opposite of addition, we can define $[a, b]-[c, d]$ to mean the number-pair $[x, y]$ such that

$$[x, y]+[c, d] = [a, b].$$

By (*b*) this gives $$[x+c, y+d] = [a, b],$$
which by (*a*) implies
$$x+c = a \quad \text{and} \quad y+d = b;$$
hence $$x = a-c, \quad y = b-d,$$
so that $$[a, b]-[c, d] = [a-c, b-d].$$

Similarly, the quotient $[a, b]\div[c, d]$ can be defined in terms of multiplication to be the number-pair $[x, y]$ such that

$$[x, y]\times[c, d] = [a, b].$$

By (*c*) this gives $$[xc-yd, yc+xd] = [a, b],$$
and so by (*a*) $$xc-yd = a \quad \text{and} \quad yc+xd = b.$$

If c, d are not both zero, we can solve these simultaneous equations for x, y:

$$x = \frac{ac+bd}{c^2+d^2}, \quad y = \frac{bc-ad}{c^2+d^2}.$$

Hence if $[c, d] \neq [0, 0]$,

$$[a, b]\div[c, d] = \left[\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right].$$

(2) By using the above definitions we can show that complex numbers obey all the laws of ordinary algebra. For example,

if $[a, b]\times[c, d] = [0, 0]$, *then* $[a, b]$ *or* $[c, d]$ *or both will be* $[0, 0]$.

For the hypothesis and the definition of 'multiplication' show that

$$ac-bd = 0 \quad \text{and} \quad bc+ad = 0,$$

and so $$0 = (ac-bd)^2+(bc+ad)^2 = a^2c^2+b^2d^2+b^2c^2+a^2d^2$$
$$= (a^2+b^2)(c^2+d^2).$$

Hence at least one of the real numbers a^2+b^2, c^2+d^2 must be 0; and this implies $a = 0 = b$, or $c = 0 = d$, or both. The result follows.

See also Ex. 13(*a*), nos. 19–25.

(3) Application of definitions (*b*), (*c*) to number-pairs of the form $[a, 0]$, $[c, 0]$ gives

$$[a, 0]+[c, 0] = [a+c, 0],$$

$$[a, 0]\times[c, 0] = [ac, 0].$$

Apart from the presence of the brackets and the second part 0 in each pair, these results are exactly like the laws of addition and multiplication of the ordinary numbers a, c: *the laws of combination of complex numbers* $[a, 0]$ *and real numbers* a *are structurally the same.*†

Owing to this fact, we agree to abbreviate $[a, 0]$ to a, and in particular we write 1 for $[1, 0]$. (There is a precedent for this: we agree to write 2 for $+2$, etc., because the (unsigned) natural numbers and the positive signed numbers behave exactly the same algebraically.)

In definition (*c*) let us now choose $a = 0$, $c = 0$, $b = 1$, $d = 1$. Then

$$[0, 1]\times[0, 1] = [-1, 0],$$

i.e.

$$[0, 1]^2 = [-1, 0];$$

and according to our agreement about abbreviation of the right-hand side, this can be written

$$[0, 1]^2 = -1.$$

Thus in the algebra of number-pairs there is an element whose square is equal to -1, viz. $[0, 1]$. Let us abbreviate $[0, 1]$ to i; then we have *number-pairs* i and -1 for which $i^2 = -1$.

By definition (*b*),

$$\begin{aligned}[a, b] &= [a, 0]+[0, b]\\ &= [a, 0]+[b, 0][0, 1] \quad \text{by definition } (c),\\ &= a+bi \quad \text{by our abbreviations.}\end{aligned}$$

If we now rewrite definitions (*a*), (*b*), (*c*), replacing $[a, b]$ by $a+bi$, etc., we recover the results (iii), (i), (ii) respectively in 13.12; but now these results are consequences of our definitions and agreement about abbreviations, and *all* numbers in them are actually number-pairs although they appear to be ordinary numbers.

† We say that there is an *isomorphism* between the complex numbers of the form $[a, 0]$ and the real numbers a.

Since $[b, 0][0, 1] = [0, 1][b, 0]$, as is easily verified, we can write $[a, b]$ as $a + ib$ instead of $a + bi$ when convenient. Our conventions amount to writing a for $a + 0i$, i for $0 + 1i$, and bi for $0 + bi$.

(4) *Conclusions.* In the algebra of real numbers ('ordinary algebra'), $\sqrt{(-1)}$ does not exist. In complex algebra (the algebra of ordered number-pairs) there are pairs abbreviated to i, -1 which do satisfy $i^2 = -1$, so that this pair i has the property which $\sqrt{(-1)}$ would have (if it existed) in real algebra.

Correct results in complex algebra are obtained by applying the laws of ordinary algebra to the number-pairs in their abbreviated form $a + bi$, with the replacement of i^2 by -1 whenever it occurs. That is, the intuitive work of 13.12 is correct, provided we realise that we are handling *abbreviations for pairs.*

Remark. The relations 'less than', 'greater than' are not defined in complex algebra; e.g. it is meaningless to write $3 + 2i < 4 + 5i$. In particular, the terms 'positive', 'negative' cannot be applied to complex numbers.

13.15 Importance of complex algebra

Complex algebra provides a method for proving results of real algebra. This is so for two reasons.

(i) A single equation in complex algebra is equivalent to two equations in real algebra, by definition (*a*) in 13.14. Hence *pairs* of results can be obtained from one calculation, thus making for economy.

(ii) From any general relation between complex numbers there can be obtained a special case by taking the second parts of all numbers to be zero. Because of the exact correspondence between complex numbers $[x, 0]$ and real numbers x (13.14(3)), we deduce a relation in real algebra.

These remarks will be illustrated in this chapter and the next.

Complex algebra offers generality and completeness. For example, we can assert that in complex algebra *every* quadratic equation has two roots (possibly equal).

Since complex numbers arose from the attempt to solve all quadratic equations (and in particular the equation $x^2 + 1 = 0$), we may wonder whether consideration of equations having degree greater than 2 would lead to some higher form of complex number, say 'super-complex numbers'. The answer is NO, because in 13.42 it will be proved that in complex algebra every polynomial equation of degree n

has n roots (i.e. the roots are themselves complex numbers). Thus complex algebra has a finality which the real number system lacks, and no further generalisations can be obtained in this direction.

13.16 Further possible generalisations of 'number'

(1) Ordinary algebra is 1-dimensional, i.e. any real number can be represented by a point on a line, say the axis of x.

Complex algebra is 2-dimensional, since a complex number is represented by a point in a plane, with reference to two axes.

Is there a 3-dimensional algebra—an algebra of *ordered triplets* $[a,b,c]$? Hamilton (1805–65) considered this problem and showed that no genuine 'algebra' could be constructed, although a system of elements (which may be called *vectors*) of the form $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ where a, b, c are real numbers and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ would obey some of the usual laws, and is useful in 3-dimensional mechanics, for example.

However, Hamilton did construct an algebra of ordered quadruplets $[a,b,c,d]$ or *quaternions*, in which all the usual laws are obeyed except that $xy \neq yx$. We could similarly think of an 'algebra' of n-vectors $[a_1, a_2, \ldots, a_n]$ or even of *infinite* sequences $[a_1, a_2, a_3, \ldots]$.

(2) Further, we may consider arrays with more than one row, e.g.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}, \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad \text{etc.,}$$

where the elements are real numbers. With suitable laws of combination, the algebra of rectangular arrays or *matrices* would be obtained. Although such an algebra was originally investigated for its own sake, it is now widely used, e.g. in theoretical physics and much geometry. Nor is there need to restrict the elements to be real.

The reader must seek elsewhere for development of these suggestions.

Exercise 13(*a*)

Simplify and express in the form $X + Yi$:

1 $(2+7i)+(3+4i)$. 2 $(2+7i)-(3+4i)$. 3 $(2+6i)+(5-3i)$.

4 $(2+6i)-(-5+3i)$. 5 $(3+5i)(2+3i)$. 6 $(4+5i)(4-5i)$.

7 $(3+5i)\div(2+3i)$. 8 $(6+5i)/(3+2i)$.

9 $(1+i)^2$, and hence $(1+i)^4$, $(1+i)^6$.

10 $(1+i)(1+2i)(1+3i)$. 11 $(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)$.

12 $(\cos\theta+i\sin\theta)/(\cos\phi+i\sin\phi)$.

13 *Calculate* $(a+bi)/(c+di)$ *by the process analogous to 'rationalising the denominator' with surds:*

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2},$$

which agrees with 13.14(1).

Solve nos. 7, 8, 12 by this method.

14 If $Z = X + Yi$ and $Z = (z+1)/(z-1)$ where $z = x+yi$, express X, Y in terms of x and y.

15 Complex numbers z_1, z_2 are given by

$$z_1 = R_1 + i\omega L, \quad z_2 = R_2 - \frac{i}{\omega C},$$

and z is defined by
$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}.$$

Find the values of ω for which z is of the form $x + 0i$.

16 If $\sqrt{(a+bi)} = x+yi$ and $x > 0$, prove $x^2 - y^2 = a^2$ and $2xy = b$. Hence express $\sqrt{(3+4i)}$ in the form $x+yi$.

17 Express $(x+yi)^3$ in the form $X+Yi$. *Write down* the corresponding expression for $(x-yi)^3$.

18 Prove $\{\frac{1}{2}(-1 \pm i\sqrt{3})\}^3 = 1$.

By direct appeal to the definitions (a)–(c) *in* 13.14(1), *verify that complex numbers satisfy the following algebraic laws.*

19 $[a,b]+[c,d] = [c,d]+[a,b]$ (*commutative law* for addition).

20 $[a,b]\times[c,d] = [c,d]\times[a,b]$ (*commutative law* for multiplication).

21 $([a,b]+[c,d])+[e,f] = [a,b]+([c,d]+[e,f])$ (*associative law* for addition).

22 $([a,b]\times[c,d])\times[e,f] = [a,b]+([c,d]\times[e,f])$ (*associative law* for multiplication).

23 $([a,b]+[c,d])\times[e,f] = [a,b]\times[e,f]+[c,d]\times[e,f]$ (*distributive law*).

24 $[a,b]+[a,b]+\dots$ to n terms $= [a,b]\times[n,0]$, each being equal to $[na,nb]$.

25 By using the multiplication rule, verify that $-i = [0,-1]$ satisfies $x^2+1 = 0$.

26 Consider ordered number-pairs $\{a,b\}$ of integers (positive, negative or zero, but with $b \neq 0$) which satisfy the following laws:

(a) $\{a,b\} = \{c,d\}$ if and only if $ad = bc$;

(b) $\{a,b\}+\{c,d\} = \{ad+bc, bd\}$;

(c) $\{a,b\}\times\{c,d\} = \{ac, bd\}$.

Obtain expressions for $\{a,b\}-\{c,d\}$ and $\{a,b\}\div\{c,d\}$. Verify that this algebra is isomorphic to (i.e. the same in structure as) the algebra of rational numbers (fractions). (We could *define* fractions abstractly in terms of integers in this way.)

13.2 The modulus-argument form of a complex number

13.21 Modulus and argument

We saw in 13.13 that the complex number $x+yi$ (or $x+iy$) can be represented uniquely by the point $P(x,y)$ in the xy-plane. If P has polar coordinates (r,θ), then (see 1.62, and fig. 25)

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{i}$$

and so
$$x+yi = r(\cos\theta + i\sin\theta). \tag{ii}$$

When $x+yi$ is so expressed, *and r is restricted to be positive* (cf. 1.62), then r is called the *modulus* and θ the *argument* of $x+yi$, respectively written $|x+yi|$, $\arg(x+yi)$.† We may think of $x+yi$ as 'the number r in direction θ', a generalisation of the idea of 'directed number' as presented in elementary algebra.

It is customary to write z for $x+yi$, and to write $r = |z|$, $\theta = \arg z$; z may be called the *number* of the point P in Argand's diagram, and sometimes P is called the *affix* of z.

Frequently it is necessary to express a complex number, given in the form $x+yi$, in terms of its modulus and argument. From equations (i),

$$r = +\sqrt{(x^2+y^2)} \tag{iii}$$

since r is restricted to be positive; and

$$\cos\theta : \sin\theta : 1 = x : y : +\sqrt{(x^2+y^2)}, \tag{iv}$$

which gives a *unique* value of θ in the range $-\pi < \theta \leqslant +\pi$. Any value differing from this by any positive or negative integral multiple of 2π would give the same point P and consequently is a possible value of $\arg z$. Thus $\arg z$ is a many-valued function, and we define its *principal value* to be that in the range $-\pi < \theta \leqslant \pi$. (The single equation $\tan\theta = y/x$ does not determine $\arg z$ because it gives *two* values for θ in this range.)

Examples

(i) *Express* $(5+i)/(2+3i)$ *in modulus-argument form.*

$$\frac{5+i}{2+3i} = \frac{(5+i)(2-3i)}{(2+3i)(2-3i)} = \frac{13-13i}{4+9} = 1-i.$$

We require r, θ such that $r\cos\theta = 1$, $r\sin\theta = -1$. Hence $r = +\sqrt{2}$, and θ is given by $\cos\theta:\sin\theta:1 = 1:-1:\sqrt{2}$; the principal value of the argument is therefore $-\frac{1}{4}\pi$. The required result is

$$\sqrt{2}\{\cos(-\tfrac{1}{4}\pi)+i\sin(-\tfrac{1}{4}\pi)\}.$$

(ii) *Express* $1-\cos\theta-i\sin\theta$ *in modulus-argument form.*

$$\begin{aligned}1-\cos\theta-i\sin\theta &= 2\sin^2\tfrac{1}{2}\theta-2i\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\\ &= 2\sin\tfrac{1}{2}\theta(\sin\tfrac{1}{2}\theta-i\cos\tfrac{1}{2}\theta)\\ &= 2\sin\tfrac{1}{2}\theta\{\cos(\tfrac{1}{2}\theta-\tfrac{1}{2}\pi)+i\sin(\tfrac{1}{2}\theta-\tfrac{1}{2}\pi)\}.\end{aligned}$$

This will be the required result if $\sin\frac{1}{2}\theta > 0$, i.e. if $2n\pi < \frac{1}{2}\theta < (2n+1)\pi$.

If $(2n-1)\pi < \frac{1}{2}\theta < 2n\pi$, then $\sin\frac{1}{2}\theta < 0$ and we must write

$$\begin{aligned}1-\cos\theta-i\sin\theta &= -2\sin\tfrac{1}{2}\theta\{-\cos(\tfrac{1}{2}\theta-\tfrac{1}{2}\pi)-i\sin(\tfrac{1}{2}\theta-\tfrac{1}{2}\pi)\}\\ &= -2\sin\tfrac{1}{2}\theta\{\cos(\tfrac{1}{2}\theta+\tfrac{1}{2}\pi)+i\sin(\tfrac{1}{2}\theta+\tfrac{1}{2}\pi)\}.\end{aligned}$$

† Sometimes θ is called the *amplitude* of $x+yi$ and is written $\mathrm{am}(x+yi)$ or $\mathrm{amp}(x+yi)$.

In numerical cases like ex. (i) the modulus and argument can usually be written down easily after drawing a sketch.

We remark here that the complex number z, represented by the point $P(x, y)$, can also be associated with the displacement OP in the sense *from O towards P*. Given the axes, there is an exact correspondence between the complex number $x+yi$, the point $P(x, y)$, and the *vector* $\boldsymbol{OP}$. Also see 13.32.

13.22 Further definitions, notation, and properties

(1) We agreed in 13.14 (3) to abbreviate a complex number z of the form $x+0i$ to x. It is customary to go further and call z '*real*', although this double use of 'real' is most unfortunate. Similarly a number of the form $0+yi$ is shortened to iy and called *purely imaginary.*

If z is 'real', then it is represented by a point on the x-axis; and $\arg z = 0$ or π according as z (i.e. x) is positive or negative.

If z is purely imaginary, it corresponds to some point on the y-axis; and $\arg z = \pm\frac{1}{2}\pi$ according as $y \gtrless 0$.

(2) *Real and imaginary parts.* In the complex number $z = x+yi$, x is called the *real part* (or first part) and is sometimes written $x = \mathscr{R}(z)$; and y is called the *imaginary part* (or second part), written $y = \mathscr{I}(z)$. The process of deducing from the relation $z_1 = z_2$ that $x_1 = x_2$ and $y_1 = y_2$ is called *equating real and imaginary parts*, respectively.

If z_1, z_2 are expressed in modulus-argument form as

$$r(\cos\theta+i\sin\theta), \quad s(\cos\phi+i\sin\phi),$$

then from

$$r(\cos\theta+i\sin\theta) = s(\cos\phi+i\sin\phi)$$

it follows by equating real and imaginary parts that

$$r\cos\theta = s\cos\phi \quad \text{and} \quad r\sin\theta = s\sin\phi.$$

Squaring and adding gives $r^2 = s^2$, so that $r = s$ since r, s are positive. The two equations now become

$$\cos\theta = \cos\phi, \quad \sin\theta = \sin\phi,$$

so that *in general* θ, ϕ *differ by some integral multiple of* 2π. If θ, ϕ denote the *principal* values of $\arg z_1$, $\arg z_2$, then $\theta = \phi$.

The process of deducing from $z_1 = z_2$ that $|z_1| = |z_2|$ is called *taking moduli*. That of deducing $\arg z_1 = \arg z_2$ is called *taking arguments.* This equation holds for principal values of 'arg'; but since 'arg' is many-valued, we may write $\arg z_1 \approx \arg z_2$ to emphasise that any value of the left-hand side is also a value of the right, and conversely.

(3) *Conjugate complex numbers.* Given a complex number $z = x+yi$, the number $x-yi$ obtained by changing the sign of i is called the *conjugate* of z and is written $\bar{z}$ (or sometimes z^* when the previous notation is required to denote 'average value'). Conjugate complex numbers have the following properties.

(i) *The product* $z\bar{z} = (x+yi)(x-yi) = x^2+y^2 = |z|^2$ *is 'real'.*

(ii) *The sum* $z+\bar{z} = (x+yi)+(x-yi) = 2x = 2\mathscr{R}(z)$ *is 'real'.*

(iii) *The difference* $z-\bar{z} = (x+yi)-(x-yi) = 2yi = 2i\mathscr{I}(z)$ *is purely imaginary.*

(iv) $|z| = |\bar{z}|$; but (for principal values) $\arg\bar{z} = -\arg z$, except when z is 'real and negative', i.e. of the form $x+0i$ where $x < 0$. In the xy-plane the points which represent $z, \bar{z}$ are images in the x-axis.

(v) If $z = \bar{z}$, then z is 'real'; if $z = -\bar{z}$, then z is purely imaginary.

If $z_1 = z_2$, then $x_1 = x_2$ and $y_1 = y_2$; hence $x_1-iy_1 = x_2-iy_2$, i.e. $\bar{z}_1 = \bar{z}_2$. The process of deducing from $z_1 = z_2$ that $\bar{z}_1 = \bar{z}_2$ is called *taking conjugates.*

Finally, the definition of 'conjugate' shows that $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$,

$$\overline{(z_1z_2)} = \bar{z}_1\bar{z}_2, \quad \overline{(z_1/z_2)} = \bar{z}_1/\bar{z}_2, \quad \overline{(\bar{z})} = z.$$

(4) *Properties of* $|z|$. From the relation $|z|^2 = z\bar{z}$ in (3) it follows that

$$|z_1z_2|^2 = z_1z_2\overline{(z_1z_2)} = z_1\bar{z}_1 . z_2\bar{z}_2 = |z_1|^2 . |z_2|^2.$$

Hence $$|z_1z_2| = |z_1| . |z_2|,$$

since both sides are positive by definition. Similarly, if $z_2 \neq 0$,

$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right)\overline{\left(\frac{z_1}{z_2}\right)} = \frac{z_1\bar{z}_1}{z_2\bar{z}_2} = \frac{|z_1|^2}{|z_2|^2},$$

so that $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

13.23 The cube roots of unity†

If z is a number whose cube is $+1$, then $z^3-1 = 0$, i.e.

$$(z-1)(z^2+z+1) = 0;$$

hence either $z-1 = 0$ or $z^2+z+1 = 0$ by 13.14 (2). From the latter,

$$(z+\tfrac{1}{2})^2 = -\tfrac{3}{4}, \quad z+\tfrac{1}{2} = \pm i\tfrac{1}{2}\sqrt{3} \quad \text{and} \quad z = \tfrac{1}{2}(-1\pm i\sqrt{3}).$$

Since $\cos\frac{2}{3}\pi = -\frac{1}{2}$ and $\sin\frac{2}{3}\pi = \frac{1}{2}\sqrt{3}$, the first of these roots has modulus-argument form $\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi$. We denote this by ω.

† A simpler approach will be given in 14.13, ex. (ii).

Also, $\cos\frac{4}{3}\pi = -\frac{1}{2}$ and $\sin\frac{4}{3}\pi = -\frac{1}{2}\sqrt{3}$, so the second root is $\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi$. It is also $\bar{\omega}$, as is clear from the original formula for z.

Now
$$\omega^2 = (\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi)^2$$
$$= \cos^2\tfrac{2}{3}\pi - \sin^2\tfrac{2}{3}\pi + 2i\sin\tfrac{2}{3}\pi\cos\tfrac{2}{3}\pi$$
$$= \cos\tfrac{4}{3}\pi + i\sin\tfrac{4}{3}\pi.$$

Hence the roots of $z^2+z+1 = 0$ can be written $z = \omega$, ω^2. There are thus three cube roots of $+1$ in complex algebra, viz. 1, ω, ω^2. The relations
$$\omega^3 = 1, \quad \omega^2+\omega+1 = 0, \quad \omega^2 = \bar{\omega}$$
often simplify calculations.

Example

Direct expansion shows that
$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a^3+b^3+c^3-3abc.$$
The operation $\mathbf{c}_1 \to \mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_3$ reveals the factor $a+b+c$.

Since the expression on the right can be written
$$a^3+(b\omega)^3+(c\omega^2)^3-3a(b\omega)(c\omega^2),$$
similar reasoning reveals the factor $a+b\omega+c\omega^2$. Likewise, since the expression is also $a^3+(b\omega^2)^3+(c\omega)^3-3a(b\omega^2)(c\omega)$, $a+b\omega^2+c\omega$ is a factor.

We now have three complex linear factors, so that any further factor must be numerical. Comparison of the coefficients of a^3 shows that it is $+1$. Hence
$$a^3+b^3+c^3-3abc \equiv (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega).$$

By comparing this result with
$$a^3+b^3+c^3-3abc \equiv (a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$
(see 10.22 (2), example; or Ex. 11 (*a*), no. 20), it follows that
$$a^2+b^2+c^2-bc-ca-ab \equiv (a+b\omega+c\omega^2)(a+b\omega^2+c\omega).$$

Exercise 13(*b*)

Express in modulus-argument form:

1 $1+i\sqrt{3}$. 2 $1-i\sqrt{3}$. 3 $-1+i$. 4 $1-i$.

5 $(1+3i\sqrt{3})/(\sqrt{3}+2i)$. 6 $\dfrac{5+i\sqrt{3}}{4-2i\sqrt{3}}$. 7 $(1+i)\dfrac{2+i}{3-i}$.

8 $1+\cos\theta+i\sin\theta$. 9 $(\cos\alpha+i\sin\alpha)+(\cos\beta+i\sin\beta)$.

10 If $a+ib = (x+iy)^n$ where a, b, x, y are real, express a^2+b^2 in terms of x and y.

11 If the coefficients in the equation $z^4+pz^3+qz^2+rz+s = 0$ are 'real', and there is a purely imaginary root, prove $r^2+p^2s = pqr$.

12 If $|(z-1)/(z+1)| = 2$, prove that the point P which represents z lies on the circle
$$x^2+y^2+\tfrac{10}{3}x+1 = 0.$$

13 If $w = z/(z+3)$ where $w = u+iv$ and $z = x+iy$, and if z lies on the circle $(x+3)^2+y^2 = 1$, find the locus of w.

14 Prove that
$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}^2 = -27.$$

15 Prove $\omega^{2n}+\omega^n+1 = 3$ or 0 according as n is or is not a multiple of 3.

16 Express x^3+y^3 as the product of three linear factors, and deduce the factors of x^2-xy+y^2. [Method of 13.23, ex.]

17 Prove $(a+\omega b+\omega^2 c)^3-(a+\omega^2 b+\omega c)^3 = -3i\sqrt{3}.(b-c)(c-a)(a-b)$. [Use factors of x^3-y^3.]

18 Prove that $(a^2+ab+b^2)(x^2+xy+y^2)$ can be written in the form
$$X^2+XY+Y^2.$$
[Use $a^2+ab+b^2 \equiv (a-\omega b)(a-\omega^2 b)$, etc.]

*19 Prove that $(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)$ can be written in the form $X^3+Y^3+Z^3-3XYZ$, where either

(i) $X = ax+by+cz, \quad Y = cx+ay+bz, \quad Z = bx+cy+az,$

or (ii) $X = ax+cy+bz, \quad Y = cx+by+az, \quad Z = bx+ay+cz.$

[In (i), $(a+b\omega+c\omega^2)(x+y\omega^2+z\omega) = X+Y\omega^2+Z\omega$, etc.]

*20 If α_r $(r = 1, 2, 3)$ denote the cube roots of -1, prove
$$(1+x\alpha_1+x^2\alpha_1^2)(1+x\alpha_2+x^2\alpha_2^2)(1+x\alpha_3+x^2\alpha_3^2) = (1+x^3)^2.$$
[A typical factor is $(1-x^3\alpha^3)/(1-x\alpha) = (1+x^3)/(1-x\alpha)$.]

21 If the equations $x^3 = 1$, $ax^5+bx+c = 0$ have a common root, prove
$$a^3+b^3+c^3-3abc = 0.$$
[One of the three roots $1, \omega, \omega^2$ of $x^3 = 1$ satisfies the other equation, so
$$(a+b+c)(a\omega^5+b\omega+c)(a\omega^{10}+b\omega^2+c) = 0.]$$

13.3 Applications of the Argand representation

13.31 Geometrical interpretation of modulus and argument

If z is represented by P, then $|z| = \sqrt{(x^2+y^2)}$ is represented by OP. Also
$$\begin{aligned}|z_1-z_2| &= |(x_1+iy_1)-(x_2+iy_2)| \\ &= |(x_1-x_2)+i(y_1-y_2)| \\ &= \sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2\}} \\ &= P_1P_2\end{aligned}$$
by the 'distance formula' of coordinate geometry.

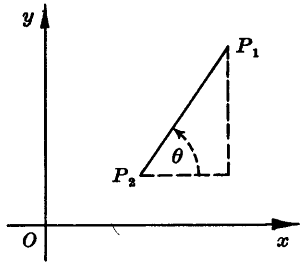

Fig. 128

If θ is the principal value of $\arg z$, then it is given by

$$\cos\theta : \sin\theta : 1 = x : y : \sqrt{(x^2+y^2)}$$

and is the angle of turn from the direction Ox towards the direction OP, which can be written $\angle(Ox, OP)$. Similarly, $\arg(z_1 - z_2)$ is the angle defined by

$$\cos\theta : \sin\theta : 1 = x_1 - x_2 : y_1 - y_2 : \sqrt{\{(x_1-x_2)^2 + (y_1-y_2)^2\}},$$

and represents the angle of turn from Ox towards P_2P_1, written $\angle(Ox, P_2P_1)$.

Examples (*some loci*)

If P_1 is the fixed point z_1 and P is the variable point z, then

(i) *the locus $|z - z_1| = c$ is the circle with centre P_1 and radius c;*

(ii) *the locus $\arg(z - z_1) = \alpha$ is the half-line P_1P for which $\angle(Ox, P_1P)$ is α.* (The other half is given by $\arg(z - z_1) = \pi + \alpha$.)

(iii) *If P_1, P_2 are fixed points z_1, z_2, the locus $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line P_1P_2.* For $PP_1 = PP_2$, and the result follows by pure geometry. Alternatively, it can be proved algebraically from

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2$$

by simplifying.

(iv) *If P, Q are the points z, $2z + 3 - 4i$ and P moves on a circle of centre O and radius r, find the locus of Q.*

Write $w = 2z + 3 - 4i$. Then $w - 3 + 4i = 2z$, so

$$|w - (3 - 4i)| = |2z| = 2r$$

since $|z| = r$. Therefore Q lies on the circle of centre $(3, -4)$ and radius $2r$.

13.32 Constructions for the sum and difference of z_1, z_2

(1) *Sum.* Let P_1, P_2 correspond to z_1, z_2, and complete the parallelogram $P_2OP_1P_3$. Then

$$\begin{aligned} x_3 &= \text{projection of } OP_3 \text{ on } Ox \\ &= \text{projection of } OP_1 + \text{projection of } P_1P_3 \\ &= \text{projection of } OP_1 + \text{projection of } OP_2\dagger \\ &= x_1 + x_2. \end{aligned}$$

Similarly, by projecting on the y-axis we find $y_3 = y_1 + y_2$. Hence

$$x_3 + iy_3 = (x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2),$$

i.e. $z_3 = z_1 + z_2$, so that P_3 represents the sum $z_1 + z_2$.

† Since OP_2 is equal and parallel to P_1P_3, and in the same sense.

Observe that the construction for P_3 is the familiar parallelogram law for compounding ('adding') two displacements $\mathbf{OP_1}$, $\mathbf{OP_2}$. To the sum of two complex numbers corresponds the sum of the associated vectors.

(2) *Difference.* Since $z_1 - z_2 = z_1 + (-z_2)$, we first construct the image P_2' of P_2 in O, which represents $-z_2$. The above sum-construction performed with P_1, P_2' will give P_4, corresponding to $z_1 - z_2$.

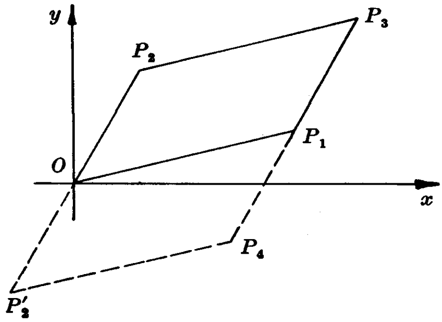

Fig. 129

As OP_4 is equal to, parallel to, and in the same sense as P_2P_1, we can represent $z_1 - z_2$ by the displacement $\mathbf{P_2P_1}$.

13.33 The triangle inequalities

By applying the theorem that 'the sum of two sides of a triangle is greater than the third side' to triangle OP_1P_3, we have

$$OP_3 \leqslant OP_1 + P_1P_3,$$

i.e.†
$$OP_3 \leqslant OP_1 + OP_2,$$

and hence
$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

From this we deduce the following, exactly as for real numbers in 1.14:

$$|z_1 - z_2| \geqslant \big||z_1| - |z_2|\big|,$$

$$|z_1 + z_2| \geqslant \big||z_1| - |z_2|\big|.$$

These inequalities should also be verified geometrically from fig. 129.

The results can all be proved algebraically, without appeal to a figure. To do so, we first observe that

$$\mathscr{R}(z) = x \leqslant \sqrt{(x^2 + y^2)} = |z|,$$

† Equality is included in case O, P_1, P_2 (and hence also P_3) are collinear.

with equality when and only when z is 'real and positive', i.e. of the form $x+0i$ where $x \geqslant 0$. (Similarly $\mathscr{I}(z) \leqslant |z|$.) By 13.22 (3),

$$|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$
$$= z_1\bar{z}_1+z_1\bar{z}_2+\bar{z}_1z_2+z_2\bar{z}_2$$
$$= |z_1|^2+2\mathscr{R}(z_1\bar{z}_2)+|z_2|^2$$

since $z_1\bar{z}_2$ and $\bar{z}_1z_2$ are conjugate complex numbers. As $\mathscr{R}(z_1\bar{z}_2) \leqslant |z_1\bar{z}_2|$,

$$|z_1+z_2|^2 \leqslant |z_1|^2+2|z_1\bar{z}_2|+|z_2|^2$$
$$= (|z_1|+|z_2|)^2, \quad \text{since} \quad |z_1\bar{z}_2| = |z_1|.|\bar{z}_2| = |z_1|.|z_2|.$$

Therefore $$|z_1+z_2| \leqslant |z_1|+|z_2|,$$

positive signs being chosen because the modulus of a complex number is positive by definition. Equality occurs if and only if $z_1\bar{z}_2$ is 'real and positive', i.e. $z_1 = a^2z_2$ where a is 'real'.

The other results can also be established similarly.

13.34 Constructions for the product and quotient of z_1, z_2

It is convenient to use the modulus-argument form.

(1) *Product*. Let $z_1 = r(\cos\theta+i\sin\theta)$, $z_2 = s(\cos\phi+i\sin\phi)$, so that P_1, P_2 have polar coordinates (r,θ), (s,ϕ). Then

$$z_1z_2 = rs(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)$$
$$= rs\{(\cos\theta\cos\phi-\sin\theta\sin\phi)+i(\sin\theta\cos\phi+\cos\theta\sin\phi)\}$$
$$= rs\{\cos(\theta+\phi)+i\sin(\theta+\phi)\}.$$

Therefore z_1z_2 is represented by the point which has polar coordinates $(rs, \theta+\phi)$.

To construct this point, let A be the point whose polar coordinates are $(1, 0)$. Draw triangle OAP_1, and construct triangle OP_2P_3 directly similar to OAP_1 (fig. 130). Then

$$\frac{OP_3}{OP_2} = \frac{OP_1}{OA}, \quad \text{so} \quad OP_3 = OP_1.OP_2 = rs.$$

Also $$x\hat{O}P_3 = x\hat{O}P_2+P_2\hat{O}P_3 = x\hat{O}P_2+A\hat{O}P_1 = \phi+\theta.$$

Hence P_3 is the required point.

The above calculation verifies the result $|z_1z_2| = |z_1|.|z_2|$, and also shows that (cf. the end of 13.22 (2))

$$\arg(z_1z_2) \approx \arg z_1+\arg z_2.$$

Ordinary equality may not hold when all three terms have their principal values; e.g. if

$$z_1 = z_2 = \cos\tfrac{2}{3}\pi+i\sin\tfrac{2}{3}\pi, \quad \text{then} \quad z_1z_2 = \cos\tfrac{4}{3}\pi+i\sin\tfrac{4}{3}\pi,$$

and the principal values are

$$\arg z_1 = \arg z_2 = \tfrac{2}{3}\pi, \quad \arg(z_1 z_2) = -\tfrac{2}{3}\pi.$$

We observe that multiplication of a complex number z_2 by $r(\cos\theta + i\sin\theta)$ turns the corresponding vector $\mathbf{OP_2}$ through angle θ and multiplies its length by r.

(2) *Quotient.* If $z_4 = z_1/z_2$, then $z_1 = z_4 z_2$, and so we can use the product-construction in (1), making P_1 take the role of P_3 there.

We therefore construct triangle OP_1P_2, and then make triangle OP_4A directly similar to this (fig. 131).

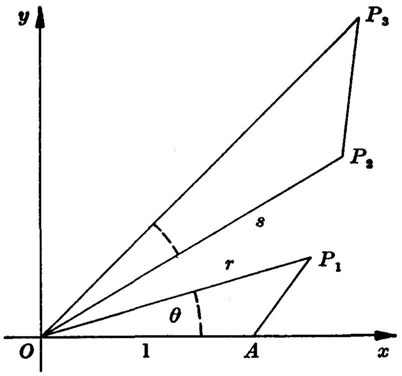

Fig. 130

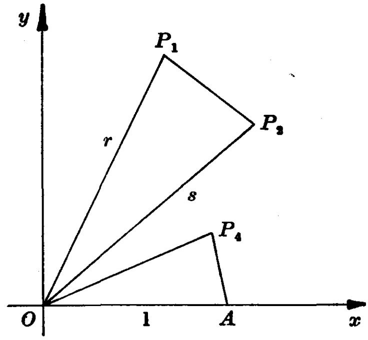

Fig. 131

Alternatively, the construction can be obtained from

$$\frac{z_1}{z_2} = \frac{r}{s}\frac{\cos\theta + i\sin\theta}{\cos\phi + i\sin\phi}$$

$$= \frac{r}{s}\frac{(\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi)}{(\cos\phi + i\sin\phi)(\cos\phi - i\sin\phi)}$$

$$= \frac{r}{s}\frac{(\cos\theta\cos\phi + \sin\theta\sin\phi) + i(\sin\theta\cos\phi - \cos\theta\sin\phi)}{\cos^2\phi + \sin^2\phi}$$

$$= \frac{r}{s}\{\cos(\theta - \phi) + i\sin(\theta - \phi)\}.$$

This calculation verifies that $|z_1/z_2| = |z_1|/|z_2|$, and shows that

$$\arg\left(\frac{z_1}{z_2}\right) \approx \arg z_1 - \arg z_2,$$

with the usual understanding about many-valuedness.

Remark. It also follows that

$$\arg\frac{z_1-z_2}{z_3-z_2} \approx \arg(z_1-z_2)-\arg(z_3-z_2)$$
$$= \angle(Ox, P_2P_1)-\angle(Ox, P_2P_3)$$
$$= \angle(P_2P_3, P_2P_1),$$

i.e. the angle $P_3P_2P_1$ in the sense *from* P_2P_3 *towards* P_2P_1.

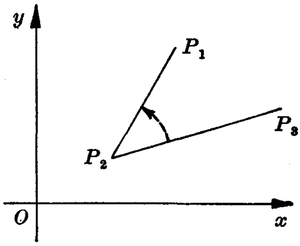

Fig. 132

13.35 Harder examples on the Argand representation

*(i) *Interpret geometrically the equations*

$$(a)\ \arg\frac{z-z_1}{z-z_2} = \alpha; \quad (b)\ \left|\frac{z-z_1}{z-z_2}\right| = k \quad (k \neq 1).$$

(*a*) By the Remark in 13.34, we have $P_2\hat{P}P_1 = \alpha$. Hence *the locus of P is an arc of a circle through P_1, P_2 and containing angle α.*

(*b*) We shall prove that *the locus of P is a circle* (a *circle of Apollonius* with respect to P_1, P_2).

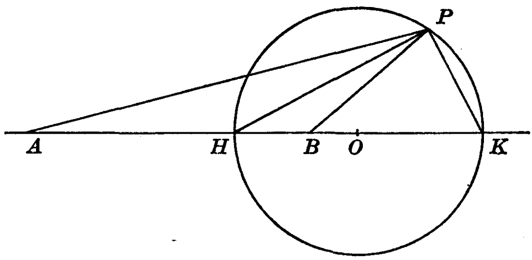

Fig. 133

Geometrical proof. We are given that $AP/PB = k \neq 1$, where A, B, represent z_1, z_2.†

Divide AB internally at H and externally at K in the ratio $k:1$. (The figure is drawn assuming $k > 1$; the reader should illustrate the case $k < 1$.) Then

$$\frac{AP}{PB} = \frac{AH}{HB} \quad \text{and} \quad \frac{AP}{PB} = \frac{AK}{KB},$$

and by a theorem of pure geometry it follows that PH, PK are the internal and external bisectors of $A\hat{P}B$. Hence $H\hat{P}K$ is a right-angle. Since H, K are fixed points, P lies on the circle having HK for diameter.

Algebraical proof. From the hypothesis,

$$|z-z_1|^2 = k^2|z-z_2|^2,$$

i.e. $$(x-x_1)^2+(y-y_1)^2 = k^2(x-x_2)^2+k^2(y-y_2)^2;$$

and since $k \neq 1$, this represents a circle (see 15.61).

† The implied change of notation from P_1, P_2 to A, B respectively is made for convenience in writing.

It is easy to prove that, if the centre is O, *the radius OH of the circle is given by*

$$OH^2 = OA.OB.$$

For since $AH/HB = AK/KB$, it follows that $AH/AK = HB/KB$, and so

$$\frac{OA-OH}{OA+OH} = \frac{OH-OB}{OH+OB}, \quad \text{i.e.} \quad \frac{2OA}{2OH} = \frac{2OH}{2OB}$$

by properties of equal ratios, which gives the result.

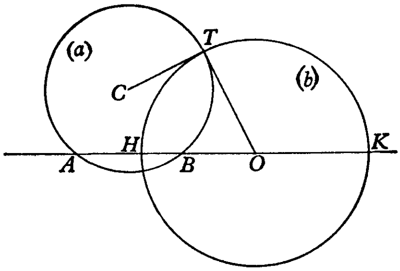

Fig. 134

We can now show that *the circles in* (a), (b) *cut orthogonally* (see 15.66 (1)). If T is a common point, then $OT^2 = OA.OB$. By the converse of the tangent-chord theorem, OT touches the circle (a). Hence the radius CT of (a) is perpendicular to OT, and so touches (b) at T. Thus the tangents to (a), (b) at T are perpendicular, i.e. the circles are orthogonal.

*(ii) *Prove that the triangles $P_1P_2P_3$, $P_4P_5P_6$ are directly similar if and only if*

$$\frac{z_1-z_2}{z_3-z_2} = \frac{z_4-z_5}{z_6-z_5}.$$

(a) If the condition holds, then by taking moduli, $P_1P_2/P_3P_2 = P_4P_5/P_6P_5$; and by taking arguments and using the Remark in 13.34,

$$P_3\hat{P}_2P_1 = P_6\hat{P}_5P_4$$

and are in the *same sense*. Hence by the test 'common angle and containing sides proportional', triangles $P_1P_2P_3$, $P_4P_5P_6$ are *directly* similar.

(b) Conversely, if the triangles are directly similar, then the above two relations hold, so that $(z_1-z_2)/(z_3-z_2)$, $(z_4-z_5)/(z_6-z_5)$ have the same modulus and the same argument. They are therefore equal.

The condition can be written

$$\begin{vmatrix} z_1-z_2 & z_3-z_2 \\ z_4-z_5 & z_6-z_5 \end{vmatrix} = 0, \quad \text{i.e.} \quad \begin{vmatrix} z_1-z_2 & 0 & z_3-z_2 \\ z_4-z_5 & 0 & z_6-z_5 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

By $\mathbf{r}_1 \to \mathbf{r}_1+z_2\mathbf{r}_3$ followed by $\mathbf{r}_2 \to \mathbf{r}_2+z_5\mathbf{r}_3$ we obtain

$$\begin{vmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Exercise 13(c)

1 Indicate in a diagram the points representing $\frac{1}{2}(z_1+z_2)$, z_1+2z_2, z_1-2z_2.

2 If collinear points P_1, P_2, P_3 are such that $P_1P_2 = 2P_2P_3$, find the relation between z_1, z_2, z_3.

3 Write down the complex number represented by the point dividing P_1P_2 in the ratio $k:l$.

4 What is the locus of P if $2 < |z+1-2i| < 3$?

5 Find the greatest and least values of (i) $|z-4|$ if $|z| \leqslant 1$; (ii) $|z+1|$ if $|z-3| \leqslant 5$.

6 Prove that $|z_1+z_2|^2+|z_1-z_2|^2 = 2\{|z_1|^2+|z_2|^2\}$ (i) geometrically, (ii) algebraically. [For (i) use the theorem of Apollonius on triangle OP_3P_4 in fig. 129.]

7 (i) If $|z_1+z_2| = |z_1-z_2|$, prove that $\arg z_1$ and $\arg z_2$ differ by $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ (principal values intended). (ii) If $\arg\{(z_1+z_2)/(z_1-z_2)\} = \frac{1}{2}\pi$, prove $|z_1| = |z_2|$. [Treat geometrically.]

8 Verify the following construction for the roots of $z^2-2az+b^2 = 0$, where a, b are real and $0 < a < b$. With the origin O for centre, draw a circle of radius b. From the point A on Ox at distance a from O draw a perpendicular to Ox cutting the circle at P, Q. Then P, Q represent the roots.

9 *Verify the following construction for* $\sqrt{z}$. Let A be $(1, 0)$; produce PO to B so that $OB = 1$. Through O draw a line parallel to AB to meet the circle PAB in Q and R; these are the required points.

10 Given the points representing z_1, z_2, construct those representing the two values of $\sqrt{(z_1z_2)}$.

11 P_1 is any point, and on the circle on OP_1 as diameter points P_2, P_3 are chosen so that $P_1\hat{O}P_2 = P_2\hat{O}P_3 = \phi$. Prove $z_2^2\cos 2\phi = z_1z_3\cos^2\phi$.

12 If G is the centroid of triangle $P_1P_2P_3$ and $4z_1+z_2+z_3 = 0$, prove that O bisects P_1G. [Use Ex. 15(*a*), no. 1.]

13 If λ is real and $z = 4\lambda+3i(1-\lambda)$, prove that P lies on a straight line. As λ varies, prove that the least value of $|z|$ is $2{\cdot}4$, and interpret geometrically.

14 If $z_1-z_2 = z_4-z_3$, prove that $P_1P_2P_3P_4$ is a parallelogram and that the point $\frac{1}{4}(z_1+z_2+z_3+z_4)$ is its centre.

15 Show that the point P_2, where $z_2 = (1+\lambda i)z_1$ and λ is real, lies on the perpendicular at P_1 to OP_1. Find λ if $\arg z_2 = \arg z_1+\frac{1}{3}\pi$, principal values being intended.

16 If z_1, z_2 are the roots of $z^2-az+a^2 = 0$, where a is a complex number, prove that the points z_1, z_2 are the vertices of the equilateral triangles drawn on opposite sides of the line OA.

17 ABC is equilateral, with sides of length 1 and centroid at O. If A represents z, what do B, C represent? Show that the sum of these three numbers is always zero.

18 If A, B, C, D, E are the vertices of a regular pentagon inscribed in a circle whose centre is O and radius is r, and if OA makes angle θ with Ox, write down the numbers which these vertices represent.

19 If $2/z_1 = 1/z_2+1/z_3$, prove that $OP_2P_1P_3$ is a cyclic quadrilateral. [The relation is $(z_1-z_2)/(0-z_2) = (z_1-z_3)/(z_3-0)$; take arguments.]

20 Interpret geometrically the equation

$$\mathscr{R}\left(\frac{z-z_1}{z_2-z_3}\right) = 0.$$

*21 On the sides of triangle ABC are drawn triangles BCX, CAY, ABZ directly similar to each other. Prove that the centroids of ABC, XYZ coincide. [By 13.35, ex. (ii) and properties of equal ratios,

$$\frac{x-c}{b-c} = \frac{y-a}{c-a} = \frac{z-b}{a-b} = \frac{\Sigma x - \Sigma a}{0},$$

so $\Sigma x = \Sigma a$.]

*22 For any complex number z prove that the triangle with vertices zz_1, zz_2, zz_3 is directly similar to the triangle with vertices z_1, z_2, z_3.

23 If P_1P_3 and P_2P_4 are equal and perpendicular, the sense of rotation from P_1P_3 to P_2P_4 being clockwise, prove $z_1 - iz_2 - z_3 + iz_4 = 0$, *and conversely*. [$|z_4 - z_2| = |z_3 - z_1|$ and $\arg\{(z_3-z_1)/(z_4-z_2)\} = +\frac{1}{2}\pi$. Hence $(z_3-z_1)/(z_4-z_2) = i$.]

24 On the sides of a plane convex quadrilateral, squares are drawn externally. Prove that the centres of these squares are the vertices of a quadrilateral whose diagonals are equal and perpendicular. [Let the vertices in clockwise order be z_1, z_2, z_3, z_4. The centres of the squares are $\frac{1}{2}(z_1+z_2)+\frac{1}{2}i(z_2-z_1)$, etc. Use no. 23.]

25 By considering the modulus of the left-hand side, prove that all the roots of

$$z^n \sin n\alpha + z^{n-1}\sin(n-1)\alpha + \ldots + z\sin\alpha = 1$$

lie outside the circle $|z| = \frac{1}{2}$.

[$1 = |z^n \sin n\alpha + \ldots + z\sin\alpha| \leqslant |z|^n + |z|^{n-1} + \ldots + |z|$. If a root z satisfies $|z| \leqslant \frac{1}{2}$, then for this z, $1 \leqslant (\frac{1}{2})^n + (\frac{1}{2})^{n-1} + \ldots + \frac{1}{2} = 1 - (\frac{1}{2})^n$, a contradiction.]

13.4 Factorisation in complex algebra

13.41 'The fundamental theorem of algebra'

We remarked in 13.15 that complex numbers cannot be generalised by attempting to solve equations of degree higher than 2. This is a consequence of the following theorem.

In complex algebra the polynomial equation

$$p(z) = p_0 z^n + p_1 z^{n-1} + \ldots + p_n = 0 \quad (p_0 \neq 0) \tag{i}$$

has AT LEAST ONE *root.*

This result, known also as d'Alembert's or Gauss's theorem, has not yet been proved by any strictly *algebraical* argument. We shall assume it here; the reader may later study a proof which depends on the complex integral calculus.

13.42 Roots of the general polynomial equation

Defining 'repeated root of order r' as in real algebra (10.43), and reckoning such a root as r roots, it is now easy to prove the following.†

† This result should be compared with the weaker Theorem I in 10.13. Both the Remainder Theorem and Theorem I are valid in complex algebra, their proofs being unchanged.

In complex algebra the equation (i) *has exactly n roots.*

Proof. By the fundamental theorem there is a root $z = \alpha_1$, i.e.

$$p_0\alpha_1^n + p_1\alpha_1^{n-1} + \dots + p_n = 0.$$

The equation (i) is therefore equivalent to

$$p_0(z^n - \alpha_1^n) + p_1(z^{n-1} - \alpha_1^{n-1}) + \dots + p_{n-1}(z - \alpha_1) = 0;$$

i.e. since $z - \alpha_1$ divides each bracket, to

$$(z - \alpha_1) f_1(z) = 0,$$

where $f_1(z)$ is a polynomial of degree $n-1$ in z, whose first term is $p_0 z^{n-1}$ (as is clear by starting the division).

Again by the fundamental theorem, $f_1(z) = 0$ has a root $z = \alpha_2$ (not necessarily distinct from α_1), and by similar reasoning

$$f_1(z) \equiv (z - \alpha_2) f_2(z),$$

where $f_2(z)$ is a polynomial in z of degree $n-2$, beginning with $p_0 z^{n-2}$.

Continuing thus, we find

$$p(z) \equiv (z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_{n-1}) f_{n-1}(z),$$

where $f_{n-1}(z)$ is a polynomial of degree 1 in z, and begins with $p_0 z$; it must therefore be of the form $p_0 z + k$, or $p_0(z - \alpha_n)$ say. Therefore

$$p(z) \equiv p_0(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n). \qquad \text{(ii)}$$

This shows that $p(z)$ has n factors linear in z; or equivalently, that the equation $p(z) = 0$ has n roots $\alpha_1, \alpha_2, \dots, \alpha_n$ (which may or may not all be distinct). Also, (ii) shows that these are the *only* values of z which make $p(z)$ zero; for if β is such a value, then

$$p_0(\beta - \alpha_1)(\beta - \alpha_2) \dots (\beta - \alpha_n) = 0,$$

and since $p_0 \neq 0$ by hypothesis, at least one of the factors $\beta - \alpha_1$, $\beta - \alpha_2$, ..., $\beta - \alpha_n$ must be zero (13.14 (2)), i.e. β coincides with an α. Hence $p(z) = 0$ has *exactly n* roots.

If the roots are not all distinct, (ii) takes the form

$$p(z) \equiv p_0(z - \alpha_1)^{n_1}(z - \alpha_2)^{n_2} \dots (z - \alpha_k)^{n_k}, \qquad \text{(iii)}$$

where $n_1 + n_2 + \dots + n_k = n$ and $\alpha_1, \alpha_2, \dots, \alpha_k$ are all distinct. *This decomposition is unique* (apart perhaps from the arrangement of the factors). For if also

$$p(z) \equiv p_0(z - \beta_1)^{m_1}(z - \beta_2)^{m_2} \dots (z - \beta_l)^{m_l}, \qquad \text{(iv)}$$

where $\beta_1, \beta_2, \dots, \beta_l$ are distinct, then each β must be an α, since otherwise from (iii) $p(\beta_r) \neq 0$ while from (iv) $p(\beta_r) = 0$; similarly each α must be a β. Suppose the notation has been chosen so that $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, ...; then (iv) can be written

$$p(z) \equiv p_0(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\dots(z-\alpha_k)^{m_k}. \qquad \text{(v)}$$

It now follows that $m_r = n_r$ $(r = 1, 2, \dots, k)$. E.g. if $m_1 < n_1$, then from (iii) and (v) we should obtain, after dividing out $p_0(z-\alpha_1)^{m_1}$:

$$(z-\alpha_1)^{n_1-m_1}(z-\alpha_2)^{n_2}\dots(z-\alpha_k)^{n_k} \equiv (z-\alpha_2)^{m_2}\dots(z-\alpha_k)^{m_k},$$

which is impossible since the left-hand side is zero when $z = \alpha_1$ but the right-hand side is not.

13.43 Principle of equating coefficients

We can now deduce results for complex polynomials corresponding to Theorem II and its corollaries in 10.13. The enunciations and proofs are the same. In particular, the 'principle of equating coefficients' is valid for polynomials in complex algebra.

Example

Solve the following three simultaneous equations in z_1, z_2, z_3, where a_1, a_2, a_3 are all different:

$$\frac{z_1}{a_1+\lambda_r}+\frac{z_2}{a_2+\lambda_r}+\frac{z_3}{a_3+\lambda_r}-1 = 0 \quad (r = 1, 2, 3).$$

Consider the expression

$$(a_1+\lambda)(a_2+\lambda)(a_3+\lambda)\left\{\frac{z_1}{a_1+\lambda}+\frac{z_2}{a_2+\lambda}+\frac{z_3}{a_3+\lambda}-1\right\}.$$

When expanded, it is a polynomial in λ of degree 3, with leading term $-\lambda^3$. The given equations imply that this expression is zero when $\lambda = \lambda_1, \lambda_2, \lambda_3$. Hence it must be identical with

$$-(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3).$$

Putting $\lambda = -a_1$ in the identity, we have

$$(a_2-a_1)(a_3-a_1)z_1 = (-1)^4(a_1+\lambda_1)(a_1+\lambda_2)(a_1+\lambda_3),$$

which gives z_1. Similarly the substitutions $\lambda = -a_2$, $\lambda = -a_3$ give z_2, z_3.

13.44 Repeated roots and the derived polynomial

The definitions of 'repeated factor', 'repeated root' in 10.43 are retained for complex polynomials. Continuing to write

$$p(z) \equiv p_0 z^n + p_1 z^{n-1} + \dots + p_n,$$

we define the *derived polynomial* of $p(z)$ to be

$$p'(z) \equiv p_0 n z^{n-1} + p_1(n-1) z^{n-2} + \dots + p_{n-1}.$$

The reader may wonder why we do not give a definition like the following: '$p'(z)$ denotes

$$\lim_{h\to 0}\frac{p(z+h)-p(z)}{h},$$

where h is complex (say $h = \xi + i\eta$) and $h \to 0$ means that ξ and η tend to zero separately and in any manner.' There are difficulties underlying this procedure; they can only be fully examined in dealing with general functions of a complex variable, a subject which may be studied at a later stage.

From the definition given it is straightforward to verify the usual rules of the differential calculus, and in particular the product rule, in this 'differential calculus of complex polynomials'. The theorems on repeated roots given in 10.43 remain valid in complex algebra since their proofs are exactly as before.

13.45 Equations with 'real' coefficients; conjugate complex roots

(1) *If the coefficients* $p_0, p_1, \ldots, p_n$ *in the polynomial* $p(z)$ *are 'real', and if* $\alpha + \beta i$ (*where* $\beta \neq 0$) *is a root of* $p(z) = 0$, *then its conjugate* $\alpha - \beta i$ *is also a root.*

Proof. The condition for $\alpha + \beta i$ to be a root is $p(\alpha + \beta i) = 0$, i.e.

$$p_0(\alpha+\beta i)^n + p_1(\alpha+\beta i)^{n-1} + \ldots + p_{n-1}(\alpha+\beta i) + p_n = 0.$$

On expanding the binomials and replacing i^2 by -1 whenever it occurs, this equation can be written

$$P + Qi = 0,$$

where P, Q are *real* polynomials in (α, β).

This condition is equivalent to the separate equations $P = 0$, $Q = 0$. Hence also $P - Qi = 0 - 0i = 0$, which (*because* $p(z)$ *has* REAL *coefficients*) is equivalent to the statement $p(\alpha - \beta i) = 0$, i.e. $\alpha - \beta i$ is a root of $p(z) = 0$. See Ex. 13 (*d*), no. 7 for an alternative proof.

If the coefficients in $p(z)$ are not all real, the argument breaks down at the last step. For example, if $p(z)$ is the linear polynomial $z - (1 + i)$, then clearly $z = 1 + i$ is a root of $p(z) = 0$; but $1 - i$ is not a root since

$$p(1-i) = (1-i) - (1+i) = -2i \neq 0.$$

The correct statement in the general case is that $\alpha - \beta i$ is a root of $\bar{p}(z) = 0$, where $\bar{p}(z)$ is the polynomial obtained from $p(z)$ by changing the sign of i in all the *coefficients*. For if $p(\alpha + \beta i) = P + Qi$, then we must still get a true result by changing the sign of i *everywhere* on both sides† (because the equality was obtained by using only the property $i^2 = -1$ of i, and this continues to hold when i is replaced by $-i$); this would give $\bar{p}(\alpha - \beta i) = P - Qi$. In the case of entirely *real* coefficients, $\bar{p}(z) \equiv p(z)$.

It is also easy to prove the following.

† In short, by 'taking conjugates'.

Conjugate complex roots occur to the same order.

For if $p(z) = 0$ has an r-fold root $z = \alpha+\beta i$ ($\beta \neq 0$), then it also has a root $\alpha-\beta i$. Remove the factor $(z-\alpha-\beta i)(z-\alpha+\beta i)$ from $p(z)$, obtaining $f_1(z)$, say. Then $f_1(z) = 0$ has an $(r-1)$-fold root $\alpha+\beta i$, and therefore also a root $\alpha-\beta i$; and by removing the corresponding factors from $f_1(z)$ we obtain $f_2(z)$, say. Proceeding thus, we arrive at a polynomial $f_r(z)$, where $\alpha+\beta i$ is *not* a root of $f_r(z) = 0$; hence $\alpha-\beta i$ cannot be a root (otherwise its conjugate $\alpha+\beta i$ *would* be a root). Thus $\alpha-\beta i$ is also an r-fold root of $p(z) = 0$.

Example

Prove that the equation

$$f(x) \equiv \frac{b_1^2}{x-a_1}+\frac{b_2^2}{x-a_2}+\dots+\frac{b_n^2}{x-a_n}+k = 0$$

has n roots in real algebra if $a_1, a_2, \dots, a_n$ are all different, the b's are non-zero, and $k \neq 0$.

Consider the corresponding equation in complex algebra, the a's, b's and k all being real. Denote any root† by $\alpha+\beta i$; then $\alpha-\beta i$ is also a root, and hence

$$\sum_{r=1}^{n} b_r^2\left(\frac{1}{\alpha-\beta i-a_r}-\frac{1}{\alpha+\beta i-a_r}\right) = 0,$$

i.e.

$$2i\beta\sum_{r=1}^{n}\frac{b_r^2}{(\alpha-a_r)^2+\beta^2} = 0.$$

Since the sum is non-zero, we have $\beta = 0$. Hence all the roots are of the form $\alpha+0i$. In real algebra there are thus n roots.

(2) There is a theorem in real algebra on surd roots of an equation with *rational* coefficients which can be proved by an argument like that in (1).

If the polynomial equation $p(x) = 0$ (in real algebra) has rational coefficients and if $x = a+b\sqrt{c}$ is a root (where $\sqrt{c}$ is a surd), then also $a-b\sqrt{c}$ is a root.

Proof. The condition for $a+b\sqrt{c}$ to be a root of $p(x) = 0$ is, when simplified, of the form $P+Q\sqrt{c} = 0$, where P, Q are rational numbers or zero. This implies that $P = 0$ and $Q = 0$, for otherwise if $Q \neq 0$ the *surd* $\sqrt{c}$ would be equal to the *rational* number $-P/Q$, which is a contradiction. Since

$$p(a-b\sqrt{c}) = P-Q\sqrt{c} = 0-0\sqrt{c} = 0,$$

$a-b\sqrt{c}$ is also a root. Also see Ex. 13 (d), no. 8.

$a+b\sqrt{c}$, $a-b\sqrt{c}$ are called *conjugate surds*.

† That is, root of the corresponding *polynomial* equation

$$(z-a_1)(z-a_2)\dots(z-a_n)f(z) = 0.$$

13.5 Relations between roots and coefficients

13.51 Symmetrical relations

In 13.42 we proved that

$$p_0 z^n + p_1 z^{n-1} + \dots + p_n \equiv p_0(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n),$$

so that $\alpha_1, \alpha_2, \dots, \alpha_n$ are the n roots of the equation $p(z) = 0$. On multiplying out the right-hand side, it becomes (cf. the proof in 12.11)

$$p_0\{z^n - (\Sigma\alpha)z^{n-1} + (\Sigma\alpha_1\alpha_2)z^{n-2} - \dots + (-1)^n \alpha_1\alpha_2\dots\alpha_n\}.$$

Hence on equating coefficients of $z^{n-1}, z^{n-2}, \dots$, and the constant terms,

$$\Sigma\alpha = -\frac{p_1}{p_0}, \quad \Sigma\alpha_1\alpha_2 = \frac{p_2}{p_0},$$

$$\Sigma\alpha_1\alpha_2\alpha_3 = -\frac{p_3}{p_0}, \quad \dots, \quad \alpha_1\alpha_2\dots\alpha_n = (-1)^n\frac{p_n}{p_0};$$

and in general,

$$(-1)^r p_r/p_0 = \text{sum of the products of the roots taken } r \text{ at a time.}$$

This work generalises the results of 10.3, which are theorems of real algebra, and are valid only if the equation under consideration has the maximum possible number of roots. The above results in complex algebra hold without any such restriction because *every* polynomial equation of degree n has exactly n roots.

13.52 Unsymmetrical relations

We remarked in 10.32 that the symmetrical relations do not help us to solve the given equation because they are equivalent to the information that '$\alpha_1, \alpha_2, \dots, \alpha_n$ are the roots of $p(z) = 0$'. However, if some non-symmetrical relation between some or all of the roots is given, then it may be possible to solve the equation.

Example

Solve (in complex algebra) the equation

$$3x^4 + 17x^3 - 5x^2 + 8x + 12 = 0,$$

given that the product of two roots is 4.

Let the roots be $\alpha, \beta, \gamma, \delta$, and suppose $\alpha\beta = 4$. We have

$$\alpha+\beta+\gamma+\delta = -\tfrac{17}{3}, \quad \beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta+\alpha\beta\gamma = -\tfrac{8}{3}, \quad \alpha\beta\gamma\delta = 4.$$

(There is also the relation $\Sigma\alpha\beta = -\tfrac{5}{3}$, but we shall find that this is not needed.)

From $\alpha\beta = 4$ and the third relation, $\gamma\delta = 1$.

The second relation can be written

$$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) = -\tfrac{8}{3},$$

so that

$$4(\gamma+\delta)+(\alpha+\beta) = -\tfrac{8}{3}.$$

This together with the first relation can be solved to give

$$\alpha+\beta = -\tfrac{20}{3}, \quad \gamma+\delta = 1.$$

Hence α, β are the roots of $x^2+\frac{20}{3}x+4 = 0$, viz. $-\frac{2}{3}$, -6; and γ, δ are the roots of $x^2-x+1 = 0$, viz. $-\omega$, $-\omega^2$.

13.53 Transformation of equations

Properties of the roots of an equation are often conveniently discussed by considering another equation whose roots are related to those of the given equation in a known manner. We now show how to form the equations whose roots are (i) reciprocals of, (ii) k times, (iii) less by k than, (iv) squares of the roots of the polynomial equation† $p(x) = 0$. The process (ii) is known as 'multiplying the roots by k', (iii) as 'diminishing the roots by k', and (iv) as 'squaring the roots'. Applications are given in 13.72, 13.73.

(i) Write $y = 1/x$. If α is a root of $p(x) = 0$, then since $x = 1/y$, the equation $p(1/y) = 0$ is satisfied by $y = 1/\alpha$. The required polynomial equation is therefore

$$p_n y^n + p_{n-1}y^{n-1} + \dots + p_1 y + p_0 = 0.$$

(ii) Write $y = kx$. When α satisfies $p(x) = 0$, $k\alpha$ will satisfy $p(y/k) = 0$. Hence the required equation reduces to

$$p_0 y^n + p_1 k y^{n-1} + p_2 k^2 y^{n-2} + \dots + p_n k^n = 0.$$

The case $k = -1$ gives the equation whose roots are the negatives of those of $p(x) = 0$ ('changing the sign of the roots').

(iii) Write $y = x-k$. When α satisfies $p(x) = 0$, $\alpha-k$ will satisfy $p(y+k) = 0$.

(iv) Write $y = x^2$. When α satisfies $p(x) = 0$, α^2 will satisfy $p(\sqrt{y}) = 0$. This can be rationalised (i.e. freed of square roots) as in ex. (ii) below.

Examples

(i) *Transform the equation* $3x^4-4x^3+6x+5 = 0$ *so that the coefficient of the leading term becomes* 1, *without introducing fractional coefficients.*

Multiply the roots by k, i.e. put $y = kx$, so that $x = y/k$. Then

$$3y^4-4ky^3+6k^3y+5k^4 = 0,$$

i.e.

$$y^4-\tfrac{4}{3}ky^3+2k^3y+\tfrac{5}{3}k^4 = 0.$$

† The method is equally applicable to equations other than polynomial ones.

By inspection, the coefficients will be integers when $k = 3$ (this is the smallest such value). The new equation is then

$$y^4 - 4y^3 + 54y + 135 = 0,$$

where $y = 3x$.

(ii) *Find the equation whose roots are the squares of the roots of*

$$3x^3 - x^2 + 2x - 3 = 0. \qquad (a)$$

Write the given equation as

$$x(3x^2 + 2) = x^2 + 3, \qquad (b)$$

and put $y = x^2$: $\quad \sqrt{y}\,(3y+2) = y+3.$

Squaring, $\quad y(9y^2 + 12y + 4) = y^2 + 6y + 9,$

i.e. $\quad 9y^3 + 11y^2 - 2y - 9 = 0. \qquad (c)$

Remark. Strictly, we should show that *every* root of equation (c) is the square of a root of equation (a),† as follows. If y satisfies (c), then (by reversing the working) it satisfies one of the equations

$$\pm\sqrt{y}\,.\,(3y+2) = y+3.$$

Thus either $+\sqrt{y}$ or $-\sqrt{y}$ satisfies (b), i.e. the given equation (a). In either event, y is the square of a root of (a).

Further examples of the transformation process are given in 10.32, ex. (ii) and Ex. 10(c).

Exercise 13(d)

The algebra is complex.

1 Prove that the equation

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!} = 0$$

has no repeated roots.

2 Solve $z^4 - 4z^2 + 8z - 4 = 0$, given that $1 + i$ is a root.

3 Given that $2 + \sqrt{3}$ is a root of $x^3 - 2x^2 - 7x + 2 = 0$, solve the equation completely.

4 Prove that ω is a repeated root of $3x^5 + 2x^4 + x^3 - 6x^2 - 5x - 4 = 0$, and hence solve the equation.

5 One root of $x^6 + x^5 - 9x^4 - 10x^3 - 9x^2 + x + 1 = 0$ is $\sqrt{2} + \sqrt{3}$. Find all the roots.

6 'The linear equation $az + b = 0$ with real coefficients a, b has a root $\alpha + \beta i$, and therefore also a root $\alpha - \beta i$: two roots for an equation of degree 1.' Explain this apparent paradox.

7 (i) Verify that $(z - \alpha - \beta i)(z - \alpha + \beta i) \equiv z^2 - 2\alpha z + \alpha^2 + \beta^2$, where α, β are real.

(ii) Let $q(z)$, $az + b$ be the quotient and remainder when $p(z)$ is divided by $z^2 - 2\alpha z + \alpha^2 + \beta^2$. *If $p(z)$ has real coefficients*, explain why a, b must be real.

† This may not always be the case; e.g. the square of each root of $x^3 - x = 0$ satisfies $y^3 - y = 0$, yet the root $y = -1$ is not the square of a root of the x-equation. Indeed, there is not a unique result in this instance: the method of ex. (ii) gives $y(y-1)^2 = 0$, and clearly $y^2(y-1) = 0$ also satisfies the requirements. Such ambiguity can arise whenever the squares of the roots of the given equation are not distinct.

(iii) If $\alpha+\beta i$ is a root of $p(z) = 0$, show by putting $z = \alpha+\beta i$ in the identity†

$$p(z) \equiv (z^2-2\alpha z+\alpha^2+\beta^2)\,q(z)+az+b$$

that $a\alpha+b = 0$ and $a\beta = 0$. If $\beta \neq 0$, deduce that $a = 0$ and $b = 0$, and hence that $z = \alpha-\beta i$ is also a root of $p(z) = 0$.

*8 Prove the theorem about conjugate surd roots in 13.45(2) by a method similar to that of no. 7. Also prove that such roots occur to the same order.

9 If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4+px^3+qx^2+rx+s = 0$, find $\Sigma\alpha\beta\gamma(\alpha+\beta+\gamma)$.

10 One root of $x^4+px^3+qx^2+rx+s = 0$ is equal to the product of the other three. Prove $(ps+r)^2 = s(q+s+1)^2$.

*11 Find the sum of the cubes of the roots of $x^5+x^2+x-1 = 0$.

Solve the following equations.

12 $x^3-5x^2-16x+80 = 0$ if the sum of two roots is zero.

13 $x^4+3x^3-5x^2-6x-8 = 0$ if the sum of two roots is -2.

14 $x^4-2x^3-21x^2+22x+40 = 0$ if the roots are in A.P.

15 Increase the roots of $x^3+3ax^2+3bx+c = 0$ by k, and find for what value of k the new equation contains no quadratic term. (This process is called 'removing the second term from the given equation'.)

16 Find two substitutions of the form $y = x-k$ which remove the third term from $x^4+4x^3-18x^2-100x-112 = 0$, and use one of them to solve the equation.

17 Find a substitution $y = kx$ which will transform the equation

$$8x^4+8x^3-18x^2-16x-3 = 0$$

into one with integral coefficients of which that of the leading term is $+1$.

18 If α, β, γ are the roots of $x^3+px^2+qx+r = 0$, find the equation whose roots are (i) $\alpha(\beta+\gamma), \beta(\gamma+\alpha), \gamma(\alpha+\beta)$; (ii) $(\alpha-1)/\alpha, (\beta-1)/\beta, (\gamma-1)/\gamma$.

19 If the roots of $x^n-1 = 0$ are $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, prove that

$$(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_n) = n.$$

[Construct the equation whose roots are $1-1, 1-\alpha_1, \ldots$.]

*20 If α, β, γ are the roots of $x^3+px+q = 0$, form the equation whose roots are

$$\frac{\beta}{\gamma}+\frac{\gamma}{\beta},\quad \frac{\gamma}{\alpha}+\frac{\alpha}{\gamma},\quad \frac{\alpha}{\beta}+\frac{\beta}{\alpha}.$$

[$y+2 = (\beta^2+\gamma^2+2\beta\gamma)/\beta\gamma = \alpha^2/\beta\gamma = -\alpha^3/q = p\alpha/q+1$.]

13.6 Factorisation in real algebra

13.61 Roots of the general polynomial equation

(1) We have proved that complex roots of an equation with *real* coefficients occur in conjugate pairs (13.45(1)). Since

$$(z-\alpha-\beta i)(z-\alpha+\beta i) \equiv (z-\alpha)^2+\beta^2,$$

the result (ii) of 13.42 shows that *any polynomial $p(z)$ of degree n with real coefficients can be expressed in the form*

$$p(z) \equiv p_0(z-a_1)(z-a_2)\ldots(z-a_k)\{(z-b_1)^2+c_1^2\}\ldots\{(z-b_l)^2+c_l^2\}, \quad \text{(vi)}$$

where

$$k+2l = n,$$

† Cf. 10.11, Remark (β).

and all constants a, b, c are real. The factors shown need not be distinct, and may be all linear (the case $l = 0$) or all quadratic (the case $k = 0$).

(2) Writing $\quad p(x) \equiv p_0 x^n + p_1 x^{n-1} + \ldots + p_n,$

there is an identity

$$p(x) \equiv p_0(x-a_1)(x-a_2)\ldots(x-a_k)\{(x-b_1)^2+c_1^2\}\ldots\{(x-b_l)^2+c_l^2\} \quad \text{(vii)}$$

in real algebra corresponding to (vi), owing to the exact correspondence between real numbers x and complex numbers of the form $x+0i$.

Since $k+2l = n$, we see that

(a) if n is odd, k must also be odd, so that there is at least one factor of the form $x-a$ in (vii), and certainly an odd number of such factors;

(b) if n is even, k must be even, and so either all factors will be quadratic or an even number of linear ones will be present.

Remembering that in real algebra no quadratic factor can be zero, we have the following result.

In real algebra an equation of EVEN *degree has either no roots or an even number of roots; and an equation of* ODD *degree has an odd number of roots* (*and therefore at least one*).

Remark. Just as equation (ii) in 13.42 shows that, in complex algebra, every polynomial can be resolved into linear factors, so equation (vii) above shows that factorisation in real algebra requires no more than linear and irreducible quadratic factors. Both results are *existence theorems*, i.e. they tell us that the factorisation *can be done* in a certain manner, but they provide no process for actually doing it. The results are of great theoretical importance.

13.62 Location of roots in real algebra

(1) *Rational roots of an equation with rational coefficients.* By the method of 13.53, ex. (i), the given polynomial equation can be transformed into one in which the coefficient of the leading term is $+1$ and all other coefficients are *integers*. We suppose that this has been done.

If the polynomial $p(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n$ *has* INTEGRAL *coefficients, then every rational root must be an integer which is a factor of* p_n.

This result is well known and is often used (e.g. in 10.12, ex. (i)).

Proof. Suppose $x = h/k$ is a rational root of $p(x) = 0$. Without loss of generality we can assume that h, k are integers with no common factor and that $k > 0$. Then

$$\left(\frac{h}{k}\right)^n + p_1\left(\frac{h}{k}\right)^{n-1} + \ldots + p_n = 0,$$

so
$$\frac{h^n}{k} = -(p_1 h^{n-1} + p_2 h^{n-1} k + \ldots + p_n k^{n-1})$$
$$= \text{an integer.}$$

Hence h^n/k is an integer, so that $k = 1$. Thus the rational root is in fact an *integer* h. Since
$$h^n + p_1 h^{n-1} + \ldots + p_n = 0,$$
$$\therefore \quad p_n = -h(h^{n-1} + p_1 h^{n-2} + \ldots + p_{n-1}),$$

and so h is a factor of p_n.

(2) *Change of sign of a polynomial.*

If the polynomial $p(x)$ is non-zero when $x = x_1$ and when $x = x_2$, then the number of roots of $p(x) = 0$ between x_1 and x_2 is

odd if $p(x_1)$, $p(x_2)$ have opposite signs,

and even or zero if $p(x_1)$, $p(x_2)$ have the same signs.

These results are 'intuitively obvious' properties of all continuous functions (cf. 2.65). For a *polynomial* $p(x)$ they can be proved as follows.

By result (vii) in 13.61,

$$p(x) \equiv p_0(x-a_1)(x-a_2)\ldots(x-a_k)\,g(x),$$

where $g(x)$ is a product of factors of the form $(x-b)^2+c^2$, or else (in the case when $p(x)$ consists entirely of linear factors) is 1. In either event, $g(x)$ is positive for all values of x.

If $p(x_1)$, $p(x_2)$ have opposite signs, the linear factors cannot be all absent, and $(x_1-a_1)(x_1-a_2)\ldots(x_1-a_k)$, $(x_2-a_1)(x_2-a_2)\ldots(x_2-a_k)$ must have opposite signs. Now x_1-a, x_2-a have like signs unless a lies between x_1 and x_2. Hence an *odd* number of $a_1, a_2, \ldots, a_k$ lie between x_1 and x_2.

Conversely, if an odd number of roots lie between x_1 and x_2, then the above products must have opposite signs, so that also $p(x_1)$, $p(x_2)$ have opposite signs.

If $p(x_1)$, $p(x_2)$ have *like* signs, it follows that the number of roots between x_1 and x_2 *cannot be odd*, i.e. it must be even or zero.

Examples

(i) Let p_s be the numerically greatest coefficient (or one such coefficient if two or more are numerically equal) in

$$p(x) \equiv x^n + p_1 x^{n-1} + \dots + p_n.$$

Then
$$\begin{aligned}|p_1x^{n-1}+\dots+p_{n-1}x+p_n| &\leqslant |p_1|\cdot|x|^{n-1}+\dots+|p_{n-1}|\cdot|x|+|p_n|\\ &\leqslant |p_s|\{|x|^{n-1}+\dots+|x|+1\}\\ &= |p_s|\frac{|x|^n-1}{|x|-1} \quad \text{if} \quad |x| \neq 1.\end{aligned}$$

If $|x| > 1$ it follows that

$$\left|\frac{p(x)}{x^n}-1\right| < \frac{|p_s|}{|x|-1}.$$

Given any positive proper fraction, e.g. $\frac{1}{2}$, we shall have $|p_s|/\{|x|-1\} \leqslant \frac{1}{2}$ if x is such that $|x| \geqslant 2|p_s|+1 = K'$, say. Hence if $K \geqslant K'$,

$$p(x) = x^n(1+\theta) \quad \textit{for all} \quad |x| \geqslant K, \quad \textit{where} \quad |\theta| < \tfrac{1}{2}.$$

This shows that $p(x)$ has the same sign as x^n for all $x \geqslant K$ and all $x \leqslant -K$: we say that $p(x)$ is *dominated* by x^n for large (positive or negative) x. Thus $p(K)$, $p(-K)$ have opposite signs if n is odd, and the same sign if n is even; since K can be as large as we please, *we have proved the theorem in* 13.61 (2).

The symbols $p(\infty)$, $p(-\infty)$ are customarily used to denote '$p(K)$, $p(-K)$ where K is arbitrarily large'.

(ii) *Solve the example in* 13.45 (1) *by a 'change of sign' argument.*

We may suppose the notation to be chosen so that $a_1 < a_2 < \dots < a_n$. Consider the polynomial

$$\begin{aligned}p(x) &\equiv (x-a_1)(x-a_2)\dots(x-a_n)f(x)\\ &\equiv b_1^2(x-a_2)\dots(x-a_n)+b_2^2(x-a_1)(x-a_3)\dots(x-a_n)+\dots\\ &\quad +b_n^2(x-a_1)\dots(x-a_{n-1})+k(x-a_1)(x-a_2)\dots(x-a_n).\end{aligned}$$

When x takes the values

$$-\infty \qquad a_1 \qquad a_2 \qquad \dots \qquad a_{n-1} \quad a_n \quad +\infty,$$

$p(x)$ has the same sign as

$$k(-1)^n \quad (-1)^{n-1} \quad (-1)^{n-2} \quad \dots \quad -1 \quad +1 \quad k.$$

In this sequence there are exactly n changes of sign, whether k is positive or negative. Hence $p(x) = 0$ has at least n roots. Since $p(x)$ is of degree n when $k \neq 0$, it cannot have more than n roots. The number of roots is therefore precisely n.

(3) *Rolle's theorem for polynomials.* This is a special case of the general theorem in 6.21; see 6.23 (1), and also the deductions (2), (3) there. An algebraic proof is indicated in Ex. 3 (*e*), no. 14; also see no. 15.

Examples

(iii) *Find the number of roots of $x^4-2x^3+6x^2-10x-8=0$ in real algebra, and locate them between consecutive integers.*

Writing $$p(x) \equiv x^4-2x^3+6x^2-10x-8,$$

then $$p'(x) \equiv 4x^3-6x^2+12x-10 \equiv 2(x-1)(2x^2-x+5),$$

the factor $x-1$ being discovered by trial.

Hence $p'(x)=0$ has one root in real algebra. By 6.23 (3), $p(x)=0$ cannot have more than two roots; and if $p(x)=0$ actually has two roots, the root $x=1$ of $p'(x)=0$ must lie between them. We therefore consider the sign of $p(x)$ for $-\infty$, 1, $+\infty$.

x	$-\infty$	1	$+\infty$
Sign of $p(x)$	+	−	+

Thus $p(x)=0$ has two roots, one less than 1 and the other greater than 1. By trial we find that $p(0)<0$, $p(-1)>0$, so there is a root between 0 and -1. Similarly we find $p(2)<0$, $p(3)>0$, so the other root lies between 2 and 3.

(iv) *Find the range of values of k for which the equation*

$$p(x) \equiv x^4-26x^2+48x-k=0$$

has 4 distinct roots.

$$p'(x) \equiv 4(x^3-13x+12) \equiv 4(x-1)(x-3)(x+4).$$

Possible roots of $p(x)=0$ must be separated by roots of $p'(x)=0$, viz. -4, 1, 3.

x	$-\infty$	-4	1	3	$+\infty$
Value of $p(x)$	$+\infty$	$-352-k$	$23-k$	$-9-k$	$+\infty$

In order that $p(x)=0$ shall have 4 distinct roots we require there to be four changes of sign in the above sequence; the signs must therefore be

$$+ \quad - \quad + \quad - \quad +.$$

This will be so if and only if $-9<k<23$.

The reader should sketch the graph of $y=x^4-26x^2+48x$, which has a maximum at $x=1$ and a minimum at $x=3$ (and at $x=-4$). The line $y=k$ cuts the curve in four distinct points if and only if this line lies between the levels of these two turning points.

(v) *Prove that the equation $d^4(x^2-1)^4/dx^4=0$ has 4 distinct roots which all lie between -1 and $+1$.*

Putting $f(x)\equiv(x^2-1)^4$, then $f(x)=0$ has roots $+1$, -1 each four-fold. Hence $f'(x)=0$ has roots $+1$, -1 each three-fold (by 10.43) and a root $x=\alpha$ between $+1$, -1 (by Rolle's theorem). Then $f''(x)=0$ has double roots $+1$, -1, a root β between -1 and α, and a root γ between α and $+1$. Similarly, $f'''(x)=0$ has simple roots $+1$, -1, and roots between -1 and β, β and γ, and γ and $+1$: 5 distinct roots altogether. Therefore $f^{(4)}(x)=0$ has a root between each of these, i.e. it has at least 4 distinct roots all lying between $+1$ and -1. Since $f^{(4)}(x)$ is of degree 4, it can have no other roots.

The expression $\{d^n(x^2-1)^n/dx^n\}/2^n n!$ is denoted by $P_n(x)$ and called the *Legendre polynomial of degree n*. An argument similar to the above shows that $P_n(x)=0$ has exactly n distinct roots, all lying between -1 and $+1$. These polynomials arise in Mathematical Physics.

Exercise 13(e)

The algebra is real.

Find the rational roots (if any) of the following equations.

1 $3x^3-11x^2+9x-2=0$. 2 $2x^4+5x^3+14x^2-3x-54=0$.

3 $3x^4-4x^3+6x+5=0$.

Determine the number of (real) roots of the following equations, and locate each between consecutive integers.

4 $x^4+2x^2+3x-1=0$. 5 $x^6-x^3+x^2-2=0$.

6 $x^5-2x^3+x-10=0$.

7 If $a_1<a_2<a_3<a_4<a_5<a_6$, prove that the equation

$$(x-a_1)(x-a_3)(x-a_5)+k^2(x-a_2)(x-a_4)(x-a_6)=0$$

has 3 distinct roots for any real k.

8 If $a>0$, $b>0$ and $p<q$, prove that $a/(x-p)+b/(x-q)=x$ has three roots α, β, γ such that $\alpha<p<\beta<q<\gamma$.

9 If $a_1, a_2, \ldots, a_n$ are all different, prove that the equation $\sum_{r=1}^{n}\dfrac{1}{x-a_r}=0$ has exactly $n-1$ roots (i) by considering changes of sign of

$$p(x)\equiv(x-a_1)\ldots(x-a_n)\sum_{r=1}^{n}\frac{1}{x-a_r};$$

(ii) by applying Rolle's theorem to $f(x)\equiv(x-a_1)\ldots(x-a_n)$.
[We may assume $a_1<a_2<\ldots<a_n$ without loss of generality.]

10 (i) If $p_n<0$ and n is even, prove that the equation

$$p(x)\equiv p_0x^n+p_1x^{n-1}+\ldots+p_n=0$$

has at least one positive and at least one negative root.

(ii) Prove that the number of positive roots of $p(x)=0$ is odd if and only if $p_n<0$.

Determine the number of roots of the following equations by first finding the zeros of the derived polynomial; and locate each between consecutive integers.

11 $x^4+4x^3-8x^2-1=0$. 12 $8x^5-5x^4-40x^3-50=0$.

13 Prove that $x^3+3x-3=0$ has only one root α, and hence that

$$x^4+6x^2-12x-9=0$$

has just two roots.

14 If $a>1$, prove $x^5-5a^4x+4=0$ has 3 roots. What happens when $a<1$?

Find the range of k for which the following equations have 4 distinct roots. Illustrate the results by sketches.

15 $x^4-14x^2+24x-k=0$. 16 $3x^4-16x^3+6x^2+72x-k=0$.

17 If $p^2>q$ and α, β are the roots of $x^2-2px+q=0$, prove that

$$f(x)\equiv x^3-3px^2+3qx-r=0$$

has 3 roots if r lies between $pq-2(p^2-q)\alpha$ and $pq-2(p^2-q)\beta$. [By long division, $f(x) \equiv (x-p)(x^2-2px+q)+2(q-p^2)x+(pq-r)$, so

$$f(\alpha) = 2(q-p^2)\alpha+(pq-r), \quad \text{etc.}]$$

18 Prove that $x^3+ax+b = 0$ has 3 distinct roots if and only if $27b^2+4a^3 < 0$.

*19 Use the argument in ex. (i) of 13.62 to prove that $p(x) = 0$ has every root numerically less than $1+|p_s|$. [If x is a root, $-x^n = p_1x^{n-1}+\ldots+p_n$; by taking moduli, $|x|^n \leqslant |p_s|\{|x|^n-1\}/\{|x|-1\}$ if $|x| \neq 1$. If $|x| > 1$, the right-hand side $< |p_s|\cdot|x|^n/\{|x|-1\}$, so $1 < |p_s|/\{|x|-1\}$, i.e. $|x| < 1+|p_s|$. Thus any root numerically greater than 1 must be numerically less than $1+|p_s|$.]

*20 Replacing 'numerical value' by 'modulus', verify that the results of 13.62, ex. (i) and no. 19 hold in complex algebra, even when the coefficients p_r are not real.

13.7 Approximate solution of equations (further methods)

We have already explained Newton's method (6.73), which applies to equations in general. The following method also has this advantage, although it usually does not give an approximation of a required order as quickly as Newton's.

13.71 The method of proportional parts

For a straight line, the increment of the ordinate is proportional to the increment of the abscissa. For a curve, a small arc will usually not depart far from the chord joining its ends, so that *over this arc* the ratio

(increment of ordinate)/(increment of abscissa)

will not differ much from the gradient of the corresponding chord.

These geometrical considerations lead to the 'principle of proportional parts': *the increment of any continuous function is approximately proportional to the increment of the variable,* the range of this variable being small. An analytical formulation and an estimate of the error are given in Ex. 6 (e), no. 15.

Suppose now that numbers a, b have been found such that $f(a) = A > 0$, $f(b) = B < 0$. Geometrically, the points (a, A), (b, B) are on opposite sides of Ox, and preferably *close* to Ox (a wise choice of a, b will secure this). The line joining these points has equation

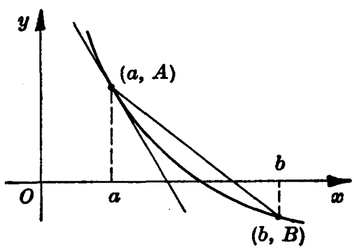

Fig. 135

$$y-A = \frac{B-A}{b-a}(x-a),$$

and cuts Ox at $x = c$ where $c = a+(b-a)A/(A-B)$. This expression lies in value between a, b and is certainly a closer approximation to the root of $f(x) = 0$ than either a or b.

Newton's method is equivalent to replacing the arc by the tangent at one extremity (6.73 (2)). The present method replaces the arc by its chord.

Example

Find the root of $x^3-2x-2=0$ correct to 3 places of decimals.

By trial we find $f(1)=-3, f(2)=2$, so there is a root between 1 and 2. If this root is $1+h$, then by 'proportional parts'

$$\frac{f(2)-f(1)}{2-1} \doteqdot \frac{f(1+h)-f(1)}{(1+h)-1};$$

and since $f(1+h)=0$, this gives $5 \doteqdot 3/h$, i.e. $h \doteqdot 0{\cdot}6$.

A better approximation is therefore $x = 1{\cdot}6$. Since

$$f(1{\cdot}6) = -1{\cdot}1, \quad f(1{\cdot}7) = -0{\cdot}487, \quad f(1{\cdot}8) = +0{\cdot}232,$$

the root actually lies between $1{\cdot}7$ and $1{\cdot}8$. If the root is $1{\cdot}7+h$, then

$$\frac{f(1{\cdot}8)-f(1{\cdot}7)}{1{\cdot}8-1{\cdot}7} \doteqdot \frac{f(1{\cdot}7+h)-f(1{\cdot}7)}{(1{\cdot}7+h)-1{\cdot}7},$$

i.e.
$$\frac{0{\cdot}719}{0{\cdot}1} \doteqdot \frac{0{\cdot}487}{h},$$

so that $h \doteqdot 0{\cdot}0677$, and a better approximation is $x = 1{\cdot}768$.

Applying the process once again,

$$\frac{f(1{\cdot}8)-f(1{\cdot}768)}{1{\cdot}8-1{\cdot}768} \doteqdot \frac{f(1{\cdot}768+h)-f(1{\cdot}768)}{h},$$

i.e.
$$\frac{0{\cdot}241}{0{\cdot}032} \doteqdot \frac{0{\cdot}009}{h}$$

since $f(1{\cdot}768) \doteqdot 0{\cdot}009$. Hence $h \doteqdot 0{\cdot}001195$, and $x \doteqdot 1{\cdot}76919 \doteqdot 1{\cdot}769$ to three places of decimals. (Taking $a = 1{\cdot}768$, $b = 1{\cdot}8$, $c = 1{\cdot}769$ in Ex. 6(*e*), no. 15, shows that the error is less than $0{\cdot}0002$.)

13.72 Horner's method

Suppose that the equation $p(x) = 0$ has a root $2{\cdot}76\ldots$ and has no other root whose integral part is 2. Then $p(2), p(3)$ have opposite signs; this fact, discovered by trial, gives the first figure of the root.

The first figure 2 is now removed from subsequent calculations by diminishing the roots of $p(x) = 0$ by 2. The new equation then has a root $0{\cdot}76\ldots$. To avoid decimals, we multiply the roots by 10, so that the new equation has a root $7{\cdot}6\ldots$.

The figure 7 is discovered by showing that the last equation has a root between 7 and 8. The roots are then diminished by 7, and next multiplied by 10. The figure 6 is then found by trial. The process is repeated until the required number of decimal places has been found.

Example

Solve $x^3-2x-2=0$ correct to 3 places of decimals.

By trial, there is a root between 1 and 2. Diminish the roots by 1 by putting $y = x-1$, i.e. $x = y+1$; we get

$$y^3+3y^2+y-3=0.$$

Multiply the roots by 10:

$$y^3+30y^2+100y-3000=0.$$

By trial, this has a root between 7 and 8. Diminish the roots by 7 by putting $z = y-7$, i.e. $y = z+7$; we get

$$z^3+51z^2+667z-487 = 0.$$

Multiply the roots by 10:

$$z^3+510z^3+66{,}700z-487{,}000 = 0.$$

By trial this has a root between 6 and 7. Diminish the roots by 6, putting $t = z-6$, i.e. $z = t+6$; we obtain

$$t^3+528t^2+72{,}928t-68{,}224 = 0.$$

Multiply the roots by 10:

$$t^3+5280t^2+7{,}292{,}800t-68{,}224{,}000 = 0.$$

The process can be continued; but since the numbers in the last two terms are large in comparison with those in the other terms, clearly a good approximation is given by

$$7{,}292{,}800t-68{,}224{,}000 \doteqdot 0,$$

i.e. $t \doteqdot 9$.

Hence the required root is $x \doteqdot 1{\cdot}769$.

Remarks

(α) The method used in the last step gives a rough estimate for the trial root when applied at the previous stage: $66{,}700z-487{,}000 \doteqdot 0$ gives $z \doteqdot 7$. Actually this is too large; writing

$$\phi(z) \equiv z^3+510z^2+66{,}700z-487{,}000,$$

we should find that $\phi(7)$ and $\phi(8)$ have the same sign. This indicates that we should calculate $\phi(6)$, which is found to have the opposite sign. Although this estimate is rather rough, it is better than none, and often saves much futile trial. Estimates thus made increase in accuracy the further we have got in the process; in fact we may approximately double the number of significant figures already obtained by using this 'division estimate'.

(β) Horner's method will also give rational roots. Since these can always be tested for exhaustively (13.62(1)), they should be removed from the equation before applying the method. The same applies to repeated roots: see 10.51, ex. (ii).

(γ) To approximate to a *negative* root, first change the sign of all the roots (by putting $y = -x$), and then approximate to the corresponding positive root of the new equation.

(δ) *Case of nearly equal roots.* If two roots lie between consecutive integers n and $n+1$, then $p(n)$ and $p(n+1)$ have the same sign, and detection of these roots is difficult; more refined methods than 'change of sign' are required. Assuming that these roots have been approximately located, Horner's method can still be used.

(ϵ) Although the method applies in principle to equations in general, its use is practicable only with *polynomial* equations.

13.73 von Graeffe's method of root-squaring

This method approximates to all the roots in one process, although the *signs* have to be sorted out at the end. It can also be used to find complex roots, but is applicable to polynomial equations only.

Given a cubic $x^3+ax^2+bx+c=0$, whose roots are α, β, γ (say), we construct the cubic whose roots are $\alpha^2, \beta^2, \gamma^2$ by putting $y=x^2$: the equation becomes

$$\sqrt{y}\,(y+b) = -(ay+c),$$

and by squaring, $$y^3+(2b-a^2)\,y^2+(b^2-2ac)\,y-c^2=0,$$

i.e. $$y^3+Ay^2+By+C=0,$$

where $$A=2b-a^2,\quad B=b^2-2ac,\quad C=-c^2.$$

If the root-squaring process is applied n times, we obtain an equation whose roots are $\alpha^{2^n}, \beta^{2^n}, \gamma^{2^n}$. Suppose that $|\alpha|>|\beta|>|\gamma|$; then for n sufficiently large, α^{2^n} will be very large compared with β^{2^n} and γ^{2^n}, so that α^{2^n} is approximately 'minus the coefficient of the quadratic term' in this equation.

Since β^{2^n} is large compared with γ^{2^n}, the coefficient of the linear term will be approximately $(\alpha\beta)^{2^n}$. Similarly the constant term is approximately $-(\alpha\beta\gamma)^{2^n}$.

Example

Solve $x^3-2x-2=0$ in complex algebra.

Here $a=0$, $b=-2$, $c=-2$. Proceeding step by step, we construct the following table.

	$A=2b-a^2$	$B=b^2-2ac$	$C=-c^2$
α^2	-4	4	-4
α^4	-8	-16	-16
α^8	-96	0	-256
α^{16}	-9216	$-49{,}152$	$-65{,}536$

By trial the given equation is found to have a root α near 1·7; and this is the *only* real root (e.g. see Ex. 13(*e*), no. 18). Since $\alpha\beta\gamma=2$, it follows that $\beta\gamma \doteqdot 2\div 1{\cdot}7 \doteqdot 1{\cdot}2$, i.e. $r^2 \doteqdot 1{\cdot}2$ where $\beta = r(\cos\theta+i\sin\theta)$, $\gamma=r(\cos\theta-i\sin\theta)$; hence $r\doteqdot 1{\cdot}1$. Thus $\alpha>r=|\beta|=|\gamma|$.

From the theory just explained,

$$\alpha^{16}\doteqdot -A = 9216, \quad \text{whence} \quad \alpha\doteqdot 1{\cdot}7692;$$

and $$\alpha^{16}(r^2)^{16}\doteqdot -C = 65{,}536, \quad \text{so} \quad r\doteqdot 1{\cdot}0632.$$

Referring now to the *given* equation and considering the sum of its roots, $\alpha+2r\cos\theta=0$, from which $\cos\theta\doteqdot -0{\cdot}83215$ and $\theta\doteqdot 146^\circ\,19'$. Hence $r(\cos\theta\pm i\sin\theta)$ can be found. The three roots are approximately

$$1{\cdot}7692, \quad -0{\cdot}8847\pm 0{\cdot}5896i.$$

Exercise 13(*f*)

The algebra is real.

Solve the following equations approximately, correct to at least 2 places of decimals.

1 $x^3-2x-5=0$.

2 $3x^3-9x+2=0$.

3 $x^3-2x+5=0$ [negative root].

4 $x^3+x^2+x-100=0$ [two places of decimals].

5 $x^5-4x-2=0$ [positive root, two places].

6 $x^3-6x+1=0$ [3 roots].

7 $x^3-7x+7=0$ [*two* roots lie between 1 and 2].

8 $x = e^{-x}$ [three places].

9 $10^x = 20x$ [2 roots, each to three places].

*10 $x^x = 1{\cdot}5$. [This has a root between 1 and 2; write the equation as $x \log x = \log 1{\cdot}5$.]

*11 Find the smallest positive root of $x = \tan^{-1} x$, correct to four places of decimals.

*12 Find the smallest positive root of $\sin x = \operatorname{th} x$, correct to two places of decimals.

*13 If ϵ is small, prove that $\theta + \sin\theta\cos\theta = 2\epsilon\cos\theta$ has a small root which is approximately $\epsilon - \frac{1}{6}\epsilon^3$. [Use Newton's method; assume the series for $\sin x$, $\cos x$.]

*14 Show (graphically or otherwise) that $f(x) \equiv 1 + x^{-2} - \tan x = 0$ has a root near $x = K$, where $K = \frac{1}{4}\pi + n\pi$ and n is a large positive integer. Prove that a better approximation is $K + \lambda$ where $f(K) + \lambda f'(K) = 0$, and that $\lambda \doteqdot 1/2K^2$.

Find an approximation correct to terms of order $1/K^4$.

*15 Prove that large roots of $\sec x = 1 + 1/x$ are given approximately by $x = 2n\pi$ where n is a large positive integer. Show that a better approximation is $x = 2n\pi \pm 1/\sqrt{(n\pi)}$. [Assume the series for $\cos\theta$.]

Miscellaneous Exercise 13(*g*)

1 If c is real, and the number $(1+i)/(2+ci) + (2+3i)/(3+i)$ is represented in the xy-plane by a point on the line $y = x$, prove $c = -5 \pm \sqrt{21}$.

2 If $c^2 + s^2 = 1$, prove $(1+c+is)/(1+c-is) = c+is$, and write down a similar result for $(1+s+ic)/(1+s-ic)$.

3 If $$l^2 + m^2 + n^2 = 1 \quad \text{and} \quad m + in = (1+l)z,$$
prove $$\frac{l+im}{1+n} = \frac{1+iz}{1-iz}.$$

4 If $z = r(\cos\theta + i\sin\theta)$ and $a = \rho(\cos\alpha + i\sin\alpha)$, calculate $|z-a|^2$ in terms of r, ρ, θ, α. Deduce that $|1-\bar{a}z|^2 - |z-a|^2 = (1-r^2)(1-\rho^2)$.

5 If $|z_1 - z_2| \leqslant \frac{1}{2}|z_2|$, prove (i) $|z_1| \geqslant \frac{1}{2}|z_2|$; (ii) $|z_1 + z_2| \geqslant \frac{3}{2}|z_2|$.

6 The number z is represented by a point on the circle whose centre is $1 + 0i$ and radius is 1.

(i) Represent the number $z - 2$, and prove $(z-2)/z = i\tan(\arg z)$.

(ii) Construct the point representing z^2, and prove that

$$(a)\ |z^2 - z| = |z| \quad \text{and} \quad (b)\ \arg(z-1) = \arg(z^2) = \tfrac{2}{3}\arg(z^2 - z).$$

7 (i) If P_1, P_2 represent z_1, z_2, and the line P_1P_2 is turned through a positive right-angle about P_1 into the position P_1P_3, prove that P_3 represents $z_1 + i(z_2 - z_1)$.

(ii) Two opposite vertices of a square are $1 + 2i$, $3 - 5i$. Find the numbers of the other vertices.

8 (i) If k is a *real* constant, interpret $z_1 + k(z_2 - z_1)$ geometrically.

(ii) The internal and external bisectors of $P_1\hat{O}P_2$ meet P_1P_2 at I, E, and M is the mid-point of IE. If $z_1 = \cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi$, $z_2 = 2(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)$, show that I represents $\frac{1}{3}(1+\sqrt{3})(1+i)$, and find the numbers of E and M.

9 The numbers z_1, z_2, z_3 are represented by the vertices P_1, P_2, P_3 of an isosceles triangle, the angles at P_2, P_3 each being $\frac{1}{2}(\pi - \alpha)$. Prove that

$$(z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\tfrac{1}{2}\alpha.$$

10 Interpret the following as loci:

(i) $|z+3i|^2-|z-3i|^2 = 12$; (ii) $|z+ki|^2+|z-ki|^2 = 10k^2 \quad (k>0)$;

(iii) $\arg\dfrac{z-z_1}{z-z_2} = \frac{1}{3}\pi$; *(iv) $|z-z_1|-|z-z_2| = 1$.

11 The numbers z_1, z_2 are connected by the equation $z_1 = z_2+1/z_2$.

(i) If P_2 describes a circle of radius $a \neq 1$ and centre O, show that P_1 describes the ellipse

$$\frac{x^2}{(1+a^2)^2}+\frac{y^2}{(1-a^2)^2} = \frac{1}{a^2}.$$

(ii) If $z_2 = r(\cos\theta+i\sin\theta)$, determine the locus of P_1 when $\theta = \frac{1}{4}\pi$ and r varies.

12 If $u+iv = a/z$ where a is real, show that the curves in the xy-plane which correspond to u = constant, v = constant are (in general) orthogonal systems of circles.

13 If $\{(z+c)/(z-c)\}^2 = (w+2c)/(w-2c)$ and c is real, prove that when $z = c(\cos\theta+i\sin\theta)$, then $w = 2c\cos\theta$. Hence show that if z describes the circle $|z| = c$, then w describes the segment of the x-axis between $\pm 2c$, once in each direction.

14 If z, w are represented by P, Q and $zw+w-z+1 = 0$, prove that when $z = \cos\theta+i\sin\theta$, then $|w| = \pm\tan\frac{1}{2}\theta$ according as $0 \leqslant \theta \leqslant \pi$, $-\pi \leqslant \theta \leqslant 0$. If P describes the circle $|z| = 1$, prove that Q describes the y-axis, and indicate corresponding directions of motion.

*15 An ellipse has foci $(\pm ae, 0)$ and z_1, z_2 correspond to the ends of conjugate semi-diameters. Prove $z_1^2+z_2^2 = a^2e^2$. [Using eccentric angles, if

$$z_1 = a\cos\phi+ib\sin\phi \quad \text{then} \quad z_2 = \pm(a\sin\phi-ib\cos\phi).]$$

*16 If $u = (z^2+az+b)/(z^2+cz+d)$ and a, b, c, d are real, prove that the points z for which u is real lie either on the real axis or on a circle whose centre is on the real axis. [Clear fractions, equate real and imaginary parts, and eliminate u.]

In the following, ω and ω^2 denote the cube roots of $+1$, $\omega \neq 1$.

17 If $x = a+b$, $y = a+b\omega$, $z = a+b\omega^2$, prove that $\Sigma x = 3a$, $\Sigma yz = 3a^2$, $xyz = a^3+b^3$, $\Sigma x^2 = 3a^2$, $\Sigma x^3 = 3(a^3+b^3)$.

18 (i) If $f(x) = \sum_{r=1}^{3n} a_r x^r$, what does $f(x)+f(\omega x)+f(\omega^2 x)$ represent?

*(ii) Calculate similarly $f(x)+\omega f(\omega x)+\omega^2 f(\omega^2 x)$ and $f(x)+\omega^2 f(\omega x)+\omega f(\omega^2 x)$.

19 If $z^n+p_1z^{n-1}+\dots+p_n \equiv (z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n)$, prove that

$$(1+\alpha_1^2)(1+\alpha_2^2)\dots(1+\alpha_n^2) = (1-p_2+p_4-\dots)^2+(p_1-p_3+p_5-\dots)^2.$$

[$1+p_1t+p_2t^2+\dots+p_nt^n \equiv (1-\alpha_1 t)\dots(1-\alpha_n t)$. Put $t = \pm i$, then multiply.]

20 Find conditions for $z^2+(a+bi)z+(c+di) = 0$ to have a real root, where a, b, c, d are real.

21 If $z = \alpha+\beta i$ is a root of $z^3+pz+q = 0$ (p, q real), prove α is a real root of $8x^3+2px-q = 0$.

22 Find the equation of lowest degree with rational coefficients which has $1, 2-\sqrt{3}, \sqrt{2}-1$ for roots.

23 Prove $3x^5-5x^3+k = 0$ has three real roots if $-2 < k < 2$, and one real root if $k > 2$ or $k < -2$.

24 Find the condition for $3x^4+4px^3+s = 0$ to have no real roots (p, q real).

25 Prove that for n *odd*, $a+x+\frac{1}{2}x^2+\ldots+x^n/n = 0$ has one root; and that for n *even* there are 0 or 2 roots according as $a \gtrless b$, where b is a certain number which is to be determined.

By using the substitution $y = x+1/x$, solve the following equations† *in complex algebra.*

26 $12x^4-4x^3-41x^2-4x+12 = 0.$

27 $x^5-5x^4+9x^3-9x^2+5x-1 = 0.$ [Remove the obvious root $x = 1$ first.]

28 $2x^5+5x^4+8x^3+8x^2+5x+2 = 0.$

29 Sketch the graph of $y = \sec x$. Deduce that large roots of $x\cos x = 2$ are approximately $(n+\frac{1}{2})\pi$ where n is a large positive integer. Find a closer approximation.

30 Show graphically that $\sin x = \operatorname{th} x$ has an infinity of roots. If n is a large positive integer, show that pairs of roots lie in the neighbourhood of $x = (2n+\frac{1}{2})\pi$, and that a closer approximation to these roots is

$$x = (2n+\tfrac{1}{2})\pi \pm 2e^{-(2n+\frac{1}{2})\pi}.$$

[Express $\operatorname{th} x$ in terms of e.]

† Since each is unaltered when x is replaced by $1/x$, they are called *reciprocal equations*.

14

DE MOIVRE'S THEOREM AND SOME APPLICATIONS

14.1 de Moivre's theorem

14.11 *If n is an integer (positive or negative), then*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta;$$

if n is rational, then $\cos n\theta + i\sin n\theta$ *is one of the values*† *of*

$$(\cos\theta + i\sin\theta)^n.$$

Proof. (i) *Let n be a positive integer.* By direct multiplication,

$$\begin{aligned}&(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &\quad= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\\ &\quad= \cos(\theta+\phi) + i\sin(\theta+\phi).\end{aligned}$$

Similarly,

$$\begin{aligned}&(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\\ &\quad= \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}(\cos\theta_3 + i\sin\theta_3)\\ &\quad= \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)\end{aligned}$$

by two applications of the above result. Proceeding step by step in this way, we find

$$(\cos\theta_1 + i\sin\theta_1)\dots(\cos\theta_n + i\sin\theta_n) = \cos(\Sigma\theta) + i\sin(\Sigma\theta).$$

Putting $\theta_1 = \theta_2 = \dots = \theta_n = \theta$, this becomes

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

(ii) *Let n be a negative integer,* say $n = -m$. Then

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m}\\ &= \frac{1}{(\cos\theta + i\sin\theta)^m}\\ &= \frac{1}{\cos m\theta + i\sin m\theta}\end{aligned}$$

† See the definition in Case (iii) below.

by Case (i), since m is a positive integer. Now

$$(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta) = \cos^2 m\theta + \sin^2 m\theta = 1,$$

so
$$\begin{aligned}\frac{1}{\cos m\theta + i\sin m\theta} &= \cos m\theta - i\sin m\theta \\ &= \cos(-m\theta) + i\sin(-m\theta) \quad \text{by trigonometry,} \\ &= \cos n\theta + i\sin n\theta \quad \text{since } n = -m.\end{aligned}$$

Hence again $\quad (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

(iii) *Let n be rational*, say $n = p/q$ where (without loss of generality) we can suppose that p and q are integers and $q > 0$. So far no meaning has been assigned to the expression $z^{p/q}$ when z is complex. We now *define* the values of $z^{p/q}$ to be the roots of the equation $\zeta^q = z^p$.

Consider

$$\begin{aligned}&\left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)^q \\ &= \cos p\theta + i\sin p\theta \quad \text{by Case (i), since } q \text{ is a positive integer,} \\ &= (\cos\theta + i\sin\theta)^p \quad \text{by Case (i) or (ii) according as the integer } p \text{ is positive or negative.}\end{aligned}$$

Hence by the definition just given,

$$\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q} \quad \text{is a value of} \quad (\cos\theta + i\sin\theta)^{p/q}.$$

14.12 The values of $(\cos\theta + i\sin\theta)^{p/q}$

Still supposing that p, q are integers and $q > 0$, let $s(\cos\phi + i\sin\phi)$ be any one of the values of $(\cos\theta + i\sin\theta)^{p/q}$. This means that, on raising each expression to the qth power,

$$s^q(\cos\phi + i\sin\phi)^q = (\cos\theta + i\sin\theta)^p,$$

i.e.
$$s^q(\cos q\phi + i\sin q\phi) = \cos p\theta + i\sin p\theta.$$

It now follows as in 13.22 (2) that $s^q = 1$ and $q\phi = p\theta + 2k\pi$, where k is an integer or zero. Since s is the modulus of a complex number, it is by definition real and positive, so that $s^q = 1$ implies $s = +1$.

Taking $k = 0, 1, 2, \ldots, q-1$, we obtain the q expressions

$$\cos\left(\frac{p\theta}{q} + \frac{2k\pi}{q}\right) + i\sin\left(\frac{p\theta}{q} + \frac{2k\pi}{q}\right) \tag{i}$$

as possible values of $(\cos\theta + i\sin\theta)^{p/q}$.

These q values are distinct, because angles given by $k = k_1$, $k = k_2$ differ by $2\,|k_1 - k_2|\,\pi/q$, which is less than 2π since $|k_1 - k_2| < q$.

No further values are given by taking other values of k, because any other value of k differs from one of the values $0, 1, 2, \ldots, q-1$ by some integral multiple of q.

Consequently, $(\cos\theta + i\sin\theta)^{p/q}$ has exactly q different values, viz. those in (i) given by $k = 0, 1, 2, \ldots, q-1$. Any other set of q integral values of k could be chosen, provided they give q distinct values of (i); e.g. we often take $k = 0, \pm 1, \pm 2, \ldots$.

The q values (i) are represented in the Argand diagram by points

$$P_1, P_2, \ldots, P_q$$

on the unit circle $|z| = 1$:

$$\angle(Ox, OP_1) = p\theta/q,$$

and arcs P_1P_2, P_2P_3, $\ldots$, $P_{q-1}P_q$ subtend angles $2\pi/q$ at the centre O. Thus $P_1P_2P_3 \ldots P_q$ is a regular q-sided polygon inscribed in the unit circle.

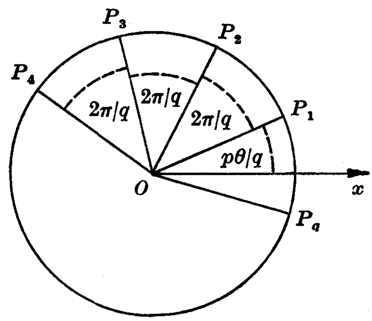

Fig. 136

Since every complex number z can be written in the form

$$r(\cos\theta + i\sin\theta), \quad \text{where} \quad -\pi < \theta \leqslant \pi,$$

the above shows that the values of $z^{p/q}$ are

$$r^{p/q}\left\{\cos\left(\frac{p\theta}{q} + \frac{2k\pi}{q}\right) + i\sin\left(\frac{p\theta}{q} + \frac{2k\pi}{q}\right)\right\},$$

where $k = 0, 1, 2, \ldots, q-1$ and (as elsewhere in this book) $r^{p/q}$ denotes the *positive* qth root of r^p. They are represented by points equally spaced around the circle $|z| = r^{p/q}$.

Definition. It will often be convenient to abbreviate $\cos\theta + i\sin\theta$ to $\operatorname{cis}\theta$.

14.13 Examples

(i) *Solve the equation* $(z+1)^n = z^n$.

If the left-hand side were expanded, this would reduce to an equation of degree $n-1$ in z. Hence in complex algebra there will be $n-1$ roots.

Since $z = 0$ is not a solution, the equation can be written

$$\left(\frac{z+1}{z}\right)^n = 1 \qquad (a)$$

$$= \cos 0 + i\sin 0.$$

By taking nth roots of both sides, and using the general result (i) in 14.12,

$$\frac{z+1}{z} = \cos\left(0+\frac{2k\pi}{n}\right)+i\sin\left(0+\frac{2k\pi}{n}\right) \quad (k = 0, 1, 2, \ldots, n-1).$$

$$\therefore \quad z\left(\cos\frac{2k\pi}{n}-1+i\sin\frac{2k\pi}{n}\right) = 1,$$

i.e.† $$z\left(-2\sin^2\frac{k\pi}{n}+2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n}\right) = 1,$$

and $$2iz\sin\frac{k\pi}{n}\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right) = 1.$$

Hence by the theorem (with index -1)

$$2iz\sin\frac{k\pi}{n} = \cos\frac{k\pi}{n}-i\sin\frac{k\pi}{n} \quad (k = 0, 1, 2, \ldots, n-1).$$

If $k \neq 0$, $$z = \frac{1}{2i}\left(\cot\frac{k\pi}{n}-i\right)$$

$$= -\frac{1}{2}\left(1+i\cot\frac{k\pi}{n}\right) \quad (k = 1, 2, \ldots, n-1).$$

These are the $n-1$ solutions.

Alternatively, instead of quoting the general theory, we may proceed from stage (a) above by writing

$$\left(\frac{z+1}{z}\right)^n = \cos 2k\pi+i\sin 2k\pi;$$

hence by de Moivre's theorem,

$$\frac{z+1}{z} = \cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n},$$

which takes distinct values for $k = 0, 1, 2, \ldots, n-1$; etc.

(ii) *Show that the roots of* $z^n = 1$ *can be written* $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

From $$z^n = 1 = \cos 2k\pi+i\sin 2k\pi,$$

$$z = \cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n},$$

which takes distinct values when $k = 0, 1, 2, \ldots, n-1$.

The value $k = 0$ gives $z = 1$. Write

$$\alpha = \cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$$

for the case $k = 1$; then since

$$\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n} = \left(\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)^k = \alpha^k,$$

where $k = 2, 3, \ldots, n-1$, the n roots are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

Remark. Since the sum of the roots of $z^n-1 = 0$ is zero (13.51), we have

$$1+\alpha+\alpha^2+\ldots+\alpha^{n-1} = 0.$$

† This step of passing to half-angles is often useful.

Compare the brevity of this work with the algebraical treatment of the equation $z^3 = 1$ in 13.23. Observe also that

$$\alpha^{n-r} = \text{cis}\left\{\frac{2(n-r)\pi}{n}\right\} = \text{cis}\left\{2\pi - \frac{2r\pi}{n}\right\}$$

$$= \text{cis}\left\{-\frac{2r\pi}{n}\right\} = \cos\frac{2r\pi}{n} - i\sin\frac{2r\pi}{n} = \overline{\alpha^r},$$

and hence $\alpha^{n-1} = \overline{\alpha}$, $\alpha^{n-2} = \overline{\alpha}^2$, ... Also see Ex. 14(*a*), no. 21.

*(iii) *If p, q are coprime integers (i.e. have no common factor other than* ± 1), *the q values of* $(\cos\theta + i\sin\theta)^{p/q}$ *can be written*

$$\cos\frac{p}{q}(\theta + 2k\pi) + i\sin\frac{p}{q}(\theta + 2k\pi), \qquad \text{(ii)}$$

where $k = 0, 1, 2, \ldots, q-1$.

These q expressions each satisfy $z^q = \cos p\theta + i\sin p\theta$, and are therefore values of $(\cos\theta + i\sin\theta)^{p/q}$. They are all *distinct*; for if

$$\text{cis}\frac{p}{q}(\theta + 2k_1\pi) = \text{cis}\frac{p}{q}(\theta + 2k_2\pi),$$

then as in 13.22 (2),

$$\frac{p}{q}(\theta + 2k_1\pi) - \frac{p}{q}(\theta + 2k_2\pi) = 2m\pi$$

for some integer m, i.e. $p(k_1 - k_2) = mq$. Since $|k_1 - k_2| < q$, this shows that q has a factor in common with p, which is impossible because p, q are coprime.

Remarks

(α) Since the equation $z^q = \text{cis}\, p\theta$ has exactly q roots, *the set of values of* (ii) *must be the same as the set of values of* (i), *in some order.*

(β) If p, q are not coprime, the function $(\cos\theta + i\sin\theta)^{p/q}$ is understood to mean $\{(\cos\theta + i\sin\theta)^p\}^{1/q}$, and therefore has q distinct values, viz. the values (i) of $(\cos p\theta + i\sin p\theta)^{1/q}$. However, in this case the expression $\text{cis}\{(p/q)(\theta + 2k\pi)\}$ does not take q distinct values, and consequently does not represent all the values of $(\cos\theta + i\sin\theta)^{p/q}$. See Ex. 14(*a*), no. 8.

Exercise 14(*a*)

1 If $x = \text{cis}\,\theta$, $y = \text{cis}\,\phi$, and m, n are integers, prove

$$\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi).$$

2 Simplify $$\frac{(1 + \cos\theta + i\sin\theta)^5}{(\cos\theta - i\sin\theta)^4}.$$

3 Prove $\left(\dfrac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n = \text{cis}(\tfrac{1}{2}n\pi - n\theta)$, n being an integer.

4 If $x = \text{cis}\,\theta$, prove $(x^{2n} - 1)/(x^{2n} + 1) = i\tan n\theta$, where n is an integer.

5 Find the modulus and the principal value of the argument of $(-1 + i\sqrt{3})^5$.

6 Find the three cube roots of $2i - 2$.

7 Obtain all the values of $\{(\sqrt{2} + 1 + i)/(\sqrt{2} + 1 - i)\}^{\frac{1}{4}}$ in the form $a + bi$.

8 Write down the six values of $(\cos\theta+i\sin\theta)^{\frac{3}{6}}$ and the two values of $(\cos\theta+i\sin\theta)^{\frac{1}{2}}$. Observe that $\text{cis}\{\frac{3}{6}(\theta+2k\pi)\}$ has only two distinct values, and therefore cannot completely represent the first function.

9 Simplify $(\cos\theta+i\sin\theta)^3$ in two ways. By equating real and imaginary parts, deduce that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \quad \text{and} \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta,$$

and express $\tan 3\theta$ in terms of $\tan\theta$.

10 If $z = \text{cis}\,\theta$, express in terms of θ:

$$\text{(i) } \frac{1}{z}; \quad \text{(ii) } z+\frac{1}{z}; \quad \text{(iii) } z-\frac{1}{z}; \quad \text{(iv) } z^n+\frac{1}{z^n}; \quad \text{(v) } z^n-\frac{1}{z^n}.$$

Solve completely the following equations.

11 $z^n = -1$.

12 $(z+1)^n+(z-1)^n = 0$.

13 $(z-1)^n = z^n$.

14 $(1+z)^n = (1-z)^n$.

15 $z^{2n}-2z^n\cos n\alpha+1 = 0$. [First solve as a quadratic in z^n.]

16 By expanding $(1+z)^n$, where n is a positive integer, and putting $z = \text{cis}\,\theta$, prove

$$(2\cos\tfrac{1}{2}\theta)^n\,\text{cis}\,\tfrac{1}{2}n\theta = \sum_{r=0}^{n} {}^nC_r\,\text{cis}\,r\theta.$$

Deduce that
$$\sum_{r=0}^{n} {}^nC_r\cos\frac{r\pi}{2n} = 2^{n-\frac{1}{2}}\left(\cos\frac{\pi}{4n}\right)^n.$$

17 If n is a positive integer, prove $(1+i)^n+(1-i)^n = 2^{\frac{1}{2}n+1}\cos\frac{1}{4}n\pi$. Writing

$$(1+x)^n \equiv c_0+c_1x+c_2x^2+\ldots+c_nx^n,$$

prove
$$c_0-c_2+c_4-\ldots = 2^{\frac{1}{2}n}\cos\tfrac{1}{4}n\pi$$

and
$$c_1-c_3+c_5-\ldots = 2^{\frac{1}{2}n}\sin\tfrac{1}{4}n\pi.$$

18 If $\alpha = \text{cis}\,\frac{2}{7}\pi$ and $\beta = \alpha+\alpha^2+\alpha^4$, $\gamma = \alpha^3+\alpha^5+\alpha^6$, prove $\beta+\gamma = -1$ and $\beta\gamma = 2$. Write down the quadratic having roots β, γ, and deduce that

$$\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7} = +\tfrac{1}{2}\sqrt{7}.$$

*19 If $\cos\theta+\cos\phi+\cos\psi = 0$ and $\sin\theta+\sin\phi+\sin\psi = 0$, prove that $\cos 3\theta+\cos 3\phi+\cos 3\psi-3\cos(\theta+\phi+\psi) = 0$ and a similar result for sines. [Put $x = \text{cis}\,\theta$, $y = \text{cis}\,\phi$, $z = \text{cis}\,\psi$ and use Ex. 10 (*f*), no. 4 (iii).]

20 What conditions have to be satisfied by z in order that the points representing all integral powers of z should (i) lie on a circle with centre the origin, and (ii) be finite in number? Mark in a diagram the points which represent a number z such that there are only three distinct points given by the sequence $z, z^2, z^3, \ldots$.

*21 If n is any *prime* number and λ denotes any complex root of $z^n = 1$, prove that the numbers $1, \lambda, \lambda^2, \ldots, \lambda^{n-1}$ are some arrangement of the numbers $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ in 14.13, ex. (ii). [Use the argument in 14.13, ex. (iii).]

14.2 Use of the binomial theorem

14.21 $\cos^m\theta \sin^n\theta$ in terms of multiple angles (m, n being positive integers or zero)

This transformation is sometimes needed for the integration of circular functions: see 4.82. Writing $z = \cos\theta + i\sin\theta$, then

$$\frac{1}{z} = \cos\theta - i\sin\theta,$$

so
$$2\cos\theta = z+\frac{1}{z}, \quad 2i\sin\theta = z-\frac{1}{z}. \tag{i}$$

Also $z^n = \cos n\theta + i\sin n\theta$ and $1/z^n = \cos n\theta - i\sin n\theta$, so that

$$z^n+\frac{1}{z^n} = 2\cos n\theta, \quad z^n-\frac{1}{z^n} = 2i\sin n\theta. \tag{ii}$$

By means of (i) and the binomial theorem, the given function can be expanded in powers of z and $1/z$. By (ii) the resulting expansion can be expressed in multiple angles.

Examples

(i) *Express* $\cos^6\theta$ *in terms of circular functions of multiple angles.*

From (i),

$$\begin{aligned}(2\cos\theta)^6 &= \left(z+\frac{1}{z}\right)^6\\ &= z^6+6z^4+15z^2+20+\frac{15}{z^2}+\frac{6}{z^4}+\frac{1}{z^6}\\ &= \left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20\\ &= 2\cos 6\theta+6(2\cos 4\theta)+15(2\cos 2\theta)+20 \quad \text{by (ii).}\end{aligned}$$

$$\therefore \quad \cos^6\theta = \tfrac{1}{32}(\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10).$$

(ii) *Express* $\cos^5\theta\sin^4\theta$ *in terms of multiple angles.*

$$\begin{aligned}(2\cos\theta)^5(2i\sin\theta)^4 &= \left(z+\frac{1}{z}\right)^5\left(z-\frac{1}{z}\right)^4\\ &= \left(z+\frac{1}{z}\right)\left(z^2-\frac{1}{z^2}\right)^4\\ &= \left(z+\frac{1}{z}\right)\left(z^8-4z^4+6-\frac{4}{z^4}+\frac{1}{z^8}\right)\\ &= \left(z^9+\frac{1}{z^9}\right)+\left(z^7+\frac{1}{z^7}\right)-4\left(z^5+\frac{1}{z^5}\right)-4\left(z^3+\frac{1}{z^3}\right)+6\left(z+\frac{1}{z}\right)\\ &= 2\cos 9\theta+2\cos 7\theta-4(2\cos 5\theta)-4(2\cos 3\theta)+6(2\cos\theta).\end{aligned}$$

$$\therefore \quad \cos^5\theta\sin^4\theta = \tfrac{1}{256}(\cos 9\theta+\cos 7\theta-4\cos 5\theta-4\cos 3\theta+6\cos\theta).$$

We observe that $\cos^n\theta$ *can always be expanded in terms of cosines* because it depends on the expansion of $(z+1/z)^n$. Since $\sin^m\theta$ depends on $(z-1/z)^m$, which involves terms like z^r+1/z^r if m is even, and terms like z^r-1/z^r if m is odd, therefore $\sin^m\theta$ *can be expressed in terms of cosines or sines according as m is even or odd.*

14.22 cos *n*θ, sin *n*θ, tan *n*θ as powers of circular functions (*n* being an integer)

We now reverse the process in 14.21:

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = (c+is)^n, \quad \text{say,}$$

$$= c^n + \binom{n}{1}c^{n-1}is + \binom{n}{2}c^{n-2}(is)^2 + \binom{n}{3}c^{n-3}(is)^3 + \binom{n}{4}c^{n-4}(is)^4 + \dots$$

$$= \left\{c^n - \binom{n}{2}c^{n-2}s^2 + \binom{n}{4}c^{n-4}s^4 - \dots\right\} + i\left\{\binom{n}{1}c^{n-1}s - \binom{n}{3}c^{n-3}s^3 + \dots\right\}.$$

Hence $$\cos n\theta = c^n - \binom{n}{2}c^{n-2}s^2 + \binom{n}{4}c^{n-4}s^4 - \dots \quad \text{(iii)}$$

and $$\sin n\theta = \binom{n}{1}c^{n-1}s - \binom{n}{3}c^{n-3}s^3 + \dots. \quad \text{(iv)}$$

Taking the ratio of $\sin n\theta$ to $\cos n\theta$, and dividing top and bottom of the right-hand side by c^n, we obtain

$$\tan n\theta = \frac{\binom{n}{1}t - \binom{n}{3}t^3 + \dots}{1 - \binom{n}{2}t^2 + \binom{n}{4}t^4 - \dots}, \quad \text{(v)}$$

where $t = \tan\theta$.

The formulae (iii), (iv) can be transformed by means of the relation $s^2+c^2 = 1$ to show that $\cos n\theta$ can be expressed entirely in powers of $\cos\theta$, and so on; see for example Ex. 14 (*b*), no. 20, and ex. (iii) below.

Examples

(i) *Express* $\cos 5\theta$ *and* $\sin 6\theta/\sin\theta$ *in terms of* $\cos\theta$.

$$\cos 5\theta = c^5 - \binom{5}{2}c^3s^2 + \binom{5}{4}cs^4$$
$$= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2$$
$$= 16c^5 - 20c^3 + 5c.$$

$$\sin 6\theta = \binom{6}{1}c^5s - \binom{6}{3}c^3s^3 + \binom{6}{5}cs^5$$
$$= s\{6c^5 - 20c^3s^2 + 6cs^4\}$$
$$= s\{6c^5 - 20c^3(1-c^2) + 6c(1-c^2)^2\},$$

$$\therefore \quad \frac{\sin 6\theta}{\sin\theta} = 32c^5 - 32c^3 + 6c.$$

(ii) *Write down the formula for* $\tan 5\theta$. *What equation is satisfied by* $\tan\theta$ *if* (*a*) $\tan 5\theta = 0$; (*b*) $5\theta = \frac{1}{4}\pi$? *Solve each of these equations.*

$$\tan 5\theta = \frac{\binom{5}{1}t - \binom{5}{3}t^3 + \binom{5}{5}t^5}{1 - \binom{5}{2}t^2 + \binom{5}{4}t^4} = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}.$$

(*a*) If $\tan 5\theta = 0$, then $5t - 10t^3 + t^5 = 0$, so $t \equiv \tan\theta$ satisfies

$$x^5 - 10x^3 + 5x = 0.$$

Now if $\tan 5\theta = 0$, then $5\theta = r\pi$ and $\theta = \frac{1}{5}r\pi$, so that $\tan\theta = \tan\frac{1}{5}r\pi$, and the values $r = 0, 1, 2, 3, 4$ give distinct values of $\tan\frac{1}{5}r\pi$. Therefore the roots of the above equation are $x = 0$ (corresponding to $r = 0$) and

$$\tan\tfrac{1}{5}\pi, \quad \tan\tfrac{2}{5}\pi, \quad \tan\tfrac{3}{5}\pi = -\tan\tfrac{2}{5}\pi, \quad \tan\tfrac{4}{5}\pi = -\tan\tfrac{1}{5}\pi,$$

i.e.
$$0, \quad \pm\tan\tfrac{1}{5}\pi, \quad \pm\tan\tfrac{2}{5}\pi.$$

(*b*) If $5\theta = \frac{1}{4}\pi$, then $\tan 5\theta = 1$ and so

$$5t - 10t^3 + t^5 = 1 - 10t^2 + 5t^4;$$

hence $t \equiv \tan\theta$ satisfies

$$x^5 - 5x^4 - 10x^3 + 10x^2 + 5x - 1 = 0.$$

But if $\tan 5\theta = 1$, then $5\theta = \frac{1}{4}\pi + r\pi$ and $\theta = \frac{1}{20}\pi + \frac{1}{5}r\pi$, so that

$$\tan\theta = \tan\left(\frac{4r+1}{20}\pi\right),$$

and this takes distinct values for $r = 0, 1, 2, 3, 4$. The required roots are therefore

$$x = \tan\left(\frac{4r+1}{20}\pi\right) \quad (r = 0, 1, 2, 3, 4).$$

The equation in (*b*) could also be solved algebraically by the method in Ex. 13 (*g*), nos. 26–28.

*(iii) *If n is odd, prove that* $\sin n\theta$ *can be expressed in the form*

$$\sin n\theta = b_1 s + b_3 s^3 + \ldots + b_n s^n,$$

and find b_1, b_n. *Also prove that* $(r+1)(r+2)b_{r+2} = (r^2 - n^2)b_r$, *and hence find* b_3.
From formula (iv) and the relation $c^2 = 1 - s^2$,

$$\sin n\theta = \binom{n}{1}(1-s^2)^{\frac{1}{2}(n-1)}s - \binom{n}{3}(1-s^2)^{\frac{1}{2}(n-3)}s^3 + \binom{n}{5}(1-s^2)^{\frac{1}{2}(n-5)}s^5 - \ldots.$$

Since each of $n-1$, $n-3$, ... is *even*, this can be expanded as a polynomial in s involving only odd powers of s and having degree n, for

$$\begin{aligned} b_n &= \text{coefficient of } s^n \\ &= (-1)^{\frac{1}{2}(n-1)}\binom{n}{1} - (-1)^{\frac{1}{2}(n-3)}\binom{n}{3} + (-1)^{\frac{1}{2}(n-5)}\binom{n}{5} - \ldots \\ &= (-1)^{\frac{1}{2}(n-1)}\left\{\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots\right\} \\ &= (-1)^{\frac{1}{2}(n-1)}2^{n-1} \quad \text{by Ex. 12}(b)\text{, no. 2.} \end{aligned}$$

Also

$$b_1 = \text{coefficient of } s = \binom{n}{1} = n.$$

By deriving the identity $\sin n\theta = \sum_{r=1}^{n} b_r s^r$ twice wo θ, we obtain first

$$n \cos n\theta = \sum_{r=1}^{n} r b_r s^{r-1} c$$

and then

$$-n^2 \sin n\theta = \sum_{r=1}^{n} [r(r-1)\, b_r s^{r-2} c^2 - r b_r s^{r-1} s]$$

$$= \Sigma\, [r(r-1)\, b_r s^{r-2} - \{r(r-1)\, b_r + r b_r\}\, s^r]$$

$$= \Sigma\, [r(r-1)\, b_r s^{r-2} - r^2 b_r s^r];$$

hence

$$-n^2 \sum_{r=1}^{n} b_r s^r \equiv \sum_{r=1}^{n} [r(r-1)\, b_r s^{r-2} - r^2 b_r s^r].$$

Equating coefficients of s^r,

$$-n^2 b_r = (r+2)(r+1)\, b_{r+2} - r^2 b_r,$$

from which the required relation follows. Taking $r = 1$, we have

$$2.3\, b_3 = (1-n^2)\, n, \quad \text{i.e.} \quad b_3 = \tfrac{1}{6} n(1-n^2).$$

14.23 $\tan(\theta_1+\theta_2+\ldots+\theta_n)$

$\cos(\theta_1+\theta_2+\ldots+\theta_n) + i \sin(\theta_1+\theta_2+\ldots+\theta_n)$

$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$

as in 14.11, proof (i),

$= \cos\theta_1 \cos\theta_2 \ldots \cos\theta_n (1+it_1)(1+it_2)\ldots(1+it_n)$ where $t_r = \tan\theta_r$,

$= \cos\theta_1 \cos\theta_2 \ldots \cos\theta_n (1 + i\Sigma_1 + i^2\Sigma_2 + i^3\Sigma_3 + \ldots),$

where Σ_r denotes the sum of the products of $t_1, t_2, \ldots, t_n$ taken r at a time; this last step follows from the argument used in proving the binomial theorem, 12.11. Equating real and imaginary parts,

$$\cos(\theta_1+\theta_2+\ldots+\theta_n) = \cos\theta_1 \ldots \cos\theta_n (1 - \Sigma_2 + \Sigma_4 - \ldots), \qquad \text{(vi)}$$

and

$$\sin(\theta_1+\theta_2+\ldots+\theta_n) = \cos\theta_1 \ldots \cos\theta_n (\Sigma_1 - \Sigma_3 + \ldots). \qquad \text{(vii)}$$

By division,

$$\tan(\theta_1+\theta_2+\ldots+\theta_n) = \frac{\Sigma_1 - \Sigma_3 + \Sigma_5 - \ldots}{1 - \Sigma_2 + \Sigma_4 - \Sigma_6 + \ldots}. \qquad \text{(viii)}$$

The formulae (vi)–(viii) include (iii)–(v) as the special cases when $\theta_1 = \theta_2 = \ldots = \theta_n = \theta$. All these can be written down easily for any particular value of n, the terms continuing until they would cease to have a meaning.

Exercise 14(*b*)

Express the following in terms of multiple angles.

1 $\cos^4\theta$. 2 $\sin^4\theta$. 3 $\sin^5\theta$.

4 $\cos^4\theta\sin^5\theta$. 5 $\cos^5\theta\sin^3\theta$.

Calculate

6 $\int\sin^7\theta\,d\theta$. 7 $\int\cos^4\theta\sin^6\theta\,d\theta$. 8 $\int_0^{\frac{1}{2}\pi}\sin^4\theta\cos^6\theta\,d\theta$.

9 Express $\cos 6\theta$ in terms of (i) $\cos\theta$; (ii) $\sin\theta$.

10 Express $\sin 7\theta$ in terms of $\sin\theta$.

11 Express $\cos 7\theta/\cos\theta$ in terms of $\sin\theta$.

12 Solve completely the equation $\cos 5\theta+5\cos 3\theta+10\cos\theta=\frac{1}{2}$.

13 Solve completely $16\sin^5\theta=\sin 5\theta$.

14 Prove $\cos 7\theta/\cos\theta=1-2(2\cos 2\theta)-(2\cos 2\theta)^2+(2\cos 2\theta)^3$. Hence prove that the roots of $x^3-x^2-2x+1=0$ are $x=2\cos\{\frac{1}{7}(2k+1)\pi\}$, $k=0,1,2$.

15 If $x=2\cos\theta$, prove $(1+\cos 7\theta)/(1+\cos\theta)=(x^3-x^2-2x-1)^2$. [Express in half-angles.]

*16 (i) Give the last terms in formulae (iii), (iv) when n is (*a*) even; (*b*) odd.
(ii) State the last terms in numerator and denominator of formula (v) when n is (*a*) even; (*b*) odd.

17 Write down the expression for $\tan 3\theta$ in terms of $t\equiv\tan\theta$. By first solving $\tan\theta=1$, show that the roots of $x^3-3x^2-3x+1=0$ are $\tan\frac{1}{12}\pi$, $\tan\frac{5}{12}\pi$ and $\tan\frac{3}{4}\pi$. What are the roots of $x^2-4x+1=0$?

18 (i) If $\theta_1+\theta_2+\theta_3+\theta_4=n\pi$, where n is an integer, state the relation between t_1, t_2, t_3, t_4, where $t_r=\tan\theta_r$.
(ii) If $\tan\theta_1, \ldots, \tan\theta_4$ are the roots of $t^4+at^3+bt^2+ct+d=0$, express $\tan(\theta_1+\theta_2+\theta_3+\theta_4)$ in terms of the coefficients. What can be said about the angles if $c=a$?

19 If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are the roots of $ax^3+x^2+bx+1=0$, prove that $\alpha+\beta+\gamma$ is an integral multiple of π.

*20 (i) Prove that $\cos n\theta$ can be expressed as a polynomial of degree n in $c\equiv\cos\theta$ of the form $a_nc^n+a_{n-2}c^{n-2}+\ldots+a_{n-2r}c^{n-2r}+\ldots$, the last term being a_0 or a_1c according as n is even or odd. (ii) Prove that $a_n=2^{n-1}$. (iii) If n is even, put $\theta=\frac{1}{2}\pi$ to show that $a_0=(-1)^{\frac{1}{2}n}$; if n is odd, consider $\lim_{\theta\to\frac{1}{2}\pi}(\cos n\theta/\cos\theta)$ to prove $a_1=n(-1)^{\frac{1}{2}(n-1)}$. (iv) By deriving twice wo θ and equating coefficients of c^r, prove $(r+1)(r+2)a_{r+2}=(r^2-n^2)a_r$. (v) Hence expand $\cos n\theta$ (*a*) in descending powers of c; (*b*) in ascending powers of c, first for n *even* and then for n *odd*.

14.3 Factorisation

We now give further applications of the results in 10.12 and of Theorem I in 10.13, which remain valid in complex algebra. Just as we have used $\sum_{r=1}^{n} f(r)$ to denote $f(1)+f(2)+\ldots+f(n)$, here it is convenient to write

$$\prod_{r=1}^{n} f(r) = f(1)\,f(2)\ldots f(n).$$

As with the Σ-notation, the range of values of r can be omitted when clear from the context.

14.31 $x^n - 1$

We have $x^n - 1 = 0$ if $x^n = 1 = \operatorname{cis} 2r\pi$, i.e. if $x = \operatorname{cis}(2r\pi/n)$. For $r = 1, 2, \ldots, n$ it follows that $x - \operatorname{cis}(2r\pi/n)$ are distinct factors of $x^n - 1$.

Since $r = -k$ and $r = n-k$ give the same value to $\operatorname{cis}(2r\pi/n)$, we may use $r = 0, -1, -2, \ldots$ instead of $r = n, n-1, n-2, \ldots$.

Case (i): n even. Take $r = 0, \pm 1, \pm 2, \ldots, \pm(\frac{1}{2}n - 1), \frac{1}{2}n$. This gives $\frac{1}{2}n - 1$ pairs of factors, together with the single factors corresponding to 0 and $\frac{1}{2}n$; in all, there are $2(\frac{1}{2}n - 1) + 2 = n$ factors.

The factors corresponding to $r = \pm k$ are

$$x - \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right), \quad x - \left(\cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}\right),$$

and their product is

$$\left(x - \cos\frac{2k\pi}{n}\right)^2 + \left(\sin\frac{2k\pi}{n}\right)^2 = x^2 - 2x\cos\frac{2k\pi}{n} + 1.$$

The factors corresponding to $r = 0, \frac{1}{2}n$ are respectively $x-1, x+1$. Hence

$$\textit{for } n \textit{ even}, \quad x^n - 1 = (x-1)(x+1)\prod_{k=1}^{\frac{1}{2}n-1}\left(x^2 - 2x\cos\frac{2k\pi}{n} + 1\right). \tag{i}$$

Case (ii): n odd. Take $r = 0, \pm 1, \pm 2, \ldots, \pm\frac{1}{2}(n-1)$; this gives $1 + 2\{\frac{1}{2}(n-1)\} = n$ factors altogether. Arguing as before, we find that

$$\textit{for } n \textit{ odd}, \quad x^n - 1 = (x-1)\prod_{k=1}^{\frac{1}{2}(n-1)}\left(x^2 - 2x\cos\frac{2k\pi}{n} + 1\right). \tag{ii}$$

14.32 $x^n + 1$

$x^n + 1 = 0$ if $x^n = -1 = \operatorname{cis}(2r-1)\pi$, i.e. if $x = \operatorname{cis}\{(2r-1)\pi/n\}$, where r is an integer or zero. Distinct values of x are given by any n consecutive values of r.

Case (i): n even. As in 14.31, we wish to pair off conjugate complex linear factors so as to get a real quadratic product. Now the factors

$$x - \operatorname{cis}\frac{2r-1}{n}\pi, \quad x - \operatorname{cis}\frac{2r'-1}{n}\pi$$

are conjugate if $(2r-1)+(2r'-1) = 0$, i.e. if $r' = -(r-1)$; therefore with the values

$$r = 1, \quad 2, \quad 3, \ldots, \tfrac{1}{2}n$$

we pair

$$r' = 0, -1, -2, \ldots, -(\tfrac{1}{2}n - 1).$$

This gives $\frac{1}{2}n + \{1 + (\frac{1}{2}n - 1)\} = n$ factors altogether. The factors corresponding to $r = k$, $r' = -(k-1)$ are

$$x - \left(\cos\frac{2k-1}{n}\pi + i\sin\frac{2k-1}{n}\pi\right), \quad x - \left(\cos\frac{2k-1}{n}\pi - i\sin\frac{2k-1}{n}\pi\right),$$

and their product is $x^2 - 2x\cos\{(2k-1)\pi/n\} + 1$. Hence

$$\textit{for } n \textit{ even}, \quad x^n + 1 = \prod_{k=1}^{\frac{1}{2}n}\left(x^2 - 2x\cos\frac{2k-1}{n}\pi + 1\right). \qquad \text{(iii)}$$

Case (ii): n odd. We pair off the values

$$r = 1, \quad 2, \quad 3, \ldots, \tfrac{1}{2}(n-1)$$

with

$$r' = 0, -1, -2, \ldots, -\tfrac{1}{2}(n-3).$$

The value $r = \frac{1}{2}(n+1)$ gives $2r - 1 = n$, to which corresponds the real factor $x+1$. Then as before we have

$$\textit{for } n \textit{ odd}, \quad x^n + 1 = (x+1)\prod_{k=1}^{\frac{1}{2}(n-1)}\left(x^2 - 2x\cos\frac{2k-1}{n}\pi + 1\right). \qquad \text{(iv)}$$

14.33 $x^{2n} - 2x^n \cos n\alpha + 1$

The equation $x^{2n} - 2x^n\cos n\alpha + 1 = 0$ is quadratic in x^n:

$$(x^n - \cos n\alpha)^2 = -1 + \cos^2 n\alpha = -\sin^2 n\alpha,$$

$$\therefore \quad x^n = \cos n\alpha \pm i\sin n\alpha.$$

Hence the $2n$ values of x which satisfy it are

$$\cos\left(\alpha + \frac{2r\pi}{n}\right) \pm i\sin\left(\alpha + \frac{2r\pi}{n}\right),$$

where $r = 0, 1, 2, \ldots, n-1$; these are distinct unless $n\alpha$ is zero or an integral multiple of π.

The product of the conjugate factors corresponding to $r = k$ is

$$\left\{x-\cos\left(\alpha+\frac{2k\pi}{n}\right)-i\sin\left(\alpha+\frac{2k\pi}{n}\right)\right\}\left\{x-\cos\left(\alpha+\frac{2k\pi}{n}\right)+i\sin\left(\alpha+\frac{2k\pi}{n}\right)\right\}$$

$$=\left\{x-\cos\left(\alpha+\frac{2k\pi}{n}\right)\right\}^2+\left\{\sin\left(\alpha+\frac{2k\pi}{n}\right)\right\}^2$$

$$=x^2-2x\cos\left(\alpha+\frac{2k\pi}{n}\right)+1.$$

$$\therefore \quad x^{2n}-2x^n\cos n\alpha+1=\prod_{k=0}^{n-1}\left\{x^2-2x\cos\left(\alpha+\frac{2k\pi}{n}\right)+1\right\}. \qquad \text{(v)}$$

Examples

Various deductions can be made from the result (v).

(i) Putting $\alpha = 0$, we obtain

$$(x^n-1)^2=\prod_{k=0}^{n-1}\left\{x^2-2x\cos\frac{2k\pi}{n}+1\right\},$$

and if n is *even* this is equal to

$$(x-1)^2(x+1)^2\prod_{k=1}^{\frac{1}{2}n-1}\left\{x^2-2x\cos\frac{2k\pi}{n}+1\right\}^2$$

because the values $k = 0$, $k = \frac{1}{2}n$ give the factors $(x-1)^2$, $(x+1)^2$ respectively, while factors corresponding to k and $n-k$ are equal. Taking square roots (in which the positive sign is chosen since in real algebra all the quadratic factors are positive and x^n-1, x^2-1 always have the same sign when n is even), we obtain formula (i) of 14.31. If n is *odd*, we likewise deduce formula (ii).

Similarly, by putting $\alpha = \pi/n$ in (v), we obtain formulae (iii), (iv).

(ii) Divide both sides of (v) by x^n:

$$x^n+\frac{1}{x^n}-2\cos n\alpha=\prod_{k=0}^{n-1}\left\{x+\frac{1}{x}-2\cos\left(\alpha+\frac{2k\pi}{n}\right)\right\}.$$

Putting $x = \operatorname{cis}\theta$,

$$2\cos n\theta-2\cos n\alpha=\prod_{k=0}^{n-1}\left\{2\cos\theta-2\cos\left(\alpha+\frac{2k\pi}{n}\right)\right\},$$

i.e.

$$\cos n\theta-\cos n\alpha=2^{n-1}\prod_{k=0}^{n-1}\left\{\cos\theta-\cos\left(\alpha+\frac{2k\pi}{n}\right)\right\}.$$

(iii) In (v) put $x = 1$ and $\alpha = 2\beta$:

$$2(1-\cos 2n\beta)=\prod_{k=0}^{n-1}2\left\{1-\cos\left(2\beta+\frac{2k\pi}{n}\right)\right\},$$

$$\therefore \quad \sin^2 n\beta=2^{2n-2}\prod_{0}^{n-1}\sin^2\left(\beta+\frac{k\pi}{n}\right),$$

$$\therefore \quad \sin n\beta=\pm 2^{n-1}\prod_{0}^{n-1}\sin\left(\beta+\frac{k\pi}{n}\right).$$

To decide the sign of the right-hand side, first suppose $0 < \beta < \pi/n$; then each factor on the right is positive, and so is $\sin n\beta$. Hence the sign $+$ is appropriate for this range of β. As β increases, $\sin n\beta$ changes sign whenever β passes through a value $k\pi/n$, and simultaneously *one* factor on the right changes. Hence the sign is always $+$:

$$\sin n\beta = 2^{n-1}\prod_{k=0}^{n-1} \sin\left(\beta+\frac{k\pi}{n}\right).$$

(iv) By taking logarithms of the modulus of each side of this last result, assuming that β is not zero or an integral multiple of 2π,

$$\log|\sin n\beta| = (n-1)\log 2+\sum_{k=0}^{n-1}\log\left|\sin\left(\beta+\frac{k\pi}{n}\right)\right|.$$

Deriving wo β, $$n\cot n\beta = \sum_{k=0}^{n-1}\cot\left(\beta+\frac{k\pi}{n}\right).$$

(v) *de Moivre's and Cotes's properties*† *of the circle.*

$A_0A_1A_2\ldots A_{n-1}$ is a regular n-sided polygon inscribed in the circle of centre O and radius a; P is a point such that $OP = x$ and $\angle(OP, OA_0) = \theta$.

Since sides of the polygon subtend angle $2\pi/n$ at O, we have

$$\angle(OP, OA_r) = \theta+2r\pi/n.$$

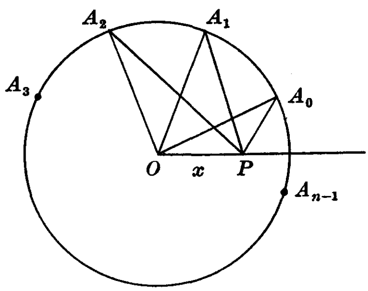

Fig. 137

Also, by the cosine rule,

$$PA_r^2 = x^2+a^2-2ax\cos\left(\theta+\frac{2r\pi}{n}\right).$$

$$\therefore \quad PA_0^2.PA_1^2\ldots PA_{n-1}^2 = \prod_{r=0}^{n-1}\left\{x^2-2ax\cos\left(\theta+\frac{2r\pi}{n}\right)+a^2\right\}$$
$$= x^{2n}-2a^nx^n\cos n\theta+a^{2n}$$

by a slight extension of formula (v). Hence de Moivre's property:

$$PA_0.PA_1\ldots PA_{n-1} = \sqrt{(x^{2n}-2a^nx^n\cos n\theta+a^{2n})}.$$

In particular, if P lies on OA_0, then $\theta = 0$ and so

$$PA_0.PA_1\ldots PA_{n-1} = |x^n-a^n|.$$

If OP bisects $A_{n-1}\hat{O}A_0$, then $\theta = \pi/n$, and so

$$PA_0.PA_1\ldots PA_{n-1} = x^n+a^n.$$

These last two results are known as Cotes's properties.

14.34 $\sin n\theta$, n odd

By ex. (iii) of 14.22, $\sin n\theta$ can be expressed as a polynomial in $\sin\theta$ when n is odd; the leading term is $b_n\sin^n\theta$, where $b_n = (-1)^{\frac{1}{2}(n-1)}2^{n-1}$. The polynomial is zero when $\sin n\theta = 0$, i.e. when $n\theta = r\pi$, $\theta = r\pi/n$ and so $\sin\theta = \sin(r\pi/n)$.

† These illustrate geometrically the results of 14.31–14.33.

Hence for $r = 0, 1, 2, \ldots, n-1$, $\sin\theta - \sin(r\pi/n)$ are distinct factors of $\sin n\theta$; consequently

$$\sin n\theta = b_n \sum_{r=1}^{n-1}\left(\sin\theta - \sin\frac{r\pi}{n}\right). \tag{vi}$$

This can be expressed alternatively as follows. The n distinct values of $\sin(r\pi/n)$ are given by $r = 0, \pm 1, \pm 2, \ldots, \pm\frac{1}{2}(n-1)$. The factors which correspond to $r = k$, $r = -k$ are $\sin\theta - \sin(k\pi/n)$, $\sin\theta + \sin(k\pi/n)$, and have product $\sin^2\theta - \sin^2(k\pi/n)$. Hence

$$\sin n\theta = b_n \sin\theta \prod_{k=1}^{\frac{1}{2}(n-1)}\left(\sin^2\theta - \sin^2\frac{k\pi}{n}\right). \tag{vii}$$

Divide each factor following $\sin\theta$ in (vii) by $-\sin^2(k\pi/n)$; then

$$\sin n\theta = A\sin\theta \prod_{1}^{\frac{1}{2}(n-1)}\left(1 - \frac{\sin^2\theta}{\sin^2(k\pi/n)}\right),$$

where A is independent of θ and can be determined by dividing both sides by $\sin\theta$ and then letting $\theta \to 0$. We obtain

$$A = \lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta} = n,$$

and so

$$\sin n\theta = n\sin\theta \prod_{k=1}^{\frac{1}{2}(n-1)}\left(1 - \frac{\sin^2\theta}{\sin^2(k\pi/n)}\right). \tag{viii}$$

Examples

(i) *Comparison of series and product.*

By identifying the product (viii) with the expansion of $\sin n\theta$ obtained in ex. (iii) of 14.22, we have *for n odd*:

$$ns \prod_{k=1}^{\frac{1}{2}(n-1)}\left(1 - s^2\operatorname{cosec}^2\frac{k\pi}{n}\right) \equiv ns + \tfrac{1}{6}n(1-n^2)s^3 + \ldots + b_n s^n.$$

Equating coefficients of the various powers of s gives identities involving $\operatorname{cosec}^2(k\pi/n)$; e.g. from those of s^3,

$$-n\sum_{k=1}^{\frac{1}{2}(n-1)}\operatorname{cosec}^2\frac{k\pi}{n} = \tfrac{1}{6}n(1-n^2),$$

i.e.

$$\sum_{k=1}^{\frac{1}{2}(n-1)}\operatorname{cosec}^2\frac{k\pi}{n} = \tfrac{1}{6}(n^2-1) \quad \textit{for } n \textit{ odd}. \tag{ix}$$

(ii) *Proof that* $\sum_{r=1}^{\infty}\frac{1}{r^2} = \frac{1}{6}\pi^2$.

If $0 < \theta < \frac{1}{2}\pi$, then $\sin\theta < \theta < \tan\theta$ and so

$$\frac{1}{\theta^2} < \operatorname{cosec}^2\theta = 1 + \cot^2\theta < 1 + \frac{1}{\theta^2}.$$

Put $\theta = r\pi/n$ where n is odd, and sum for $r = 1, 2, \ldots, \frac{1}{2}(n-1)$:

$$\frac{n^2}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{\frac{1}{4}(n-1)^2}\right\} < \sum_{r=1}^{\frac{1}{2}(n-1)}\operatorname{cosec}^2\frac{r\pi}{n}$$

$$< \tfrac{1}{2}(n-1) + \frac{n^2}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{\frac{1}{4}(n-1)^2}\right\}.$$

Writing
$$s_m = \frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{m^2}$$
and using formula (ix), we have
$$s_{\frac{1}{2}(n-1)} < \frac{\pi^2}{n^2}\frac{n^2-1}{6} < \frac{\pi^2}{n^2}\frac{n-1}{2}+s_{\frac{1}{2}(n-1)}.$$

Since the series $\Sigma(1/r^2)$ is known to converge by 12.41 (3), therefore s_m tends to some limit s when $m \to \infty$ in any manner, and in particular through the sequence $\frac{1}{2}(n-1)$ for odd n. Letting $n \to \infty$, the above inequality gives
$$s \leqslant \tfrac{1}{6}\pi^2 \leqslant s,$$
so that $s = \frac{1}{6}\pi^2$.

Exercise 14(*c*)

1 Obtain real quadratic factors of x^6-x^3+1. By equating coefficients of x^5 and of x^3, prove that
$$\cos\tfrac{1}{9}\pi+\cos\tfrac{5}{9}\pi+\cos\tfrac{7}{9}\pi = 0 \quad\text{and}\quad \cos\tfrac{1}{9}\pi\cos\tfrac{5}{9}\pi\cos\tfrac{7}{9}\pi = \tfrac{1}{8}.$$

2 Find the real quadratic factors of x^8-4x^4+16.

3 Write down real quadratic factors of $x^6+x^5+x^4+x^3+x^2+x+1$. [The expression is $(x^7-1)/(x-1)$.]

By substituting ± 1 *for* x *and* 2β *or* $2\beta+\pi/n$ *for* α *in formula* (v) *of* 14.33, *prove the following.*

4 $\cos n\beta = 2^{n-1}\prod_{r=0}^{n-1}\sin\left\{\beta+\frac{(2r+1)\pi}{2n}\right\}.$

5 $\sin n\beta = (-1)^{\frac{1}{2}n}\,2^{n-1}\prod_{r=0}^{n-1}\cos\left\{\beta+\frac{r\pi}{n}\right\}$ if n is even.

6 $\cos n\beta = (-1)^{\frac{1}{2}n}\,2^{n-1}\prod_{r=0}^{n-1}\cos\left\{\beta+\frac{(2r+1)\pi}{2n}\right\}$ if n is even.

7 Obtain a result by deriving no. 4 logarithmically wo β.

8 Prove $\sum_{r=0}^{n-1}\operatorname{cosec}^2\left(\beta+\frac{r\pi}{n}\right) = n^2\operatorname{cosec}^2 n\beta.$

9 Prove $\operatorname{ch} nx-\cos n\alpha = 2^{n-1}\prod_{r=0}^{n-1}\left\{\operatorname{ch} x-\cos\left(\alpha+\frac{2r\pi}{n}\right)\right\}$. [Divide formula (v) by x^n and then put $x = e^y$.]

10 Express $x^{n-1}/(x^{2n}-2x^n\cos n\alpha+1)$ as a sum of n partial fractions. [Derive (v) logarithmically wo α.]

11 Prove that
$$x^n+\frac{1}{x^n} = \prod_{r=0}^{n-1}\left\{x+\frac{1}{x}-2\cos\left(\frac{2r+1}{2n}\pi\right)\right\},\quad x^n-\frac{1}{x^n} = \left(x-\frac{1}{x}\right)\prod_{r=1}^{n-1}\left(x+\frac{1}{x}-2\cos\frac{r\pi}{n}\right).$$
[Use formulae (iii), (i) with $2n$ instead of n.]

12 By putting $x = \operatorname{cis}\theta$ in the results of no. 11, prove
$$\text{(i)}\ \cos n\theta = 2^{n-1}\prod_{r=0}^{n-1}\left(\cos\theta-\cos\frac{2r+1}{2n}\pi\right);$$
$$\text{(ii)}\ \sin n\theta = 2^{n-1}\sin\theta\prod_{r=1}^{n-1}\left(\cos\theta-\cos\frac{r\pi}{n}\right).$$

13 (i) Obtain a result from no. 12 (i) by putting $\theta = 0$.

(ii) By dividing both sides of no. 12 (ii) by $\sin\theta$ and then letting $\theta \to 0$, prove

$$\sqrt{n} = +2^{n-1}\prod_{r=1}^{n-1}\sin\frac{r\pi}{n}.$$

14 (i) By deriving no. 12 (i) logarithmically, prove

$$\frac{n\tan n\theta}{\sin\theta} = \sum_{r=0}^{n-1}\frac{1}{\cos\theta-\cos\{(2r+1)\pi/2n\}}.$$

(ii) What result is obtainable similarly from no. 12 (ii)?

15 With the notation of 14.33, ex. (v), prove that if P lies on the circumference, then $PA_0.PA_1\dots PA_{n-1} = 2a^n\,|\sin\tfrac{1}{2}n\theta|$.

16 Show that the solutions of $(1+x)^{2n}+(1-x)^{2n} = 0$ are

$$x = \pm i\tan\frac{2r-1}{4n}\pi, \quad \text{where} \quad r = 1, 2, \dots, n.$$

Deduce that $$(1+x)^{2n}+(1-x)^{2n} = 2\prod_{r=1}^{n}\left(x^2+\tan^2\frac{(2r-1)\pi}{4n}\right),$$

and hence prove that $$\sum_{r=1}^{n}\sec^2\frac{(2r-1)\pi}{4n} = 2n^2.$$

*17 Prove that

$$\cos n\theta-\cos n\alpha = 2^{n-1}\prod_{r=0}^{n-1}\left\{\cos\theta-\cos\left(\alpha+\frac{2r\pi}{n}\right)\right\}$$

by using Ex. 14 (*b*), no. 20 and arguing as in 14.34.

14.4 Roots of equations

14.41 Construction of equations with roots given trigonometrically

(i) *Form the equation whose roots are* $\cos\frac{2}{9}\pi$, $\cos\frac{4}{9}\pi$, $\cos\frac{8}{9}\pi$.

$\theta = \frac{2}{9}\pi, \frac{4}{9}\pi, \frac{8}{9}\pi$ all satisfy $\cos 3\theta = -\frac{1}{2}$. Since $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, the equation $4x^3-3x = -\frac{1}{2}$ is satisfied by $x = \cos\frac{2}{9}\pi, \cos\frac{4}{9}\pi, \cos\frac{8}{9}\pi$. The required equation is therefore

$$8x^3-6x+1 = 0.$$

(ii) *Form the equation whose roots are* $\cos\frac{2}{7}\pi$, $\cos\frac{4}{7}\pi$, $\cos\frac{6}{7}\pi$.

Consider the equation $\cos 4\theta = \cos 3\theta$. It is satisfied by

$$4\theta = 2m\pi \pm 3\theta,$$

where m is any integer or zero, i.e. by $\theta = \frac{2}{7}m\pi$ (this includes both solutions obtained from the double sign $\pm$).

Writing $x = \cos\theta$, the equation becomes

$$8x^4-8x^2+1 = 4x^3-3x,$$

since $\cos 4\theta = 2\cos^2 2\theta - 1 = 2(2x^2-1)^2-1$. The roots of this are given by $m = 0, 1, 2, 3$, viz. $$x = \cos 0 = 1,\ \cos\tfrac{2}{7}\pi,\ \cos\tfrac{4}{7}\pi,\ \cos\tfrac{6}{7}\pi.$$

Since $8x^4-4x^3-8x^2+3x+1 = (x-1)(8x^3+4x^2-4x-1)$, the required equation is

$$8x^3+4x^2-4x-1 = 0.$$

(iii) *Form the equation whose roots are* $\tan^2\frac{1}{9}\pi$, $\tan^2\frac{2}{9}\pi$, $\tan^2\frac{1}{3}\pi$, $\tan^2\frac{4}{9}\pi$.

The equation $\tan 9\theta = 0$ is satisfied by $\theta = \frac{1}{9}n\pi$, where n is any integer or zero. Writing $t = \tan\theta$, it becomes (see formula (v) of 14.22)

$$\binom{9}{1}t - \binom{9}{3}t^3 + \binom{9}{5}t^5 - \binom{9}{7}t^7 + \binom{9}{9}t^9 = 0.$$

Removing the root $t = 0$, we see that the equation

$$t^8 - 36t^6 + 126t^4 - 84t^2 + 9 = 0$$

has roots $t = \tan(\frac{1}{9}n\pi)$, where $n = 1, 2, \ldots, 8$, i.e.

$$\pm\tan\tfrac{1}{9}\pi, \quad \pm\tan\tfrac{2}{9}\pi, \quad \pm\tan\tfrac{3}{9}\pi, \quad \pm\tan\tfrac{4}{9}\pi.$$

Putting $x = t^2$, it follows that

$$x^4 - 36x^3 + 126x^2 - 84x + 9 = 0$$

has roots $\tan^2\frac{1}{9}\pi$, $\tan^2\frac{2}{9}\pi$, $\tan^2\frac{1}{3}\pi$, $\tan^2\frac{4}{9}\pi$.

N.B.—Since $\tan^2\frac{1}{3}\pi = 3$, we may remove the root $x = 3$ from the last equation, obtaining $x^3 - 33x^2 + 27x - 3 = 0$, which consequently has roots $\tan^2\frac{1}{9}\pi$, $\tan^2\frac{2}{9}\pi$, $\tan^2\frac{4}{9}\pi$.

14.42 Results obtained by using relations between roots and coefficients

If the results of 13.51 are applied to equations like those found in 14.41, various trigonometrical relations are obtained.

Examples

(iv) *Prove* $\sec\frac{2}{9}\pi + \sec\frac{4}{9}\pi + \sec\frac{8}{9}\pi = 6$, *and find* $\sec\frac{1}{9}\pi + \sec\frac{5}{9}\pi + \sec\frac{7}{9}\pi$.

The equation whose roots are the reciprocals of the roots of that in ex. (i) will have roots $\sec\frac{2}{9}\pi$, $\sec\frac{4}{9}\pi$, $\sec\frac{8}{9}\pi$. This equation is $x^3 - 6x^2 + 8 = 0$, and the sum of its roots is 6.

Since $\sec\frac{1}{9}\pi = -\sec\frac{8}{9}\pi$, $\sec\frac{5}{9}\pi = -\sec\frac{4}{9}\pi$, and $\sec\frac{7}{9}\pi = -\sec\frac{2}{9}\pi$, hence $\sec\frac{1}{9}\pi + \sec\frac{5}{9}\pi + \sec\frac{7}{9}\pi = -6$.

(v) *Calculate* $\tan^2\frac{2}{7}\pi + \tan^2\frac{4}{7}\pi + \tan^2\frac{6}{7}\pi$.

From ex. (ii), the equation having roots $\sec\frac{2}{7}\pi$, $\sec\frac{4}{7}\pi$, $\sec\frac{6}{7}\pi$ is

$$x^3 + 4x^2 - 4x - 8 = 0.$$

Putting $y = x^2$, the equation having roots $\sec^2\frac{2}{7}\pi$, etc. is

$$y(y-4)^2 = 16(y-2)^2,$$

i.e.

$$y^3 - 24y^2 + 80y - 64 = 0.$$

Writing $z = y - 1$, i.e. $y = z + 1$, this becomes (after reduction)

$$z^3 - 21z^2 + 35z - 7 = 0,$$

which has roots $\sec^2\frac{2}{7}\pi - 1 = \tan^2\frac{2}{7}\pi$, etc. The sum of the roots is 21.

(vi) *Prove that* $\cos 3\alpha = 4\cos\alpha\cos(\alpha+\frac{2}{3}\pi)\cos(\alpha+\frac{4}{3}\pi)$.

Consider the equation $\cos 3\theta = \cos 3\alpha$, which is satisfied by $\theta = \alpha + \frac{2}{3}k\pi$ where k is any integer or zero. It can be written

$$4x^3 - 3x - \cos 3\alpha = 0,$$

where $x = \cos\theta$, and is satisfied by the distinct values $x = \cos\alpha$, $\cos(\alpha+\frac{2}{3}\pi)$ and $\cos(\alpha+\frac{4}{3}\pi)$. By taking the product of the roots, the relation follows.

Exercise 14(*d*)

1 Form the equation whose roots are $\cos\frac{1}{9}\pi$, $\cos\frac{7}{9}\pi$, $\cos\frac{13}{9}\pi$.

2 Form the equation whose roots are $\pm\cos\frac{1}{5}\pi$, $\pm\cos\frac{3}{5}\pi$. [Consider $\cos 5\theta = -1$.]

3 By considering $\cos 3\theta = \cos 2\theta$, construct the equation whose roots are $\cos\frac{2}{5}\pi$ and $\cos\frac{4}{5}\pi$. Hence find $\cos 36°$, $\cos 72°$ in surd form.

4 Show that $x = 2\cos\dfrac{2r\pi}{11}$ $(r = 1, 2, \ldots, 5)$ are the roots of

$$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = 0.$$

[Consider $\cos 6\theta = \cos 5\theta$; remove the root $x = 2$ at the end.]

*5 (i) Construct the equation whose roots are $\pm 2\sin\frac{2}{7}\pi$, $\pm 2\sin\frac{4}{7}\pi$, $\pm 2\sin\frac{8}{7}\pi$ by eliminating x from $y = 2\sqrt{(1-x^2)}$ and the result of ex. (ii) in 14.41.

(ii) Verify that $2\sin\frac{2}{7}\pi$, $2\sin\frac{4}{7}\pi$, $2\sin\frac{8}{7}\pi$ are the roots of $x^3 = +\sqrt{7}\,(x^2-1)$. [The equation in y obtained in (i) can be written

$$y^6 = 7(y^2-1)^2, \quad \text{i.e.} \quad y^3 = \pm\sqrt{7}\,(y^2-1).$$

Since $2\sin\frac{2}{7}\pi > 1$ and $2\sin\frac{4}{7}\pi > 1$, and also $2\sin\frac{8}{7}\pi = -2\sin\frac{6}{7}\pi$ lies between 0 and -1, these values of y give y^3 and y^2-1 the same sign.]

Use the equations obtained in exs. (i)–(iii) *of* 14.41 *to prove the following.*

6 $\cos\frac{2}{9}\pi\cos\frac{4}{9}\pi + \cos\frac{4}{9}\pi\cos\frac{8}{9}\pi + \cos\frac{8}{9}\pi\cos\frac{2}{9}\pi = -\frac{3}{4}$.

7 $\tan^2\frac{1}{9}\pi + \tan^2\frac{2}{9}\pi + \tan^2\frac{4}{9}\pi = 33$ and $\tan\frac{1}{9}\pi\tan\frac{2}{9}\pi\tan\frac{4}{9}\pi = +\sqrt{3}$.

8 $\sec^2\frac{1}{9}\pi + \sec^2\frac{5}{9}\pi + \sec^2\frac{7}{9}\pi = 36$ and $\sec^4\frac{1}{9}\pi + \sec^4\frac{5}{9}\pi + \sec^4\frac{7}{9}\pi = 1104$.

9 Calculate $\sin^2\frac{2}{7}\pi + \sin^2\frac{4}{7}\pi + \sin^2\frac{6}{7}\pi$.

10 Prove $\sin 3\alpha = -4\sin\alpha\sin(\alpha+\frac{2}{3}\pi)\sin(\alpha+\frac{4}{3}\pi)$. [Consider $\sin 3\theta = \sin 3\alpha$.]

11 Prove $\tan\alpha + \tan(\alpha+\frac{1}{3}\pi) + \tan(\alpha+\frac{2}{3}\pi) = 3\tan 3\alpha$. [Consider $\tan 3\theta = \tan 3\alpha$.]

*12 By considering $\tan n\theta = \tan n\alpha$, prove that the equation

$$\binom{n}{1}x^{n-1} - \binom{n}{3}x^{n-3} + \ldots = \tan n\alpha\left\{x^n - \binom{n}{2}x^{n-2} + \ldots\right\}$$

has roots $x = \cot(\alpha + r\pi/n)$, $r = 0, 1, 2, \ldots, n-1$, where α is not zero or an integral multiple of 2π. Deduce that $\sum_{r=0}^{n-1}\cot\left(\alpha+\dfrac{r\pi}{n}\right) = n\cot n\alpha$.

14.5 Finite trigonometric series: summation by $C+iS$

The method of summing certain trigonometric series which is illustrated in the following examples gives results in *pairs*, and is also applicable to infinite series (see 14.66, ex. (i)).

14.51 Cosines and sines of angles in A.P.

Write

$$C = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha+(n-1)\beta\}$$

and $$S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin\{\alpha+(n-1)\beta\}.$$

Then

$$C+iS = \operatorname{cis}\alpha + \operatorname{cis}(\alpha+\beta) + \operatorname{cis}(\alpha+2\beta) + \ldots + \operatorname{cis}\{\alpha+(n-1)\beta\}$$

$$= \operatorname{cis}\alpha + \operatorname{cis}\alpha\,\operatorname{cis}\beta + \operatorname{cis}\alpha(\operatorname{cis}\beta)^2 + \ldots + \operatorname{cis}\alpha(\operatorname{cis}\beta)^{n-1}$$

$$= \operatorname{cis}\alpha\{1-(\operatorname{cis}\beta)^n\}/\{1-\operatorname{cis}\beta\}$$

on summing the G.P., provided $\operatorname{cis}\beta \neq 1$, i.e. β is not an integral multiple of 2π. Now†

$$1-(\operatorname{cis}\beta)^n = 1-\operatorname{cis}n\beta$$

$$= 1-\cos n\beta - i\sin n\beta$$

$$= 2\sin^2 \tfrac{1}{2}n\beta - 2i\sin\tfrac{1}{2}n\beta\cos\tfrac{1}{2}n\beta$$

$$= -2i\sin\tfrac{1}{2}n\beta(\cos\tfrac{1}{2}n\beta + i\sin\tfrac{1}{2}n\beta)$$

$$= -2i\sin\tfrac{1}{2}n\beta\operatorname{cis}\tfrac{1}{2}n\beta.$$

Hence $$C+iS = \operatorname{cis}\alpha\frac{-2i\sin\frac{1}{2}n\beta\operatorname{cis}\frac{1}{2}n\beta}{-2i\sin\frac{1}{2}\beta\operatorname{cis}\frac{1}{2}\beta}$$

$$= \frac{\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}\operatorname{cis}\{\alpha+\tfrac{1}{2}(n-1)\beta\}.$$

Equating real and imaginary parts, we obtain

$$C = \frac{\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}\cos\{\alpha+\tfrac{1}{2}(n-1)\beta\},\quad S = \frac{\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}\sin\{\alpha+\tfrac{1}{2}(n-1)\beta\}.$$

If β is zero or a multiple of 2π, then from the original series clearly $C = n\cos\alpha$ and $S = n\sin\alpha$. Cf. 12.27, exs. (i), (ii).

† The following reduction is shorter than multiplying numerator and denominator by the conjugate of $1-\operatorname{cis}\beta$ to make the denominator real.

14.52 Other examples

(i) *Sum* $1+x\cos\theta+x^2\cos 2\theta+\dots$ *to n terms, where x is real.*

Write
$$C = 1+x\cos\theta+x^2\cos 2\theta+\dots+x^{n-1}\cos(n-1)\theta$$
and†
$$S = \quad x\sin\theta+x^2\sin 2\theta+\dots+x^{n-1}\sin(n-1)\theta.$$
Then
$$C+iS = 1+x\operatorname{cis}\theta+x^2\operatorname{cis}2\theta+\dots+x^{n-1}\operatorname{cis}(n-1)\theta$$
$$= 1+x\operatorname{cis}\theta+(x\operatorname{cis}\theta)^2+\dots+(x\operatorname{cis}\theta)^{n-1}$$
$$= \frac{1-(x\operatorname{cis}\theta)^n}{1-x\operatorname{cis}\theta} \quad \text{on summing the G.P., if } x\operatorname{cis}\theta \neq 1,$$
$$= \frac{1-x^n\operatorname{cis}n\theta}{1-x\operatorname{cis}\theta}$$
$$= \frac{(1-x^n\operatorname{cis}n\theta)\{1-x\operatorname{cis}(-\theta)\}}{(1-x\operatorname{cis}\theta)\{1-x\operatorname{cis}(-\theta)\}}$$
to make the denominator real,
$$= \frac{1-x^n\operatorname{cis}n\theta-x\operatorname{cis}(-\theta)+x^{n+1}\operatorname{cis}(n-1)\theta}{1-2x\cos\theta+x^2}$$
since $\operatorname{cis}\theta$, $\operatorname{cis}(-\theta)$ are conjugate. Taking the real part of this result,
$$C = \frac{1-x\cos\theta-x^n\cos n\theta+x^{n+1}\cos(n-1)\theta}{1-2x\cos\theta+x^2}.$$

N.B.—If $|x| < 1$, then $x^n \to 0$ when $n \to \infty$, and the series has a sum to infinity given by
$$\lim_{n\to\infty} C = \frac{1-x\cos\theta}{1-2x\cos\theta+x^2}.$$
If $|x| \geqslant 1$, there is no sum to infinity.

(ii) *Prove that* $\sin\alpha+\binom{n}{1}\sin(\alpha+\beta)+\binom{n}{2}\sin(\alpha+2\beta)+\dots+\sin(\alpha+n\beta)$
$$= (2\cos\tfrac{1}{2}\beta)^n\sin(\alpha+\tfrac{1}{2}n\beta).$$

Write $C = \cos\alpha+\binom{n}{1}\cos(\alpha+\beta)+\binom{n}{2}\cos(\alpha+2\beta)+\dots+\cos(\alpha+n\beta)$

and $S = \sin\alpha+\binom{n}{1}\sin(\alpha+\beta)+\binom{n}{2}\sin(\alpha+2\beta)+\dots+\sin(\alpha+n\beta)$.

Then $C+iS = \operatorname{cis}\alpha+\binom{n}{1}\operatorname{cis}(\alpha+\beta)+\binom{n}{2}\operatorname{cis}(\alpha+2\beta)+\dots+\operatorname{cis}(\alpha+n\beta)$
$$= \operatorname{cis}\alpha+\binom{n}{1}\operatorname{cis}\alpha\operatorname{cis}\beta+\binom{n}{2}\operatorname{cis}\alpha(\operatorname{cis}\beta)^2+\dots+\operatorname{cis}\alpha(\operatorname{cis}\beta)^n$$
$$= \operatorname{cis}\alpha\left\{1+\binom{n}{1}\operatorname{cis}\beta+\binom{n}{2}(\operatorname{cis}\beta)^2+\dots+(\operatorname{cis}\beta)^n\right\}$$

† We do not write $S = 1+x\sin\theta+\dots$ because at the next step $C+iS$ would start with $1+i$, which does not appear as a factor in succeeding terms.

$$= \operatorname{cis}\alpha\,(1+\operatorname{cis}\beta)^n$$
$$= \operatorname{cis}\alpha\,(1+\cos\beta+i\sin\beta)^n$$
$$= \operatorname{cis}\alpha\,(2\cos^2\tfrac{1}{2}\beta+2i\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\beta)^n$$
$$= \operatorname{cis}\alpha\,(2\cos\tfrac{1}{2}\beta\operatorname{cis}\tfrac{1}{2}\beta)^n$$
$$= (2\cos\tfrac{1}{2}\beta)^n\operatorname{cis}(\alpha+\tfrac{1}{2}n\beta).$$

$\therefore\quad S = (2\cos\tfrac{1}{2}\beta)^n\sin(\alpha+\tfrac{1}{2}n\beta)$ on equating imaginary parts.

Exercise 14(*e*)

1 Sum $\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)-\sin(\alpha+3\beta)+\dots$ to $2n$ terms.

2 Prove $1+\cos\theta\sec\theta+\cos 2\theta\sec^2\theta+\dots+\cos n\theta\sec^n\theta = \dfrac{\sin(n+1)\theta}{\sin\theta\cos^n\theta}$.

3 Calculate $\sum\limits_{r=1}^{n}\dbinom{n}{r}\sin 2r\theta$.

4 Find $(2\cos\theta)^n-\dbinom{n}{1}(2\cos\theta)^{n-1}\cos\theta+\dbinom{n}{2}(2\cos\theta)^{n-2}\cos 2\theta-\dots$ to $(n+1)$ terms.

5 Sum to n terms the series whose rth term is $(2r-1)x^{r-1}$. If $\alpha = 2\pi/n$ and n is a positive integer, prove that

$$1+3\cos\alpha+5\cos 2\alpha+\dots+(2n-1)\cos(n-1)\alpha = -n$$

and
$$3\sin\alpha+5\sin 2\alpha+\dots+(2n-1)\sin(n-1)\alpha = -n\cot\tfrac{1}{2}\alpha.$$

6 Find the sum to n terms of $\sin A+(b/c)\sin 2A+(b^2/c^2)\sin 3A+\dots$. If $b < c$ in the triangle ABC, deduce that the sum to infinity is $(c/a)\sin C$.

7 If θ is not zero or an integral multiple of π, find the sum to infinity of

$$\sin\alpha+\cos\theta\sin(\alpha+\theta)+\cos^2\theta\sin(\alpha+2\theta)+\cos^3\theta\sin(\alpha+3\theta)+\dots.$$

8 If n is a positive integer, prove that

$$1+nx\cos\theta+\binom{n}{2}x^2\cos 2\theta+\binom{n}{3}x^3\cos 3\theta+\dots+\binom{n}{n}x^n\cos n\theta = r^n\cos n\alpha,$$

where $r = +\sqrt{(1+2x\cos\theta+x^2)}$ and $\cos\alpha:\sin\alpha:1 = (1+x\cos\theta):x\sin\theta:r$.

Further examples appear in Exs. 14 (*f*), (*g*).

14.6 Infinite series of complex terms. Some single-valued functions of a complex variable

14.61 Convergence and absolute convergence

(1) *Limit of a complex function of n.* If $s_n = \sigma_n+i\tau_n$, and if $\sigma_n \to \sigma$ and $\tau_n \to \tau$ when $n \to \infty$, we say that $s_n \to \sigma+i\tau$ when $n \to \infty$ and write

$$\lim_{n\to\infty} s_n = \sigma+i\tau.$$

This definition of 'limit' is consistent with that in 2.71 for real functions of n; for, given a positive number ϵ however small, there is a number N such that, for all $n \geqslant N$,
$$|\sigma_n-\sigma| < \tfrac{1}{2}\epsilon \quad\text{and}\quad |\tau_n-\tau| < \tfrac{1}{2}\epsilon$$

and so $|s_n-(\sigma+i\tau)| = |(\sigma_n-\sigma)+i(\tau_n-\tau)| \leqslant |\sigma_n-\sigma|+|\tau_n-\tau| < \epsilon.$
Conversely, if this last condition is satisfied for all $n \geqslant N$, then since

$$|\sigma_n-\sigma| \leqslant |(\sigma_n-\sigma)+i(\tau_n-\tau)| < \epsilon,$$

we have $\sigma_n \to \sigma$ (and similarly $\tau_n \to \tau$).

(2) For the series Σz_r, where $z_r = x_r+iy_r$ and x_r, y_r are real,

$$s_n = \sum_{r=1}^{n} z_r = \sum_{r=1}^{n} x_r+i\sum_{r=1}^{n} y_r = \sigma_n+i\tau_n, \quad \text{say.}$$

By (1), s_n tends to a limit if and only if σ_n, τ_n both tend to limits when $n \to \infty$. Accordingly we give the following definitions.

Σz_r is *convergent* if Σx_r and Σy_r both converge.

If Σx_r, Σy_r converge to sums X, Y, then $Z = X+iY$ is the *sum to infinity* of Σz_r.

(3) If the series (of real positive terms) $\Sigma\,|z_r|$ converges, we say that Σz_r is *absolutely convergent* (A.C.).

It follows that *if* Σz_r *is* A.C., *then* Σz_r *is also convergent* in the sense of (2). For since $|x_r| \leqslant \sqrt{(x_r^2+y_r^2)} = |z_r|$, and $\Sigma\,|z_r|$ is convergent by hypothesis, the comparison test of 12.41 shows that $\Sigma\,|x_r|$ converges; i.e. Σx_r is A.C. and consequently (by 12.52) also converges. Similarly Σy_r converges, and hence by definition Σz_r converges.

14.62 The infinite G.P. $1+z+z^2+z^3+\dots$

Write $$s_n(z) = 1+z+z^2+\dots+z^{n-1}.$$

If we multiply by z and subtract (just as for a G.P. with a real common ratio), we find that, if $z \neq 1$,

$$s_n(z) = \frac{1}{1-z}-\frac{z^n}{1-z}.$$

Putting $z = r(\cos\theta+i\sin\theta)$, where $r > 0$, we have

$$z^n = r^n(\cos n\theta+i\sin n\theta).$$

If $r < 1$, then $\lim\limits_{n\to\infty} r^n = 0$; and since $|r^n\cos n\theta| \leqslant r^n$ and $|r^n \sin n\theta| \leqslant r^n$, therefore both $r^n\cos n\theta$ and $r^n\sin n\theta$ tend to zero when $n \to \infty$. Hence

$$\lim_{n\to\infty} z^n = 0 \quad \text{when} \quad |z| = r < 1.$$

Thus when $|z| < 1$, $\lim\limits_{n\to\infty} s_n(z) = 1/(1-z)$.

If $r > 1$, then $r^n \to \infty$ when $n \to \infty$, and so $s_n(z)$ has no limit when $n \to \infty$. If $r = 1$ and $z \neq 1$, then $z^n = \cos n\theta+i\sin n\theta$; but $\cos n\theta$,

$\sin n\theta$ oscillate between ± 1 when $n \to \infty$, and again $s_n(z)$ does not tend to a limit. If $z = 1$, clearly $s_n(z) = n$.

The G.P. $1+z+z^2+\ldots$ *therefore converges only if* $|z| < 1$, *and its sum to infinity is then* $1/(1-z)$. We can write

$$(1-z)^{-1} = 1+z+z^2+z^3+\ldots \quad (|z| < 1).$$

The G.P. is absolutely convergent when the series of moduli $1+r+r^2+\ldots$ converges. This is the case when and only when $0 \leqslant r < 1$, i.e. $|z| < 1$. Hence the ranges of convergence and of absolute convergence are the same.

14.63 The exponential series

$$1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots$$

(1) Write $z = r(\cos\theta + i\sin\theta)$. Then since the series

$$1+\frac{r}{1!}+\frac{r^2}{2!}+\frac{r^3}{3!}+\ldots$$

converges for all r, the series

$$1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots \tag{i}$$

is A.C. for all complex numbers z, and hence converges for all z.

Denoting the sum-function of series (i) by $\exp z$, or by $\exp(z)$ when necessary, we have

$$\exp z = \lim_{n\to\infty}\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}\right) \quad \textit{for all } z. \tag{ii}$$

Remark. The notation is chosen because of the analogy with the real series

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

whose sum-function e^x is occasionally written $\exp x$ for convenience in printing. We usually avoid writing e^z for the sum to infinity of series (i) because later work (not in this book) shows that e^z is a function having infinitely many values for each non-real value of z, and it is clearly undesirable to use this for the sum of a convergent series, which is necessarily *unique*. However, when the index notation e^z is used, it is to be interpreted as $\exp z$; thus in the symbolic method of 5.43, Case (iii), the notation e^{iy} is more suggestive than $\exp(iy)$.

(2) By using the definition (ii), it can now be shown that $\exp z$ obeys the functional law

$$\exp z_1 \times \exp z_2 = \exp(z_1+z_2), \tag{iii}$$

just as $\exp x$ does in real algebra.

Let $|z_1| = r_1$, $|z_2| = r_2$, and write

$$s_n(z) = 1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}.$$

$$\begin{aligned} s_n(z_1).s_n(z_2) &= \left\{1+\frac{z_1}{1!}+\frac{z_1^2}{2!}+\ldots+\frac{z_1^n}{n!}\right\}\left\{1+\frac{z_2}{1!}+\frac{z_2^2}{2!}+\ldots+\frac{z_2^n}{n!}\right\} \\ &= 1+\frac{z_1}{1!}+\frac{z_1^2}{2!}+\ldots+\frac{z_1^n}{n!} \\ &\quad +\frac{z_2}{1!}+\frac{z_1z_2}{1!\,1!}+\frac{z_1^2z_2}{2!\,1!}+\ldots+\frac{z_1^nz_2}{n!\,1!} \\ &\quad +\frac{z_2^2}{2!}+\frac{z_1z_2^2}{1!\,2!}+\frac{z_1^2z_2^2}{2!\,2!}+\ldots+\frac{z_1^nz_2^2}{n!\,2!} \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad +\frac{z_2^n}{n!}+\frac{z_1z_2^n}{1!\,n!}+\frac{z_1^2z_2^n}{2!\,n!}+\ldots+\frac{z_1^nz_2^n}{n!\,n!}. \end{aligned}$$

We now add up these terms 'by diagonals': the terms of total degree s ($s \leqslant n$) give

$$\begin{aligned} &\frac{z_1^s}{s!}+\frac{z_1^{s-1}z_2}{(s-1)!\,1!}+\frac{z_1^{s-2}z_2^2}{(s-2)!\,2!}+\ldots+\frac{z_2^s}{s!} \\ &= \frac{1}{s!}\left\{z_1^s+\frac{s}{1!}z_1^{s-1}z_2+\frac{s(s-1)}{2!}z_1^{s-2}z_2^2+\ldots+z_2^s\right\} \\ &= \frac{1}{s!}(z_1+z_2)^s, \end{aligned}$$

and consequently

$$s_n(z_1)\,s_n(z_2)-s_n(z_1+z_2) = \sum_{p,q}\frac{z_1^pz_2^q}{p!\,q!}, \tag{iv}$$

where the last summation is for all p and q for which $p+q > n$, $p \leqslant n$, $q \leqslant n$; for these are precisely the terms not included in the preceding summation by diagonals.

In exactly the same way we have

$$s_n(r_1)\,s_n(r_2)-s_n(r_1+r_2) = \sum_{p,q}\frac{r_1^pr_2^q}{p!\,q!}. \tag{v}$$

We know that, when $n \to \infty$, $s_n(r_1) \to e^{r_1}$, $s_n(r_2) \to e^{r_2}$, and $s_n(r_1+r_2) \to e^{r_1+r_2}$. Hence the left-hand side of (v) tends to zero, so also $\sum_{p,q}\frac{r_1^pr_2^q}{p!\,q!} \to 0$.

From (iv),

$$\begin{aligned} |s_n(z_1)\,s_n(z_2)-s_n(z_1+z_2)| &= \left|\sum_{p,q}\frac{z_1^pz_2^p}{p!\,q!}\right| \\ &\leqslant \sum_{p,q}\frac{|z_1|^p\,|z_2|^q}{p!\,q!} = \sum_{p,q}\frac{r_1^pr_2^q}{p!\,q!} \to 0 \quad \text{when} \quad n \to \infty. \end{aligned}$$

But when $n \to \infty$, $s_n(z_1) \to \exp z_1$, $s_n(z_2) \to \exp z_2$ and $s_n(z_1+z_2) \to \exp(z_1+z_2)$; hence the result (iii) follows for any two complex numbers z_1, z_2.

14.64 The modulus-argument form of exp z

By (iii), $$\exp z = \exp(x+iy)$$
$$= \exp x \exp(iy).$$

Now by putting $z = x$ in (i),

$$\exp x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$$
$$= e^x \quad \text{for all } x.$$

Also $$s_n(iy) = 1+\frac{iy}{1!}+\frac{(iy)^2}{2!}+\ldots+\frac{(iy)^n}{n!}$$
$$=\left\{1-\frac{y^2}{2!}+\frac{y^4}{4!}-\ldots\right\}+i\left\{y-\frac{y^3}{3!}+\frac{y^5}{5!}-\ldots\right\},$$

where the series in the brackets are *finite* series. Since by 12.61 (2)

$$\cos y = 1-\frac{y^2}{2!}+\frac{y^4}{4!}-\ldots$$

and $$\sin y = y-\frac{y^3}{3!}+\frac{y^5}{5!}-\ldots$$

for all (real) values of y, we have by letting $n \to \infty$ that

$$\exp(iy) = \lim_{n\to\infty} s_n(iy)$$
$$= \cos y + i \sin y.$$

Therefore $$\mathbf{\exp z = e^x(\cos y + i \sin y).} \tag{vi}$$

This result shows that $\exp z$ *is periodic, with period*† $2\pi i$. For

$$\exp(z+2\pi i) = \exp\{x+(y+2\pi)i\}$$
$$= e^x\{\cos(y+2\pi)+i\sin(y+2\pi)\}$$
$$= e^x(\cos y + i\sin y) = \exp z.$$

14.65 Euler's exponential forms for sine, cosine

From (vi),

$$\exp(iy) = \cos y + i\sin y, \quad \exp(-iy) = \cos y - i \sin y.$$

† Cf. the definition for real functions in 1.52(2). It can be shown from (vi) and the fact that $\cos y$ and $\sin y$ have period 2π that $2\pi i$ is the number p of smallest modulus which satisfies $\exp(z+p) = \exp z$ for all z, and that every such number p is an integral multiple of $2\pi i$.

First adding, and then subtracting, we find

$$\cos y = \tfrac{1}{2}\{\exp(iy)+\exp(-iy)\}, \quad \sin y = \frac{1}{2i}\{\exp(iy)-\exp(-iy)\}, \tag{vii}$$

where we observe the factor $1/2i$, *not* $1/2$, in the last result.

These formulae, often written

$$\cos y = \tfrac{1}{2}(e^{iy}+e^{-iy}), \quad \sin y = \frac{1}{2i}(e^{iy}-e^{-iy}),$$

are known as *Euler's exponential forms*. However, they are equivalent to the infinite series expansions of $\cos y$, $\sin y$ stated above.

14.66 Examples

(i) *Find the sum to infinity of*

$$1+\cos\theta\tan\theta+\frac{1}{2!}\cos 2\theta\tan^2\theta+\frac{1}{3!}\cos 3\theta\tan^3\theta+\ldots.$$

Write $t = \tan\theta$,

$$C = 1+t\cos\theta+\frac{t^2}{2!}\cos 2\theta+\frac{t^3}{3!}\cos 3\theta+\ldots,$$

and
$$S = \quad t\sin\theta+\frac{t^2}{2!}\sin 2\theta+\frac{t^3}{3!}\sin 3\theta+\ldots.$$

Then
$$\begin{aligned} C+iS &= 1+tz+\frac{t^2z^2}{2!}+\frac{t^3z^3}{3!}+\ldots, \quad \text{where} \quad z = \operatorname{cis}\theta, \\ &= \exp(tz) \quad \text{for all } z \text{ and all real } t, \\ &= \exp(\sin\theta+i\sin\theta\tan\theta) \\ &= e^{\sin\theta}\operatorname{cis}(\sin\theta\tan\theta). \end{aligned}$$

$$\therefore \quad C = e^{\sin\theta}\cos(\sin\theta\tan\theta).$$

(ii) *Trial exponentials: the case of failure.* In 5.33 (2) we remarked that the trial method of solving $y''+ay'+by = 0$ by the substitution $y = e^{mx}$ will fail if the quadratic $m^2+am+b = 0$ has no (real) roots, i.e. if $a^2 < 4b$. Putting $a^2+4p^2 = 4b$, we find that in complex algebra the roots are $m = -\tfrac{1}{2}a \pm ip$, so that by a formal application of 5.32 (2), II, the general solution would be

$$\begin{aligned} y &= A\,e^{(-\frac{1}{2}a+ip)x}+B\,e^{(-\frac{1}{2}a-ip)x} \\ &= e^{-\frac{1}{2}ax}(A\,e^{ipx}+B\,e^{-ipx}) \\ &= e^{-\frac{1}{2}ax}\{A(\cos px+i\sin px)+B(\cos px-i\sin px)\} \\ &= e^{-\frac{1}{2}ax}\{A'\cos px+B'\sin px\}, \end{aligned}$$

where $A' = A+B$, $B' = i(A-B)$. This solution is identical in form with that found in 5.33 (1), Case (iii), so that A', B' must be *real* constants; in fact A, B in the above calculation are conjugate complex numbers.

(iii) *Integration by $C+iS$.* From formula (vi) we have (after a slight change of notation)

$$e^{ix} = \cos x+i\sin x.$$

If u, v are derivable functions of x, and we *define* the derivative of $u+iv$ to be $u'+iv'$, then

$$\frac{d}{dx}(\cos x+i\sin x) = -\sin x+i\cos x = i(\cos x+i\sin x),$$

i.e.
$$\frac{d}{dx}(e^{ix}) = i\,e^{ix}.$$

More generally, by formula (vi) we have (for real constants a, b)

$$e^{(a+ib)x} = e^{ax}(\cos bx+i\sin bx) = e^{ax}\cos bx+i\,e^{ax}\sin bx,$$

$$\begin{aligned}\therefore\quad \frac{d}{dx}\{e^{(a+ib)x}\} &= e^{ax}(a\cos bx-b\sin bx)+i\,e^{ax}(a\sin bx+b\cos bx)\\ &= e^{ax}\{(a+ib)\cos bx+i(a+ib)\sin bx\}\\ &= (a+ib)\,e^{(a+ib)x}.\end{aligned}$$

Thus the law $d(e^{mx})/dx = m\,e^{mx}$, already known from 4.41 (5) for all real constants m, also holds for complex ones. Equivalently, we may write

$$\int e^{(a+ib)x}\,dx = \frac{1}{a+ib}\,e^{(a+ib)x}+c,$$

where c is a complex arbitrary constant.

This result permits rapid calculation of certain integrals involving combinations of exponential and circular functions. For example, put

$$C = \int e^{ax}\cos bx\,dx, \quad S = \int e^{ax}\sin bx\,dx;$$

then
$$\begin{aligned}C+iS &= \int e^{ax}(\cos bx+i\sin bx)\,dx\\ &= \int e^{(a+ib)x}\,dx = \frac{1}{a+ib}\,e^{(a+ib)x}+c\\ &= \frac{a-ib}{a^2+b^2}\,e^{ax}(\cos bx+i\sin bx)+c.\end{aligned}$$

Equating real and imaginary parts,

$$C = \frac{a\cos bx+b\sin bx}{a^2+b^2}\,e^{ax}+c_1, \quad S = \frac{a\sin bx-b\cos bx}{a^2+b^2}\,e^{ax}+c_2.$$

Cf. Ex. 4 (*e*), nos. 28, 29.

A similar method, combined with integration by parts, will apply to

$$\int x^n\,e^{ax}\,{}^{\cos}_{\sin}\,bx\,dx;$$

cf. Ex. 4 (*m*), no. 35.

Exercise 14(*f*)

Simplify the following.

1 $\exp(\pi i)$. 2 $\exp(1+\frac{1}{2}\pi i)$. 3 $\exp(\log r+i\theta)$.

4 $\exp(i)+\exp(-i)$. 5 $\exp\{\operatorname{cis}\theta\}+\exp\{\operatorname{cis}(-\theta)\}$.

6 $\exp\{(a+ib)x\}-\exp\{(a-ib)x\}$.

7 Expand $e^{x\cos\alpha}\sin(x\sin\alpha)$ in ascending powers of x. [Use no. 6.]

Find the sum to infinity of the following series.

8 $x\sin\theta+(x^2/2!)\sin 2\theta+(x^3/3!)\sin 3\theta+\dots$.

9 $\cos\alpha-\cos\beta\cos(\alpha+\beta)+(1/2!)\cos^2\beta\cos(\alpha+2\beta)-\dots$.

10 $\sum_{r=1}^{\infty}\frac{x^r}{r!}\sec^r\beta\sin r\beta$. 11 $\cos\theta+\frac{1}{3!}\cos 3\theta+\frac{1}{5!}\cos 5\theta+\dots$.

12 $\sum_{r=1}^{\infty}\frac{\sin^2 r\theta}{r!}$.

13 If $w=\exp(z/a)$ where $w=u+iv$, $z=x+iy$, and a is real, find the locus of w corresponding to each of the lines $y=0$, $y=\frac{1}{4}\pi a$ in the z-plane.

14 If $u+iv=\exp(x+iy)$, find the locus of the point (u,v) in the w-plane when (i) $x=$ constant; (ii) $y=$ constant. Verify that these two loci cut orthogonally.

15 Criticise the following argument. 'If $y=\cos x+i\sin x$, then

$$\frac{dy}{dx}=-\sin x+i\cos x=i(\cos x+i\sin x)=iy;$$

therefore by 4.41, Remark (α), $y=A\,e^{ix}$. Since $y=1$ when $x=0$, therefore $A=1$. Hence $\cos x+i\sin x=e^{ix}$.'

14.67 Generalised circular and hyperbolic functions

(1) Consistently with formulae (vii), we *define*

$$\left.\begin{aligned}\cos z&=\tfrac{1}{2}\{\exp(iz)+\exp(-iz)\},\\ \sin z&=\frac{1}{2i}\{\exp(iz)-\exp(-iz)\},\end{aligned}\right\}\qquad\text{(viii)}$$

and then $\tan z$ to be $\sin z/\cos z$, etc.

These definitions are equivalent to

$$\cos z+i\sin z=\exp(iz),\quad \cos z-i\sin z=\exp(-iz),$$

and also to

$$\left.\begin{aligned}\cos z&=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\dots\\ \sin z&=\frac{z}{1!}-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots\end{aligned}\right\}\text{(all } z).\qquad\text{(ix)}$$

Directly from these definitions it can now be shown that the formulae relating trigonometric functions of a real variable continue to hold for a complex variable. (The definitions were chosen with a view to making this so.) This can be done as in real trigonometry if we first verify the addition theorems together with results concerning special angles, e.g.

$$\cos 0=1,\quad \sin 0=0,\quad \cos(-z)=\cos z,\quad \sin(-z)=-\sin z,$$

and also periodicity properties.

For example, to prove that

$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$

we should use the definitions (viii) to translate each term on the right into exponential functions and verify that the right-hand side reduces to the translation of the left.

(2) We *define*

$$\operatorname{ch} z = \tfrac{1}{2}\{\exp(z)+\exp(-z)\}, \quad \operatorname{sh} z = \tfrac{1}{2}\{\exp(z)-\exp(-z)\}, \qquad \text{(x)}$$

and then $\operatorname{th} z$ to be $\operatorname{sh} z/\operatorname{ch} z$, etc.

These definitions are consistent with those of hyperbolic functions of a real variable (4.44), and are equivalent to

$$\left.\begin{aligned} \operatorname{ch} z &= 1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\dots \\ \operatorname{sh} z &= \frac{z}{1!}+\frac{z^3}{3!}+\frac{z^5}{5!}+\dots \end{aligned}\right\} \text{(all } z\text{)}. \qquad \text{(xi)}$$

Again it can be verified that all formulae relating hyperbolic functions of a real variable continue to hold for a complex one. However, in contrast to the real case, definition (x) and the periodicity of $\exp z$ show that $\operatorname{ch} z$ *and* $\operatorname{sh} z$ *have period* $2\pi i$; also $\operatorname{th} z$ *has period* πi, for

$$\operatorname{th}(z+\pi i) = \frac{\operatorname{sh}(z+\pi i)}{\operatorname{ch}(z+\pi i)} = \frac{-\operatorname{sh} z}{-\operatorname{ch} z}$$

by using (x) and $\exp(\pm\pi i) = -1$, so that $\operatorname{th}(z+\pi i) = \operatorname{th} z$.

14.68 Relation between circular and hyperbolic functions: Osborn's rule

We have

$$\begin{aligned} \cos(iz) &= \tfrac{1}{2}\{\exp(i.iz)+\exp(-i.iz)\} && \text{by definition (viii),} \\ &= \tfrac{1}{2}\{\exp(-z)+\exp(z)\} \\ &= \operatorname{ch} z && \text{by definition (x);} \end{aligned}$$

and

$$\begin{aligned} \sin(iz) &= \frac{1}{2i}\{\exp(i.iz)-\exp(-i.iz)\} && \text{by definition (viii),} \\ &= -\tfrac{1}{2}i\{\exp(-z)-\exp(z)\} \\ &= i\operatorname{sh} z && \text{by definition (x).} \end{aligned}$$

Hence $$\mathbf{\cos(iz) = \operatorname{ch} z, \quad \sin(iz) = i\operatorname{sh} z.} \qquad \text{(xii)}$$

In any formula relating circular functions of $z_1, z_2, \ldots$ we may replace $z_1, z_2, \ldots$ by $iz_1, iz_2, \ldots$ and then use results (xii) to translate it into hyperbolic functions of $z_1, z_2, \ldots$. In the course of this translation each product of two sines becomes i^2 multiplied by a product of two hyperbolic sines. Since the formulae all hold in particular when the variables are real, *we have the justification of Osborn's rule stated in* 4.44 (4).

An example of the detailed verification is the following:

$$\begin{aligned}\operatorname{ch}(A+B) &= \cos(iA+iB)\\ &= \cos(iA)\cos(iB)-\sin(iA)\sin(iB)\\ &= \operatorname{ch}A\operatorname{ch}B - i^2\operatorname{sh}A\operatorname{sh}B\\ &= \operatorname{ch}A\operatorname{ch}B+\operatorname{sh}A\operatorname{sh}B.\end{aligned}$$

Examples

(i)
$$\begin{aligned}\cos(x+iy) &= \cos x\cos(iy)-\sin x\sin(iy)\\ &= \cos x\operatorname{ch}y - i\sin x\operatorname{sh}y;\\ \sin(x+iy) &= \sin x\cos(iy)+\cos x\sin(iy)\\ &= \sin x\operatorname{ch}y+i\cos x\operatorname{sh}y;\end{aligned}$$

and
$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{\sin(x+iy)\cos(x-iy)}{\cos(x+iy)\cos(x-iy)},$$

where we have multiplied top and bottom by the conjugate of the denominator in order to make the new denominator real. Using the formulae for products into sums,

$$\tan(x+iy) = \frac{\sin 2x+\sin 2iy}{\cos 2x+\cos 2iy} = \frac{\sin 2x+i\operatorname{sh}2y}{\cos 2x+\operatorname{ch}2y}.$$

(ii) From ex. (i),

$$\begin{aligned}|\cos z|^2 &= \cos^2 x\operatorname{ch}^2 y+\sin^2 x\operatorname{sh}^2 y\\ &= (1-\sin^2 x)\operatorname{ch}^2 y+\sin^2 x(\operatorname{ch}^2 y-1)\\ &= \operatorname{ch}^2 y-\sin^2 x\\ &\leqslant \operatorname{ch}^2 y \quad \text{for all } x.\end{aligned}$$

Similarly $\quad |\cos z|^2 = \cos^2 x+\operatorname{sh}^2 y \geqslant \operatorname{sh}^2 y \quad$ for all x.

Thus $\quad |\operatorname{sh} y| \leqslant |\cos z| \leqslant \operatorname{ch} y \quad$ *for all* z.

The reader should verify that the same inequalities hold for $|\sin z|$.

(iii) *Find the equation of the curve described by the point* (x, y) *when* $z = x+iy$ *varies so that* $\operatorname{sh} z - z$ *is real. Sketch the part of the curve which lies between the lines* $y = \pm\pi$.

$$\begin{aligned}\operatorname{sh}(x+iy) &= \frac{1}{i}\sin(ix-y) = \frac{1}{i}\sin(ix)\cos y-\frac{1}{i}\cos(ix)\sin y\\ &= \operatorname{sh}x\cos y+i\operatorname{ch}x\sin y,\end{aligned}$$

$$\therefore\quad \operatorname{sh}z-z = (\operatorname{sh}x\cos y-x)+i(\operatorname{ch}x\sin y-y),$$

which is real if $\operatorname{ch} x\sin y = y$.

Clearly $y = 0$ satisfies this equation which, if y is not an integral multiple of π, can be written

$$\operatorname{ch} x = \frac{y}{\sin y}.$$

The curve is symmetrical about Ox and Oy. When $y \to 0$ (through positive or negative values), $\operatorname{ch} x \to 1$ and so $x \to 0$ (through positive or negative values). When $y \to \pi-$, $\operatorname{ch} x \to +\infty$ and so $|x| \to \infty$; when $y \to (-\pi)+$, $|x| \to \infty$. The reader should verify that the branches have gradients $\pm\sqrt{3}$ at the origin.

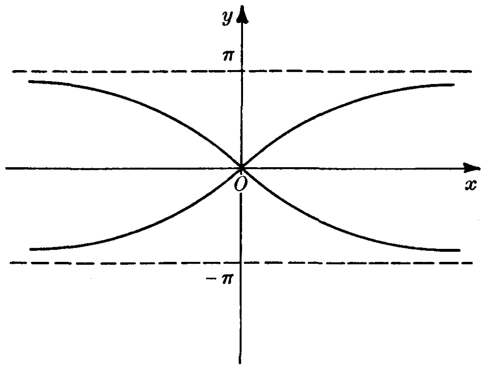

Fig. 138

Exercise 14(*g*)

1 Prove that $\operatorname{ch}(iz) = \cos z$, $\operatorname{sh}(iz) = i \sin z$.

2 By using results (xii) in 14.68, prove that

$$\operatorname{sh}(A+B) = \operatorname{sh} A \operatorname{ch} B + \operatorname{ch} A \operatorname{sh} B$$

and
$$\operatorname{th}(A+B) = (\operatorname{th} A + \operatorname{th} B)/(1 + \operatorname{th} A \operatorname{th} B)$$

follow from the corresponding properties of circular functions.

3 Use the definitions (viii) in 14.67 to prove that $\cos^2 z + \sin^2 z = 1$ and

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

Express the following in the form $a + ib$.

4 $\operatorname{ch}(x+iy)$. 5 $\sec(x+iy)$. 6 $\operatorname{th}(x+iy)$.

7 If $y > 0$, prove that $\operatorname{th} y \leqslant |\tan(x+iy)| \leqslant \coth y$.

8 Determine the general forms of z for which (i) e^z; (ii) $\cos z$ are real.

9 Find the general solution of $\sin z = 3i \cos z$.

10 Solve completely $\cos z = \{e(1-i) + (1+i)/e\}/2\sqrt{2}$.

11 If $\operatorname{ch}(x+iy) \cos(u+iv) = 1$ and $\cos y \cos u \neq 0$, prove

$$\tan u \operatorname{th} v = \operatorname{th} x \tan y.$$

[Equate imaginary parts.]

12 If $\cos(x+iy) = u+iv$, prove

(i) $u^2 \sec^2 x - v^2 \operatorname{cosec}^2 x = 1$; (ii) $u^2 \operatorname{sech}^2 y + v^2 \operatorname{cosech}^2 y = 1$.

Interpret these as loci corresponding to the lines x = constant, y = constant respectively.

13 Expand $\frac{1}{2}(\operatorname{sh} z - \sin z)$ in ascending powers of z.

Use $C+iS$ to find the sum to infinity of

14 $\dfrac{\sin 2\theta}{2!}+\dfrac{\sin 4\theta}{4!}+\dfrac{\sin 6\theta}{6!}+\ldots$. 15 $\cos\theta+\dfrac{\cos 5\theta}{5!}+\dfrac{\cos 9\theta}{9!}+\ldots$.

Miscellaneous Exercise 14(*h*)

1 If $x = \operatorname{cis}\alpha$ and $y = \operatorname{cis}\beta$, prove that

$$\frac{x-y}{x+y} = i\tan\tfrac{1}{2}(\alpha-\beta) \quad\text{and}\quad \frac{(x+y)(xy-1)}{(x-y)(xy+1)} = \frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}.$$

2 If the real and imaginary parts of $(1+ix)^n$ are equal, n being a positive integer and x real, prove that $x = \tan\{(4r+1)\pi/4n\}$ where r is any integer or zero. [Put $x = \cot\theta$.]

3 Prove

$$\{(\cos\alpha+i\sin\alpha)-(\cos\beta+i\sin\beta)\}^n+\{(\cos\alpha-i\sin\alpha)-(\cos\beta-i\sin\beta)\}^n$$
$$= \lambda_n 2^{n+1}\sin^n\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}n(\alpha+\beta),$$

where $\lambda_n = (-1)^{\frac{1}{2}n}$ if n is even, and $\lambda_n = (-1)^{\frac{1}{2}(n+1)}$ if n is odd.

4 (i) Solve $x^5+x^4+x^3+x^2+x+1 = 0$.
(ii) Find the solutions of $x^6 = 1$ for which $1+x+x^2 \neq 0$.

5 If n is a positive integer, show that every root of $(z+1)^{2n}+(z-1)^{2n} = 0$ is purely imaginary. If the roots are represented by $P_1, P_2, \ldots, P_{2n}$ and O is the origin, prove $OP_1^2+OP_2^2+\ldots+OP_{2n}^2 = 2n(2n-1)$.

6 Show that the roots of $(z-1)^5 = 32(z+1)^5$ are represented by points on a circle of radius $\frac{4}{3}$, and that the roots are

$$\left(-3+4i\sin\frac{2r\pi}{5}\right)\bigg/\left(5-4\cos\frac{2r\pi}{5}\right) \quad (r = 0, 1, 2, 3, 4).$$

Deduce that $\displaystyle\prod_{r=1}^{5}\left(-3+4i\sin\frac{2r\pi}{5}\right) = -\frac{33}{31}\prod_{r=1}^{5}\left(5-4\cos\frac{2r\pi}{5}\right)$.

7 If $\alpha = \operatorname{cis}(2\pi/n)$ and n, k are integers, prove that $1+\alpha^k+\alpha^{2k}+\ldots+\alpha^{(n-1)k}$ is equal to n if k is a multiple of n, and is zero otherwise. [Sum the G.P. when $\alpha^k \neq 1$.]

8 Express $(1-t^2)/\{(1-at)(1-bt)\}$ in partial fractions when the non-zero constants a, b are (i) unequal; (ii) equal. Taking $a = b^{-1} = \operatorname{cis}\theta$ and a suitable value of t, deduce that if $0 < \phi < \frac{1}{2}\pi$,

$$\frac{\cos\phi}{1-\sin\phi\cos\theta} = 1+2\sum_{r=1}^{\infty}\tan^r\tfrac{1}{2}\phi\cos r\theta.$$

9 Prove that

$$\frac{d^n}{dx^n}\left(\frac{x}{x^2+1}\right) = (-1)^n n!\cos(n+1)\theta\sin^{n+1}\theta,$$

where $x = \cot\theta$. [Method of 6.61, (vii).]

10 If $\omega^3 = 1$, $\omega \neq 1$, prove

$$\frac{3}{x^3-1} = \frac{1}{x-1} + \frac{1}{\omega x-1} + \frac{1}{\omega^2 x-1}.$$

By taking $x = \operatorname{cis} 2\theta$, deduce that

$$3\cot 3\theta = \cot\theta + \cot(\theta+\tfrac{1}{3}\pi) + \cot(\theta+\tfrac{2}{3}\pi).$$

11 Express

$$\cos 6\theta + 6\cos 4\theta - 9\cos 3\theta + 15\cos 2\theta - 27\cos\theta + 14 = 0$$

as a polynomial equation in $\cos\theta$, and hence find all solutions between 0 and 2π inclusive.

12 If $\alpha_1+\alpha_2+\ldots+\alpha_{2n} = 2\pi$, prove that (with the notation in 14.23)

$$\sec\alpha_1 \sec\alpha_2 \ldots \sec\alpha_{2n} = 1-\Sigma_2+\Sigma_4-\ldots+(-1)^n\Sigma_{2n}.$$

13 If θ_1, θ_2, θ_3 are roots of $\tan(\theta+\alpha) = \lambda\tan 2\theta$, no two differing by an integral multiple of π, and if $\lambda \neq 1$, prove $\theta_1+\theta_2+\theta_3 = n\pi-\alpha$ for some integer n.

*14 If θ_1, θ_2, θ_3, θ_4 are values of θ between 0 and 2π which satisfy

$$a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta + 2g\cos\theta + 2f\sin\theta + c = 0,$$

prove $\tan\frac{1}{2}(\theta_1+\theta_2+\theta_3+\theta_4) = 2h/(a-b)$. [Put $t = \tan\frac{1}{2}\theta$ to obtain a quartic in t.]

15 Factorise $u_n \equiv x^{2n} - 2x^n\cos n\alpha + 1$ by methods of real algebra as follows.

(i) Prove $u_{n+1} \equiv 2xu_n\cos\alpha - x^2u_{n-1} + (x^{2n}+1)u_1$.

(ii) Use Mathematical Induction to prove that $u_1 \equiv x^2-2x\cos\alpha+1$ is a factor of u_n.

(iii) Deduce that $x^2-2x\cos(\alpha+2r\pi/n)+1$ is also a factor of u_n, for any integer r. [u_n is unaltered by replacing α by $\alpha+2r\pi/n$.]

(iv) Verify that $r = 0, 1, 2, \ldots, n-1$ give *distinct* factors.

16 Solve $(z+1)^8-z^8 = 0$, and prove that

$$(z+1)^8-z^8 = \tfrac{1}{16}(2z+1)\prod_{r=1}^{3}(4z^2+4z+\operatorname{cosec}^2\tfrac{1}{8}r\pi).$$

Hence show that

$$16(\cos^{16}\theta-\sin^{16}\theta) = \cos 2\theta\prod_{r=1}^{3}(\cos^2 2\theta+\cot^2\tfrac{1}{8}r\pi).$$

17 *ABCDEF* is a regular hexagon inscribed in the circle $|z| = a$, A being the point $(a, 0)$. If P, representing the number z, is any point on the circle, write down the complex numbers represented by the six points obtained by drawing lines from the origin equal to, parallel to, and in the same sense as the lines AP, BP, ..., FP, and prove that their product is z^6-a^6. Hence prove that

$$AP.BP.CP.DP.EP.FP \leqslant 2a^6.$$

18 Prove

$$\sin 5\theta = 16\sin\theta\sin(\theta+\tfrac{1}{5}\pi)\sin(\theta+\tfrac{2}{5}\pi)\sin(\theta+\tfrac{3}{5}\pi)\sin(\theta+\tfrac{4}{5}\pi),$$

and by considering $\lim\limits_{\theta\to 0}(\sin 5\theta/\sin\theta)$ deduce that $\sin\frac{1}{5}\pi\sin\frac{2}{5}\pi = +\frac{1}{4}\sqrt{5}$.

19 If $w = u+iv$, $z = x+iy$, $w = (\exp z-1)/(\exp z+1)$, and $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$, prove that the point (u, v) lies *inside* the circle $|w| = 1$.

20 Prove that if a is real, the equation $\exp z = z+a$ has no purely imaginary root. If $x+iy$ $(y \neq 0)$ is a solution, prove that $x > 0$. [Use $\sin y/y < 1$.]

21 If $u+iv = (z-1)\exp(-i\alpha)+(z-1)^{-1}\exp(i\alpha)$ where $z = x+iy$ and α is real, find u and v in terms of x, y, α. Prove that the locus of z when $v = 0$ consists of a circle with centre $(1, 0)$ and unit radius, and a straight line through the centre of this circle.

22 If $\exp(u+iv) = r \operatorname{cis}\theta$, prove $u+iv = \log r + i(\theta+2n\pi)$, where n is any integer or zero.

*23 If $x+iy = \cos(u+iv)$, prove

$$(1+x)^2+y^2 = (\operatorname{ch} v+\cos u)^2 \quad \text{and} \quad (1-x)^2+y^2 = (\operatorname{ch} v-\cos u)^2.$$

If $x = \cos\theta$, $y = \sin\theta$ $(0 < \theta < \pi)$ in the preceding, find $\cos u$ and $\operatorname{ch} v$ in terms of $\cos\frac{1}{2}\theta$, $\sin\frac{1}{2}\theta$, justifying the choice of sign when square roots are taken.

24 If $u+iv = \coth(x+iy)$, prove $v = -\sin 2y/(\operatorname{ch} 2x-\cos 2y)$. Show that

$$iv = \{1-\exp(2x-2iy)\}^{-1}-\{1-\exp(2x+2iy)\}^{-1}$$

and deduce that if $x < 0$, then $v = -2\sum_{r=1}^{\infty} e^{2rx}\sin 2ry$.

25 If $w = z^2$, sketch in the Argand diagram the path of w when z describes the sides of the rectangle whose vertices are $\pm a$, $\pm a+ia$ (a real), starting at O and moving counterclockwise.

26 (i) Sketch the graph of $y = \operatorname{th} x$. Show that the equation $\operatorname{th} x = \lambda x$ has two real non-zero roots if $0 < \lambda < 1$.

(ii) Prove that the equation $\tan z = \lambda z$ has two purely imaginary roots if $0 < \lambda < 1$; but that if λ has any other real value, all roots of the equation are real.

Find the sum to infinity of

27 $\cos^2\alpha+\frac{1}{2!}\cos^2 2\alpha+\frac{1}{3!}\cos^3 3\alpha+\ldots$ (α real).

28 $x\sin\theta+\frac{x^3}{3!}\sin 3\theta+\frac{x^5}{5!}\sin 5\theta+\ldots$ (x, θ real).

29 Expand $e^{ax}\cos bx$ as a power series in x. [Consider $e^{ax}\operatorname{cis} bx = \exp(ax+ibx)$ and put $a+ib = r\operatorname{cis}\theta$.]

30 (i) By considering $\exp(x)+\exp(\omega x)+\exp(\omega^2 x)$ where $\omega^3 = 1$, $\omega \neq 1$, prove

$$1+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\ldots = \tfrac{1}{3}\{e^x+2e^{-\frac{1}{2}x}\cos(\tfrac{1}{2}x\sqrt{3})\} \quad \text{for all real } x.$$

*(ii) What results are obtained by considering

$$(a)\ \exp(x)+\omega\exp(\omega x)+\omega^2\exp(\omega^2 x);$$

$$(b)\ \exp(x)+\omega^2\exp(\omega x)+\omega\exp(\omega^2 x)?$$

[Use Ex. 13 (g), no. 18.]

31 By putting $z = x+iy$, show that the two differential equations for the functions x, y of t (arising from the dynamics of Foucault's pendulum)

$$\ddot{x}-2k\dot{y}+n^2x = 0, \quad \ddot{y}+2k\dot{x}+n^2y = 0$$

are equivalent to $$\ddot{z}+2ik\dot{z}+n^2z = 0,$$

and solve the latter. Writing $z' = z\,e^{ikt}$, prove that $x' = A\cos\mu t$, $y' = B\sin\mu t$, where $\mu^2 = n^2+k^2$ and A, B are real arbitrary constants.

32 s, ψ are the intrinsic coordinates of a point on a plane curve, and x, y are the cartesian coordinates. Writing $z = x+iy$, prove that

$$\frac{dz}{ds} = e^{i\psi}, \quad \frac{d^2z}{ds^2} = i\kappa\, e^{i\psi}.$$

[Assume equations (iii) of 8.12.] Deduce that

$$\kappa = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}^{\frac{1}{2}} = \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}.$$

(Cf. Ex. 8 (d), no. 8.)

33 Verify that the formulae for rotation of axes through angle θ in 15.73 (3) can be written $z = z'\,e^{i\theta}$, where $z = x+iy$ and $z' = x'+iy'$. Deduce from this the formulae for the reverse transformation.

15

SURVEY OF ELEMENTARY COORDINATE GEOMETRY

15.1 Oblique axes

15.11 Advantage of oblique axes

Although it is usual to refer cartesian coordinates to two *perpendicular* axes Ox, Oy, there are problems in which use of oblique axes is more convenient. For example, in proving properties of the triangle by coordinate methods, two of the sides can be chosen for coordinate axes. The coordinates (x, y) of a point P are then its signed distances from Oy, Ox measured parallel to Ox, Oy respectively. The angle xOy (measured counterclockwise from Ox) is denoted by ω.

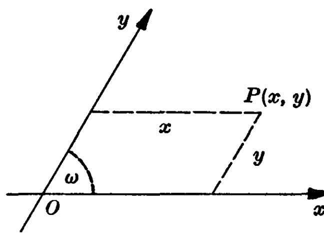

Fig. 139

In sections 15.1–15.5 we revise the main results about 'the straight line', and extend them to the case of oblique axes when this does not introduce any essential complication. *The reader should observe which results remain the same in form whether or not the axes are oblique*; they are marked by the sign $\{\omega\}$.

Notation. In Chapters 15–19 we use the convention that P_1 has coordinates (x_1, y_1), and so on.

15.12 Cartesian and polar coordinates

We first obtain the relations analogous to those in 1.62 for rectangular axes, taking O as pole and Ox as initial line (fig. 140). Draw PN perpendicular to Ox; then

$$r\cos\theta = ON = OM + MN = x + y\cos\omega,$$

$$r\sin\theta = NP = y\sin\omega.$$

Hence also

$$\tan\theta = \frac{y\sin\omega}{x + y\cos\omega},$$

$$r^2 = x^2 + y^2 + 2xy\cos\omega.$$

Example

If the line P_1P_2 is of length r and makes angle θ with Ox (fig. 141), then

$$r\cos\theta = (x_2-x_1)+(y_2-y_1)\cos\omega,$$
$$r\sin\theta = (y_2-y_1)\sin\omega.$$

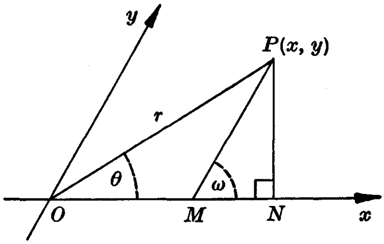

Fig. 140

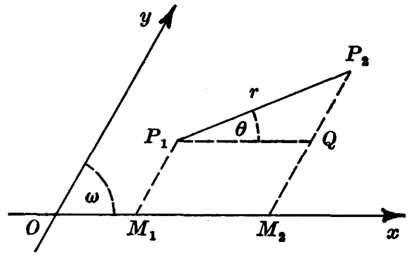

Fig. 141

15.13 Distance formula

From the above example, or directly from the cosine rule applied to triangle P_1P_2Q (fig. 141),

$$P_1P_2^2 = r^2 = (x_2-x_1)^2+(y_2-y_1)^2+2(x_2-x_1)(y_2-y_1)\cos\omega.$$

15.14 Section formulae {ω}

These give the coordinates of the point dividing P_1P_2 in the ratio $k:l$. Let $P(x,y)$ be the required point, so that $P_1P/PP_2 = k/l$.

(1) *Internal division.* With the construction shown in fig. 142, triangles P_1PQ, PP_2R are similar; hence

$$\frac{P_1P}{PP_2} = \frac{P_1Q}{PR}.$$

Since $P_1Q = x-x_1$ and $PR = x_2-x$, this becomes

$$\frac{k}{l} = \frac{x-x_1}{x_2-x},$$

from which $$x = \frac{lx_1+kx_2}{l+k}.$$

Similarly, from

$$\frac{P_1P}{PP_2} = \frac{PQ}{P_2R} = \frac{y-y_1}{y_2-y}$$

we find $$y = \frac{ly_1+ky_2}{l+k}.$$

In each expression observe that the number k, which corresponds to the segment P_1P, is multiplied by the coordinate of P_2; while l is multiplied by that of P_1.

(2) *External division.* Lettering fig. 143 (which illustrates the case $k > l$) as in fig. 142, we still have similar triangles P_1PQ, PP_2R, so that

$$\frac{P_1P}{PP_2} = \frac{P_1Q}{PR},$$

i.e.

$$\frac{k}{l} = \frac{x-x_1}{x-x_2},$$

from which

$$x = \frac{lx_1 - kx_2}{l-k}.$$

Similarly

$$y = \frac{ly_1 - ky_2}{l-k}.$$

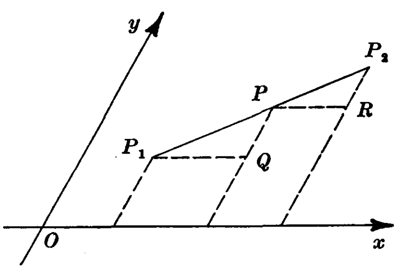

Fig. 142

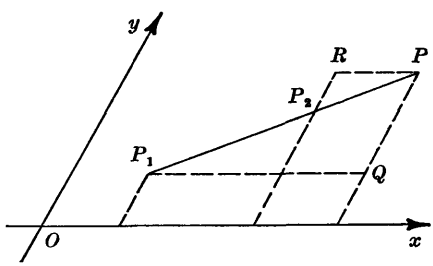

Fig. 143

(3) *Summary.* In both cases above, k and l were understood to be positive numbers. However, the result of (2) can be written

$$x = \frac{lx_1 + (-k)x_2}{l+(-k)}, \quad \text{etc.},$$

which is formally the same as that of (1) for division in the negative ratio $-k:l$. Hence if we agree that $P_1P:PP_2$ shall be reckoned positive when P divides P_1P_2 internally and negative when the division is external, both cases are covered by the *same* formulae, viz.

$$x = \frac{lx_1 + kx_2}{l+k}, \quad y = \frac{ly_1 + ky_2}{l+k}.$$

15.15 Gradient of a line {ω}

The *gradient* of a straight line is defined to be the tangent of the angle which the line makes with the positive x-axis. Since this definition makes no reference to Oy, we have (as for rectangular axes) that if two lines with gradients m_1, m_2 are parallel, then $m_1 = m_2$; if they are perpendicular, then $m_1 m_2 = -1$; and in the general case an angle between them is given by

$$\tan^{-1}\frac{m_1-m_2}{1+m_1 m_2}.$$

For $\alpha = \theta_1-\theta_2$, where

$$\tan\theta_1 = m_1,\ \tan\theta_2 = m_2,$$

and so

$$\tan\alpha = \frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2} = \frac{m_1-m_2}{1+m_1 m_2}.$$

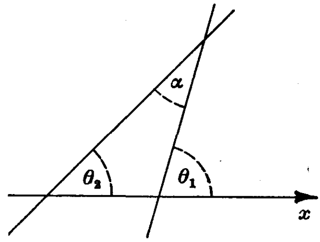

Fig. 144

Example

From the example in 15.12, the gradient of P_1P_2 is

$$\tan\theta = \frac{(y_2-y_1)\sin\omega}{(x_2-x_1)+(y_2-y_1)\cos\omega}.$$

15.16 Area of a triangle

(1) *One vertex at O.* Let the polar coordinates of P_2, P_3 be (r_2, θ_2), (r_3, θ_3); see fig. 145. Then by 15.12,

$$r_2\cos\theta_2 = x_2+y_2\cos\omega, \quad r_2\sin\theta_2 = y_2\sin\omega,$$

and similarly for P_3.

$$\begin{aligned}\text{Area of triangle } OP_2P_3 &= \tfrac{1}{2}OP_2 . OP_3 \sin P_2\hat{O}P_3 \\ &= \tfrac{1}{2}r_2r_3\sin(\theta_3-\theta_2) \\ &= \tfrac{1}{2}r_2r_3\sin\theta_3\cos\theta_2 - \tfrac{1}{2}r_2r_3\cos\theta_3\sin\theta_2 \\ &= \tfrac{1}{2}(x_2+y_2\cos\omega)(y_3\sin\omega) \\ &\qquad - \tfrac{1}{2}(y_2\sin\omega)(x_3+y_3\cos\omega) \\ &= \tfrac{1}{2}\sin\omega(x_2y_3-x_3y_2) \quad \text{after reduction} \\ &= \tfrac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}\sin\omega.\end{aligned}$$

Remark. According as the vertices O, P_2, P_3 are named in counter-clockwise or clockwise order, the angle $\theta_3-\theta_2$ is positive or negative and hence the expression for the area is positive or negative respectively. If we consider a *signed* area which is positive when the vertices are named counterclockwise and is negative otherwise, then the above result gives the area of OP_2P_3 in magnitude and sign.

(2) *Triangle* $P_1P_2P_3$. We may choose new axes through one of the vertices, say P_1, which are parallel to the original axes Ox, Oy. The new coordinates of P_2 are then (fig. 146)

$$x_2' = x_2 - x_1, \quad y_2' = y_2 - y_1,$$

and of P_3 are
$$x_3' = x_3 - x_1, \quad y_3' = y_3 - y_1.$$

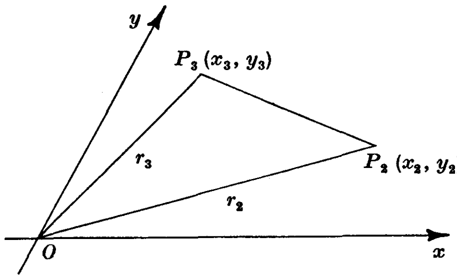

Fig. 145

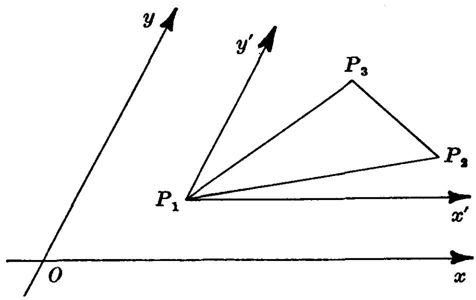

Fig. 146

By (1), area of $P_1P_2P_3$ is

$$\tfrac{1}{2}\begin{vmatrix} x_2' & y_2' \\ x_3' & y_3' \end{vmatrix}\sin\omega = \tfrac{1}{2}\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}\sin\omega$$

$$= \tfrac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x_2-x_1 & y_2-y_1 & 1 \\ x_3-x_1 & y_3-y_1 & 1 \end{vmatrix}\sin\omega$$

$$= \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\sin\omega$$

by the operation $\mathbf{c}_1 \to \mathbf{c}_1 + x_1\mathbf{c}_3$ followed by $\mathbf{c}_2 \to \mathbf{c}_2 + y_1\mathbf{c}_3$; the middle step is verified by expanding from the first row.

Example

The points P_1, P_2, P_3 will be collinear if and only if the area of triangle $P_1P_2P_3$ is zero, i.e. *if and only if*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

15.2 Forms of the equation of a straight line

15.21 Line with gradient m through (x_1, y_1)

Assuming the axes are rectangular, let $P(x, y)$ be any point of the line. Then the gradient is $(y-y_1)/(x-x_1)$, and hence

$$\frac{y-y_1}{x-x_1} = m,$$

i.e. $$y-y_1 = m(x-x_1).$$

This equation, which is satisfied by the coordinates of *any* point P on the required line, is what we mean by 'the equation of the line'.

The result has been used in previous chapters when finding equations of tangents and normals to a curve. It can be written

$$y = mx+c,$$

where $c = y_1 - mx_1$. There is no simple analogue for oblique axes.

15.22 Gradient form

(1) *Rectangular axes.* The above shows that any line with gradient m has an equation of the form $y = mx+c$. *Conversely*, any equation which can be put into the form $y = mx+c$ (where m, c are constants) represents a straight line of gradient m making intercept c on Oy. For when $x = 0$, the equation shows that $y = c$, so $(0, c)$ lies on the locus; also $(y-c)/(x-0) = m$, so that the gradient is constant and equal to m.

(2) *Oblique axes.* If the line makes angle θ with Ox and intercept c on Oy, its gradient is $\tan\theta$; and if $P(x, y)$ is any point on the line, then by the example in 15.15 its gradient is

$$\frac{(y-c)\sin\omega}{x+(y-c)\cos\omega}, \qquad \text{(i)}$$

since $(0, c)$ lies on the line. Hence

$$\tan\theta = \frac{(y-c)\sin\omega}{x+(y-c)\cos\omega},$$

from which $$(y-c)\sin(\omega-\theta) = x\sin\theta$$

and
$$y = \frac{\sin\theta}{\sin(\omega-\theta)}x + c.$$

The equation of the line is therefore still of the *form* $y = mx + c$, but now m *no longer represents the gradient of the line.*

If the equation $y = mx + c$ is given, then $m = (y-c)/x$, and the expression (i) for the gradient of the join of $P(x, y)$ and $(0, c)$ becomes $m\sin\omega/(1+m\cos\omega)$, which is independent of (x, y). Hence this equation represents a straight line of gradient
$$\frac{m\sin\omega}{1+m\cos\omega};$$
clearly it makes y-intercept c.

Example

Find an angle between the lines $y = m_1x + c_1$, $y = m_2x + c_2$, the axes being inclined at angle ω.

The gradients of the lines are
$$\mu_1 = \frac{m_1\sin\omega}{1+m_1\cos\omega}, \quad \mu_2 = \frac{m_2\sin\omega}{1+m_2\cos\omega}.$$
By 15.15, an angle between the lines is given by $\tan^{-1}\{(\mu_1-\mu_2)/(1+\mu_1\mu_2)\}$ unless $\mu_1\mu_2 = -1$ (in which case they are perpendicular). After reduction this is found to be
$$\tan^{-1}\frac{(m_1-m_2)\sin\omega}{1+(m_1+m_2)\cos\omega+m_1m_2};$$
the lines are perpendicular if $1+(m_1+m_2)\cos\omega+m_1m_2 = 0$, and *are parallel if* $m_1 = m_2$.

15.23 General linear equation $Ax+By+C=0$ {ω}

If $B \neq 0$ the equation can be written $y = -(A/B)x - C/B$, which by 15.22 is the equation of a straight line.

If $B = 0$ but $A \neq 0$, the equation can be written $x = -C/A$, which is the equation of a straight line parallel to Oy.

Since these are the only possibilities, it follows that *every linear equation in x, y represents a straight line.*

15.24 Intercept form: line making intercepts *a*, *b* on *Ox*, *Oy* {ω}

Let the gradient form of the equation of the required line be $y = mx + c$. Since the line passes through the points $(a, 0)$, $(0, b)$, we have
$$0 = ma + c \quad\text{and}\quad b = c.$$
Hence $c = b$ and $m = -b/a$, and the required equation is
$$y = -\frac{b}{a}x + b,$$
i.e.
$$\frac{x}{a}+\frac{y}{b} = 1.$$

Example

ABC is a triangle with sides CA, CB along given lines, and $1/CA+1/CB$ is constant. Prove that AB passes through a fixed point.

Choose the lines along which CA, CB lie to be Ox, Oy. Let the coordinates of A, B be $(a, 0)$, $(0, b)$; then by hypothesis $1/a+1/b = k$, where k is constant.

The equation of AB is $x/a+y/b = 1$. The above condition shows that this line passes through the point $(1/k, 1/k)$, which is independent of a, b.

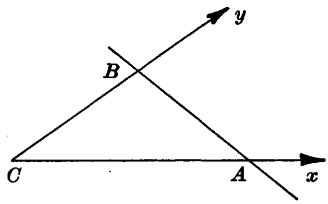

Fig. 147

15.25 Line joining P_1, P_2 {ω}

(1) For rectangular axes, if $P(x, y)$ is any point on the line, then

$$\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2}$$

since each expression is equal to the gradient of the line.

(2) This result is also true for oblique axes. For if the required line is $y = mx+c$, then $y_1 = mx_1+c$ and $y_2 = mx_2+c$, so that by subtraction

$$y-y_1 = m(x-x_1) \quad \text{and} \quad y_1-y_2 = m(x_1-x_2);$$

and the equation is obtained by division.

(3) In either case the result can be obtained as follows. Consider the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

It is linear in x and y (as is clear by expanding from the first row), and therefore by 15.23 represents a *line*. It is satisfied by (x_1, y_1) and by (x_2, y_2) (since then two rows become identical). Therefore it represents the line P_1P_2. (This form of the equation could also be seen from the example in 15.16.)

15.26 Parametric form for the line through (x_1, y_1) in direction θ

Let P be any point on the line, and let $P_1P = r$; here r is a *signed* length, so it will be positive for points of the line on one side of P_1 and negative for points on the other. The axes being rectangular,

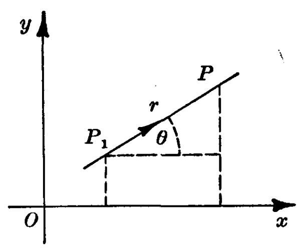

Fig. 148

$$x-x_1 = r\cos\theta \quad \text{and} \quad y-y_1 = r\sin\theta.$$

Hence the coordinates of any point on the line are expressible in terms of the parameter r as $(x_1+r\cos\theta,\ y_1+r\sin\theta)$.

Later we shall use this form in the treatment of conics. There is no simple analogue for oblique axes.

15.27 General parametric form {ω}

More generally, *the equations* $x = at+b$, $y = ct+d$ (where a, b, c, d are constants of which a, c are not both zero, and t is a variable parameter) *always represent a straight line.* For elimination of t shows that

$$c(x-b) = a(y-d),$$

which is linear in x, y and therefore represents some line (by 15.23).

15.28 Perpendicular form

Let the perpendicular from O to the line have length p (essentially positive) and make angle α with Ox. Then the foot N of the perpendicular has polar

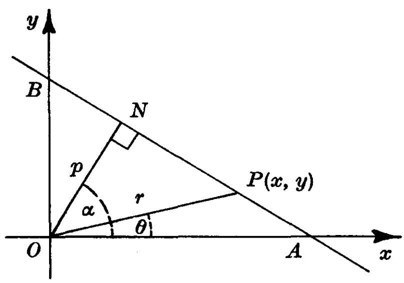

Fig. 149

coordinates (p,α). Let any other point P of the line have cartesian coordinates (x,y) and polar coordinates (r,θ); then

$$p = r\cos(\theta-\alpha) = r\cos\theta\cos\alpha+r\sin\theta\sin\alpha$$
$$= x\cos\alpha+y\sin\alpha,$$

so that the line has equation

$$x\cos\alpha+y\sin\alpha = p.$$

Alternatively, the intercepts on the axes are $OA = p\sec\alpha$, $OB = p\operatorname{cosec}\alpha$, and the result follows from 15.24.

15.3 Further results

15.31 Sides of a line {ω}

The point $P\big((lx_1+kx_2)/(l+k),\ (ly_1+ky_2)/(l+k)\big)$ lies on the line λ whose equation is $ax+by+c = 0$ if

$$a(lx_1+kx_2)+b(ly_1+ky_2)+c(l+k) = 0,$$

i.e. if
$$\frac{k}{l} = -\frac{ax_1+by_1+c}{ax_2+by_2+c}. \tag{i}$$

If P_1, P_2 lie on opposite sides of λ (fig. 150), then P divides P_1P_2 internally and so k/l is positive; hence ax_1+by_1+c, ax_2+by_2+c *have opposite signs.*

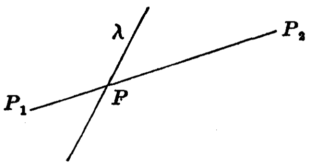

Fig. 150

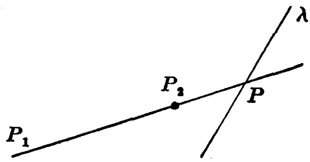

Fig. 151

Similarly if P_1, P_2 lie on the *same side* (fig. 151), k/l is negative and so ax_1+by_1+c, ax_2+by_2+c have the *same sign.*

It follows that the converses of these statements are also true.

Examples

(i) *Menelaus's theorem.* Let the line $ax+by+c=0$ divide the sides P_2P_3, P_3P_1, P_1P_2 of the triangle $P_1P_2P_3$ in the ratios $k_1:l_1$, $k_2:l_2$, $k_3:l_3$. Writing
$$u_n \equiv ax_n+by_n+c \quad (n=1,2,3)$$
we have by (i) above:
$$\frac{k_1}{l_1} = -\frac{u_2}{u_3}, \quad \frac{k_2}{l_2} = -\frac{u_3}{u_1}, \quad \frac{k_3}{l_3} = -\frac{u_1}{u_2}.$$

Multiplying,
$$\frac{k_1}{l_1}\cdot\frac{k_2}{l_2}\cdot\frac{k_3}{l_3} = -1.$$

(ii) *Ceva's theorem.* If lines P_1Q_1, P_2Q_2, P_3Q_3 through the vertices of triangle $P_1P_2P_3$ are concurrent at (h,k) and divide P_2P_3, P_3P_1, P_1P_2 in the ratios $k_1:l_1$, $k_2:l_2$, $k_3:l_3$, then since the equation of P_1Q_1 is (see 15.25 (3))
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ h & k & 1 \end{vmatrix} = 0,$$

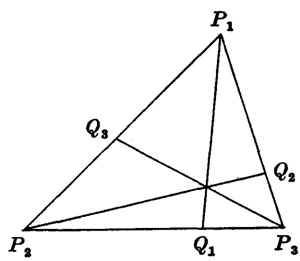

Fig. 152

with similar equations for P_2Q_2, P_3Q_3, we have as in ex. (i) that
$$k_1:l_1 = -\begin{vmatrix} x_2 & y_2 & 1 \\ x_1 & y_1 & 1 \\ h & k & 1 \end{vmatrix} : \begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ h & k & 1 \end{vmatrix}, \quad \text{etc.}$$

On multiplying,

$$\frac{k_1}{l_1}\cdot\frac{k_2}{l_2}\cdot\frac{k_3}{l_3} = +1.$$

15.32 Perpendicular distance of a point from a line

Let d denote the length (essentially positive) of the perpendicular from P_1 to $ax+by+c=0$, and let the foot be P_2. Then

$$d^2 = (x_2-x_1)^2+(y_2-y_1)^2. \qquad \text{(ii)}$$

Since P_1P_2 is perpendicular to

$$ax+by+c=0,$$

we have

$$\frac{y_2-y_1}{x_2-x_1}\left(-\frac{a}{b}\right) = -1,$$

i.e. $\quad b(x_2-x_1)-a(y_2-y_1) = 0. \qquad$ (iii)

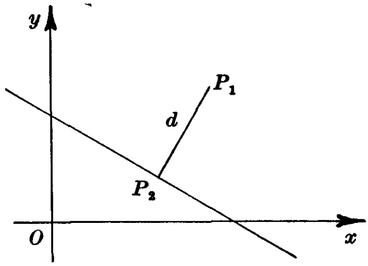

Fig. 153

Because P_2 lies on $ax+by+c=0$,

$$ax_2+by_2+c=0;$$

and this can be written (analogously to (iii)) in the form

$$a(x_2-x_1)+b(y_2-y_1) = -(ax_1+by_1+c). \qquad \text{(iv)}$$

Square and add (iii) and (iv), using (ii):

$$d^2(a^2+b^2) = (ax_1+by_1+c)^2.$$

$$\therefore \quad d = \pm\frac{ax_1+by_1+c}{\sqrt{(a^2+b^2)}},$$

where the sign is chosen to make the expression positive: + if $ax_1+by_1+c>0$, − if $ax_1+by_1+c<0$. Also see Ex. 15 (*a*), no. 15.

If $c \neq 0$ and the equation of the line is written so that c is positive, then a_1x+b_1y+c will be positive when P_1 lies on the origin side of the line, and negative otherwise, by 15.31.

Exercise 15(*a*)†

1 Prove that the point which divides in the ratio 2:1 the median P_1Q_1 of the triangle $P_1P_2P_3$ has coordinates $\left(\frac{1}{3}(x_1+x_2+x_3), \frac{1}{3}(y_1+y_2+y_3)\right)$. From the symmetry of this result deduce that the medians of a triangle are concurrent.

2 Show that the equation of the line through (h,k) which is perpendicular to Ox is

$$x+y\cos\omega = h+k\cos\omega.$$

† The main purpose of this exercise is to illustrate the use of oblique axes in solving locus problems.

3 P is the point (h, k) referred to axes at angle ω. Prove that the line joining the feet of the perpendiculars from P onto the axes is $\sin\omega\sqrt{(h^2+k^2+2hk\cos\omega)}$.

4 Two fixed lines Ox, Oy are cut by a variable line at A, B respectively; P, Q are the feet of perpendiculars from A, B onto Oy, Ox. If AB passes through a fixed point, prove that PQ will also pass through a fixed point.

5 If the equal sides AB, AC of an isosceles triangle are produced to E, F so that $BE.CF = AB^2$, prove that EF always passes through a fixed point.

Perpendiculars PM, PN are drawn from P onto Ox, Oy. Find the locus of P if

6 $PM+PN = 2c$. 7 $OM+ON = 2c$. 8 $MN = 2c$ [use no. 3].

9 From a point P on one side of a fixed triangle, perpendiculars PM, PN are drawn to the other sides. As P varies on this side, find the locus of the mid-point of MN.

10 Through a fixed point E any two straight lines are drawn to cut one fixed line Ox at A, B and another fixed line Oy at C, D. Prove that the locus of the meet of BC, AD is a straight line through O.

11 If a straight line passes through a fixed point, find the locus of the mid-point of the part intercepted between two fixed intersecting lines.

*12 A line AB of constant length c slides between two given oblique lines which meet at O. Find the locus of the orthocentre of triangle OAB.

*13 Using similar triangles, give a proof of Menelaus's theorem (15.31, ex. (i)) by the methods of 'pure' geometry.

*14 Using areas, prove Ceva's theorem (15.31, ex. (ii)) by 'pure' methods.

15 Obtain the result of 15.32 as follows. Let $ax+by+c = 0$ cut Ox, Oy at L, M. Equate $\frac{1}{2}p.LM$ to the area of triangle P_1LM. Verify the result when the line is parallel to Ox or Oy. (This method could be used when the axes are oblique.)

15.4 Concurrence of straight lines

15.41 Lines through the meet of two given lines {ω}

If L, L' are linear functions of x, y, then the equation

$$L+kL' = 0$$

is satisfied by any point which satisfies both $L = 0$ and $L' = 0$. Hence if the lines $L = 0$, $L' = 0$ intersect, the locus $L+kL' = 0$ passes through their common point.

If k is constant, $L+kL'$ is linear in x, y, and so $L+kL' = 0$ is then the equation of some straight line through the meet of $L = 0$, $L' = 0$.

If $L = 0$, $L' = 0$ are parallel and k is constant, then $L+kL' = 0$ is a straight line parallel to both (as is clear by expressing each line in 'gradient form').

One condition will determine the constant k. For example, the line may be required to pass through a given point, or to be parallel to a given line.

Conversely, if the linear equation $L = 0$ can be written in the form $L_1 + kL_2 = 0$, where k is independent of x, y and L_1, L_2 are linear with constant coefficients, then $L = 0$ passes through a fixed point, viz. the meet of the lines $L_1 = 0$, $L_2 = 0$.

Example

The mid-points of the diagonals of a complete quadrilateral are collinear.

If $ABCD$ is any quadrilateral, let AD, BC be produced to meet at E; and BA, CD to meet at O. The resulting figure is called a *complete quadrilateral* and AC, BD, OE are its *diagonals*.

Choose OAB for Ox and ODC for Oy. Call $A(2a, 0)$, $B(2b, 0)$, $C(0, 2c)$, $D(0, 2d)$, and let M be the mid-point of OE. Through M draw lines parallel to EA, EB;

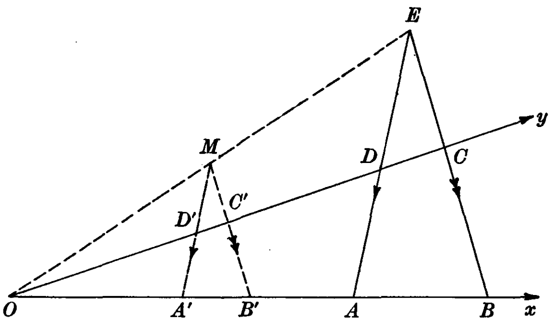

Fig. 154

by the intercept theorem of elementary geometry these will cut Ox, Oy at the points $A'(a, 0)$, $B'(b, 0)$, $C'(0, c)$, $D'(0, d)$, and their equations will be

$$\frac{x}{a} + \frac{y}{d} = 1, \quad \frac{x}{b} + \frac{y}{c} = 1.$$

Thus M lies on the lines

$$dx + ay = ad \quad \text{and} \quad cx + by = bc,$$

and therefore also on

$$(dx + ay - ad) - (cx + by - bc) = 0,$$

i.e.

$$(d - c)\,x + (a - b)\,y = ad - bc.$$

The mid-points of BD, AC are (b, d), (a, c), and these clearly lie on this last line.

15.42 Condition for concurrence of three given lines {ω}

Let the lines be

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0, \quad a_3x + b_3y + c_3 = 0.$$

If two of these intersect, say the second and third, the common point (obtained by solving for x, y) is

$$\left(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}, \frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}\right). \qquad \text{(i)}$$

This lies on the first line if

$$a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2) = 0,$$

i.e. if

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

If no two of the lines intersect, then they are all parallel; this means that (with the notation of 11.31) $C_1 = C_2 = C_3 = 0$, and so

$$\Delta \equiv c_1C_1+c_2C_2+c_3C_3 = 0.$$

Hence *if the lines are concurrent or parallel, then* $\Delta = 0$.

Conversely, if $\Delta = 0$, the point (i) lies on the first line because then

$$a_1\left(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}\right)+b_1\left(\frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}\right)+c_1 = \frac{\Delta}{a_2b_3-a_3b_2} = 0.$$

This converse argument fails if $a_2b_3-a_3b_2 = 0$; but a similar argument applies to the lines taken in a different order unless also

$$a_3b_1-a_1b_3 = 0 \quad \text{and} \quad a_2b_3-a_3b_2 = 0,$$

in which case the three lines are parallel. Hence *when* $\Delta = 0$ *the lines are either concurrent or else all parallel.*

Thus the condition $\Delta = 0$ is necessary and sufficient for concurrence or parallelism; see also 11.43, Corollaries I (b), (c).

Exercise 15(b)

1 Find the equation of the line concurrent with $2x-3y-3 = 0$ and $x+3y-15 = 0$ and passing through $(-2, -1)$; verify that it passes through the origin.

2 Find the equation of the line of gradient $\frac{3}{5}$ which is concurrent with $5x-2y = 4$, $8x-7y+5 = 0$.

3 Find the equation of the perpendicular from the meet of the lines $3x-7y = 2$, $4x+5y = 1$ to the line $9x+10y+15 = 0$.

4 Find the equations of the lines which make *numerically* equal intercepts on the axes and which are concurrent with $3x+2y-1 = 0$, $2x-y+3 = 0$.

5 (i) Show that for all numbers k the equation

$$2x+3y-1+k(3x-y+4) = 0$$

represents a line passing through a fixed point.

(ii) Find the point of intersection of the lines

$$2x+3y-1+a(3x-y+4) = 0,$$
$$b(2x+3y-1)-c(3x-y+4) = 0.$$

6 Find the equation of the line joining the meet of

$$3x+4y-7 = 0, \quad 2x-3y+1 = 0$$

to the meet of $\quad 3x-5y+37 = 0, \quad 5x-2y-8 = 0.$

Test the following sets of lines for concurrence.

7 $3x-2y+4 = 0$, $2x-y-1 = 0$, $7x-5y+13 = 0$.

8 $5x+2y-4 = 0$, $x-3y+2 = 0$, $3x+8y-6 = 0$.

9 $(p+1)x+(p-1)y+p = 0$, $(q-1)x+(q+1)y+q = 0$, $x = y$ $(p \neq q)$.

10 $x-t_1y+at_1^2 = 0$, $x-t_2y+at_2^2 = 0$, $(t_1+t_2)x+(1-t_1t_2)y = a(t_1+t_2)$ $(t_1 \neq t_2)$.

11 If the lines $x-2y+3 = 0$, $2x-3y = 0$, $ax+y+1 = 0$ are concurrent, find the value of a.

12 If the lines $ax+2y+1 = 0$, $bx+3y+1 = 0$, $cx+4y+1 = 0$ are concurrent, prove that a, b, c are in arithmetical progression.

13 Find the condition for concurrence of the distinct lines $3ax+2y = a^3$, $3bx+2y = b^3$, $3cx+2y = c^3$. [Use theory of equations.]

*14 Find the condition for concurrence of the distinct lines

$$x\sin 3\alpha+y\sin\alpha = a, \quad x\sin 3\beta+y\sin\beta = a, \quad x\sin 3\gamma+y\sin\gamma = a.$$

[If (h, k) is the common point, then $\theta = \alpha, \beta, \gamma$ must satisfy $h\sin 3\theta+k\sin\theta = a$, i.e. $4h\sin^3\theta-(k+3h)\sin\theta+a = 0$. Since the term in $\sin^2\theta$ is absent, the roots $\sin\alpha$, $\sin\beta$, $\sin\gamma$ of this cubic in $\sin\theta$ satisfy $\Sigma\sin\alpha = 0$.]

15 (i) If $a_1x+b_1y+c_1 = 0$, $a_2x+b_2y+c_2 = 0$, $a_3x+b_3y+c_3 = 0$ are *distinct* lines, and numbers l, m, n *none of which is zero* can be found such that

$$l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3) \equiv 0, \qquad \text{(A)}$$

prove the lines are concurrent or else all parallel.

*(ii) If $\Delta = 0$, use 11.43, Theorem II to show that numbers l, m, n not all zero exist such that

$$a_1l+a_2m+a_3n = 0, \quad b_1l+b_2m+b_3n = 0, \quad c_1l+c_2m+c_3n = 0$$

and hence that (A) holds. Deduce the *converse* of (i).

15.5 Line-pairs

15.51 Equations which factorise linearly

A single equation in x, y may represent two or more lines.

(i) $y^2-x^2 = 0$ can be written $y = \pm x$ and therefore represents the *pair* of lines $y = +x$, $y = -x$.

(ii) $x^2+3xy+2y^2 = 0$ can be factorised as $(x+y)(x+2y) = 0$, and so represents the line-pair $x+y = 0$, $x+2y = 0$.

(iii) $x^2+3xy+2y^2+3x+5y+2 = 0$ can be written $(x+y+2)(x+2y+1) = 0$, and therefore represents the pair $x+y+2 = 0$, $x+2y+1 = 0$.

(iv) $x^3+3x^2y+2xy^2 = 0$, i.e. $x(x+y)(x+2y) = 0$, represents the *three* lines $x = 0$, $x+y = 0$, $x+2y = 0$.

In general, the equation

$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2) = 0$$

represents the pair of lines $a_1x+b_1y+c_1 = 0$, $a_2x+b_2y+c_2 = 0$. For if (x_0, y_0) satisfies either of these equations, then it satisfies the original one; and conversely, if (x_0, y_0) satisfies the given equation, then either $a_1x_0+b_1y_0+c_1 = 0$ or $a_2x_0+b_2y_0+c_2 = 0$ or both.

15.52 The locus $ax^2+2hxy+by^2=0$

(1) If $b = 0$, the equation is $x(ax+2hy) = 0$, representing a line-pair.

If $b \neq 0$, the equation can be written

$$\left(\frac{y}{x}\right)^2+2\frac{h}{b}\left(\frac{y}{x}\right)+\frac{a}{b} = 0.$$

When $(h/b)^2 > a/b$, i.e. $h^2 > ab$, this quadratic has distinct roots $y/x = m_1, m_2$, and then $y = m_1x$ or $y = m_2x$. Hence *if $h^2 > ab$, the equation represents two distinct lines through O.*

When $h^2 = ab$, the equation is $(y/x+h/b)^2 = 0$ and therefore represents the single line $hx+by = 0$. We say conventionally that the equation represents two *coincident lines*, or a *repeated line.*

When $h^2 < ab$, the quadratic for y/x has no roots. The original equation is satisfied solely by the values $x = 0$, $y = 0$. Hence *if $h^2 < ab$, the equation represents the origin only.*

The three general results just given include the case $b = 0$ disposed of at the start.

The work of 15.5 so far is valid for oblique axes; *in the following we shall suppose the axes to be rectangular.*

(2) *Angle between the lines $ax^2+2hxy+by^2 = 0$.* If the separate lines have equations $y = m_1x$, $y = m_2x$, then the above quadratic for y/x has roots m_1, m_2. Hence

$$m_1+m_2 = -\frac{2h}{b}, \quad m_1m_2 = \frac{a}{b}, \tag{i}$$

and so $$(m_1-m_2)^2 = (m_1+m_2)^2-4m_1m_2 = \frac{4}{b^2}(h^2-ab).$$

The required angle is given by

$$\tan\theta = \frac{m_1-m_2}{1+m_1m_2} = \pm\frac{2\sqrt{(h^2-ab)}}{b(1+a/b)}$$

$$= \pm\frac{2\sqrt{(h^2-ab)}}{a+b},$$

provided $a+b \neq 0$.

If $a+b=0$, then $m_1m_2=-1$ and *the lines are perpendicular*; and conversely.

(3) *Bisectors of the angles between these lines.* By expressing equality of the perpendiculars from (x, y) to the lines $y-m_1x=0$, $y-m_2x=0$, the equations of the angle-bisectors are

$$\frac{y-m_1x}{\sqrt{(1+m_1^2)}} = \pm\frac{y-m_2x}{\sqrt{(1+m_2^2)}}.$$

Hence the two bisectors have equation

$$\frac{(y-m_1x)^2}{1+m_1^2}-\frac{(y-m_2x)^2}{1+m_2^2} = 0,$$

i.e. $$(1+m_2^2)(y-m_1x)^2-(1+m_1^2)(y-m_2x)^2 = 0,$$

i.e. $$(m_1^2-m_2^2)x^2-2(m_1-m_2)(1-m_1m_2)xy-(m_1^2-m_2^2)y^2 = 0.$$

If $m_1 \neq m_2$, this reduces to

$$(m_1+m_2)(x^2-y^2) = 2(1-m_1m_2)xy,$$

i.e. by (i), $$h(x^2-y^2) = (a-b)xy.$$

This equation satisfies the perpendicularity condition at the end of (2), as of course it should from elementary geometrical considerations. It can be written

$$\begin{vmatrix} x^2 & xy & y^2 \\ a & h & b \\ 1 & 0 & 1 \end{vmatrix} = 0.$$

15.53 The general line-pair $s=0$

(1) *Parallel line-pair through the origin.* Write

$$s \equiv ax^2+2hxy+by^2+2gx+2fy+c = 0. \qquad \text{(i)}$$

This is the standard form of the *general equation of the second degree* in x, y.

If s can be factorised† in the form

$$s \equiv (A_1x+B_1y+C_1)(A_2x+B_2y+C_2), \qquad \text{(ii)}$$

then (i) represents a line-pair. By equating coefficients of the second-degree terms in (ii),

$$a = A_1A_2, \quad 2h = A_1B_2+A_2B_1, \quad b = B_1B_2;$$

so we also have

$$ax^2+2hxy+by^2 \equiv (A_1x+B_1y)(A_2x+B_2y).$$

Therefore *if* (i) *represents a line-pair, then the equation*

$$ax^2+2hxy+by^2 = 0 \qquad \text{(iii)}$$

represents the parallel line-pair through O.

This fact can be used to calculate the angle between the lines (i), which is the same as the angle between (iii), found in 15.52 (2).

When $h^2 = ab$, the lines (i) may be *coincident or parallel*; for although this condition implies $A_1:A_2 = B_1:B_2$, these ratios may or may not be equal to $C_1:C_2$.

(2) *Necessary condition for $s = 0$ to represent a line-pair.*

If (i) represents a line-pair, s can be written in the form (ii); and if $A_1 \neq 0$ and $A_2 \neq 0$ (i.e. if $a \neq 0$, since $a = A_1A_2$), this means that (i), regarded as a quadratic in x, can be solved in the form

$$x = -\frac{B_1}{A_1}y-\frac{C_1}{A_1}, \quad x = -\frac{B_2}{A_2}y-\frac{C_2}{A_2}. \qquad \text{(iv)}$$

Now (i), written as a quadratic in x, is

$$ax^2+2(hy+g)x+(by^2+2fy+c) = 0, \qquad \text{(v)}$$

so $$x = \frac{1}{a}[-(hy+g) \pm \sqrt{\{(hy+g)^2-a(by^2+2fy+c)\}}].$$

Hence (i) can be solved in the form (iv) if and only if the quadratic in y under the square-root sign is a perfect square, i.e. if

$$(h^2-ab)y^2+2(gh-af)y+(g^2-ac)$$

is a perfect square. This is so if and only if

$$(gh-af)^2-(h^2-ab)(g^2-ac) = 0,$$

(which holds even when $h^2-ab = 0$, for the condition then implies $gh-af = 0$ also), i.e. if

$$a(abc+2fgh-af^2-bg^2-ch^2) = 0.$$

† This will not be the case in general: see 15.74.

Since we are assuming $a \neq 0$, the condition is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0. \qquad \text{(vi)}$$

If $a = 0$ but $b \neq 0$, then equation (i) can similarly be solved as a quadratic in y, and the same condition (vi) is obtained.

If $a = 0$, $b = 0$, but $h \neq 0$, equation (i) is

$$2hxy + 2gx + 2fy + c = 0,$$

i.e.
$$xy + \frac{g}{h}x + \frac{f}{h}y + \frac{c}{2h} = 0$$

which, if factorisable, is

$$\left(x + \frac{f}{h}\right)\left(y + \frac{g}{h}\right) = 0,$$

so that
$$\frac{fg}{h^2} = \frac{c}{2h},$$

i.e.
$$2fgh - ch^2 = 0,$$

and this is what condition (vi) becomes when $a = b = 0$.

These are the only cases that need be considered, since if

$$a = b = h = 0$$

then (i) would not be of *second* degree. We have therefore shown that the *necessary* condition for (i) to represent a pair of straight lines is (vi). By ex. (ii) in 11.22, (vi) can be written

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0. \qquad \text{(vii)}$$

The determinant Δ is called the *discriminant* of s, or of equation (i).

Remark (α). The necessary condition $\Delta = 0$ is *not sufficient* for (i) to represent a line-pair. For Δ is zero when

$$a = b = 1, \quad c = f = g = h = 0,$$

whereas $x^2 + y^2$ has no linear factors. (A sufficient condition is indicated in Ex. 1 (f), no. 19. Other ways of obtaining the necessary condition $\Delta = 0$ appear in Ex. 9 (f), no. 25 (ii); (3) below; and 15.73, ex. (ii).)

(3) *Point of intersection of the line-pair.* In a numerical case we should resolve the second-degree equation into linear factors (e.g. as

in 10.21, ex. (ii)). For the general case (i), equations giving the point of intersection (sometimes called the *centre* or *vertex* of the line-pair) can be found as follows.

Writing (i) in the form (v), we see that to a given value of y there correspond in general two values of x; but if y is the ordinate of the meet P, then (v) will give only one value of x, viz. the abscissa of P.

Now when (v) has equal roots, that root is (see 1.31 (*b*))

$$x = -\frac{hy+g}{a};$$

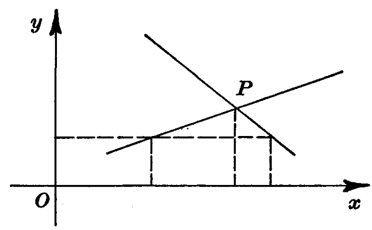

Fig. 155

so the coordinates of P satisfy

$$ax+hy+g = 0. \qquad \text{(viii)}$$

Similarly, writing (i) as a quadratic in y, viz.

$$by^2+2(hx+f)y+(ax^2+2gx+c) = 0,$$

at P we have $$y = -\frac{hx+f}{b},$$

i.e. $$hx+by+f = 0. \qquad \text{(ix)}$$

Equations (viii), (ix) suffice to determine P as the point $(G/C, F/C)$; see ex. (i) in 11.32 for the notation.

Remark (β). Since (i) can also be written

$$(ax+hy+g)x+(hx+by+f)y+(gx+fy+c) = 0,$$

it follows that the coordinates of P must also satisfy

$$gx+fy+c = 0. \qquad \text{(x)}$$

Equations (viii)–(x) are therefore consistent, and Corollary I (*b*) of 11.43 gives (vii) as a necessary condition for $s = 0$ to represent an *intersecting* line-pair. Compare (2) above, where (vii) was shown to be necessary for *any* type of line-pair, not necessarily intersecting. Also see 15.73, ex. (ii).

15.54 Line-pair joining O to the meets of the line $lx+my=1$ and the locus $s=0$

In general the equation $s = 0$ represents a curve, but the following argument holds in particular when the equation represents a line-pair.

Consider the *homogeneous* equation in x, y:

$$ax^2+2hxy+by^2+2(gx+fy)(lx+my)+c(lx+my)^2 = 0.$$

It is satisfied by the points which satisfy both the equation of the line and of the locus, and hence it represents some other locus through these points. Since the equation is homogeneous in x, y, it represents two lines through O. Hence it must be the equation of the required line-pair.

Exercise 15(*c*)

[*Rectangular axes.*]

1 Find the area enclosed by the lines $x^2+4xy+2y^2 = 0$ and the line $x+y = 1$.

2 Show that the area enclosed by the lines

$$ax^2+2hxy+by^2 = 0 \quad \text{and} \quad lx+my = 1$$

is $\sqrt{(h^2-ab)}/(am^2-2hlm+bl^2)$.

3 Prove that the product of the lengths of the perpendiculars from (x_1, y_1) to the lines

$$ax^2+2hxy+by^2 = 0 \quad \text{is} \quad \pm(ax_1^2+2hx_1y_1+by_1^2)/\sqrt{\{(a-b)^2+4h^2\}}.$$

4 Find the equation of the lines through O which are perpendicular to the lines $ax^2+2hxy+by^2 = 0$. [If (x, y) lies on either of the required lines, then $(-y, x)$ lies on one of the given lines.]

5 Find (i) the angle between, (ii) the equation of the bisectors of the angles between, the lines $5x^2+2xy-4y^2 = 0$.

6 If the lines $ax^2+2hxy+by^2 = 0$, $a'x^2+2h'xy+b'y^2 = 0$ have the same angle-bisectors, prove that $h(a'-b') = h'(a-b)$.

7 Show that any pair of lines which has the same angle-bisectors as the pair $ax^2+2hxy+by^2 = 0$ can be written $ax^2+2hxy+by^2+\lambda(x^2+y^2) = 0$. [Use no. 6.]

Find the equation of the pair of lines, one of which passes through (p, q), and whose angle-bisectors are $x^2-y^2 = 0$.

8 Show that the following equations each represent pairs of straight lines; find the angle between them, and their point of intersection.

(i) $2x^2+7xy+3y^2-4x-7y+2 = 0$;
(ii) $15x^2+xy-2y^2+3x-y = 0$;
(iii) $xy-3x+2y-6 = 0$;
*(iv) $x^2+4xy-2y^2+6x-12y-15 = 0$.

9 Write down the equations of those line-pairs through the origin which are perpendicular to the line-pairs in no. 8. [Use no. 4.]

10 Find the equation of the lines joining O to the meet of $x+2y = 3$ with the lines $4x^2+16xy-12y^2-8x+12y-3 = 0$.

*11 Show that the line $2(g-g')x+2(f-f')y+(c-c') = 0$ is a diagonal of the parallelogram formed by the line-pairs $s \equiv ax^2+2hxy+by^2+2gx+2fy+c = 0$, $s' \equiv ax^2+2hxy+by^2+2g'x+2f'y+c' = 0$. [Consider $s-s' = 0$, which is linear and is satisfied by the points common to $s = 0$, $s' = 0$.]

*12 If the line-pairs $s = 0$, $s' = 0$ in no. 11 possess a line in common, find its equation.

15.6 The circle

15.61 General equation of a circle; centre and radius

Referred to rectangular axes, the equation of the circle with centre $C(h, k)$ and radius r is
$$(x-h)^2+(y-k)^2 = r^2. \tag{i}$$
When simplified, this equation is of the form
$$x^2+y^2+2gx+2fy+c = 0, \tag{ii}$$
in which *the coefficients of x^2 and y^2 are equal, and there is no xy-term.*

Conversely, by completing the square for the terms involving x and for those involving y, equation (ii) can be written
$$(x+g)^2+(y+f)^2 = g^2+f^2-c.$$
If $g^2+f^2-c > 0$, this represents the circle with centre $(-g, -f)$ and radius $\sqrt{(g^2+f^2-c)}$. If $g^2+f^2-c = 0$, it represents the single point $(-g, -f)$; and if $g^2+f^2-c < 0$, the equation does not represent any locus.

Example*

Referred to oblique axes at angle ω, the circle with centre $C(h, k)$ and radius r has equation
$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega = r^2,$$
which is of the form
$$x^2+y^2+2xy\cos\omega+2gx+2fy+c = 0.$$

15.62 Circle on diameter P_1P_2

Let $P(x,y)$ be any point on the required circle; then $P_1P \perp P_2P$, by 'angle in a semicircle'. Since
$$\text{gradient of } P_1P = \frac{y-y_1}{x-x_1}, \quad \text{and} \quad \text{gradient of } P_2P = \frac{y-y_2}{x-x_2},$$
hence
$$\frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2} = -1,$$
i.e.
$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0. \tag{iii}$$
This equation, which is satisfied by the coordinates of *any* point P on the circle, is therefore the equation required. (See also Ex. 15 (f), no. 15.)

15.63 Tangent at P_1

Let $P_1(x_1, y_1)$ be a given point on the circle (ii) in 15.61. Then since the tangent at P_1 is perpendicular to the radius CP_1, and CP_1 has gradient $(y_1+f)/(x_1+g)$, the equation of the tangent is

$$y-y_1 = -\frac{x_1+g}{y_1+f}(x-x_1),$$

i.e. $$y(y_1+f)-y_1^2-fy_1+x(x_1+g)-x_1^2-gx_1 = 0,$$

i.e. $$xx_1+yy_1+gx+fy = x_1^2+y_1^2+gx_1+fy_1.$$

Since P_1 lies on the circle,

$$x_1^2+y_1^2+2gx_1+2fy_1+c = 0,$$

and so the equation of the tangent at P_1 can be written

$$xx_1+yy_1+gx+fy = -gx_1-fy_1-c,$$

i.e. $$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c = 0. \qquad \text{(iv)}$$

Remark. This result can be remembered by the following *rule of alternate suffixes*. 'Write xx for x^2, $x+x$ for $2x$, etc., in the equation of the circle:

$$xx+yy+g(x+x)+f(y+y)+c = 0;$$

and attach the suffix 1 to *alternate variables*.'

It should be emphasised that this rule is only an aid to memory, and certainly does not constitute a proof that (iv) is the equation of the tangent at P_1. Other applications will appear in later chapters.

Example

Find the condition for the line $lx+my = n$ to touch the circle (ii).

The line will be a tangent if and only if the perpendicular from the centre $(-g, -f)$ to it is equal to the radius $\sqrt{(g^2+f^2-c)}$:

$$\pm\frac{-lg-mf-n}{\sqrt{(l^2+m^2)}} = \sqrt{(g^2+f^2-c)},$$

i.e., on squaring, $$(lg+mf+n)^2 = (g^2+f^2-c)(l^2+m^2).$$

15.64 Chord of contact from P_1

If P_1 is outside the circle, two tangents can be drawn from P_1; if their points of contact are P_2, P_3, then the whole line P_2P_3 is called the *chord of contact* of tangents from P_1. We now find the equation of this chord.

For simplicity we consider the circle $x^2+y^2 = a^2$, whose centre is the origin. The tangent P_2P_1 at P_2 has equation

$$xx_2+yy_2 = a^2.$$

Since P_1 lies on this line,

$$x_1x_2+y_1y_2 = a^2$$

which shows that the coordinates $x = x_2$, $y = y_2$ satisfy the equation

$$x_1x+y_1y = a^2,$$

i.e. that P_2 lies on the line $xx_1+yy_1 = a^2$.

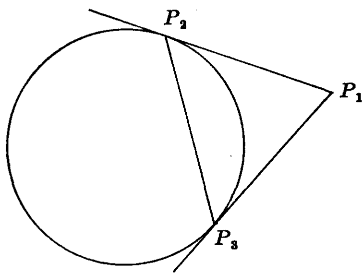

Fig. 156

Similarly, P_3 lies on this same line, which must therefore be the chord P_2P_3. Hence the chord of contact from P_1 to $x^2+y^2 = a^2$ is

$$xx_1+yy_1 = a^2.$$

The same argument applied to the general circle

$$x^2+y^2+2gx+2fy+c = 0$$

shows that the chord of contact from P_1 has equation

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c = 0,$$

i.e. $$(x_1+g)x+(y_1+f)y+(gx_1+fy_1+c) = 0.$$

Remark. The equation $xx_1+yy_1 = a^2$ has two quite distinct meanings according to whether P_1 lies on the circle $x^2+y^2 = a^2$ or outside it. When P_1 lies on the circle, it represents the *tangent at* P_1; when P_1 lies outside, it represents the *chord of contact from* P_1. Similar remarks apply to the general case. See also (1), (2) in 15.65.

Examples

(i) *Find the equation of the tangents from O to* $x^2+y^2+2gx+2fy+c = 0$.

The chord of contact from O (fig. 157) has equation

$$gx+fy+c = 0.$$

As in 15.54, consider the equation

$$x^2+y^2+2(gx+fy)\left(-\frac{gx+fy}{c}\right)+c\left(-\frac{gx+fy}{c}\right)^2=0,$$

which reduces to $$c(x^2+y^2)=(gx+fy)^2.$$

It is satisfied by the points common to the circle and line, i.e. by the points of contact of the tangents from O. It is homogeneous of degree 2 in x, y, and therefore represents a line-pair. Hence it represents the pair of tangents from O.

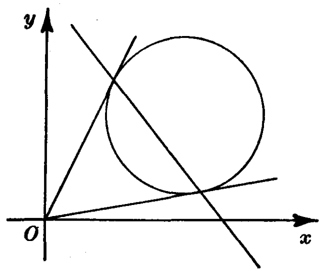

Fig. 157

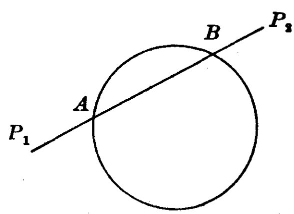

Fig. 158

(ii) *Pair of tangents from* P_1 *to* $x^2+y^2=a^2$.

When P_1 is given, let P_2 be any other point in the plane. Any point on P_1P_2 will divide it in some ratio $k:l$, and hence will have coordinates

$$\left(\frac{lx_1+kx_2}{l+k}, \frac{ly_1+ky_2}{l+k}\right).$$

This point will lie on the circle if

$$\left(\frac{lx_1+kx_2}{l+k}\right)^2+\left(\frac{ly_1+ky_2}{l+k}\right)^2=a^2,$$

i.e. $$(x_2^2+y_2^2-a^2)k^2+2(x_1x_2+y_1y_2-a^2)kl+(x_1^2+y_1^2-a^2)l^2=0. \qquad \text{(v)}$$

This quadratic gives two values for the ratio $k:l$, which correspond to the points A, B where the line P_1P_2 cuts the circle; the two values are $P_1A:AP_2$ and $P_1B:BP_2$ (fig. 158). Equation (v) is known as *Joachimsthal's ratio quadratic* for the circle $x^2+y^2=a^2$.

If P_2 lies on either *tangent* from P_1, then the points A, B will coincide, and hence (v) will have equal roots $k:l$, so that

$$(x_1^2+y_1^2-a^2)(x_2^2+y_2^2-a^2)=(x_1x_2+y_1y_2-a^2)^2.$$

This equation shows that the point $P_2(x_2, y_2)$ lies on the locus

$$(x_1^2+y_1^2-a^2)(x^2+y^2-a^2)=(xx_1+yy_1-a^2)^2,$$

which must therefore be the combined equation of the tangents from P_1, i.e. of the pair of tangents. See also 15.73, ex. (i).

15.65 Examples; polar

(1) *Tangents at the intersections of the circle* $x^2+y^2=a^2$ *with variable chords through* P_1 *meet on the line* $xx_1+yy_1=a^2$.

Let the tangents at the extremities A, B of the chord AB through P_1 meet at P_2. Then by 15.64, AB has equation

$$xx_2 + yy_2 = a^2.$$

Since P_1 lies on this line,

$$x_1x_2 + y_1y_2 = a^2;$$

and this shows that P_2 lies on the line $x_1x + y_1y = a^2$.

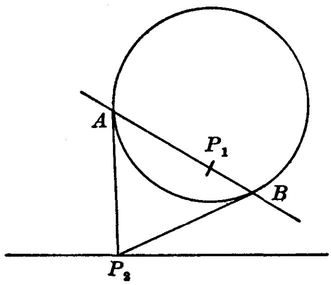

Fig. 159

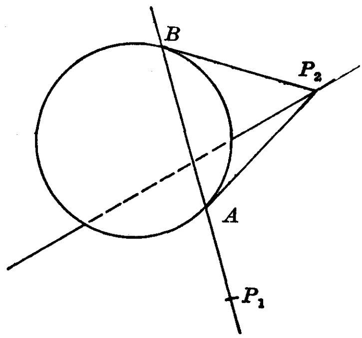

Fig. 160

If P_1 lies inside the circle, the *locus* of P_2 is the complete line, which lies entirely outside the circle. If P_1 lies outside, the *locus* of P_2 is that part of the line which is outside the circle.

(2) *A variable chord through P_1 meets the circle $x^2 + y^2 = a^2$ at A, B; P_2 is chosen so that P_1, P_2 divide AB in the same ratio (one internally, the other externally).*† *Show that P_2 lies on the line $xx_1 + yy_1 = a^2$.*

Since by hypothesis $AP_1 : P_1B = AP_2 : P_2B$, hence also $P_1A : AP_2 = P_1B : BP_2$ so that A, B divide the line P_1P_2 internally and externally in the same ratio, say $\pm k' : l'$.

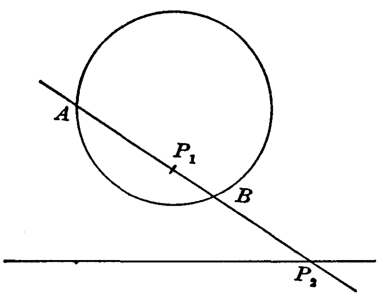

Fig. 161

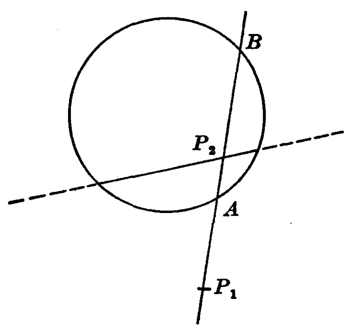

Fig. 162

Either point A, B dividing P_1P_2 in the ratio $k : l$ has coordinates of the form

$$\left(\frac{lx_1 + kx_2}{l + k}\right), \quad \left(\frac{ly_1 + ky_2}{l + k}\right),$$

† P_1 and P_2 are said to divide AB *harmonically*.

and lies on $x^2+y^2 = a^2$; hence equation (v) of 15.64, ex. (ii) holds, and its roots are $\pm k':l'$. Since their sum is zero,

$$x_1x_2+y_1y_2-a^2 = 0.$$

Hence P_2 lies on the line $x_1x+y_1y = a^2$.

If P_1 lies inside the circle, the *locus* of P_2 is the whole line, which lies completely outside the circle. If P_1 lies outside, the *locus* of P_2 is that part of the line which lies within the circle.

(3) *Definitions.* We now unify the complementary results in (1), (2).
The whole line $xx_1+yy_1 = a^2$ is called the *polar* of P_1 wo $x^2+y^2 = a^2$.
P_1 is called the *pole* of the line $xx_1+yy_1 = a^2$ wo $x^2+y^2 = a^2$.

Remarks

(α) If P_1 lies outside the circle, the polar coincides with the chord of contact from P_1.

(β) If P_1 lies on the circle, the polar coincides with the tangent at P_1.

(γ) The polar exists for every point except the centre $(0, 0)$.

(δ) *Reciprocal property.* If the polar of P_1 passes through P_2 then the polar of P_2 passes through P_1. (For if $xx_1+yy_1 = a^2$ passes through (x_2, y_2), then $x_2x_1+y_2y_1 = a^2$, which shows that P_1 lies on $xx_2+yy_2 = a^2$, the polar of P_2.)

15.66 Orthogonal circles

(1) The *angle* between two intersecting circles† is the angle between their tangents at a common point. (It is easy to prove geometrically that the angles at the two intersections are equal.)

If these tangents are perpendicular, the circles are *orthogonal.* In this case the tangent to one circle is a radius of the other.

(2) *Condition for*

$$x^2+y^2+2gx+2fy+c = 0,$$
$$x^2+y^2+2g'x+2f'y+c' = 0$$

to be orthogonal.

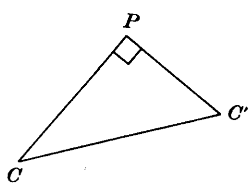

Fig. 163

The centres are $C(-g, -f)$ and $C'(-g', -f')$, and the radii are $\sqrt{(g^2+f^2-c)}$, $\sqrt{(g'^2+f'^2-c')}$.

If the circles cut orthogonally at P, the triangle CPC' is right-angled at P, and so
$$C'C^2 = CP^2+C'P^2,$$
i.e.
$$(g-g')^2+(f-f')^2 = (g^2+f^2-c)+(g'^2+f'^2-c'),$$
i.e.
$$\mathbf{2gg'+2ff' = c+c'}.$$

Conversely, when this condition is satisfied, then by adding

$$g^2+g'^2+f^2+f'^2$$

† The general definition was given in 5.72.

to both sides the previous equation is obtained; hence by the converse of the theorem of Pythagoras, triangle $C'PC$ is right-angled at P.

The condition is therefore both *necessary and sufficient* for orthogonality.

Example

Prove that in general there is only one circle orthogonal to three given circles.

If the given circles are

$$x^2+y^2+2g_rx+2f_ry+c_r = 0 \quad (r = 1, 2, 3),$$

we require to find g, f, c for which

$$2gg_1+2ff_1-c = c_1,$$
$$2gg_2+2ff_2-c = c_2,$$

and
$$2gg_3+2ff_3-c = c_3.$$

If the determinant

$$\Delta \equiv \begin{vmatrix} g_1 & f_1 & 1 \\ g_2 & f_2 & 1 \\ g_3 & f_3 & 1 \end{vmatrix}$$

is non-zero, then this system of three equations for the three unknowns g, f, c can be solved uniquely by Cramer's rule (11.41). The condition $\Delta \neq 0$ means that the three centres are not collinear (15.16, ex.) and that no two centres coincide; this is the situation 'in general'.

Exercise 15(*d*)

1 Write down the equation of the circle through O whose centre is (p, q). Prove that the tangent at O is $px+qy = 0$. [Use 'tangent $\perp$ radius'.]

2 Show that the chord of $x^2+y^2 = a^2$ whose mid-point is (x_1, y_1) has equation $xx_1+yy_1 = x_1^2+y_1^2$. [If P is any point of the required chord, $OP_1 \perp PP_1$.]

3 Prove that the mid-points of those chords of $x^2+y^2+2gx+2fy+c = 0$ which pass through P_1 lie on the circle $(x-x_1)(x+g)+(y-y_1)(y+f) = 0$. [$CM \perp CP_1$.]

4 (i) Find the equation of the chord P_1P_2 of $x^2+y^2 = a^2$. Deduce the equation of the tangent at P_1. (ii) Also find the tangent at P_1 by using Calculus.

5 If $lx+my = n$ touches $x^2+y^2 = a^2$, find its point of contact. [$xx_1+yy_1 = a^2$ and $lx+my = n$ are the same line if $x_1/l = y_1/m = a^2/n$.]

6 Find the condition for $y = mx+c$ to touch $x^2+y^2 = a^2$, and deduce that the lines $y = mx \pm a\sqrt{(1+m^2)}$ touch the circle for all values of m.

7 (i) Find the condition for the chord of contact of tangents from P_1 to $x^2+y^2 = a^2$ to subtend a right-angle at O. (ii) What is the locus of the meet of perpendicular tangents to $x^2+y^2 = a^2$?

8 (i) Show geometrically that, when they exist, the direct common tangents of two circles divide the line of centres externally in the ratio of the radii. What is the corresponding result for the transverse common tangents?

(ii) Use (i) to find the equations of the direct and transverse common tangents to $(x-15)^2+y^2 = 64$, $(x-2)^2+y^2 = 9$. [If a tangent meets the line of centres at $(\lambda, 0)$, its equation is $y = m(x-\lambda)$ where m is chosen so that the perpendicular to this line from either centre is equal to the corresponding radius.]

Prove that the direct common tangents to

$$(x-a_1)^2+y^2 = r_1^2, \quad (x-a_2)^2+y^2 = r_2^2$$

have equation $$\{(r_1-r_2)x-(a_1r_2-a_2r_1)\}^2 = \{(a_1-a_2)^2-(r_1-r_2)^2\}y^2.$$

Find the equation of the transverse common tangents.

10 If the line of centres of

$$x^2+y^2+2gx+2fy+c = 0, \quad x^2+y^2+2g'x+2f'y+c' = 0$$

cuts the axes at A, B, show that the circle on diameter AB is

$$(g-g')(f-f')(x^2+y^2) = (gf'-g'f)\{(g-g')x-(f-f')y\}.$$

*11 A variable circle passes through the meet O of two given lines, and makes intercepts OP, OQ such that $m.OP+n.OQ = 1$. Prove that this circle passes through another fixed point. [Use oblique axes.]

12 A circle passes through (h, k) and cuts orthogonally the circle

$$x^2+y^2+2gx+2fy+c = 0.$$

Prove that its centre lies on the line

$$2(h+g)x+2(k+f)y = h^2+k^2-c.$$

13 Find the equation of the circle orthogonal to $x^2+y^2+2x-2y+1 = 0$ and $x^2+y^2+4x-4y+3 = 0$ and whose centre lies on the line $3x-y-2 = 0$.

14 Find the equation of the circle orthogonal to

$$x^2+y^2 = 5, \quad x^2+y^2+6x+1 = 0 \quad \text{and} \quad x^2+y^2-4x-4y+7 = 0.$$

15 Prove that an angle between the circles

$$s \equiv x^2+y^2+2gx+2fy+c = 0, \quad s' \equiv x^2+y^2+2g'x+2f'y+c' = 0$$

(supposed to intersect) is given by

$$\cos\theta = \frac{2gg'+2ff'-c-c'}{2\sqrt{(g^2+f^2-c)}\sqrt{(g'^2+f'^2-c')}}.$$

16 If $s = 0$, $s' = 0$ are the circles in no. 15, prove that $s+\lambda s' = 0$ is also the equation of a circle for any constant λ except $\lambda = -1$. What is the interpretation when $\lambda = -1$? If s, s' intersect, explain why $s+\lambda s' = 0$ passes through their common points for *all* λ. (As λ varies we obtain a *system* or *family* of circles.)

17 If a circle $\sigma = 0$ cuts $s = 0$, $s' = 0$ orthogonally, prove that it also cuts orthogonally each member of the system $s+\lambda s' = 0$. Interpret the case $\lambda = -1$ by first showing that the centre of σ lies on the line $s-s' = 0$.

18 Prove that the length of a tangent from P_1 to $x^2+y^2+2gx+2fy+c = 0$ is $\sqrt{(x_1^2+y_1^2+2gx_1+2fy_1+c)}$.

19 (i) Prove that a point P such that the tangents from it to the circles s, s' are equal (notation of no. 15) lies on the line $2(g-g')x+2(f-f')y+(c-c') = 0$, i.e. $s-s' = 0$.

(ii) Show that the *locus* of P is the whole line if s, s' do not intersect. When they do intersect, show that $s-s' = 0$ is their common chord and that the locus of P is only that part of this line which is outside both circles.

20 Find the equation of the circle which passes through $(-3, 2)$ and the points of intersection of the circles $3x^2+3y^2+2x-7y-6=0$, $x^2+y^2+y-2=0$. [Use no. 16.]

21 Find the equation of the circle whose diameter is the common chord of the circles $x^2+y^2-2x+2y+3=0$, $5x^2+5y^2-x+7y-12=0$.

15.7 Conics

15.71 Definitions

The locus of a point P in a plane such that the ratio of its distance from a given point S to its distance from a given line d is constant is called a *conic*.

The point S is called the *focus*, the line d the *directrix*, and the constant (denoted by e) the *eccentricity* of the conic. According as $e \lesseqgtr 1$, the conic is called an *ellipse, parabola*, or *hyperbola*; e is essentially positive.

If M is the foot of the perpendicular from P to the line d, the definition is equivalent to the relation

$$SP = e.PM.$$

The name 'conic' is an abbreviation for 'conic section'. Originally the ellipse, parabola and hyperbola were obtained as the sections made by a plane drawn perpendicular to any one generator of right circular cones with acute, right and obtuse vertical angles respectively. Later they were defined by different sections of the *same* right circular cone, the ellipse, parabola and hyperbola being respectively the curves of section by a plane making an angle with the base less than, equal to, or greater than that made by the generators. If the cone is a double one, the hyperbola will consist of two separate parts or branches. A plane parallel to the base will give a circular section; one through the axis of the cone will give two intersecting lines (the generators in that plane); one touching the cone along a generator will give a single line; while a plane through the vertex but not cutting the cone elsewhere will give a single point. Hence with this approach, a circle, a line-pair, a single line, and a single point are all conic sections.

It can be proved† that, for each plane which gives an elliptic, parabolic, or hyperbolic section, there exists in that plane a fixed point S and a fixed line d such that any point P on the curve satisfies the law $SP = e.PM$; i.e. these 'conic sections' have the focus-directrix properly and are therefore 'conics' in the sense of our definition.

† We shall not do so in this book.

15.72 The equation of every conic is of the second degree

If the fixed point S is (p, q) and the fixed line d is $lx+my+n = 0$, the definition $SP^2 = e^2.PM^2$ becomes

$$(x-p)^2+(y-q)^2 = e^2\frac{(lx+my+n)^2}{l^2+m^2}.$$

When simplified, this is clearly of second degree in x, y.

In our definition we made no reference to the relative positions of S and d. In particular, if we suppose S lies on d, we may choose d for y-axis and the perpendicular at S for x-axis. Then, if $P(x, y)$ is any point on the locus, the definition $SP^2 = e^2.PM^2$ is expressed by

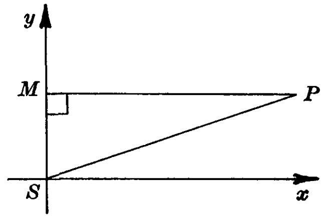

Fig. 164

$$x^2+y^2 = e^2x^2, \text{ i.e. } y^2 = (e^2-1)x^2.$$

According as $e \gtreqless 1$, this equation represents a line-pair through S, the (repeated) line $y = 0$, or the single point $S(0, 0)$.

Hence, consistently with our definitions, we must regard an intersecting line-pair as a degenerate hyperbola (since $e > 1$ for both), a single line as a degenerate parabola ($e = 1$ for both), and a single point as a degenerate ellipse ($e < 1$ for both). These conclusions agree with our remarks in 15.71 about particular 'conic sections'. The classification of a pair of *parallel* lines will be mentioned in Ex. 16 (e), no. 26 (ii); for the *circle*, see 17.17.

We now turn to the more difficult converse problem of interpreting the general second-degree equation

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

by showing that this equation can be reduced to certain standard forms. We shall eventually prove that every equation of the second degree represents a conic (including the degenerate cases just mentioned), a parallel line-pair, a circle, or nothing.

15.73 Change of coordinate axes

(1) When a locus is specified by some property (such as $SP = e.PM$), we express this as an equation between the coordinates of any point (x, y) on the locus, referred to a pair of coordinate axes. Some choices of axes (e.g. those having regard to symmetry of the locus) will lead

to a simpler equation than others. Hence, when an equation wo one set of axes is given, it is often desirable to change the axes in order to simplify the equation. This can be done by

(*a*) changing the origin, leaving the directions of the axes unaltered (a *translation* of axes);

(*b*) changing the direction of the axes, leaving the origin unaltered (a *rotation* of axes);

(*c*) both together.

Remark. When a problem involves equations of two or more loci, in general it is possible to simplify only *one* of the equations in this way; e.g. see 16.12, ex. (ii).

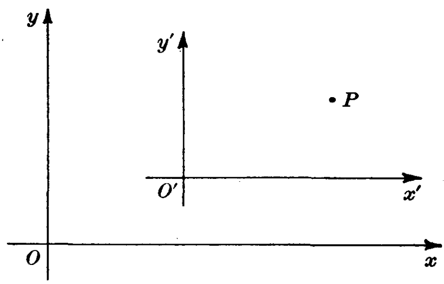

Fig. 165

(2) *Change of origin.* If the new origin O' is taken at the point (h, k), then the point P whose coordinates were (x, y) now has coordinates $x', y')$ given by

$$x' = x-h, \quad y' = y-k.$$

Thus, *to change the origin to the point* (h, k) *we substitute* $x'+h$ *for* x *and* $y'+k$ *for* y: the locus whose equation wo axes Ox, Oy is $f(x,y) = 0$ becomes $f(x'+h, y'+k) = 0$ wo axes $O'x'$, $O'y'$.

We usually omit the dashes in the new equation, and say that $f(x,y) = 0$ becomes $f(x+h, y+k) = 0$.

The above work remains valid for oblique axes; an example occurred in 15.16 (2).

Examples

(i) *Pair of tangents from* P_1 *to* $x^2+y^2 = a^2$. (Cf. 15.64, ex. (ii).)

Changing the origin to $P_1(x_1, y_1)$, the equation of the circle becomes

$$(x+x_1)^2+(y+y_1)^2 = a^2,$$

i.e.

$$x^2+y^2+2x_1x+2y_1y+(x_1^2+y_1^2-a^2) = 0.$$

The equation of the pair of tangents from the new origin is (see 15.64, ex. (i))

$$(x_1^2+y_1^2-a^2)(x^2+y^2) = (x_1x+y_1y)^2.$$

Referred to the original axes, this equation becomes

$$(x_1^2+y_1^2-a^2)\{(x-x_1)^2+(y-y_1)^2\} = \{x_1(x-x_1)+y_1(y-y_1)\}^2,$$

i.e.
$$(x_1^2+y_1^2-a^2)\{(x^2+y^2-a^2)+(x_1^2+y_1^2-a^2)-2(xx_1+yy_1-a^2)\}$$
$$= \{(xx_1+yy_1-a^2)-(x_1^2+y_1^2-a^2)\}^2,$$

i.e.
$$(x_1^2+y_1^2-a^2)(x^2+y^2-a^2) = (xx_1+yy_1-a^2)^2.$$

(ii) *Point of intersection of the line-pair* $ax^2+2hxy+by^2+2gx+2fy+c = 0$.

Suppose the lines meet at $P_1(x_1, y_1)$. Changing the origin to P_1, the equation becomes

$$a(x+x_1)^2+2h(x+x_1)(y+y_1)+b(y+y_1)^2+2g(x+x_1)+2f(y+y_1)+c = 0,$$

i.e.
$$ax^2+2hxy+by^2+2(ax_1+hy_1+g)x+2(hx_1+by_1+f)y$$
$$+(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c) = 0.$$

Since the new origin is the meet of the lines, this equation must involve only terms in x^2, xy and y^2 (15.52 (1)). Hence we must have

$$ax_1+hy_1+g = 0, \quad hx_1+by_1+f = 0,$$

and
$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c = 0.$$

The first two equations determine P_1. As in Remark (β) in 15.53 (3), the third is equivalent to $gx_1+fy_1+c = 0$.

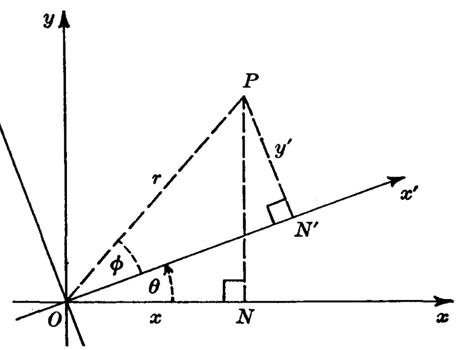

Fig. 166

(3) *Rotation of axes through angle* θ. Let the point P, whose coordinates wo Ox, Oy are (x, y), have coordinates (x', y') referred to the new axes Ox', Oy', where

$$x\hat{O}x' = \theta \quad \text{and} \quad x'\hat{O}P = \phi.$$

If $OP = r$, then from triangle OPN,

$$x = r\cos(\theta+\phi) = r\cos\theta\cos\phi - r\sin\theta\sin\phi$$
$$= x'\cos\theta - y'\sin\theta$$

since from triangle OPN', $x' = r\cos\phi$ and $y' = r\sin\phi$. Similarly,

$$y = r\sin(\theta+\phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi$$
$$= x'\sin\theta + y'\cos\theta.$$

Thus, *to rotate the axes through angle* θ *we substitute* $x'\cos\theta - y'\sin\theta$ *for* x *and* $x'\sin\theta + y'\cos\theta$ *for* y: the locus $f(x,y) = 0$ becomes

$$f(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta) = 0.$$

The reverse transformation can be obtained

either by solving the above two equations for x', y', in terms of x, y;

or by writing $-\theta$ for θ and interchanging (x', y'), (x, y) in the above equations;

or by considering

$x' = r\cos\{(\theta+\phi)-\theta\} = \ldots,\quad y' = r\sin\{(\theta+\phi)-\theta\} = \ldots.$

Both transformations can be read off from the scheme

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

(4) *Change of both origin and direction of axes.* By combining the substitutions found in (2), (3) we see that, under the most general change of rectangular axes, the locus whose equation is $f(x,y) = 0$ becomes

$$f(x\cos\theta - y\sin\theta + h,\ x\sin\theta + y\cos\theta + k) = 0.$$

It follows that *such a change of axes leaves the degree of any polynomial equation unaltered.* For, instead of the linear function x, we have substituted the linear function $x\cos\theta - y\sin\theta + h$, and similarly for y; to powers and products of linear functions correspond powers and products of the same degree.

15.74 Reduction of $s = 0$ to standard forms

By rotating the axes through angle θ, the equation

$$s \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{(i)}$$

becomes

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$$
$$+ b(x\sin\theta + y\cos\theta)^2 + 2g(x\cos\theta - y\sin\theta)$$
$$+ 2f(x\sin\theta + y\cos\theta) + c = 0, \qquad \text{(ii)}$$

in which the coefficient of xy is

$$-2a\cos\theta\sin\theta+2h(\cos^2\theta-\sin^2\theta)+2b\cos\theta\sin\theta$$
$$=2(b-a)\sin\theta\cos\theta+2h\cos 2\theta$$
$$=(b-a)\sin 2\theta+2h\cos 2\theta.$$

If $b-a \neq 0$, this will be zero when θ satisfies

$$\tan 2\theta = \frac{2h}{a-b}; \tag{iii}$$

if $b-a=0$, it will be zero when $\cos 2\theta = 0$; i.e. $\theta = \frac{1}{4}\pi$. Giving θ the value between $\pm\frac{1}{4}\pi$ satisfying the appropriate condition, equation (ii) takes the form

$$Ax^2+By^2+2Gx+2Fy+C=0, \tag{iv}$$

where A, B, C, F, G are known in terms of a, b, c, f, g, h since θ is so known. The following cases now arise.

(*a*) *If* $A \neq 0$ *and* $B \neq 0$, then by completing the square, (iv) can be written

$$A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{F}{B}\right)^2=\frac{G^2}{A}+\frac{F^2}{B}-C. \tag{v}$$

By changing the origin to the point $(-G/A,\ -F/B)$, (v) becomes

$$Ax^2+By^2=\lambda, \tag{vi}$$

where $\lambda = G^2/A+F^2/B-C$.

If $\lambda \neq 0$, this can be written in one of the forms

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1,\quad \frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1,\quad \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=-1$$

according as A/λ, B/λ are both positive, of opposite sign,† or both negative.

If $\lambda = 0$, (vi) can be put into one of the forms

$$\alpha^2x^2-\beta^2y^2=0,\quad \alpha^2x^2+\beta^2y^2=0$$

according as A, B have opposite or the same signs.

(*b*) *If* $A=0$, $B \neq 0$ *and* $G \neq 0$, equation (iv) can be written

$$B\left(y+\frac{F}{B}\right)^2+2G\left(x+\frac{C}{2G}-\frac{F^2}{2BG}\right)=0. \tag{vii}$$

Changing the origin to the point

$$\left(-\frac{C}{2G}+\frac{F^2}{2BG},\ -\frac{F}{B}\right),$$

† The case $-x^2/\alpha^2+y^2/\beta^2=1$ can be brought to the second form shown by rotating the axes through angle $\frac{1}{2}\pi$.

(vii) reduces to $$By^2+2Gx = 0,$$

i.e. $$y^2 = -\frac{2G}{B}x. \qquad \text{(viii)}$$

(*c*) *If* $A = G = 0$ *and* $B \neq 0$, equation (iv) becomes

$$By^2+2Fy+C = 0,$$

which represents two parallel lines, a repeated line, or nothing, according as $F^2 \gtreqless BC$.

(*d*) *If* $B = 0$, $A \neq 0$, $F \neq 0$, or *if* $B = F = 0$, $A \neq 0$, then by transforming equation (iv) by rotating the axes through $\frac{1}{2}\pi$, we obtain the cases considered in (*b*), (*c*).

We cannot have $A = B = 0$, for then (iv) would not be of *second* degree, and hence (see 15.73 (4)) neither would (i). Thus all admissible possibilities have been considered.

It has now been shown that, by a suitable change of axes, the general equation (i) of second degree can be reduced to one of the following *standard forms*:

$$y^2 = 4\alpha x, \quad \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2} = 1, \quad \frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2} = 1,$$
$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2} = -1, \quad \alpha x^2+\beta y^2 = 0,$$
$$y = \alpha \pm \sqrt{(\alpha^2-\beta)}.$$

Clearly the fourth equation represents nothing; the fifth represents an intersecting line-pair, a repeated line, or a single point; and the sixth represents a pair of parallel lines, a repeated line, or nothing. The following three chapters are respectively concerned with the first three loci, which will be proved to be 'conics' in the sense of the definition given in 15.71: see 16.11, 17.12 and 17.13.

Exercise 15(*e*)

1 By a change of origin, the points $(-1, 3)$, $(4, -2)$ become $(\alpha, 5)$, $(3, \beta)$; find α, β.

2 Find the new equation of the locus $x^2-xy-6y^2-3x+14y-4 = 0$ when the origin is changed to $(2, 1)$.

3 Find the new equation of the locus $x^2-y^2 = a^2$ when the axes are rotated through angle $\frac{1}{4}\pi$.

4 By rotating the axes through angle α, show that p in the equation $x\cos\alpha+y\sin\alpha = p$ is the length of the perpendicular from O to this line.

5 Find through what angle the axes must be rotated so that the equation $7x^2+4xy-y^2 = 1$ becomes of the form $ax^2+by^2 = 1$.

6 Change the origin to $(1, 0)$ and then rotate the axes through the acute angle whose tangent is $\frac{3}{4}$, for the equation

$$43x^2 - 48xy + 57y^2 + 86x - 48y + 18 = 0.$$

7 *Show that the centroid of a lamina, as defined in* 7.81, *does not depend on the choice of coordinate axes.* [Wo the new axes the centroid is given by

$$\bar{x}' = \frac{1}{A}\int x'\,dA = \frac{1}{A}\int (x\cos\theta + y\sin\theta + h)\,dA = \bar{x}\cos\theta + \bar{y}\sin\theta + h,$$

and similarly for $\bar{y}'$. Thus $(\bar{x}', \bar{y}')$ is the point $(\bar{x}, \bar{y})$ referred to the new axes.]

Miscellaneous Exercise 15(*f*)

1 If the points (at_1^2, at_1^3), (at_2^2, at_2^3), (at_3^2, at_3^3) are collinear, prove that $t_2t_3 + t_3t_1 + t_1t_2 = 0$. Conversely, show that when this condition is satisfied the three points are collinear. [Use theory of equations.]

2 Find the condition that the four distinct points $(kt_r, k/t_r)$, $r = 1, 2, 3, 4$, shall be concyclic. Show also that no three of these points can be collinear.

3 Show that a necessary condition for concurrence of the lines

$$x\cos 3\alpha + y - a\cos\alpha = 0, \quad x\cos 3\beta + y - a\cos\beta = 0, \quad x\cos 3\gamma + y - a\cos\gamma = 0$$

is $\cos\alpha + \cos\beta + \cos\gamma = 0$, and prove that this condition is also sufficient.

4 If the line-pairs $ax^2 + 2hxy + by^2 = 0$, $a'x^2 + 2h'xy + b'y^2 = 0$ have a line in common, prove $(ab' - a'b)^2 + 4(ah' - a'h)(bh' - b'h) = 0$. [Use 10.42 (1).]

5 If one of the lines $ax^2 + 2hxy + by^2 = 0$ is perpendicular to one of the lines $a'x^2 + 2h'xy + b'y^2 = 0$, prove $(aa' - bb')^2 + 4(ah' + b'h)(a'h + bh') = 0$.

6 If the lines $x^2 - 2pxy - y^2 = 0$ and $x^2 + 2qxy - y^2 = 0$ are such that one pair bisects the angles between the other pair, prove $pq = 1$.

7 If $lx + my + n = 0$ and $ax^2 + 2hxy + by^2 = 0$ form an isosceles triangle, prove that $h(l^2 - m^2) = (a - b)lm$. Find also the further condition if this triangle is equilateral. [The line is parallel to an angle-bisector of the pair, so its gradient $-l/m$ must satisfy $\{1 - (y/x)^2\}/(a - b) = (y/x)/h$.]

8 If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines meeting at A, and parallel lines are drawn through O to meet these at B, C, find the equations of the diagonals OA, BC of the parallelogram so formed. If the parallelogram is a square, prove $a + b = 0$ and $h(g^2 - f^2) = fg(a - b)$. [$OB \perp OC$ and $OA \perp BC$.]

9 Prove that the equation $m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ represents three straight lines equally inclined to one another. [In polar coordinates the equation is $m = \tan 3\theta$. Writing $m = \tan 3\alpha$, the three lines are $\theta = \alpha + \frac{1}{3}r\pi$, $r = 0, 1, 2$.]

10 If A, B, C are fixed and $PA^2 + PB^2 + PC^2$ is constant, prove that the locus of P is a circle whose centre is the centroid of triangle ABC. [Choose O at this centroid; then $\Sigma x_A = 0 = \Sigma y_A$.]

11 Find the centre and radius of the incircle of the triangle whose sides are

$$4x - 3y + 2a = 0, \quad 3x - 4y + 12a = 0, \quad 3x + 4y - 12a = 0.$$

12 Prove that for all constants λ and μ, the circle

$$(x - \alpha)(x - \alpha + \lambda) + (y - \beta)(y - \beta + \mu) = r^2$$

bisects the circumference of $(x - \alpha)^2 + (y - \beta)^2 = r^2$.

Find the equation of the circle which bisects the circumference of

$$x^2 + y^2 + 2y - 3 = 0$$

and touches the line $x - y = 0$ at the origin.

13 Show that the line $(x-a)\cos\theta+y\sin\theta = r$ touches the circle

$$(x-a)^2+y^2 = r^2,$$

and state the coordinates of the point of contact.

A pair of parallel tangents is drawn to a given circle, and another pair perpendicular to these is drawn to a second circle of equal radius. Prove that each diagonal of the square formed by the four tangents passes through a fixed point.

14 All members of a family of circles pass through two given points. Prove that the common chords of these circles and a fixed circle not belonging to the family are concurrent.

15 Write down the equation of the line-pair through P_1 and P_2 which is parallel to (i) Oy; (ii) Ox. Explain why the locus

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$

passes through the vertices of the rectangle formed by these line-pairs, and deduce that this locus is the circle on diameter P_1P_2.

16 If the orthogonal circles

$$x^2+y^2+2g_1x+2f_1y+c_1 = 0, \quad x^2+y^2+2g_2x+2f_2y+c_2 = 0$$

have centres A, B and cut at C, D, prove that the circle through A, B, C, D is

$$2(x^2+y^2)+2(g_1+g_2)x+2(f_1+f_2)y+(c_1+c_2) = 0.$$

If the equation of the circle on diameter CD is written in the form

$$x^2+y^2+2g_1x+2f_1y+c_1+\lambda\{2(g_1-g_2)x+2(f_1-f_2)y+(c_1-c_2)\} = 0,$$

prove $\lambda = -r_1^2/AB^2$ where r_1 is the radius of the first circle.

17 Find the equation of the line-pair joining O to the meets of the lines $4x^2-15xy-4y^2+39x+65y-169 = 0$ and $x+2y = 5$. Show that the quadrilateral having the first pair and also the second pair as adjacent sides is cyclic, and find the equation of its circumcircle.

18 Show that the line-pair joining O to the meets A, B of the line $lx+my = 1$ with the conic $ax^2+by^2 = 1$ has equation

$$(a-l^2)x^2-2lmxy+(b-m^2)y^2 = 0.$$

If AOB is a right-angle, deduce that AB touches the circle $(a+b)(x^2+y^2) = 1$.

19 Find the equation of the line-pair joining O to the meets of $lx+my = 1$ and the circle $x^2+y^2+2gx+2fy+c = 0$. Hence find the coordinates of the circumcentre of the triangle formed by $lx+my = 1$ and the lines

$$ax^2+2hxy+by^2 = 0.$$

If the lines $ax^2+2hxy+by^2 = 0$ vary, but remain equally inclined to the axes, show that the circumcentre varies on a fixed line through O.

20 Explain why the equation

$$(x^2+2hxy+y^2)+(\lambda x+\mu y)(x+y+1) = 0$$

represents a locus passing through the vertices of the triangle formed by the lines $x^2+2hxy+y^2 = 0$, $x+y+1 = 0$. Deduce the equation of the circumcircle of this triangle, and show that this circle is orthogonal to the circumcircle of the triangle formed by the lines $ax^2+2kxy+ay^2 = 0$, $x-y+1 = 0$.

21 Obtain the equation of the lines joining O to the points of intersection of $x-y = b$ with the curve $x^3+y^3 = 3axy$.

16

THE PARABOLA

16.1 The locus $y^2 = 4ax$

16.11 Focus-directrix property

Since the equation $y^2 = 4ax$ can be written in the form

$$(x-a)^2 + y^2 = (x+a)^2,$$

any point $P(x, y)$ on the locus is such that $SP^2 = PM^2$, where S is the point $(a, 0)$ and PM is the perpendicular from P to the line $x + a = 0$. Hence by the definition in 15.71, *the locus $y^2 = 4ax$ is a parabola with focus $(a, 0)$ and directrix $x + a = 0$.*

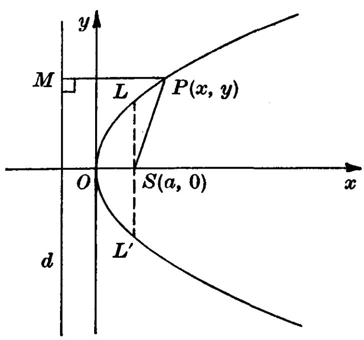

Fig. 167

It is symmetrical about Ox because the equation is unaltered by replacing y by $-y$; Ox is called the *axis* of the parabola, and O is the *vertex*. Without loss of generality we may always suppose $a > 0$; x is then positive for all points on the curve. The parabola meets Oy where $x = 0$ and hence $y = 0$; it therefore touches Oy at O, and Oy is called the *tangent at the vertex.*

The chord LL' through S at right-angles to the axis is called the *latus rectum*; its equation is $x = a$, and it cuts the curve at the points $(a, \pm 2a)$, so that its length is $4a$.

Remark. The equation $y^2 = 4ax$ also shows that *the square of the distance of $P(x, y)$ from Ox is proportional to its distance from the perpendicular line Oy.* The locus of a point satisfying this condition is therefore a parabola.

16.12 Parametric representation

For any point (x, y) of $y^2 = 4ax$ other than $(0, 0)$,

$$\frac{2x}{y} = \frac{y}{2a} = t, \quad \text{say},$$

so that $y = 2at$ and $x = at^2$; $(0, 0)$ is also given thus when $t = 0$.

Hence each point on $y^2 = 4ax$ has coordinates of the form $(at^2, 2at)$, for just one value of t. Also each value of t determines exactly one point of the curve. The equations

$$x = at^2, \quad y = 2at \tag{i}$$

therefore give a proper parametric representation (see 1.61); they can also be written

$$x:y:a = t^2:2t:1. \tag{ii}$$

The point $P(at^2, 2at)$ is briefly referred to as *the point t* of the curve; t is the *parameter* of P. By using these coordinates, the condition that P lies on the curve is automatically satisfied *without reference* to the cartesian equation; cf. the Remark in 16.21.

Examples

(i) *Find the condition for the chord joining the points t_1, t_2 to be a focal chord* (i.e. to pass through the focus).

Any line through the focus $(a, 0)$ has an equation of the form

$$l(x-a)+my = 0.$$

This cuts the curve at points t for which

$$l(at^2-a)+m.2at = 0.$$

It will be the chord t_1t_2 if the roots of this quadratic in t are t_1 and t_2. Since the product of the roots is -1, the required condition is $t_1t_2 = -1$.

This necessary condition for a focal chord is also sufficient. For if $t_1t_2 = -1$, then t_1 and t_2 are the roots of a quadratic equation of the form

$$pt^2+2qt-p = 0,$$

which shows that the points t_1, t_2 lie on the line $px+qy-ap = 0$, and this clearly goes through the focus $(a, 0)$.

(ii) *Concyclic points on the parabola.* A circle†

$$x^2+y^2+2gx+2fy+c = 0$$

cuts the parabola at points t which satisfy‡

$$a^2t^4+4a^2t^2+2agt^2+4aft+c = 0,$$

i.e.
$$a^2t^4+2a(2a+g)t^2+4aft+c = 0.$$

Since this equation is quartic in t, *a circle and parabola can intersect in at most four points.* In real algebra a quartic equation has four, two, or no roots (some or all of which may coincide); hence (ignoring possible coincidences) the number of intersections is four, two, or none.

† Since we are already taking the equation of the parabola in the simple standard form $y^2 = 4ax$, we are not entitled to choose axes so that the equation of the circle *also* takes a simple form (such as $x^2+y^2 = r^2$). Simplification of the equation of the circle would complicate that of the parabola.

‡ When discussing the intersections of two loci it is usually most convenient to employ the cartesian equation of one and the parametric equations of the other, as here.

Suppose there are four intersections, given by $t = t_1, t_2, t_3, t_4$. Then these numbers are the roots of the above quartic; and since the term in t^3 is absent, we have

$$t_1+t_2+t_3+t_4 = 0.$$

This necessary condition for concyclic points on the parabola is also sufficient. In a general way this can be seen because three points of the curve determine a unique circle, and *one* condition is required for this circle to pass through a fourth point of the curve. In detail, given the four numbers t_1, t_2, t_3, t_4, where $\Sigma t_1 = 0$, consider the circle through t_1, t_2, t_3: it cuts the curve again at a point t_4' (since a quartic with three distinct roots must have a fourth), and so by the above,

$$t_1+t_2+t_3+t_4' = 0.$$

Hence $t_4' = t_4$, and so t_1, t_2, t_3, t_4 give concyclic points.

Exercise 16(*a*)

'The parabola' in this exercise means the curve $y^2 = 4ax$.

1 Find the gradient of the chord t_1t_2.

2 If the point t is one extremity of a focal chord PQ, find the length of PQ.

3 If M is the mid-point of a focal chord PQ, prove that the distance of M from the directrix is equal to $\frac{1}{2}PQ$.

4 P, Q, R are points on the parabola such that PQ passes through the focus and PR is perpendicular to the axis. When P varies on the parabola, prove that the mid-point of QR lies on the parabola $y^2 = 2a(x+a)$.

5 The chord PQ subtends a right-angle at the vertex O. Prove that the mid-point of PQ lies on the parabola $y^2 = 2a(x-4a)$.

6 A circle is drawn on a focal chord as diameter, and cuts the axes at $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$, $(0, y_2)$. Prove that $x_1x_2 = y_1y_2 = -3a^2$.

7 (i) Prove that the line $lx+my+n = 0$ cuts the parabola in at most two points when $l \neq 0$.

(ii) If there are two common points t_1 and t_2, prove that

$$t_1+t_2 = -2m/l, \quad t_1t_2 = n/al.$$

(iii) If there is only one common point, prove that $am^2 = ln$, and conversely.

8 A circle cuts the parabola at A, B, C, D. Show that the chords AB, CD are equally inclined to the axis. [Use no. 1 and ex. (ii) in 16.12.]

9 Prove that the locus of the centre of a circle touching Oy and the circle $x^2+y^2-2ax = 0$ is the parabola $y^2 = 4ax$. [If P is the centre, show that $SP = PM$ for a suitable S and d.]

10 A variable circle passes through a given point A and touches a given line l. Prove that the locus of its centre is a parabola, and identify the focus and directrix.

16.2 Chord and tangent

16.21 Chord P_1P_2

The gradient of the chord is

$$\frac{y_1-y_2}{x_1-x_2} = \frac{4a(y_1-y_2)}{y_1^2-y_2^2} = \frac{4a}{y_1+y_2},$$

since $y_1^2 = 4ax_1$ and $y_2^2 = 4ax_2$. The equation of the chord is therefore

$$y-y_1 = \frac{4a}{y_1+y_2}(x-x_1). \qquad \text{(i)}$$

This equation is not symmetrical in the pair of coordinates (x_1, y_1), (x_2, y_2). To make it so, first clear of fractions:

$$y(y_1+y_2)-y_1^2-y_1y_2 = 4ax-4ax_1.$$

By using the condition $y_1^2 = 4ax_1$ again, this simplifies to

$$4ax-(y_1+y_2)y+y_1y_2 = 0. \qquad \text{(ii)}$$

Remark. To obtain (ii) we have appealed once to the condition $y_2^2 = 4ax_2$ that P_2 lies on the curve, and twice to $y_1^2 = 4ax_1$. Contrast the directness of the work in 16.22, where the use of parameters *automatically* ensures that P_1, P_2 lie on the curve.

For another way of proving (ii), see Ex. 16 (*b*), no. 1.

16.22 Chord $t_1 t_2$

(1) Since the gradient of the chord is

$$\frac{2at_1-2at_2}{at_1^2-at_2^2} = \frac{2}{t_1+t_2},$$

the chord has equation

$$y-2at_1 = \frac{2}{t_1+t_2}(x-at_1^2),$$

i.e.

$$x-\tfrac{1}{2}(t_1+t_2)y+at_1t_2 = 0. \qquad \text{(iii)}$$

(2) *Alternatively* (using the theory of quadratics) let the required chord be† $lx+my+a = 0$. This line cuts the curve at points t for which

$$lt^2+2mt+1 = 0.$$

Since the line is the chord t_1t_2, this quadratic in t must have roots t_1 and t_2, so $t_1+t_2 = -2m/l$ and $t_1t_2 = 1/l$. Hence

$$l = \frac{1}{t_1t_2} \quad \text{and} \quad m = -\frac{1}{2}\frac{t_1+t_2}{t_1t_2},$$

and the required chord is

$$\frac{x}{t_1t_2}-\frac{1}{2}\frac{t_1+t_2}{t_1t_2}y+a = 0,$$

which is equivalent to (iii).

† There is no loss of generality in taking the last term to be a, since the equation of a line contains only two *independent* constants.

(3) *Alternatively*, since the numbers t_1, t_2 are the roots of

$$t^2-(t_1+t_2)t+t_1t_2 = 0, \tag{iv}$$

we may use the parametric equations $x:y:a = t^2:2t:1$ to substitute for t^2, t and thereby construct the linear equation

$$x-\tfrac{1}{2}(t_1+t_2)y+at_1t_2 = 0.$$

This represents the required chord t_1t_2 because it cuts the curve at the points t given by (iv), i.e. $t = t_1$ and $t = t_2$.

16.23 Tangent at P_1

For curves other than the circle, the tangent is defined as a limit: the *tangent at* P_1 is the limit of the chord P_1P_2 when the point P_2 tends to P_1 along the curve; but see 16.25.

(1) The tangent at P_1 to $y^2 = 4ax$ is therefore obtainable from equation (ii) by letting $y_2 \to y_1$, and is

$$4ax-2y_1y+y_1^2 = 0.$$

Since $y_1^2 = 4ax_1$, this can be written

$$yy_1 = 2a(x+x_1). \tag{v}$$

Observe that this can be written down from the equation $y^2 = 4ax$ by applying the 'rule of alternate suffixes' (see 15.63, Remark).

(2) *Alternatively*, the gradient of the tangent at P_1 can be found by calculating dy/dx when $x = x_1$, $y = y_1$. Thus from $y^2 = 4ax$,

$$2y\frac{dy}{dx} = 4a \quad \text{and} \quad \frac{dy}{dx} = \frac{2a}{y},$$

and the tangent at P_1 has equation

$$y-y_1 = \frac{2a}{y_1}(x-x_1),$$

which is equivalent to the limit of (i) when $y_2 \to y_1$, and reduces to (v) above.

16.24 Tangent at the point t

(1) Letting $t_2 \to t_1$ in equation (iii), we obtain $x-t_1y+at_1^2 = 0$. Omitting the suffix, the tangent at the point t is

$$x-ty+at^2 = 0. \tag{vi}$$

(2) *Alternatively*, the gradient can be obtained by calculus:

$$\frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t};$$

and then the equation written down as

$$y - 2at = \frac{1}{t}(x - at^2).$$

Geometrically, the parameter t is the cotangent of the angle ψ made by the tangent-line at the point t with the x-axis; for $\tan\psi = 1/t$.

16.25 Tangency and repeated roots

For a curve whose equation is a polynomial in (x, y), the simultaneous solution of this equation and that of the chord P_1P_2 leads to equations for x and y containing the factors

$$(x - x_1)(x - x_2), \quad (y - y_1)(y - y_2)$$

respectively, or to an equation in a parameter t containing

$$(t - t_1)(t - t_2).\dagger$$

Since the equation of the tangent at P_1 is the limit of the equation of the chord P_1P_2 when P_2 tends to P_1 along the curve, hence if the equation of the tangent is solved with the equation of the curve, it will give equations for x, y containing the factors $(x - x_1)^2$, $(y - y_1)^2$, or an equation for t containing $(t - t_1)^2$.

Thus *tangency corresponds to repeated roots*, and is often dealt with thus rather than by direct appeal to the limit definition. For example, the line $lx + my + n = 0$ meets the parabola where $alt^2 + 2amt + n = 0$, and will be a tangent if and only if this quadratic has equal roots, i.e. if $am^2 = ln$; cf. Ex. 16 (a), no. 7.

More generally, if two loci cut at P_1 and P_2, and P_2 is made to approach P_1 along one of them, then the limit of the other will touch the first at P_1 because they have a common tangent there. For example, if $t_2 \to t_1$ in ex. (ii) of 16.12, the limiting circle will touch the parabola at the point t_1 and cut it again at points t_3, t_4; other limiting cases can be considered similarly (see Ex. 16 (b), nos. 16–18). See also Remark (α) in 6.72.

† This is not the case unless the equation of the curve is *algebraic*; e.g. the chord of the curve $y = e^x$ joining $(0, 1)$ and $(1, e)$ is $y - 1 = (e - 1)x$, and leads to $e^x - 1 = (e - 1)x$ which has no polynomial factors.

16.26 Examples

(i) *The tangents at t_1, t_2 meet at the point $(at_1t_2, a(t_1+t_2))$.*

This result can be verified by direct solution of the equations

$$x-t_1y+at_1^2=0, \quad x-t_2y+at_2^2=0.$$

Alternatively, these two equations express that t_1 and t_2 are the roots of the quadratic in t

$$at^2-ty+x=0,$$

so that

$$\frac{y}{a}=t_1+t_2 \quad \text{and} \quad \frac{x}{a}=t_1t_2,$$

giving the required intersection (x, y).

The coordinates can be remembered by an 'extension' of the rule of alternate suffixes applied at at^2, $2at$.

(ii) *The orthocentre† of the triangle formed by three tangents to a parabola lies on the directrix.*

The tangents at t_1, t_2 meet at $(at_1t_2, a(t_1+t_2))$. The perpendicular from this point to the tangent at t_3 has equation

$$t_3x+y=a(t_1+t_2+t_1t_2t_3),$$

and this line meets the directrix $x=-a$ where $y=a(t_1+t_2+t_3+t_1t_2t_3)$. The symmetry of this result shows that the other two perpendiculars meet the directrix at this same point, which must therefore be the orthocentre of the triangle of tangents.

(iii) (*a*) *Find the condition for $y=mx+c$ to touch the parabola $y^2=4ax$.*

(*b*) *If this line also touches the circle $x^2+y^2=r^2$, prove that*

$$m^4+m^2-\frac{a^2}{r^2}=0,$$

and hence find the equations of the common tangents to the parabola and the circle $x^2+y^2=\frac{1}{2}a^2$.

(*a*) The line $y=mx+c$ cuts the curve $y^2=4ax$ at points for which

$$(mx+c)^2=4ax,$$

i.e.

$$m^2x^2+2(mc-2a)x+c^2=0.$$

The line will touch the curve if and only if this quadratic in x has equal roots, i.e.

$$(mc-2a)^2=m^2c^2,$$

i.e.

$$c=\frac{a}{m}.$$

Hence *for all $m \neq 0$, the line $y=mx+a/m$ touches $y^2=4ax$.*

Alternatively, $mx-y+c=0$ will be the same line as $x-ty+at^2=0$ (the tangent at some point t) if

$$\frac{m}{1}=\frac{1}{t}=\frac{c}{at^2},$$

i.e. if $m=1/t$ and $c=at=a/m$.

† The *orthocentre* of a triangle is the point of intersection of the perpendiculars drawn from the vertices to the opposite sides.

(*b*) Similar methods will give the contact condition for the line and circle, but it is simpler to equate the radius r and the perpendicular from the centre (0, 0) to the line $mx-y+a/m = 0$:

$$r = \pm\frac{a/m}{\sqrt{(m^2+1)}},$$

from which the equation for m follows.

Taking $r^2 = \frac{1}{2}a^2$, the equation becomes

$$m^4+m^2-2 = 0, \quad \text{i.e.} \quad (m^2-1)(m^2+2) = 0,$$

so that $m = \pm 1$. The common tangents are thus

$$y = x+a, \quad y = -x-a.$$

(iv) *Tangents from P_1; the chord of contact.*

From a given point P_1 at most two tangents can be drawn to the parabola. For the tangent at the point t will pass through (x_1, y_1) if

$$x_1-ty_1+at^2 = 0,$$

and if $y_1^2 > 4ax_1$ this gives two values $t = t_1, t_2$; i.e. there are then two points the tangents at which pass through P_1.†

The argument used in 15.64 will show that the chord of contact of tangents from P_1 is

$$yy_1 = 2a(x+x_1).$$

(v) *Pair of tangents from P_1.*

The condition for the point

$$\left(\frac{lx_1+kx_2}{l+k}, \frac{ly_1+ky_2}{l+k}\right)$$

to lie on $y^2 = 4ax$ reduces to

$$(y_2^2-4ax_2)k^2+2\{y_1y_2-2a(x_1+x_2)\}kl+(y_1^2-4ax_1)l^2 = 0$$

(*Joachimsthal's ratio equation* for the parabola).

The argument in 15.64, ex. (ii) shows that the pair of tangents from P_1 has equation

$$\{y_1y-2a(x_1+x)\}^2 = (y_1^2-4ax_1)(y^2-4ax).$$

Exercise 16(*b*)

1 *Verify that, when simplified, the equation*

$$(y-y_1)(y-y_2) = y^2-4ax$$

is linear in x and y; and that it is satisfied by the points P_1, P_2 on the parabola. Deduce the equation of the chord P_1P_2.

2 Deduce from equation (iii) in 16.22 that t_1t_2 is a *focal* chord if and only if $t_1t_2 = -1$.

3 Obtain the equation of the tangent at the point t by using the theory of quadratic equations.

4 Prove that the chord whose extremities are given by the roots of $ut^2+2vt+w = 0$ has equation $ux+vy+aw = 0$.

† The set of points (x_1, y_1) for which $y_1^2 > 4ax_1$ may therefore be called the *outside* of the parabola.

5 Find the equation of each of the tangents from $(-2a, -a)$ to $y^2 = 4ax$.

6 The ordinate of the point P on the parabola is PN, and the tangent at P meets the axis at T. Prove that the vertex O bisects TN.

7 If the tangent at P meets the axis at T, prove $SP = ST$.

8 (i) If PM is the perpendicular from a point P of the parabola to the directrix, prove that the tangent at P bisects angle SPM. [Use no. 7 and pure geometry.]

(ii) Deduce the property of the parabolic reflector: rays from a source of light at S leave the surface in a beam parallel to the axis. [Angle of incidence = angle of reflection.]

9 If the tangent at P meets the directrix at Z, prove $SZ \perp SP$.

10 PQ is a variable focal chord; UP is the tangent at P, and QU is parallel to the axis. Find the locus of the mid-point of PU.

11 Show that any one of the following statements implies the other two.

(i) PQ is a focal chord;

(ii) tangents at P and Q are perpendicular;

(iii) tangents at P and Q meet on the directrix.

12 If P, Q are the points t_1, t_2, and tangents at P, Q meet at T, prove that triangle TPQ has area $\frac{1}{2}a^2(t_2-t_1)^3$. If this area is always $4a^2$, and the locus of T.

13 (i) If the chord $t_1 t_2$ subtends a right-angle at the point t, prove that

$$(t+t_1)(t+t_2)+4 = 0.$$

(ii) If the circle on chord PQ as diameter cuts the parabola again at H and K, prove that the chords PQ, HK make equal angles with the axis.

14 A circle cuts the parabola at A, B, C, D; tangents at A, B meet at T, and tangents at C, D meet at T'. Prove that the axis bisects TT'. [Use 16.12, ex. (ii).]

15 A circle passes through the vertex of the parabola and cuts the curve again at A, B, C. The tangents at B, C meet at T. Find the ratio in which AT is divided by the axis.

16 (i) *If a circle touches the parabola at A and cuts it again at C, D, prove that the tangent at A and the chord CD are equally inclined to the axis.*

(ii) If circles touch a parabola at a fixed point, prove that the common chords not through this point are parallel.

*17 (i) If $C \to A$ in no. 16(i), then by 8.42(2) the limiting circle is the *circle of curvature* of the parabola at A. If A is the point t_1, prove that the remaining intersection D has parameter $-3t_1$. What can be said about the inclination of the tangent at A and the chord AD?

(ii) A circle cuts the parabola at A, B, C, D. If the circles of curvature at these points cut the curve again at A', B', C', D' respectively, prove the latter points are also concyclic.

*18 If $t_1 \neq t_3$ in ex. (ii) of 16.12, and $t_2 \to t_1$ and $t_4 \to t_3$, the limiting circle touches the parabola at t_1 and at t_3 (*double contact*). Prove that the chord of contact $t_1 t_3$ is perpendicular to the axis, and that the centre of the circle lies on the axis and has abscissa $\geqslant 2a$.

19 Sketch the parabolas $y^2 = 4ax$, $x^2 = 4by$ $(b > 0)$, and find the equation of their common tangent.

20 Write down the equation of the chord of contact from $P(\frac{1}{3}, \frac{7}{6})$ to $y^2 = 4x$. By finding its intersections with the curve, deduce the equations of the tangents from P.

21 Prove that the pair of tangents from $T(x_1, y_1)$ cuts Oy at points A, B such that the mid-point of AB is $M(0, \frac{1}{2}y_1)$. [Use ex. (v) of 16.26.] If P, Q are the points of contact of these tangents, prove $SM \perp PQ$. [Use the chord of contact from T.]

22 If tangents are drawn to the parabola from points of a given line, prove that the corresponding chords of contact are concurrent.

*23 Prove that tangents at the extremities of a variable chord through P_1 meet on the line $yy_1 = 2a(x+x_1)$. [Method of 15.65 (1).]

*24 A variable chord through P_1 meets the parabola at A, B; P_2 is chosen on it so that P_1 and P_2 divide AB (one internally and the other externally) in the same ratio. Prove that P_2 lies on the line $yy_1 = 2a(x+x_1)$. [Method of 15.65 (2); use the ratio quadratic in 16.26, ex. (v).]

*25 Defining the *polar* of P_1 wo $y^2 = 4ax$ to be the line $yy_1 = 2a(x+x_1)$, prove that if the polar of P_1 passes through P_2, then the polar of P_2 passes through P_1. What is the polar of the focus?

*26 Use the ratio quadratic (16.26, ex. (v)) to obtain the equation of the tangent at P_1. [Since P_1 lies on $y^2 = 4ax$, the ratio quadratic has one root $k = 0$; the other root gives the remaining intersection of P_1P_2 with the curve. If P_1P_2 is the *tangent* at P_1, this intersection must coincide with P_1, and $k = 0$ is a repeated root. Hence $y_1y_2 - 2a(x_1+x_2) = 0$, which gives the locus of P_2.]

16.3 Normal

16.31 Normal at the point t

Since the tangent at the point t has gradient $1/t$, the normal there has gradient $-t$. Hence the equation of the normal at t is

$$y - 2at = -t(x - at^2),$$

i.e.

$$tx + y = 2at + at^3. \qquad \text{(i)}$$

Example

Prove that the normal at t meets the curve again at the point $-t-2/t$.

If the other intersection is the point s, then the normal at t is also the chord ts, so that

$$\text{gradient of normal at } t = \text{gradient of chord } ts,$$

i.e.

$$-t = \frac{2}{t+s},$$

from which $s = -t-2/t$. Also see Ex. 16 (*c*), no. 4.

16.32 Conormal points

Given a point (x_0, y_0), we may enquire how many normals can be drawn from it to the parabola. The normal at t will pass through (x_0, y_0) if

$$tx_0 + y_0 = 2at + t^3.$$

Since this equation is cubic in t, there are at most three points on the curve such that the normals from them pass through (x_0, y_0). Ignoring coincidences, a cubic has either one or three roots; therefore either one or three normals can be drawn from a given point. Also see Ex. 16 (c), no. 15.

Suppose that the normals at the points t_1, t_2, t_3 are concurrent at (x_0, y_0). Then these numbers are the roots of the cubic

$$at^3 + (2a - x_0)t - y_0 = 0. \qquad \text{(ii)}$$

Since the term in t^2 is absent,

$$t_1 + t_2 + t_3 = 0. \qquad \text{(iii)}$$

Three points on the curve which are such that the normals at them are concurrent are called *conormal points*. Hence (iii) is a necessary condition for the points t_1, t_2, t_3 to be conormal. The point of concurrence is given by

$$2a - x_0 = a\Sigma t_1 t_2, \quad y_0 = at_1 t_2 t_3. \qquad \text{(iv)}$$

The necessary condition $\Sigma t_1 = 0$ *for conormal points is also sufficient.* In a general way this is clear because only *one* condition is needed for concurrence of *three* lines. In detail, given t_1, t_2, t_3, we can define numbers x_0 and y_0 by equations (iv) above; then, provided

$$t_1 + t_2 + t_3 = 0,$$

we see that t_1, t_2, t_3 are the roots of the cubic (ii), which expresses that the normal at each of the corresponding points passes through (x_0, y_0).

If two roots of the cubic (ii) are equal, then two of the three normals from (x_0, y_0) coincide. If $t_2 = t_3$, then $t_1 = -2t_2$, and (x_0, y_0) is given by (iv) as

$$x_0 = 2a + 3at_2^2, \quad y = -2at_2^3.$$

By 8.53, ex. (i), these equations show that (x_0, y_0) lies on the envelope of the normals (i.e. the *evolute*) of the parabola. From any point on this locus only two distinct normals can be drawn.

Three normals from a point can coincide only when $t_1 = t_2 = t_3$, i.e. when each is zero (since $\Sigma t_1 = 0$); and then (iv) shows that (x_0, y_0) is the point $(2a, 0)$.

Examples

(i) *The circle through the feet of three concurrent normals also passes through the vertex.*

If t_1, t_2, t_3 are the feet, then $t_1 + t_2 + t_3 = 0$. If the circle through these points cuts the curve again at t_4, then by ex. (ii) of 16.12 we have $t_1 + t_2 + t_3 + t_4 = 0$. Hence $t_4 = 0$, which gives the vertex $(0, 0)$.

(ii) *Find the equation of the circle through the feet of normals which are concurrent at* (h, k). (See also 19.7, ex. (i).)

The feet of the three normals are the roots of the cubic

$$at^3 + (2a - h)t - k = 0. \qquad \text{(v)}$$

If the required circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, its meets with the curve satisfy (cf. 16.12, ex. (ii))

$$a^2t^4 + (4a^2 + 2ag)t^2 + 4aft + c = 0.$$

Since by ex. (i) $t = 0$ is one such meet, therefore $c = 0$, and the other three are given by

$$at^3 + (4a + 2g)t + 4f = 0. \qquad \text{(vi)}$$

The cubics (v), (vi) now have the same roots, and so

$$4a + 2g = 2a - h, \quad 4f = -k.$$

The required circle is therefore

$$x^2 + y^2 - (h + 2a)x - \tfrac{1}{2}ky = 0.$$

(iii) (*a*) *Prove that the normal at* P_1 *has equation* $2a(y - y_1) + y_1(x - x_1) = 0$.

(*b*) *Hence prove that the feet of the normals from* (h, k) *to the parabola lie on the curve* $xy + (2a - h)y - 2ak = 0$.

(*a*) By 16.23 the tangent at P_1 has gradient $2a/y_1$, so the normal at P_1 has gradient $-y_1/2a$, and its equation is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

(*b*) Let P_1 be the foot of a normal drawn from (h, k) to the curve. Then since (h, k) lies on the normal at P_1,

$$2a(k - y_1) + y_1(h - x_1) = 0,$$

which shows that P_1 lies on the locus

$$2a(k - y) + y(h - x) = 0.$$

Exercise 16(*c*)

1 PN is the ordinate of a point P on the parabola, and the normal at P cuts the axis at G. Prove that $NG = 2a$.

2 The tangent and normal at P meet the axis at T, G respectively. Prove $TS = SG$.

3 The line through the mid-point M of a focal chord PQ and parallel to the axis meets the normal at P in V. Find the locus of V.

4 If the normal at t meets the curve again at s, find s in terms of t by substituting $x = as^2$, $y = 2as$ in the equation of the normal and *either* (*a*) factorising the cubic in t, or (*b*) observing that the resulting quadratic in s necessarily has a root $s = t$, and that the sum of the roots is $-2/t$.

5 (i) Prove that the feet of normals which meet at the point s of the curve satisfy the quadratic $t^2 + st + 2 = 0$. [Use 16.31, ex.]

(ii) If the normals at t_1, t_2 meet on the curve, prove that $t_1t_2 = 2$.

(iii) Conversely, if $t_1t_2 = 2$, prove that the normals at t_1, t_2 meet on the curve. [Let the normals meet the curve again at s_1, s_2 respectively; use 16.31, ex. and the condition to show $s_1 = s_2$.]

(iv) Show that normals at the extremities of a focal chord can *never* meet on the curve.

6 Prove that the normals at the extremities of all chords of the form $x+ky+2a = 0$ meet on the curve. Show that all such chords pass through a fixed point, and give its coordinates. [This line cuts the curve where $at^2+2akt+2a = 0$, so that $t_1 t_2 = 2$.]

7 Find the equations of the three normals from $(15a, 12a)$ to $y^2 = 4ax$.

8 Show that two of the three normals from $(5a, 2a)$ to $y^2 = 4ax$ coincide.

9 Prove that the centroid of the triangle formed by conormal points lies on the axis.

10 PQ is a chord of a parabola drawn in a fixed direction. Prove that the locus of the meet of the normals at P and Q is a line which is itself normal to the parabola. [If t_1, t_2 are the extremities, then

$$2/(t_1+t_2) = m, \quad \text{i.e.} \quad t_1+t_2+(-2/m) = 0;$$

hence the normals at t_1, t_2 meet on the normal at the (fixed) point $-2/m$.]

11 If normals at P, Q, R are concurrent, and the chords PP', QQ', RR' are parallel to QR, RP, PQ respectively, prove that the normals at P', Q', R' are also concurrent.

12 Prove that the normals at t_1, t_2 intersect at the point (x, y), where

$$x = a(t_1^2+t_1 t_2+t_2^2+2), \quad y = -at_1 t_2(t_1+t_2).$$

[Solve directly; *or* eliminate t_3 from $\Sigma t_1 = 0$ and equations (iv) in 16.32.]

13 A variable chord PQ subtends a right-angle at the vertex. Tangents at P, Q meet at T, and the normals at P, Q meet at N; M is the mid-point of PQ. Find the loci of T, N, M.

*14 If C_r is the centre of the circle through the feet of normals from P_r, prove that when P_1, P_2, P_3 are collinear, then so are C_1, C_2, C_3, and conversely. [Use ex. (ii) of 16.32.]

*15 Show that $y^2 = 4ax$ meets the locus in 16.32, ex. (iii)(*b*) at points $P(x, y)$ for which $y^3+4a(2a-h)y-8a^2k = 0$. Deduce that at most three normals can be drawn to the parabola from a given point (h, k).

16.4 Diameters

16.41 General definition

The locus of the mid-points of a system of parallel chords of a conic is called a *diameter* of the conic.

The parallel chords are sometimes called *ordinates* to the diameter. In 17.63 the above definition will be reconciled with the usual meaning of 'diameter'.

16.42 Diameters of a parabola

The chord $t_1 t_2$ has gradient m, where $m = 2/(t_1+t_2)$. The mid-point of this chord has y-coordinate $a(t_1+t_2)$, i.e. $2a/m$. Hence the mid-points of all chords of given gradient m satisfy $y = 2a/m$, which is therefore the equation of the diameter bisecting chords of gradient m. Thus *the diameters of a parabola are straight lines parallel to the axis.* Also see Ex. 16 (d), no. 6.

Example

Prove that the tangent at the extremity P of a diameter is parallel to the chords which the diameter bisects.

The diameter $y = 2a/m$ meets $y^2 = 4ax$ where $x = a/m^2$, i.e. at the point $P(a/m^2, 2a/m)$. The tangent at P (which is the point $1/m$) has gradient

$$1/(1/m) = m,$$

and is therefore parallel to the chords of gradient m.

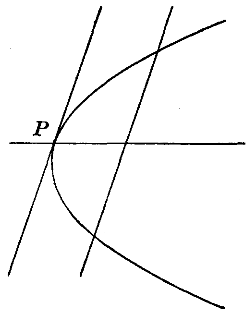

Fig. 168

Exercise 16(d)

1 Write down the equation of the diameter of $y^2 = x$ which bisects chords parallel to $2x-3y+1=0$.

2 Write down the gradient of the chords of $y^2 = 2x$ which are bisected by the diameter $2y+3=0$.

3 Prove that tangents at the extremities of any chord of a parabola meet on the diameter which bisects this chord.

4 V is the mid-point of a chord QQ' of a parabola, and TQ, TQ' are tangents. Prove that the parabola bisects TV.

*5 Prove that the circle drawn on a focal chord as diameter touches the directrix. [Use no. 3.]

6 By considering the y-coordinate of the mid-point of the chord $y = mx+c$ of $y^2 = 4ax$, show that the locus of the mid-points of chords of gradient m is the line $y = 2a/m$.

Miscellaneous Exercise 16(e)

1 A point is such that its distance from a fixed line is equal to the length of the tangent from it to a fixed circle. Prove that its locus is a parabola. Locate the focus and directrix when the given line touches the circle.

2 Obtain the equation of the circle through the points $(p, 0)$, $(q, 0)$, $(0, r)$.

A circle passes through a fixed point, and the chord cut off by it from a given line is of constant length. Prove that the locus of its centre is a parabola.

3 Two parabolas have a common axis and their concavities are in opposite senses. If any line parallel to the common axis meets the parabolas at P, Q, prove that (provided the latera recta are unequal) the locus of the mid-point of PQ is another parabola.

4 *Prove that $y = ax^2+bx+c$ represents a parabola, and find its vertex and the length of the latus rectum.*

5 Prove that the common chord of circles drawn on any two focal chords as diameters passes through the vertex.

6 A circle touches the parabola at A and cuts it again at C and D. Prove that the axis bisects the line joining A to the mid-point of CD.

7 G is the centroid of a triangle inscribed in a parabola; G' is the centroid of the corresponding triangle of tangents. Prove that GG' is parallel to the axis, and that the parabola divides GG' in the ratio $2:1$.

8 The normal at P meets the parabola again at Q, and the axis divides PQ in the ratio $k:1$. Prove that the x-coordinate of P is $2ka/(1-k)$. For what value of k is P an extremity of the latus rectum?

9 Normals at the ends of a focal chord PQ cut the curve again at P', Q'. Prove that $P'Q'$ is parallel to QP and that $P'Q' = 3QP$.

10 If normals at P, Q meet on the curve, prove that the meet of tangents at P, Q lies either on a line parallel to Oy, or on the locus $y^2(x+2a)+4a^3 = 0$.

11 A, B, C, D are concyclic points on the parabola; AB is a focal chord; AC is the normal at A. Show that the axis divides BD in the ratio $1:3$.

12 Find the locus of a point $P(h,k)$ such that two of the normals drawn from it to the parabola $y^2 = 4ax$ are perpendicular.

13 Prove that *three* normals cannot be drawn from (h,k) to $y^2 = 4ax$ unless $h > 2a$. [Prove $f(t) \equiv at^3+(2a-h)t-k$ is steadily increasing for all t if $h \leqslant 2a$, so that $f(t) = 0$ has just one root.]

14 Prove that chords of $y^2 = 4ax$ which subtend a right-angle at the point t_0 of the curve all pass through the point $\left(a(t_0^2+4), -2at_0\right)$. (Cf. Ex. 19 (c), no. 2.)

*15 A variable triangle is inscribed in $y^2 = 4ax$ so that two of its sides touch $y^2 = 4bx$. Prove that the third side touches $y^2 = 4cx$, where $(2a-b)^2c = ab^2$. [Use the condition $km^2 = ln$ for tangency of $lx+my+n = 0$ and $y^2 = 4kx$.]

16 Given a line l and a point S not on it, a variable line is drawn through S to meet l at V. A line p is drawn through V perpendicular to SV. Prove that p touches a parabola having S for focus and l for its tangent at the vertex. [Choose l for y-axis, and the perpendicular to it through S for x-axis; let these meet at O and p meet SO at T. If p has equation $lx+my+n = 0$, then $OV = -n/m$ and $OT = n/l$. By geometry $OV^2 = TO.OS$, so $am^2 = nl$ where $a = OS$.]

17 Show that the line through P_1 in direction θ (15.26) cuts $y^2 = 4ax$ at points for which r is given by

$$r^2\sin^2\theta+2(y_1\sin\theta-2a\cos\theta)\,r+(y_1^2-4ax_1) = 0.$$

(This is called the *distance quadratic* for $y^2 = 4ax$ because it gives the distances r from P_1 of the points of the parabola on the line through P_1 in direction θ.)

18 Prove that the chord of $y^2 = 4ax$ whose mid-point is (x_1, y_1) has equation $yy_1-2ax = y_1^2-2ax_1$. [For the required chord the roots of the distance quadratic are equal and opposite, so $y_1\sin\theta-2a\cos\theta = 0$, which determines $\cos\theta:\sin\theta$. The equation $x-x_1:\cos\theta = y-y_1:\sin\theta$ of the chord becomes

$$(x-x_1)/y_1 = (y-y_1)/2a.]$$

19 Deduce from no. 18 that (i) the locus of the mid-points of chords through P_1 is $y^2-2ax = yy_1-2ax_1$; (ii) the locus of the mid-points of chords of gradient m is the line $y = 2a/m$.

20 Find the locus of the mid-points of chords of $y^2 = 4ax$ which touch $y^2 = 4bx$.

21 If parallel chords are divided so that the product of their segments is constant, prove that the point of division lies on a parabola congruent to the given one.

22 Using no. 17, obtain the equation of the tangent at P_1. [Two roots $r = 0$.]

23 Obtain the equation of the pair of tangents from P_1. [Equal roots r.]

*24 Show that $(3x+4y)^2 = 54x-53y+29$ represents a parabola as follows.

(i) Verify that the equation can be written

$$(3x+4y+k)^2 = (6k+54)x+(8k-53)y+(k^2+29).$$

(ii) By choosing k so that the lines

$$3x+4y+k = 0, \quad (6k+54)x+(8k-53)y+(k^2+29) = 0$$

are perpendicular, express the equation in the form

$$\left(\frac{3x+4y+1}{5}\right)^2 = 2\left(\frac{4x-3y+2}{5}\right). \qquad (a)$$

(iii) If PM, PN are the perpendiculars from $P(x,y)$ to the (perpendicular) lines $3x+4y+1 = 0$, $4x-3y+2 = 0$, (a) shows $PM^2 = 2PN$. By the Remark in 16.11, the curve is a parabola of latus rectum 2, having axis $3x+4y+1 = 0$ and $4x-3y+2 = 0$ for tangent at the vertex.

(iv) For points on the curve, (a) shows that $4x-3y+2 \geqslant 0$; hence the curve is on the origin side of $4x-3y+2 = 0$, i.e. lies below this line. Sketch the curve.

*25 Sketch the parabola $(x+2y)^2 = 56x+12y-184$, giving the equation of the axis and the coordinates of the vertex.

*26 (i) *If* $ab = h^2$, then the terms of second degree in

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

form a perfect square, say $(px+qy)^2$. Prove by the method of no. 24 that, unless $g:f = p:q$, *the equation represents a parabola* whose axis is parallel to $px+qy = 0$.

(ii) In the exceptional case show that the equation can be written $(px+qy+\lambda)^2 = \lambda^2-c$ where $g = \lambda p$ and $f = \lambda q$. This represents a pair of parallel lines, a repeated line, or nothing according as $\lambda^2 \gtreqless c$. (Algebraically, a parallel line-pair is thus a degenerate parabola.)

*27 Show that the curve whose parametric equations are

$$x = at^2+2bt, \quad y = ct^2+2dt \quad (ad \neq bc)$$

is a parabola. [Eliminate t, and use the result of no. 26 (i).]

28 A curve is given parametrically by $x = f(t)/h(t)$, $y = g(t)/h(t)$. Prove that the chord joining the points t, $t+\epsilon$ has equation

$$\begin{vmatrix} x & y & 1 \\ f(t) & g(t) & h(t) \\ f(t+\epsilon) & g(t+\epsilon) & h(t+\epsilon) \end{vmatrix} = 0.$$

Deduce that the tangent at the point t has equation

$$\begin{vmatrix} x & y & 1 \\ f(t) & g(t) & h(t) \\ f'(t) & g'(t) & h'(t) \end{vmatrix} = 0.$$

[$\mathbf{r}_3 \to (\mathbf{r}_3-\mathbf{r}_2)/\epsilon$, and let $\epsilon \to 0$.]

17

THE ELLIPSE

17.1 The loci $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

17.11 $SP = e.PM$

We now consider the locus defined by $SP = e.PM$ ($e > 0$, $e \neq 1$), where S is a given point and M is the foot of the perpendicular from P to a given line d (15.71). Choose S for origin, and let the given line then have equation $x = k$ ($k > 0$). The equation of the locus referred to this set of axes is therefore

$$x^2+y^2 = e^2(x-k)^2,$$

i.e.
$$x^2(1-e^2)+2e^2kx+y^2 = e^2k^2.$$

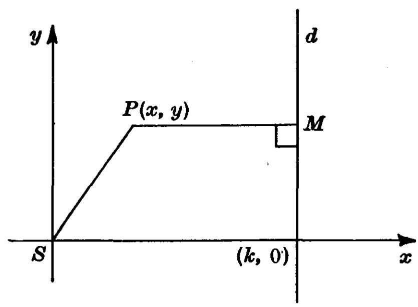

Fig. 169

On 'completing the square' with the terms involving x, this becomes

$$\left(x+\frac{e^2k}{1-e^2}\right)^2+\frac{y^2}{1-e^2} = \frac{e^4k^2}{(1-e^2)^2}+\frac{e^2k^2}{1-e^2}$$
$$= \frac{e^2k^2}{(1-e^2)^2}.$$

Change the origin to the point

$$\left(-\frac{e^2k}{1-e^2}, 0\right)$$

by the formulae

$$x = x'-\frac{e^2k}{1-e^2}, \quad y = y'.$$

Omitting dashes, the equation of the locus becomes

$$(1-e^2)x^2+y^2 = \frac{e^2k^2}{1-e^2}. \tag{i}$$

17.12 Focus-directrix property of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

In the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \tag{ii}$$

we can assume that $a > 0$ and $b > 0$, and also that $a \neq b$ (otherwise it would be the equation of the circle with centre O and radius a). Without loss of generality we can therefore always assume $a > b > 0$.

Equation (ii) will be the same as equation (i) if

$$a^2 = \frac{e^2k^2}{(1-e^2)^2} \quad \text{and} \quad b^2 = \frac{e^2k^2}{1-e^2}.$$

On dividing these we get

$$1-e^2 = \frac{b^2}{a^2}, \quad \text{i.e.} \quad e^2 = \frac{a^2-b^2}{a^2} < 1;$$

hence $0 < e < 1$.

Wo the new (dashed) axes, S has coordinates

$$\left(\frac{e^2k}{1-e^2}, 0\right) = \left(\frac{ek}{1-e^2}e, 0\right) = (ae, 0);$$

and the equation $x = k$ becomes

$$x = \frac{e^2k}{1-e^2}+k = \frac{k}{1-e^2} = \frac{a}{e}.$$

Hence (ii) is the locus of a point whose distance from $S(ae, 0)$ is e times its distance from the line $x = a/e$, where $b^2 = a^2(1-e^2)$ and so $0 < e < 1$. Therefore (ii) *is the equation of an ellipse whose eccentricity is e, whose focus is $S(ae, 0)$, and whose directrix is the line $x = a/e$.*

17.13 Focus-directrix property of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

The algebra of 17.12, with b^2 replaced by $-b^2$, shows that

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1 \tag{iii}$$

is the locus of a point whose distance from $S(ae, 0)$ is e times its distance from the line $x = a/e$, where $b^2 = a^2(e^2-1)$ and so $e > 1$. Therefore

(iii) *is the equation of a hyperbola of eccentricity* e, *focus* $S(ae, 0)$, *and directrix* $x = a/e$.

In equation (iii) we may assume $a > 0$ and $b > 0$; but we must allow $a \gtreqless b$, because (iii) is not symmetrical in x and y like (ii).

17.14 Second focus and directrix

For both curves the condition $SP = e.PM$ is expressed by

$$(x-ae)^2+y^2 = e^2\left(\frac{a}{e}-x\right)^2,$$

i.e. $$x^2-2aex+a^2e^2+y^2 = a^2-2aex+e^2x^2.$$

By adding $4aex$ to both sides, this becomes

$$(x+ae)^2+y^2 = e^2\left(\frac{a}{e}+x\right)^2,$$

which shows that the distance of $P(x,y)$ from the point $S'(-ae,0)$ is e times its distance from the line $x = -a/e$: $S'P = e.PM'$.

Hence both curves have a second focus $S'(-ae, 0)$ and a corresponding directrix $x = -a/e$.

17.15 Further definitions

From the equations, both curves are clearly symmetrical about the x-axis and about the y-axis. The origin O is the *centre* of each.

The ellipse meets Ox at the points $A(a, 0)$ and $A'(-a, 0)$, and meets Oy at $B(0, b)$ and $B'(0, -b)$. AA', BB' are called the *major* and *minor axes* of the ellipse; their lengths are $2a$, $2b$ respectively. The points A, A' are called the *vertices* of the ellipse.

The hyperbola cuts Ox at the points $A(a, 0)$ and $A'(-a, 0)$; but it does not cut Oy. AA' is called the *transverse axis*; it has length $2a$. The points A, A' are the *vertices* of the hyperbola.

For either curve, the chord through S or S' at right-angles to AA' is called a *latus rectum*.

We now consider the ellipse in detail, and leave the hyperbola until Ch. 18.

17.16 Form of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

From the equation we have

$$\frac{x^2}{a^2} = 1-\frac{y^2}{b^2} \leqslant 1,$$

so that $-a \leqslant x \leqslant a$ for all points on the curve. Similarly $-b \leqslant y \leqslant b$ for all points on the curve. Hence the ellipse lies in the rectangle whose sides are $x = \pm a$, $y = \pm b$.

When $x = a$, the equation shows that $y^2/b^2 = 0$, which has the repeated root $y = 0$. Hence the line $x = a$ *touches* the ellipse at $A(a, 0)$: it is the tangent at the vertex A. Similarly the line $x = -a$ touches the ellipse at A'; and $y = \pm b$ touch it at B, B' respectively.

Since $e < 1$, therefore $ae < a$ and so S lies between O and A; similarly S' is between O and A'. Also $a/e > a$, so the directrices cut Ox at points D, D' beyond A, A' respectively. We have

$$OS = OS' = ae, \quad OD = OD' = \frac{a}{e}. \tag{iv}†$$

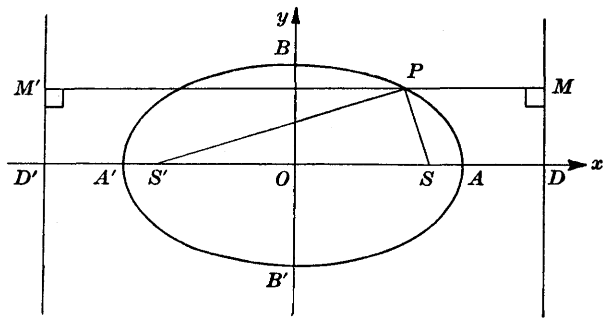

Fig. 170

17.17 Circle and ellipse

When $b = a$, the equation (ii) becomes that of the circle with centre O and radius a; but equation (i) cannot represent a circle for any choice of e and k. Hence a circle cannot be defined by the focus-directrix property, and therefore is not a 'conic' in the sense of the definition in 15.71; in particular, *a circle is not a special* (*or degenerate*) *ellipse.*

We may consider the ellipse represented by (ii) when $b \to a$. The relation $b^2 = a^2(1-e^2)$ shows that $e \to 0$; hence by (iv), S and S' approach O and the directrices recede along the x-axis. The ellipse tends to become circular in form, since

$$OP^2 = x^2+y^2 = a^2\left(1-\frac{y^2}{b^2}\right)+y^2 = a^2+\left(1-\frac{a^2}{b^2}\right)y^2 \to a^2.$$

† Thus the eccentricity $e = OS/OA$, and indeed indicates to what extent the foci are 'off-centre'.

Thus the circle with centre O and radius a is the *limit* of the ellipse; and the equation of this circle is certainly the limit when $b \to a$ of the equation of the ellipse.

17.2 Other ways of obtaining an ellipse

17.21 Auxiliary circle

The circle on diameter AA' is called the *auxiliary circle* of the ellipse. Its equation is

$$x^2+y^2 = a^2.$$

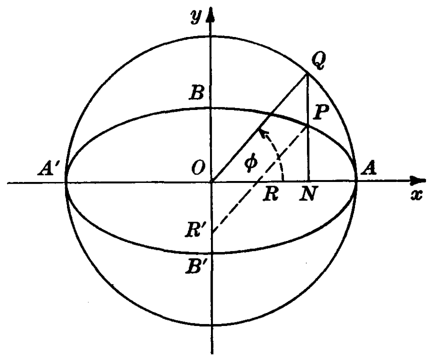

Fig. 171

Let the ordinate NP of any point P on the ellipse be produced to cut this circle at Q. Then P, Q are called *corresponding points* on the ellipse and circle. Since

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,$$

$$PN = y = \frac{b}{a}\sqrt{(a^2-x^2)}$$

$$= \frac{b}{a}\sqrt{(OQ^2-ON^2)} = \frac{b}{a}QN,$$

i.e.

$$PN:QN = b:a.$$

Conversely, *if an ordinate QN to a circle of radius a is divided at P so that $PN:QN = b:a$, then the locus of P as Q varies on the circle is an ellipse.* For if Q is the point (x, Y), then $x^2+Y^2 = a^2$; and P is the point (x, y) where $y = bY/a$, so the coordinates of P satisfy

$$x^2+\frac{a^2}{b^2}y^2 = a^2, \quad \text{i.e.} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

Example

The elliptic trammel.

Through P draw the line parallel to QO to meet Ox, Oy at R, R'; then $PR' = QO = a$. From the similar triangles RPN, OQN,

$$\frac{PR}{QO} = \frac{PN}{QN},$$

and hence $PR = b$.

If a rod PRR' has pegs on its under-side at R, R', and these pegs are made to slide in perpendicular grooves AOA', BOB', the point P will trace an ellipse whose major and minor semi-axes are PR', PR respectively.

17.22 Focal distances

The lengths SP, $S'P$ are the *focal distances* of the point P on the ellipse. Since $SP = e.PM$ and $S'P = e.PM'$ (fig. 170),

$$SP+S'P = e(PM+PM') = e.DD' = e\left(\frac{2a}{e}\right) = 2a.$$

Hence *the sum of the focal distances of a point on the ellipse is* $2a$ (the *bifocal property*).

Conversely, *the locus of a point P such that* $SP+S'P =$ *constant is an ellipse with foci* S, S'.

Proof. Choose the mid-point of SS' for origin, and take Ox along $S'S$; then let S be $(c, 0)$ and S' be $(-c, 0)$. Supposing that

$$SP+S'P = 2a,$$

any point $P(x, y)$ on the locus satisfies

$$\sqrt{\{(x-c)^2+y^2\}}+\sqrt{\{(x+c)^2+y^2\}} = 2a,$$

i.e. $$\sqrt{\{(x+c)^2+y^2\}} = 2a-\sqrt{\{(x-c)^2+y^2\}}.$$

Squaring, $$(x+c)^2+y^2 = (x-c)^2+y^2+4a^2-4a\sqrt{\{(x-c)^2+y^2\}},$$

i.e. $$4a\sqrt{\{(x-c)^2+y^2\}} = 4a^2-4cx.$$

Again squaring,

$$a^2\{x^2-2cx+c^2+y^2\} = a^4-2a^2cx+c^2x^2,$$

hence $$\left(1-\frac{c^2}{a^2}\right)x^2+y^2 = a^2-c^2,$$

i.e. $$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} = 1.$$

From triangle PSS', $SS' < PS+PS' = 2a$, i.e. $c < a$ and so a^2-c^2 is positive. The above equation therefore represents an ellipse with

$b^2 = a^2 - c^2$. If e is its eccentricity, then since $a^2 - b^2 = a^2e^2$ by 17.12, we have $c = ae$. Hence S, S' are the points $(\pm ae, 0)$, i.e. the foci.

Example

Mechanical description of the ellipse: 'pin-construction'.

Fix pins S, S' at distance $2ae$ apart. Tie the ends of a string of length $2a$ at S and S'. A pencil point P moving so that the string $S'PS$ is kept taut will trace the ellipse with foci S, S', eccentricity e, and major axis $2a$.

17.23 Orthogonal projection of a circle

Given two planes α, α' which intersect in a line l, consider perpendiculars dropped from points of α onto α'. If the perpendicular from P in α has foot P' in α', then P' is called the *orthogonal projection* of P on α'. If P lies on a locus Σ in α, then the corresponding locus of P' in α' is called the orthogonal projection of Σ on α'.

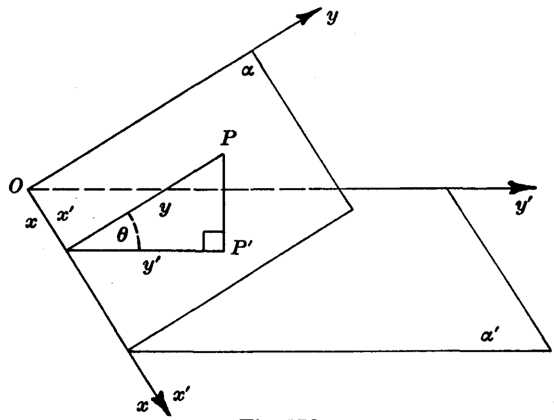

Fig. 172

We now obtain formulae relating the coordinates of P and P'. Choose the common line l for x-axis in both planes, and take any point O on it for origin. Draw lines Oy, Oy' in α, α' at right-angles to Ox. Wo these axes, suppose $P(x, y)$ in α projects into $P'(x', y')$ in α'. If θ is the angle between the planes α, α', i.e. between Oy and Oy', then

$$x' = x \quad \text{and} \quad y' = y\cos\theta.$$

The circle $x^2 + y^2 = a^2$ in α therefore projects into the curve

$$x'^2 + \frac{y'^2}{\cos^2\theta} = a^2$$

in α', i.e. into

$$\frac{x'^2}{a^2} + \frac{y'^2}{a^2\cos^2\theta} = 1.$$

Choosing θ so that $\cos\theta = b/a$ ($b < a$), this equation becomes

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2} = 1.$$

Hence *the ellipse*
$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

is the orthogonal projection of a circle of radius a onto a plane making angle $\cos^{-1}(b/a)$ *with the plane of the circle.*

If we rotate the plane xOy about Ox until it coincides with plane $x'Oy'$, we obtain fig. 171 of the ellipse and its auxiliary circle.

17.24 General properties of orthogonal projection

(1) *Effect on angles.* A line projects into a line, since $lx+my+n = 0$ becomes $lx'+m\sec\theta y'+n = 0$. This line and its projection cut Ox at the same point, viz. $(-n/l, 0)$, and make angles ϕ, ϕ' with Ox, where

$$\tan\phi = -\frac{l}{m} \quad\text{and}\quad \tan\phi' = -\frac{l}{m\sec\theta}.$$
$$\therefore \quad \tan\phi' = \tan\phi\cos\theta.$$

Hence *parallel lines project into parallel lines*, but *in general angles are changed.* However, a line perpendicular or parallel to Ox projects into another such line.

(2) *A tangent to a curve projects into the tangent to the projected curve at the corresponding point.* The tangent to $y=f(x)$ at (x_1, y_1) is

$$y-y_1 = f'(x_1).(x-x_1),$$

and projects into
$$(y'-y_1')\sec\theta = f'(x_1').(x'-x_1'),$$

i.e.
$$y'-y_1' = f'(x_1')\cos\theta.(x'-x_1').$$

This is the tangent at (x_1', y_1') to the curve $y' = f(x')\cos\theta$, which is the projection of $y=f(x)$.

By (1), *a normal does not in general project into a normal.*

(3) *Effect on lengths.* If PQ in α has length r and makes angle ϕ with Ox, let its projection $P'Q'$ on α' make angle ϕ' with Ox. Through O draw OP_1 equal and parallel to PQ; then P_1 is $(r\cos\phi, r\sin\phi)$ and so its projection P_1' is

$$(r\cos\phi, r\sin\phi\cos\theta).$$

Hence
$$OP_1'^2 = r^2(\cos^2\phi+\sin^2\phi\cos^2\theta)$$
$$= r^2(1-\sin^2\theta\sin^2\phi);$$

and also $OP_1' = r\cos P_1OP_1'$.

Since by (1) OP_1' is parallel to $P'Q'$, therefore P_1OP_1' is the angle between PQ and $P'Q'$, so that $P'Q' = PQ\cos P_1OP_1'$, i.e.

$$P'Q' = PQ\sqrt{(1-\sin^2\theta\sin^2\phi)}. \qquad \text{(i)}$$

Hence, *in general, lengths are changed by orthogonal projection*; they are conserved only on lines parallel to Ox.

Formula (i) also shows that the *ratio* of lengths on parallel lines (or on the same line) is unaltered; in particular, *mid-points project into mid-points*. On intersecting lines the ratio of lengths is altered.

(4) *Effect on areas*. Since

$$A' = \int_a^b y'\,dx' = \int_a^b (y\cos\theta)\,dx = A\cos\theta,$$

orthogonal projection reduces areas in the ratio $\cos\theta : 1$.

(5) By means of these results and that in 17.23, properties of the ellipse can be deduced from certain properties of the circle by orthogonal projection. In general only *central* properties can be obtained in this way; we cannot expect to prove focal properties thus, since foci are foreign to the circle (see 17.17).

Example

PN is the ordinate of P; the tangent at P meets the major axis at T. Prove $ON.OT = OA^2$.

First sketch the corresponding figure for the circle. In it, $OP \perp PT$, and by geometry $ON.OT = OP^2 = OA^2$, which can be written $ON:OA = OA:OT$ and involves the ratios of lengths *on the same line*.

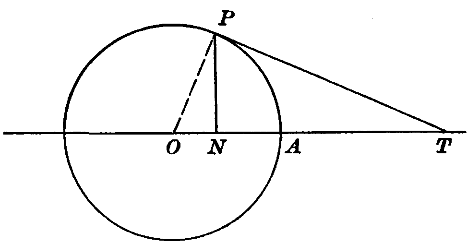

Fig. 173

If the figure is projected orthogonally onto a plane through the line OT, this relation will hold for the new figure, and is the required property of an ellipse; for since $PN \perp OT$, the ordinate PN to the circle will project into an ordinate to the ellipse.

N.B.—For the circle it is true that $ON.OT = OP^2$, but this cannot be generalised into a property of an ellipse because the lengths concerned do not lie along the same or parallel lines.

17.3 Parametric representation

17.31 Eccentric angle ϕ

Let Q be the point on the auxiliary circle corresponding to the point P on the ellipse (17.21), and let $\phi = N\hat{O}Q$ (called the *eccentric* *angle* of P). Then $x = ON = a\cos\phi$; and since $NQ = a\sin\phi$, we have $y = (b/a)\,NQ = b\sin\phi$. Hence *the coordinates of each point on the ellipse* $x^2/a^2+y^2/b^2 = 1$ *can be expressed in the form* $(a\cos\phi, b\sin\phi)$.

As ϕ increases from 0 to 2π, Q describes the auxiliary circle and P describes the ellipse. Two values of ϕ which differ by an integral multiple of 2π give the same point of the ellipse.

Conversely, for each ϕ the point $(a\cos\phi, b\sin\phi)$ lies on the ellipse, as is clear by substituting in the equation. The point whose coordinates are $(a\cos\phi, b\sin\phi)$ is referred to as *the point* ϕ.

17.32 The point t

By putting $t = \tan\frac{1}{2}\phi$ in the coordinates of the point ϕ,

$$x = a\cos\phi = a\frac{1-t^2}{1+t^2}, \quad y = b\sin\phi = b\frac{2t}{1+t^2}.$$

Hence for each t, the point

$$\left(a\frac{1-t^2}{1+t^2}, \frac{2bt}{1+t^2}\right)$$

lies on the ellipse $x^2/a^2+y^2/b^2 = 1$; we refer to it as *the point* t.

Each value of t determines just one point of the ellipse; but the point $A'(-a, 0)$ is not given by any value of t, although it can be obtained as the limit when $t \to \infty$.

This algebraic representation can also be obtained directly; for since

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2} = \left(1-\frac{x}{a}\right)\left(1+\frac{x}{a}\right),$$

therefore

$$\frac{y/b}{1+x/a} = \frac{1-x/a}{y/b} = t, \quad \text{say.}$$

Hence

$$\left(1-\frac{x}{a}\right):\frac{y}{b}:\left(1+\frac{x}{a}\right) = t^2:t:1, \qquad \text{(i)}$$

and so

$$\frac{x}{a}:\frac{y}{b}:1 = (1-t^2):2t:(1+t^2).$$

Example

Concyclic points on the ellipse.

The circle $x^2+y^2+2gx+2fy+c = 0$ cuts the ellipse at the points t for which

$$a^2\left(\frac{1-t^2}{1+t^2}\right)^2+b^2\left(\frac{2t}{1+t^2}\right)^2+2ga\frac{1-t^2}{1+t^2}+2fb\frac{2t}{1+t^2}+c = 0,$$

i.e. $$(a^2-2ga+c)\,t^4+4fbt^3+2(2b^2-a^2+c)\,t^2+4fbt+(a^2+2ga+c) = 0. \qquad \text{(ii)}$$

Since this is quartic in t, *a circle and ellipse can cut in at most four points*; the actual number of intersections will be 4, 2, or 0 (some of which may coincide), since a quartic has 4, 2 or no roots.

Suppose the above circle cuts the ellipse at the points t_1, t_2, t_3, t_4; then these numbers are the roots of (ii). Since the coefficients of t^3 and t are equal, therefore

$$-\Sigma t_1 = -\Sigma t_2 t_3 t_4.$$

Now by 14.23, (viii),

$$\tan \tfrac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4) = \frac{\Sigma t_1 - \Sigma t_2 t_3 t_4}{1-\Sigma t_1 t_2 + t_1 t_2 t_3 t_4}, \qquad \text{(iii)}$$

which is zero by the above condition. Hence $\Sigma \frac{1}{2}\phi_1 = n\pi$ for some integer n (positive, negative, or zero). Thus *if the points* $\phi_1, \phi_2, \phi_3, \phi_4$ *of the ellipse are concyclic, then* $\Sigma\phi_1 = 2n\pi$.

This necessary condition for concyclic points is also sufficient: see Ex. 17 (*a*), no. 11.

Exercise 17(*a*)

'The ellipse' means the locus $x^2/a^2 + y^2/b^2 = 1$.

1 Find the length of a latus rectum of the ellipse.

2 With the notation of 17.15, P is any point on the ellipse and the perpendicular from P to AA' has foot N; prove that $PN^2 : A'N.NA = OB^2 : OA^2$. Find a similar relation involving N', the foot of the perpendicular from P to BB'.

3 If P is the point (x, y) on the ellipse, prove $SP = a - ex$ and $S'P = a + ex$.

4 Use the bifocal property to prove that $BS = BS' = a$.

Given an ellipse and its major and minor axes AA', BB', construct the foci S, S' geometrically.

5 A rod AB moves with its ends on two fixed perpendicular lines; P is fixed in the rod, and $BP = a$, $PA = b$. Find the locus of P.

6 Prove that the gradient of the chord $\theta\phi$ is $-(b/a)\cot\frac{1}{2}(\theta+\phi)$.

7 Prove that the gradient of the chord $t_1 t_2$ is $-b(1-t_1 t_2)/a(t_1+t_2)$.

8 Show that the locus of the mid-point of SP is an ellipse whose centre is the mid-point of OS. State the lengths of the semi-axes.

9 Prove that the mid-point of the chord $\theta\phi$ has coordinates

$$\left(a\cos\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi),\ b\sin\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)\right).$$

10 Find the locus of the mid-point of the chord $\theta\phi$ when (i) $\theta+\phi = 2\alpha$; (ii) $\theta-\phi = 2\alpha$, where α is constant.

11 Prove that the condition $\Sigma\phi_1 = 2n\pi$ is *sufficient* for the points $\phi_1, \phi_2, \phi_3, \phi_4$ of the ellipse to be concyclic. [The circle through ϕ_1, ϕ_2, ϕ_3 cuts the ellipse again at ϕ_4'; prove $\phi_4' - \phi_4$ is a multiple of 2π, so that ϕ_4' and ϕ_4 give the same point. Cf. 16.12, ex. (ii).]

12 A circle cuts an ellipse at A, B, C, D. Prove that the chords AB, CD are equally inclined to the major axis (and therefore also to the minor axis). [Use 17.32, ex. and no. 6.]

13 *If a circle touches an ellipse at A and cuts it again at C and D, prove that AC, AD are equally inclined to each axis.*

14 Circles touch an ellipse at a given point. Prove that their common chords not through that point are parallel.

*15 (i) Find the point where the circle of curvature at ϕ meets the ellipse again.

(ii) If the points $\phi_1, \phi_2, \phi_3, \phi_4$ are concyclic, and the circles of curvature at these points cut the ellipse again at $\phi_1', \phi_2', \phi_3', \phi_4'$, prove these points are also concyclic.

*16 With coordinate axes as in 17.23, show that the orthogonal projection of the parabola $y^2 = 4ax$ is another parabola.

*17 Prove that the tangent to the circle $x^2+y^2 = a^2$ at the point $(a\cos\phi, a\sin\phi)$ has equation $x\cos\phi + y\sin\phi = a$. By projection deduce the equation of the tangent to the ellipse at the point ϕ.

*18 M is the mid-point of a chord HK of an ellipse whose centre is O; OM meets the ellipse at P; N is the mid-point of PH; and ON, HK meet at Q. Prove that OH bisects the chord which passes through P and Q. [In the circle figure, Q is the orthocentre of triangle OPH.]

*19 The chord PQ has mid-point M; the tangents at P, Q meet at T. Prove that O, M, T are collinear. If this line meets the ellipse at R between O and T, prove $OM.OT = OR^2$.

*20 Deduce the area of the ellipse with semi-axes a, b from that of a circle of radius a.

Further examples using orthogonal projection appear in Ex. 17 (*d*).

17.4 Chord and tangent

17.41 Chord P_1P_2

The line joining P_1 and P_2 has equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Since P_1 and P_2 lie on the ellipse,

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} = 1 \quad\text{and}\quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2} = 1.$$

By subtracting and then rearranging,

$$\frac{y_2^2 - y_1^2}{b^2} = -\frac{x_2^2 - x_1^2}{a^2},$$

so that
$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{b^2}{a^2}\frac{x_2 + x_1}{y_2 + y_1}.$$

The equation of the chord P_1P_2 therefore becomes

$$y - y_1 = -\frac{b^2}{a^2}\frac{x_2 + x_1}{y_2 + y_1}(x - x_1),$$

i.e.
$$\frac{(x-x_1)(x_2+x_1)}{a^2}+\frac{(y-y_1)(y_2+y_1)}{b^2} = 0,$$

i.e.
$$\frac{x}{a^2}(x_1+x_2)+\frac{y}{b^2}(y_1+y_2) = \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+1. \tag{i}$$

Alternatively, consider the equation

$$\frac{(x-x_1)(x-x_2)}{a^2}+\frac{(y-y_1)(y-y_2)}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1.$$

When simplified it is linear in x and y, and therefore represents a line. It is satisfied by the points (x_1, y_1), (x_2, y_2) on the ellipse, and so must be the chord P_1P_2.

17.42 Chord $\theta\phi$

The chord joining $(a\cos\phi,\ b\sin\phi)$ and $(a\cos\theta,\ b\sin\theta)$ has equation

$$y-b\sin\phi=\frac{b(\sin\theta-\sin\phi)}{a(\cos\theta-\cos\phi)}(x-a\cos\phi)$$

$$=\frac{2b\cos\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)}{-2a\sin\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)}(x-a\cos\phi)$$

$$=-\frac{b\cos\tfrac{1}{2}(\theta+\phi)}{a\sin\tfrac{1}{2}(\theta+\phi)}(x-a\cos\phi),$$

i.e.

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\phi)+\frac{y}{b}\sin\tfrac{1}{2}(\theta+\phi)=\cos\phi\cos\tfrac{1}{2}(\theta+\phi)+\sin\phi\sin\tfrac{1}{2}(\theta+\phi)$$

$$=\cos\{\phi-\tfrac{1}{2}(\theta+\phi)\},$$

i.e.
$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\phi)+\frac{y}{b}\sin\tfrac{1}{2}(\theta+\phi)=\cos\tfrac{1}{2}(\theta-\phi). \qquad \text{(ii)}$$

17.43 Chord t_1t_2

The line $ux+vy+1=0$ cuts the ellipse at points t for which

$$ua\frac{1-t^2}{1+t^2}+vb\frac{2t}{1+t^2}+1=0,$$

i.e.
$$(1-ua)t^2+2vbt+(1+ua)=0.$$

This line will be the chord t_1t_2 if the above quadratic has roots t_1 and t_2, and then

$$t_1t_2=\frac{1+ua}{1-ua},\quad t_1+t_2=-\frac{2vb}{1-ua}.$$

Hence
$$\frac{t_1t_2-1}{t_1t_2+1}=ua,$$

and
$$vb=-\tfrac{1}{2}(t_1+t_2)\frac{2}{t_1t_2+1}=-\frac{t_1+t_2}{t_1t_2+1}.$$

The required chord is therefore

$$(1-t_1t_2)\frac{x}{a}+(t_1+t_2)\frac{y}{b} = 1+t_1t_2. \qquad \text{(iii)}$$

Alternatively, the chord t_1t_2 cuts the ellipse at the points given by $t = t_1$, $t = t_2$ which are the roots of the quadratic $(t-t_1)(t-t_2) = 0$, i.e.

$$t^2-(t_1+t_2)t+t_1t_2 = 0. \qquad \text{(iv)}$$

Consider the equation constructed from (iv) by substituting for $t^2:t:1$ from (i) in 17.32:

$$\left(1-\frac{x}{a}\right)-(t_1+t_2)\frac{y}{b}+t_1t_2\left(1+\frac{x}{a}\right) = 0.$$

It is linear in x and y, and so represents a line. It cuts the ellipse at points t given by (iv), i.e. at $t = t_1$, t_2, and therefore is the chord t_1t_2.

17.44 Tangent at P_1

This can be obtained by making $x_2 \to x_1$ and $y_2 \to y_1$ in equation (i), giving

$$\frac{2xx_1}{a^2}+\frac{2yy_1}{b^2} = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+1 = 2,$$

since P_1 lies on the ellipse. The tangent at P_1 is therefore

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1, \qquad \text{(v)}$$

which can be written down by the usual 'rule of alternate suffixes'.

Alternatively, equation (v) can be found directly by calculus.

17.45 Tangent at ϕ

Letting $\theta \to \phi$ in equation (ii), we obtain

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1. \qquad \text{(vi)}$$

This could also be obtained directly by calculus. Cf. Ex. 17 (*a*), no. 17.

17.46 Tangent at t

Letting $t_2 \to t_1$ in equation (iii), we obtain (after omitting the suffix 1)

$$(1-t^2)\frac{x}{a}+2t\frac{y}{b} = 1+t^2 \qquad \text{(vii)}$$

as the equation of the tangent at the point t.

Alternatively, this can be found by using the theory of quadratic equations, or by calculus, or as a particular case of the result in Ex. 16(*e*), no. 28.

17.47 Examples

(i) *Condition for* $y = mx + c$ *to touch* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The line cuts the ellipse at points for which

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1,$$

i.e.
$$(a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0.$$

The line will touch the ellipse if and only if this quadratic in x has equal roots, i.e.

$$a^4m^2c^2 = a^2(a^2m^2 + b^2)(c^2 - b^2),$$

i.e.
$$c^2 = a^2m^2 + b^2.$$

Hence for all values of m the lines

$$y = mx \pm \sqrt{(a^2m^2 + b^2)}$$

touch the ellipse.

(ii) *Locus of the intersection of perpendicular tangents: director circle.*

If a tangent found in ex. (i) passes through the point (x_0, y_0), then

$$y_0 = mx_0 \pm \sqrt{(a^2m^2 + b^2)},$$

i.e.
$$(x_0^2 - a^2)m^2 - 2x_0y_0m + (y_0^2 - b^2) = 0.$$

Since this is quadratic in m, *in general two tangents can be drawn from a given point* (x_0, y_0) *to the ellipse*. The roots m_1, m_2 are the gradients of such tangents.

These tangents will be perpendicular if $m_1m_2 = -1$, i.e. if

$$\frac{y_0^2 - b^2}{x_0^2 - a^2} = -1,$$

i.e. if (x_0, y_0) lies on the locus

$$x^2 + y^2 = a^2 + b^2.$$

This is a circle concentric with the ellipse, and having radius $\sqrt{(a^2+b^2)} = AB$ (fig. 170). It is the locus of points from which perpendicular tangents can be drawn to the ellipse, and is called the *director circle* of the ellipse (by analogy with the parabola, for which the corresponding locus is the directrix: see Ex. 16(*b*), no. 11(ii), (iii)).

(iii) *Chord of contact of tangents from* P_1. The argument used in 15.64 will show that the chord of contact from P_1 to the ellipse has equation

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

Exercise 17(*b*)

1 Use equation (i) in 17.41 to prove that the mid-points of all chords of given gradient m lie on the line $y = -b^2x/a^2m$.

2 Find the condition for the chord $\theta\phi$ to pass through the centre of the ellipse.

3 Prove that $\theta\phi$ is a focal chord if and only if $\pm e\cos\frac{1}{2}(\theta+\phi) = \cos\frac{1}{2}(\theta-\phi)$. Verify that this is equivalent to $\tan\frac{1}{2}\theta\tan\frac{1}{2}\phi = (e-1)/(e+1)$ or $(e+1)/(e-1)$.

4 Prove that the chord PP' of the ellipse and the chord joining the corresponding points Q, Q' of the auxiliary circle meet on the major axis. Deduce a property of tangents at corresponding points P, Q.

5 If $\theta+\phi = 2\alpha$, prove that the chord $\theta\phi$ is parallel to the tangent at α, and conversely.

6 For a system of parallel chords prove that the sum of the eccentric angles of the extremities is constant.

7 Find the envelope of the chords joining points of the ellipse whose eccentric angles differ by a constant. [Let the angles be $\phi-\alpha$, $\phi+\alpha$; the chord is $(x/a)\cos\phi+(y/b)\sin\phi = \cos\alpha$, which is a tangent to the ellipse $x^2/a^2+y^2/b^2 = \cos^2\alpha$.]

8 *Prove that* $lx+my+n = 0$ *touches* $x^2/a^2+y^2/b^2 = 1$ *if and only if* $a^2l^2+b^2m^2 = n^2$. [Method of 17.47, ex. (i); *or* compare with 17.45, equation (vi).]

9 If m is the gradient of a common tangent to the circle $x^2+y^2 = c^2$ and the ellipse $x^2/a^2+y^2/b^2 = 1$, prove $m^2 = (c^2-b^2)/(a^2-c^2)$. Deduce that four common tangents exist only if $b < c < a$.

10 Prove that tangents from $(-2, -3)$ to $4x^2+9y^2 = 36$ are perpendicular.

11 PN is the ordinate of a point P on the ellipse, and the tangent at P meets the major axis at T. Prove $ON.OT = a^2$, and obtain the corresponding property for the minor axis. (Cf. 17.24 (5), ex.)

12 The tangent at P meets the directrix at Z. Prove $P\hat{S}Z$ is a right-angle.

13 (i) If Q is the point of contact of the other tangent from Z in no. 12, prove geometrically that PSQ is a straight line.

(ii) Deduce that tangents at the extremities of a focal chord meet on the corresponding directrix.

14 *Prove that the focal radii SP, $S'P$ are equally inclined to the tangent at P.* (Cf. 8.14, ex. (iv); and also Ex. 17 (*c*), no. 3.)

15 If, p, p' are the lengths of the perpendiculars from the foci S, S' to any tangent, prove that $pp' = b^2$.

16 Using the similar triangles SPY, $S'PY'$, prove $p/r = p'/r'$, where Y, Y' are the feet of the perpendiculars in no. 15 and $r = SP$, $r' = S'P$.

17 From nos. 15, 16 and the bifocal property $r+r' = 2a$ deduce that *the* (p, r) *equation* (8.2) *of the ellipse wo the focus S as pole is* $b^2/p^2 = 2a/r-1$.

18 Prove that the points Y, Y' in no. 16 lie on the auxiliary circle. [The perpendicular from $S(ae, 0)$ to

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1 \quad \text{is} \quad \frac{x}{b}\sin\phi-\frac{y}{a}\cos\phi = \frac{ae}{b}\sin\phi;$$

square and add these equations.]

19 The chord PQ subtends a right-angle at the vertex $A(a, 0)$. Prove that PQ passes through a fixed point on the major axis. [Use equation (iii) in 17.43.]

20 Prove that the chord joining the points given by the roots of $ut^2+vt+w = 0$ has equation $(w-u)x/a+vy/b+(w+u) = 0$.

21 Prove that the tangents at the points t_1, t_2 meet at (x, y), where

$$\frac{1-x/a}{t_1t_2} = \frac{1+x/a}{1} = \frac{y/b}{\frac{1}{2}(t_1+t_2)}.$$

[Use equation (vii) in 17.46, and theory of quadratics.]

22 Write down the equation of the chord of contact from $(1, 2)$ to $\frac{1}{8}x^2+\frac{1}{3}y^2 = 1$, and deduce the equations of the tangents from this point.

23 The chord PQ of the ellipse $x^2/a^2+y^2/b^2 = 1$ touches the circle $x^2+y^2 = c^2$. Prove that the tangents at P, Q meet at a point T on the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4} = \frac{1}{c^2}.$$

[The chord of contact from $T(x_1, y_1)$ to the ellipse touches the circle; use no. 8.]

24 Prove that the tangents at the extremities of a variable chord through P_1 meet on the line $xx_1/a^2+yy_1/b^2 = 1$. What follows by taking P_1 at a focus?

*25 Given the points P_1, P_2, prove that the point dividing P_1P_2 in the ratio $k:l$ will be on $x^2/a^2+y^2/b^2=1$ if

$$\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-1\right)k^2+2\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1\right)kl+\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)l^2=0.$$

(*Joachimsthal's ratio quadratic* for the ellipse).

*26 By taking P_1 on the ellipse, deduce the equation of the tangent at P_1. [See Ex. 16(*b*), no. 26.]

*27 Obtain from no. 25 the equation of the pair of tangents from P_1. [See 15.64, ex. (ii).] Deduce the equation of the director circle of the ellipse by using the condition for these lines to be perpendicular (15.52 (2)).

*28 A variable chord through P_1 cuts the ellipse at A and B, and P_2 is chosen so that P_1 and P_2 divide AB (one internally and the other externally) in the same ratio. Prove that P_2 lies on the line $xx_1/a^2+yy_1/b^2 = 1$. [Method of 15.65 (2).]

*29 (i) Defining the *polar* of P_1 wo $x^2/a^2+y^2/b^2 = 1$ to be the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1,$$

prove that if the polar of P_1 passes through P_2, then the polar of P_2 passes through P_1.

(ii) Prove that the polar of a focus is the corresponding directrix.

(iii) Verify that the centre has no polar.

17.5 Normal

17.51 Equation of the normal

The reader should verify that the normals to the ellipse at P_1, ϕ, and t have respective equations

$$\frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{y_1/b^2}, \tag{i}$$

$$ax\sec\phi - by\operatorname{cosec}\phi = a^2-b^2, \tag{ii}$$

$$2atx - b(1-t^2)y = 2(a^2-b^2)\frac{t(1-t^2)}{1+t^2}. \tag{iii}$$

17.52 Conormal points

The normal at t passes through the point (h, k) if

$$2ath - b(1-t^2)k = 2(a^2-b^2)\frac{t(1-t^2)}{1+t^2},$$

i.e.
$$bkt^4 + 2(ah+a^2-b^2)t^3 + 2(ah-a^2+b^2)t - bk = 0. \tag{iv}$$

Since this equation is quartic in t, *at most four normals can be drawn from a given point* (h, k) *to the ellipse*; and (apart from coincidences) there will be either 4, 2, or no such normals.

Suppose that the normals at t_1, t_2, t_3, t_4 meet at (h, k); then these numbers must be the roots of (iv). Since the term in t^2 is absent, and the constant term is minus the coefficient of t^4, we have

$$\Sigma t_1 t_2 = 0 \quad \text{and} \quad t_1 t_2 t_3 t_4 = -1. \tag{v}$$

These *necessary* conditions for the points t_1, t_2, t_3, t_4 of the ellipse to be conormal (i.e. such that the normals at them are concurrent) can also be shown to be *sufficient*. This is reasonable, because *two* independent conditions are required in order that *four* lines shall pass through the same point.

Example

From formula (iii) in 17.32 it follows that, when conditions (v) are satisfied, then $\Sigma\frac{1}{2}\phi_1 = (n+\frac{1}{2})\pi$ for some integer n (positive, negative, or zero). Hence $\Sigma\phi_1 = (2n+1)\pi$ *is a necessary condition for the points* $\phi_1, \phi_2, \phi_3, \phi_4$ *to be conormal.* Clearly this single condition cannot be sufficient. The example in 17.32 shows that four distinct conormal points can never be concyclic.

Exercise 17(c)

1 The normal at P meets the major axis at G, and PN is the ordinate at P. Prove that $OG = e^2.ON$. What is the corresponding result for the minor axis?

2 Prove that $SG = e.SP$.

3 Use no. 2 to prove $SG/S'G = SP/S'P$, and hence that PG bisects $S\hat{P}S'$. Deduce that the tangent at P is equally inclined to SP, $S'P$. (Cf. Ex. 17 (b), no. 14.)

4 Find the locus of the mid-point of PG.

5 Prove that $3x-y-1=0$ is a normal to $2x^2+3y^2=14$, and find the coordinates of its foot.

6 If $lx+my+n=0$ is a normal to $x^2/a^2+y^2/b^2=1$, prove that

$$\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}.$$

7 PN, PN' are the perpendiculars to the axes from a point P of the ellipse $x^2/a^2+y^2/b^2=1$. Prove that NN' is always normal to a fixed concentric ellipse, and give the equation of this curve.

*8 (i) If the perpendicular from the vertex A to the normal at the point ϕ meets the ellipse again at the point ϕ', use Ex. 17 (b), no. 5 to prove $\phi' = 2\phi$.

(ii) Deduce that the perpendiculars from a vertex to four concurrent normals meet the ellipse again in four concyclic points.

*9 Prove that the circle through any three of four conormal points cuts the ellipse again at the point diametrically opposite to the fourth. [If ϕ_4, ϕ_4' are diametrically opposite, then $\phi_4' = \phi_4+\pi$ by Ex. 17 (b), no. 2. Since

$$\Sigma\phi_1 = (2n+1)\pi, \quad \text{therefore} \quad \phi_1+\phi_2+\phi_3+\phi_4' = (2n+2)\pi.]$$

10 Prove that the feet of normals drawn from P_1 to $x^2/a^2+y^2/b^2=1$ lie on the curve $(a^2-b^2)xy+b^2y_1x-a^2x_1y=0$. Verify that this locus passes through P_1 and the centre of the ellipse. [Method of 16.32, ex. (iii) (b).]

*11 Prove that the curves $a^2x_1y=(a^2-b^2)xy+b^2y_1x$ and $b^2x^2+a^2y^2=a^2b^2$ intersect at points for which y satisfies

$$b^2(a^2x_1y)^2 = b^2x^2\{(a^2-b^2)y+b^2y_1\}^2 = a^2(b^2-y^2)\{(a^2-b^2)y+b^2y_1\}^2,$$

and that to each root of this quartic in y corresponds a single x given by $\{(a^2-b^2)y+b^2y_1\}x = a^2x_1y$. Deduce that in general four normals can be drawn from P_1 to the ellipse.

17.6 The distance quadratic

17.61 The line through $P_1(x_1, y_1)$ in direction θ (15.26) has parametric equations

$$x = x_1+r\cos\theta, \quad y = y_1+r\sin\theta.$$

It cuts the ellipse $x^2/a^2+y^2/b^2=1$ at points for which r satisfies

$$b^2(x_1+r\cos\theta)^2+a^2(y_1+r\sin\theta)^2 = a^2b^2,$$

i.e. $$(a^2\sin^2\theta+b^2\cos^2\theta)r^2+2(a^2y_1\sin\theta+b^2x_1\cos\theta)r + (b^2x_1^2+a^2y_1^2-a^2b^2) = 0. \quad \text{(i)}$$

This quadratic in r gives the (signed) distances from P_1 of the points on the ellipse which lie on the line through P_1 in direction θ, and is called the *distance quadratic* for the ellipse.

Example*

Newton's theorem.

The product of the roots of (i) is

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\Big/\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right).$$

For chords P_1QR, $P_1Q'R'$ drawn in directions θ, θ' it follows that

$$\frac{P_1Q.P_1R}{P_1Q'.P_1R'} = \left(\frac{\cos^2\theta'}{a^2}+\frac{\sin^2\theta'}{b^2}\right)\Big/\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right),$$

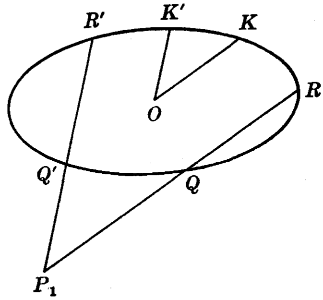

Fig. 174

which is independent of x_1 and y_1. Hence if chords P_1QR, $P_1Q'R'$ are drawn through P_1 *in fixed directions* θ, θ', the ratio $P_1Q.P_1R:P_1Q'.P_1R'$ is independent of P_1.

In particular, when P_1 is at the centre O the ratio becomes $OK^2:OK'^2$, where the radii OK, OK' lie in directions θ, θ'. Hence *Newton's theorem*:

$$\frac{P_1Q.P_1R}{P_1Q'.P_1R'} = \frac{OK^2}{OK'^2}.$$

It is a generalisation of the 'product property' of chords of a circle, and reduces to this property when $b = a$. Also see Ex. 17 (*d*), no. 16.

By letting $R \to Q$ and $R' \to Q'$ we see that the ratio of the tangents from P_1 is equal to the ratio of the parallel radii.

17.62 Chord having mid-point P_1

If the chord through P_1 in direction θ is bisected at P_1, then the roots of the distance quadratic will be equal and opposite, so that

$$b^2x_1\cos\theta + a^2y_1\sin\theta = 0.$$

This determines the direction $\cos\theta:\sin\theta$ of the chord which is bisected at P_1. Its equation

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta}$$

therefore becomes

$$b^2x_1(x-x_1)+a^2y_1(y-y_1) = 0,$$

i.e.

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}. \qquad \text{(ii)}$$

17.63 Diameters

If the chord (ii) above has gradient m, then $m = -b^2x_1/a^2y_1$. Hence the mid-points of all chords of gradient m satisfy

$$y = -\frac{b^2}{a^2m}x.$$

Cf. Ex. 17 (*b*), no. 1. According to the definition in 16.41, this locus is the diameter† bisecting all chords of given gradient m. It is a straight line which passes through the centre O of the ellipse, and is thus a 'diameter' in the usual sense.

Examples

(i) *Tangents at the extremities of a diameter are parallel to the chords which the diameter bisects* (the *ordinates* to the diameter).

Let P_1 be a point where the ellipse is met by the diameter which bisects all chords of gradient m; then $y_1 = -b^2x_1/a^2m$.

The tangent at P_1 (17.44) has gradient $-b^2x_1/a^2y_1$, i.e. m. The result follows.

(ii) *Tangents at the extremities of any chord meet on the diameter which bisects that chord.*

Let the chord PQ have gradient m, and let the tangents at P and Q meet at (x_1, y_1). Since PQ is the chord of contact from (x_1, y_1), its equation is $xx_1/a^2+yy_1/b^2 = 1$, and hence its gradient is $m = -b^2x_1/a^2y_1$. Therefore (x_1, y_1) lies on $y = -b^2x/a^2m$, which is the diameter bisecting PQ.

(iii) *The sum of the squared reciprocals of two perpendicular semi-diameters is constant.*

Choosing O for pole and Ox for initial line, the ellipse $x^2/a^2+y^2/b^2 = 1$ has polar equation

$$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right) = 1.$$

If P on the ellipse has polar coordinates (r_1, θ_1), then the coordinates of an extremity Q of the perpendicular semi-diameter are $(r_2, \theta_1+\frac{1}{2}\pi)$, where from the equation,

$$\frac{1}{r_1^2} = \frac{\cos^2\theta_1}{a^2}+\frac{\sin^2\theta_1}{b^2}$$

and
$$\frac{1}{r_2^2} = \frac{\cos^2(\theta_1+\frac{1}{2}\pi)}{a^2}+\frac{\sin^2(\theta_1+\frac{1}{2}\pi)}{b^2} = \frac{\sin^2\theta_1}{a^2}+\frac{\cos^2\theta_1}{b^2}.$$

Adding,
$$\frac{1}{r_1^2}+\frac{1}{r_2^2} = \frac{1}{a^2}+\frac{1}{b^2}.$$

17.64 Conjugate diameters

By 17.63, the diameter bisecting all chords parallel to the diameter $y = mx$ (i.e. having gradient m) is $y = m'x$, where $m' = -b^2/a^2m$,

† Strictly, the diameter is that part of the locus which is *inside* the ellipse.

i.e. $mm' = -b^2/a^2$. The symmetry of this relation shows that the diameter which bisects all chords parallel to the diameter $y = m'x$ is $y = mx$.

Two diameters which are such that each bisects all chords parallel to the other are said to be *conjugate*.

Hence $y = mx$, $y = m'x$ *are conjugate diameters if and only if*

$$mm' = -\frac{b^2}{a^2}.$$

It is clear from the definition that the axes of the ellipse are a pair of conjugate diameters; they are the only perpendicular pair.

Examples

(i) *If* θ, ϕ *are the extremities* P, D *of a pair of conjugate semi-diameters, then* $\theta \sim \phi = \frac{1}{2}\pi$.

For the gradients of OP, OD are respectively $(b/a)\tan\theta$, $(b/a)\tan\phi$, and the conjugacy condition is

$$\frac{b^2}{a^2}\tan\theta\tan\phi = -\frac{b^2}{a^2},$$

from which $\tan\theta = -\cot\phi = \tan(\phi \pm \frac{1}{2}\pi)$ and hence $\theta = \phi \pm \frac{1}{2}\pi$.

The result shows that the corresponding points P', D' on the auxiliary circle subtend a right-angle at O; i.e. *to conjugate diameters of the ellipse correspond perpendicular diameters of the auxiliary circle.*

*(ii) *Perpendicular diameters of a circle project into conjugate diameters of an ellipse.*

If $y = mx$, $y = m'x$, where $mm' = -1$, are the diameters of the circle $x^2 + y^2 = a^2$, they project into the lines $y = m_1 x$, $y = m_1' x$ where (see 17.23) $m_1 = m\cos\theta$, $m_1' = m'\cos\theta$, and so $m_1 m_1' = -\cos^2\theta = -b^2/a^2$.

Alternatively, if λ, μ are perpendicular diameters of the circle, then all chords parallel to μ are bisected by λ. Hence in the projected figure, all chords of the ellipse which are parallel to μ' are bisected by λ' (see 17.24); i.e. λ' and μ' are conjugate diameters of the ellipse.

It follows that properties of diameters and conjugate diameters can often be conveniently proved by orthogonal projection from a circle.

*(iii) *The tangent at* T *to an ellipse meets the diameter* OP *at* H, *and the line through* T *parallel to the conjugate diameter* OD *meets* OP *at* K. *Prove that*

$$OH.OK = OP^2.$$

First draw the figure for a circle, replacing 'conjugate' by 'perpendicular'. Then (fig. 175) $OK.OH = OT^2 = OP^2$, and since this can be written in terms of ratios of lengths on the same line, the result follows by projection on any plane through O. (The example in 17.24 (5) is a special case of the property just proved.)

*(iv) *Ellipse referred to a pair of its conjugate diameters as (oblique) axes.*

With the notation of fig. 176, we have by taking P_1 at V in Newton's theorem (17.61, ex.) that

$$\frac{QV^2}{PV.VP'} = \frac{OD^2}{OP^2}.$$

Choose OP, OD for axes of x and y, and then let Q be the point (x, y). Since

$$PV.VP' = (OP-OV)(OP+OV) = a_1^2 - x^2,$$

where $a_1 = OP$, the above equation becomes (on writing $b_1 = OD$)

$$\frac{y^2}{a_1^2 - x^2} = \frac{b_1^2}{a_1^2},$$

i.e.

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1.$$

This has the same form as the standard equation of the ellipse, which corresponds to the choice of the *perpendicular* conjugate diameters OA, OB for coordinate axes.

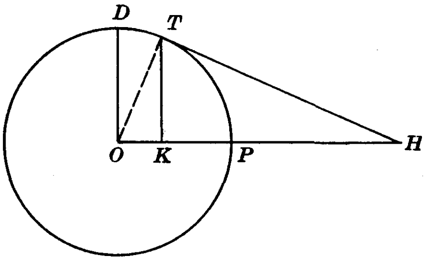

Fig. 175

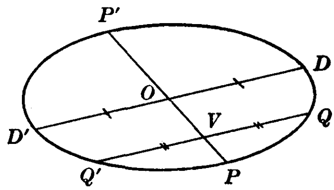

Fig. 176

Exercise 17(*d*)

1 Write down the equation of the diameter of $2x^2 + 3y^2 = 1$ which bisects the chords parallel to $2x - 5y + 3 = 0$.

2 Write down the gradient of the chords of $3x^2 + 4y^2 = 2$ which are bisected by the diameter $y = 3x$.

In the following, OP and OD are conjugate semi-diameters whose other extremities are P', D'.

3 If P is the point ϕ in fig. 176, show that the eccentric angles of D, P', D', Q, Q' can be taken as $\phi + \frac{1}{2}\pi$, $\phi + \pi$, $\phi - \frac{1}{2}\pi$, $\phi + \alpha$, $\phi - \alpha$ respectively. Write out in full the coordinates of P, D, P', D'.

4 If $P(a\cos\phi, b\sin\phi)$ and $D(-a\sin\phi, b\cos\phi)$ are extremities of *equal* conjugate diameters, prove that $\tan^2\phi = 1$. Deduce that the equations of the *equi-conjugate diameters* are $y = \pm bx/a$, and show that they lie along the diagonals of the rectangle which circumscribes the ellipse. State the length of OP.

5 Prove that $OP^2 + OD^2 = a^2 + b^2$.

6 (i) Prove that tangents at the extremities of a diameter are parallel to the conjugate diameter. [Use ex. (i) of 17.63.]

(ii) If the tangents at P, D meet at T, prove $OPTD$ is a parallelogram which has area ab. (Cf. no. 15.)

(iii) If p is the distance of P from OD, prove that $p.OD = ab$.

7 Deduce from nos. 5, 6(iii) that *the (p, r) equation of the ellipse wo its centre as pole is $a^2b^2/p^2 = a^2 + b^2 - r^2$.* [$OP = r$; eliminate OD.]

8 Prove $PS.PS' = OD^2$. [Use Ex. 17 (*a*), no. 3.]

9 The two chords joining any point of the ellipse to the extremities of a diameter are called *supplemental*. Prove that supplemental chords are parallel to conjugate diameters.

10 (i) Prove that the lines $px^2+2rxy+qy^2 = 0$ are conjugate diameters of $x^2/a^2+y^2/b^2 = 1$ if and only if $pa^2+qb^2 = 0$.

(ii) Find the condition for $lx+my+n = 0$ to be the join of extremities of two conjugate diameters of $x^2/a^2+y^2/b^2 = 1$. [Use 15.54.]

(iii) Prove that the chord joining the extremities of conjugate diameters of an ellipse touches another ellipse. [Use Ex. 17 (*b*), no. 8.]

11 If a tangent to the ellipse cuts the director circle at U, V, prove that OU, OV lie along conjugate diameters.

Nos. 12–15 *are easily solved by orthogonal projection.*

*12 QR is a diameter, and P is any point on the ellipse. The tangent at Q meets PR at T. Prove that the tangent at P bisects TQ.

*13 OP, OD are conjugate semi-diameters; the tangent at P meets the axes of the ellipse at Q and R. Prove $PQ.PR = OD^2$.

*14 PQ is a diameter and HK is any chord; PH, QK cut at X; PK, QH cut at Y. Prove that XY is parallel to the diameter conjugate to PQ.

*15 By projecting a circle and a circumscribing square, prove that the tangents at the extremities of conjugate diameters PP', DD' form a parallelogram of constant area $4ab$. (Cf. no. 6 (ii).)

*16 Prove Newton's theorem (17.61, ex.) by projection. [When written in the form

$$\frac{P_1Q}{OK}.\frac{P_1R}{OK} = \frac{P_1Q'}{OK'}.\frac{P_1R'}{OK'},$$

it involves ratios of lengths on *parallel* lines.]

*17 If parallel chords P_1QR, $P_2Q'R'$ are drawn through the *fixed points* P_1, P_2, prove that the ratio $P_1Q.P_1R:P_2Q'.P_2R'$ is independent of the direction. [Use the distance quadratic.]

*18 Prove that the mid-points of variable chords through P_1 lie on

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = \frac{x^2}{a^2}+\frac{y^2}{b^2}.$$

Show that this is an ellipse whose axes are proportional to those of the original ellipse and whose centre is $(\frac{1}{2}x_1, \frac{1}{2}y_1)$.

*19 Use the distance quadratic to obtain the equation of (i) the tangent at P_1 [two roots $r = 0$]; (ii) the pair of tangents from P_1 [equal roots r].

*20 Obtain the equation of the chord having mid-point P_1 from the ratio quadratic in Ex. 17 (*b*), no. 25. [If the chord is AP_1P_2, the roots $k:l$ are the ratios $P_1P_2:P_2P_2$ and $P_1A:AP_2$, and hence are $l = 0$, $k/l = -\frac{1}{2}$.]

Miscellaneous Exercise 17(*e*)

1 The lines $bx+aty-ab = 0$, $btx-ay+abt = 0$ meet at P. Find the locus of P when t varies.

2 With a given point S and line d for focus and directrix, ellipses are drawn. Prove that the locus of the ends of the minor axes is a parabola. [Choose S for origin.]

3 Prove that the tangents at the ends of a latus rectum are concurrent with the major axis and corresponding directrix.

*4 P, Q are the points θ, ϕ on the ellipse $x^2/a^2+y^2/b^2 = 1$, and the tangents at P, Q meet at T. Prove that triangle TPQ has area

$$ab\sin^3\tfrac{1}{2}(\phi-\theta)\sec\tfrac{1}{2}(\phi-\theta).$$

[The corresponding area for the circle is $2\triangle OPT-\triangle OPQ$; use projection.]

5 Y is the foot of the perpendicular from S to the tangent at P. Prove OP and SY meet on the directrix corresponding to S.

6 Circles with centres S, S' pass through a point P on the ellipse. Prove that the common tangents to these circles touch the auxiliary circle, and that their points of contact with this circle lie on the common chord of the original circles. [Use pure geometry and the bifocal property.]

7 S is a given point *inside* a given circle of centre O and radius a; Y is a variable point on the circumference, and p is the line through Y perpendicular to SY.

(i) Choosing OS for x-axis, let p meet Ox, Oy at T, T'. Explain why triangles TYO, TST' are similar, and prove $OY.TT' = OT.ST'$.

(ii) If S is $(c, 0)$ and p has equation $lx+my+n = 0$, prove that the equation in (i) reduces to $a^2l^2+b^2m^2 = n^2$, where $b^2 = a^2-c^2$.

(iii) Deduce that p touches an ellipse having S for focus and the given circle for auxiliary circle.

*8 Give the result corresponding to no. 7 (iii) when S lies *outside* the given circle. What happens when S lies *on* the circle?

9 Find the common tangents to $x^2+y^2 = 25$ and $x^2/169+y^2/16 = 1$.

10 The centre of an ellipse coincides with the vertex of a parabola, and they have a focus in common. Prove that the common tangents meet the common axis at its intersection with the other directrix.

11 (i) Prove that two tangents can be drawn from P_1 to $x^2/a^2+y^2/b^2 = 1$ if and only if $x_1^2/a^2+y_1^2/b^2-1 > 0$. [Use 17.47, ex. (i).]

(ii) From the ratio quadratic (Ex. 17 (*b*), no. 25) prove that if

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-1\right) < 0,$$

then P_1 and P_2 lie on opposite sides of the ellipse. (Accordingly, the set of points P_1 for which $x_1^2/a^2+y_1^2/b^2-1 > 0$ is the *outside* of the ellipse.)

12 Using the contact condition $c^2 = a^2m^2+b^2$ for the line $y = mx+c$ and the ellipse $x^2/a^2+y^2/b^2 = 1$, show that the equation of the pair of tangents from $P(x_1, y_1)$ can be written $(y_1x-x_1y)^2 = a^2(y-y_1)^2+b^2(x-x_1)^2$.

Let these tangents cut Ox at E, F. Find the locus of P (i) if PE, PF are perpendicular; (ii) if EF has fixed mid-point $(k, 0)$.

13 TP, TP' are tangents to the ellipse. Find the equations of the line-pairs through O parallel to TP, TP' and to ST, $S'T$. By showing that they have the same angle-bisectors, deduce that ST, $S'T$ are equally inclined to TP, TP' respectively.

14 A point P varies so that the chord of contact from P to $x^2/a^2+y^2/b^2 = 1$ touches the ellipse $x^2/a^2+y^2/b^2 = 1/k^2$. Find the locus of P.

15 The chord PQ of $x^2/a^2+y^2/b^2 = 1$ touches the parabola $y^2 = 4cx$. Prove that the tangents at P, Q to the ellipse meet on the parabola $a^2cy^2+b^4x = 0$.

16 Prove that points whose chords of contact are normals to the ellipse $x^2/a^2+y^2/b^2 = 1$ lie on the curve $a^6/x^2+b^6/y^2 = (a^2-b^2)^2$.

17 If the normal at P meets the major and minor axes at G, G', and if OF is the perpendicular from the centre O to this normal, prove that $PF.PG = b^2$ and $PF.PG' = a^2$.

18 If the normal at an extremity of a latus rectum passes through an extremity of the minor axis, prove that $e^4+e^2-1 = 0$.

19 P and Q are variable points on $x^2/a^2+y^2/b^2 = 1$ such that the mid-point of PQ lies on $x^2/a^2+y^2/b^2 = k^2$. Prove that the tangents at P, Q meet at a point T on $x^2/a^2+y^2/b^2 = 1/k^2$. [The chord of contact from $T(x_1, y_1)$ is the same line as the chord PQ having mid-point (x_2, y_2), where $x_2^2/a^2+y_2^2/b^2 = k^2$.]

20 The tangents to $x^2/a^2+y^2/b^2 = 1$ at P and Q meet at $T(h, k)$. Obtain the equation of the line-pair OP, OQ. Find the locus of T when the diameters OP, OQ are (i) perpendicular; (ii) conjugate. Show that these two loci meet at four points which are the vertices of a rectangle.

21 OP, OD are conjugate semi-diameters of the ellipse. The circles with diameters OP, OD meet again at Q. Prove that Q lies on the curve

$$2(x^2+y^2)^2 = a^2x^2+b^2y^2.$$

22 Lines are drawn through O perpendicular to the tangents from P_1 to $x^2/a^2+y^2/b^2 = 1$. If these lines are conjugate diameters of the ellipse, prove that P_1 lies on the curve $a^2x^2+b^2y^2 = a^4+b^4$. [Use the equation of the pair of tangents in no. 12.]

Show that each of the following equations represents an ellipse, and find the centre, semi-axes, eccentricity, foci, and directrices.

23 $\frac{1}{4}(x+2)^2+\frac{1}{3}(y-1)^2 = 1$.

24 $8x^2+9y^2-16x-6y = 63$.

25 $3(x-y+1)^2+4(x+y-1)^2 = 12$.

18

THE HYPERBOLA

18.1 The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; asymptotes

18.11 Form of the curve

We now resume the discussion, begun in 17.13–17.15, of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1. \qquad \text{(i)}$$

From the equation, $$\frac{x^2}{a^2} = 1+\frac{y^2}{b^2} \geqslant 1,$$

so that $|x| \geqslant a$ for all points on the curve; i.e. *no part of the curve lies between the lines* $x = \pm a$. Further, since

$$\frac{y^2}{b^2} = \frac{x^2}{a^2}-1,$$

we see that when $x \to +\infty$ or $x \to -\infty$, then also $y \to \pm\infty$. The curve is therefore unbounded.

When $x = a$, the equation gives $y^2/b^2 = 0$ which has the repeated root $y = 0$. Hence the line $x = a$ *touches* the hyperbola at $A(a, 0)$: it is the tangent at the vertex A. Similarly, the line $x = -a$ is the tangent at the vertex $A'(-a, 0)$.

Since $e > 1$, we have $ae > a$ and $a/e < a$. Hence S lies on OA produced beyond A; and if the directix $x = a/e$ cuts Ox at D, then D lies between O and A. Similarly S' lies on OA' produced beyond A', and D' lies between O and A'.

18.12 Asymptote: general definition

(1) We say that $P_1 \to \infty$ *along a curve* if one or both of its coordinates x_1, y_1 tend to $\pm\infty$ while satisfying the equation of the curve.

For example, $P_1 \to \infty$ along the parabola $y^2 = 4ax$ when $x_1 \to \infty$ and $y_1 \to \infty$, or when $x_1 \to \infty$ and $y_1 \to -\infty$; $P_1 \to \infty$ along the curve $y = 1/x^2$ when $x_1 \to \infty$ and $y_1 \to 0$, or when $x_1 \to -\infty$ and $y_1 \to 0$, or when $x_1 \to 0$ and $y_1 \to \infty$; but P_1 cannot tend to infinity along the ellipse $x^2/a^2+y^2/b^2 = 1$ because for every point P_1 on this curve we have $|x_1| \leqslant a$ and $|y_1| \leqslant b$.

(2) An *asymptote* of a curve is a line such that the distance of a point P_1 of the curve from this line tends to zero when $P_1 \to \infty$ along the curve.

In this definition, 'distance' may mean the usual 'perpendicular distance' or the 'oblique distance' measured parallel to some fixed direction, e.g. the y-axis. The informal use of the term 'asymptote' in 1.41, Remark (α), is consistent with our definition.

18.13 Asymptotes of the hyperbola

If P_1 lies on the curve, then $x_1^2/a^2-y_1^2/b^2 = 1$ and so

$$\frac{x_1}{a}-\frac{y_1}{b} = 1\Big/\left(\frac{x_1}{a}+\frac{y_1}{b}\right). \tag{ii}$$

The perpendicular distance of P_1 from the line $x/a-y/b = 0$ is

$$\left(\frac{x_1}{a}-\frac{y_1}{b}\right)\Big/\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{\frac{1}{2}},$$

which by (ii) is equal to

$$1\Big/\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{\frac{1}{2}}\left(\frac{x_1}{a}+\frac{y_1}{b}\right).$$

When $P_1 \to \infty$ along the curve, this expression tends to zero. Hence *the line $x/a-y/b = 0$ is an asymptote to the hyperbola* (i). Similarly *the line $x/a+y/b = 0$ is an asymptote.*

The equation of the asymptotes, regarded as a line-pair, is therefore

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 0.$$

Alternatively, the equation (i) can be written

$$y = \pm\frac{b}{a}x\left(1-\frac{a^2}{x^2}\right)^{\frac{1}{2}}.$$

If $y = mx+c$ is an asymptote to the part

$$y = +\frac{b}{a}x\left(1-\frac{a^2}{x^2}\right)^{\frac{1}{2}},$$

then
$$y_{\text{curve}}-y_{\text{asymptote}} \to 0 \quad \text{when} \quad x \to \infty.$$

Using the binomial series, we have for large x that

$$y_c-y_a = \frac{b}{a}x\left\{1-\frac{1}{2}\frac{a^2}{x^2}+O\left(\frac{1}{x^4}\right)\right\}-(mx+c)$$
$$= -c+\left(\frac{b}{a}-m\right)x+O\left(\frac{1}{x}\right);$$

and this will tend to zero when $x \to \infty$ only if $c = 0$ and $b/a - m = 0$. Hence the line $y = bx/a$ is an asymptote. Similarly, $y = -bx/a$ is an asymptote to the other part

$$y = -\frac{b}{a}x\left(1-\frac{a^2}{x^2}\right)^{\frac{1}{2}}.$$

Since
$$y^2 = b^2\left(\frac{x^2}{a^2}-1\right) < \frac{b^2}{a^2}x^2$$

for any point on the hyperbola, therefore $y_c^2 < y_a^2$; hence *the hyperbola lies entirely within that double angle between the lines* $y = \pm bx/a$ *which contains the x-axis.*

The angle between the asymptotes is $2\tan^{-1}(b/a)$. Figures 177, 178, 179 illustrate the cases $a < b$, $a = b$, $a > b$ respectively.

When $b = a$, the asymptotes are perpendicular and the curve is called a *rectangular hyperbola*. The equation (i) becomes $x^2 - y^2 = a^2$; and the eccentricity, given by $b^2 = a^2(e^2-1)$, is $e = \sqrt{2}$.

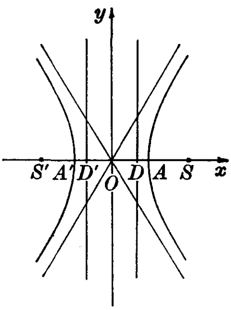

Fig. 177

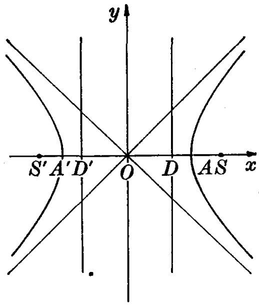

Fig. 178

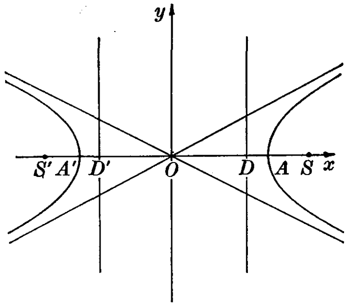

Fig. 179

18.14 The bifocal property: $SP - S'P = \pm 2a$

If P lies on the branch which contains the focus S, then

$$\begin{aligned} S'P - SP &= e.PM' - e.PM \\ &= e.MM' \\ &= e\left(\frac{2a}{e}\right) \\ &= 2a. \end{aligned}$$

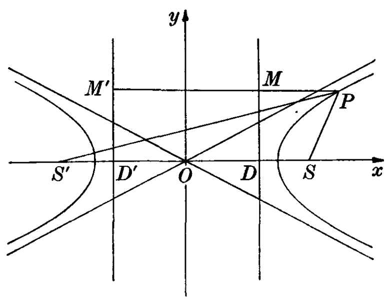

Fig. 180

If P lies on the branch enclosing S', then

$$SP - S'P = e.PM - e.PM' = e.MM' = e\left(\frac{2a}{e}\right) = 2a.$$

Conversely, *the locus of a point P such that $S'P-SP = 2a$ is one branch of a hyperbola having foci S, S' and transverse axis of length $2a$, viz. that branch which contains S.*

Proof. The algebra of 17.22 shows that, with the same choice of axes,

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} = 1.$$

From triangle PSS' we have $PS' < PS+SS'$, i.e.

$$2a = PS'-PS < SS' = 2c,$$

so that $a < c$. Hence we can write $b^2 = c^2-a^2$, and the equation becomes that of the standard hyperbola (i). Since $b^2 = a^2(e^2-1)$, we have $c^2 = a^2+b^2 = a^2e^2$, and so $c = ae$. Therefore S, S' are the points $(\pm ae, 0)$, i.e. the foci. Finally, since $SP < S'P$, P lies on the branch which contains S.

Example

Mechanical construction of a hyperbola.

A rod APB is movable about the fixed end A, and a string BPC passing through a small ring P which slides along the rod is tied to the other end B and to a fixed point C. If the string is kept taut, prove that in general P moves on one branch of a hyperbola with foci A, C.

Let l be the length of the string. Then $BP+PC = l$. Since

$$AP = AB-BP = AB-(l-PC)$$
$$= (AB-l)+PC,$$

therefore $AP-PC$ is constant, viz. $AB-l$.

If $AB \neq l$, the locus of P is that branch of a hyperbola with foci A, C which encloses C.

If $AB = l$, then $AP = PC$, and P lies on the perpendicular bisector of AC.

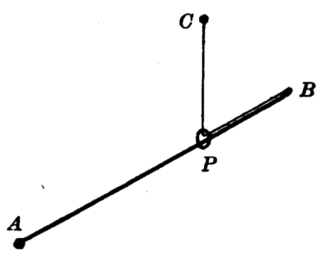

Fig. 181

18.2 Properties analogous to those of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Since the equations of the ellipse and hyperbola differ only in the sign of b^2, it follows that many properties of the hyperbola can be written down from the corresponding results for the ellipse by replacing b^2 by $-b^2$. All such properties can be proved independently by the methods used for the ellipse; the following is a summary.

(i) *Chord P_1P_2:*

$$\frac{x}{a^2}(x_1+x_2)-\frac{y}{b^2}(y_1+y_2) = \frac{x_1x_2}{a^2}-\frac{y_1y_2}{b^2}+1.$$

(ii) *Tangent* at P_1: $$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

(iii) *Normal* at P_1: $$\frac{x-x_1}{x_1/a^2}+\frac{y-y_1}{y_1/b^2}=0.$$

(iv) The *contact condition* for $y=mx+c$ is $c^2=a^2m^2-b^2$ provided $|m|<b/a$. The condition remains significant when $m=\pm b/a$, for then $c=0$; but the lines $y=\pm bx/a$ do not cut the hyperbola anywhere (see 18.13).

(v) If $-b/a<m<b/a$, the lines
$$y=mx\pm\sqrt{(a^2m^2-b^2)}$$
touch the hyperbola.

(vi) *Director circle.* The locus of the intersections of perpendicular tangents to the hyperbola is the circle
$$x^2+y^2=a^2-b^2$$
provided $a>b$. If $a\leqslant b$, there are no perpendicular tangents.

(vii) *Chord of contact* from P_1:
$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

(viii) *Pair of tangents* from P_1:
$$\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2.$$

(ix) *Chord having mid-point* P_1:
$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}.$$

(x) *Diameters.* The mid-points of all chords of gradient m lie on $y=b^2x/a^2m$. The properties of diameters stated and proved for the ellipse in 17.63, exs. (i), (ii) remain true for the hyperbola.

(xi) *Conjugate diameters.* $y=mx$ and $y=m'x$ are conjugate if and only if
$$mm'=\frac{b^2}{a^2}.$$

18.3 Parametric representation

18.31 Hyperbolic functions

The equation (i) in 18.11 is satisfied by $x=a\operatorname{ch}\phi$, $y=b\operatorname{sh}\phi$ for all ϕ; but since $\operatorname{ch}\phi>0$, these equations represent only one branch of the hyperbola.

Analogously to the treatment in 17.32, put $\tau = \operatorname{th}\frac{1}{2}\phi$; then

$$x = a\frac{1+\tau^2}{1-\tau^2}, \quad y = b\frac{2\tau}{1-\tau^2}.$$

If τ is now unrestricted, these equations represent the whole of the curve except the point $A'(-a, 0)$; but this can be obtained by letting $\tau \to \infty$. (If τ still denotes $\operatorname{th}\frac{1}{2}\phi$, it can take only values between -1 and $+1$.)

18.32 The point *t*

The general representation above can be found directly. For since the equation can be written

$$\frac{y^2}{b^2} = \left(\frac{x}{a}+1\right)\left(\frac{x}{a}-1\right),$$

therefore $$\frac{x/a-1}{y/b} = \frac{y/b}{x/a+1} = t, \quad \text{say.}$$

Hence $$\left(\frac{x}{a}-1\right) : \frac{y}{b} : \left(\frac{x}{a}+1\right) = t^2 : t : 1,$$

i.e. $$\frac{x}{a} : \frac{y}{b} : 1 = (1+t^2) : 2t : (1-t^2).$$

Each value of t except $t = \pm 1$ gives just one point of the curve; and each point of the curve except $A'(-a, 0)$ corresponds to just one value of t. We refer to the point

$$\left(a\frac{1+t^2}{1-t^2}, \frac{2bt}{1-t^2}\right)$$

as *the point t*.

18.33 The point ϕ

The equation (i) is also satisfied by $x = a\sec\phi, y = b\tan\phi$ for all ϕ. We refer to $(a\sec\phi, b\tan\phi)$ as *the point* ϕ.

If we put $t = \tan\frac{1}{2}\phi$, we obtain the point t in 18.32.

18.34 Another algebraic representation

Since equation (i) can be written

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right) = 1,$$

we may put $\dfrac{x}{a}+\dfrac{y}{b}=t$ and hence $\dfrac{x}{a}-\dfrac{y}{b}=\dfrac{1}{t}$.

From these we find

$$x=\tfrac{1}{2}a\left(t+\frac{1}{t}\right),\quad y=\tfrac{1}{2}b\left(t-\frac{1}{t}\right).$$

These equations represent the *whole* of the curve; $t=+1$ gives $A(a,0)$, and $t=-1$ gives $A'(-a,0)$. When t increases from $-\infty$ to $+\infty$, the parts traced are:

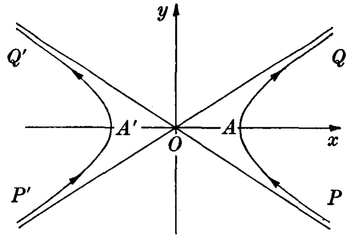

Fig. 182

$P'A'$	$A'Q'$	PA	AQ
$t<-1$	$-1<t<0$	$0<t<1$	$1<t$

18.4 Chord, tangent, and normal

To each of the preceding representations corresponds a standard form of equation for chord, tangent, and normal. By methods already illustrated in Ch. 17, the following results can be obtained.

Chord $\theta\phi$: $\dfrac{x}{a}\cos\tfrac{1}{2}(\theta-\phi)-\dfrac{y}{b}\sin\tfrac{1}{2}(\theta+\phi)=\cos\tfrac{1}{2}(\theta+\phi)$.

t_1t_2: $(1+t_1t_2)\dfrac{x}{a}-(t_1+t_2)\dfrac{y}{b}=1-t_1t_2$.

Tangent at ϕ: $\dfrac{x}{a}\sec\phi-\dfrac{y}{b}\tan\phi=1$.

t: $(1+t^2)\dfrac{x}{a}-2t\dfrac{y}{b}=1-t^2$.

Normal at ϕ: $ax\cos\phi+by\cot\phi=a^2+b^2$.

t: $2atx+b(1+t^2)y=2(a^2+b^2)\dfrac{t(1+t^2)}{1-t^2}$.

The reader should verify these; also see nos. 13–15 of the following Exercise, which contains examples similar to those already given for the ellipse together with some properties of the asymptotes of the hyperbola.

Exercise 18(a)

1 If P is the point (x,y) on the hyperbola, prove $SP=ex-a$ and $S'P=ex+a$.

2 A shot is fired from A so as to strike an object B. The sound of the firing is heard at P, n seconds before the sound of the destruction of B. Find the locus

of P. [Let u = velocity of sound, v = velocity component of the shot parallel to AB; then

$$\frac{AP}{u}+n = \frac{AB}{v}+\frac{PB}{u}, \quad \text{so that} \quad AP-PB = u\left(\frac{AB}{v}-n\right) = \text{constant.}]$$

3 Find the locus of the centre of a circle which touches externally two given circles whose centres are A, B.

4 (i) State the order in which the hyperbola is traced by the point

$$\left(a\frac{1+t^2}{1-t^2}, \frac{2bt}{1-t^2}\right)$$

as t increases from $-\infty$ to $+\infty$.

(ii) What is the condition for the points t_1, t_2 to (*a*) lie on the same branch; (*b*) be diametrically opposite?

5 P, P' are the points ϕ, $-\phi$ on the ellipse $x^2/a^2+y^2/b^2 = 1$ whose vertices are A, A'. If AP meets $A'P'$ at Q, prove that the locus of Q is the hyperbola $x^2/a^2-y^2/b^2 = 1$.

6 If the points θ, ϕ are the extremities of a focal chord of $x^2/a^2-y^2/b^2 = 1$, prove that

$$\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi = \frac{1-e}{1+e} \quad \text{or} \quad \frac{1+e}{1-e}.$$

7 Prove that $lx+my+n = 0$ touches $x^2/a^2-y^2/b^2 = 1$ if and only if $a^2l^2-b^2m^2 = n^2$.

8 (i) The tangent at any point P of the hyperbola meets a directrix at Z. Prove that PZ subtends a right-angle at the corresponding focus.

(ii) Deduce that tangents at the extremities of a focal chord meet on the corresponding directrix.

9 Y is the foot of the perpendicular from S to the tangent at any point P of the hyperbola. Prove that Y lies on the circle $x^2+y^2 = a^2$ (the *auxiliary circle*).

10 Prove that the foot F of the perpendicular from a focus to an asymptote lies on the auxiliary circle and on the corresponding directrix.

[$OF = OS\cos F\hat{O}S = \ldots = a$; also $OS.OD = a^2 = OF^2$, so $O\hat{D}F$ is a right-angle.]

11 The line joining the focus S to the point P on a hyperbola is parallel to an asymptote. Prove that this asymptote, the directrix, and the tangent at P are concurrent.

12 The point P on the hyperbola is such that the tangent at P, the latus rectum through S, and an asymptote are concurrent. Prove that SP is parallel to the other asymptote.

With the representation in 18.34, *obtain the equation of*

13 the chord t_1t_2. 14 the tangent at t. 15 the normal at t.

16 Prove that if the normals at the points t_1, t_2, t_3, t_4 of 18.32 are concurrent, then $\Sigma t_1t_2 = 0$ and $t_1t_2t_3t_4 = -1$.

17 Prove that the feet of four concurrent normals cannot all lie on the same branch. [If they do, then by no. 4 (ii) (*a*) $|t_r|$ are *either* all less than 1 *or* all greater than 1, contradicting $|t_1t_2t_3t_4| = 1$ in no. 16.]

18 If the normals at ϕ_1, ϕ_2, ϕ_3, ϕ_4 are concurrent, prove $\Sigma\phi_1 = (2n+1)\pi$ where n is some integer (positive, negative, or zero).

19 Show that the feet of the normals from P_1 to $x^2/a^2 - y^2/b^2 = 1$ lie on the curve $a^2y(x_1 - x) + b^2x(y_1 - y) = 0$.

20 If the points t_1, t_2, t_3, t_4 are concyclic, prove $\Sigma(t_1 + t_2t_3t_4) = 0$.

18.5 Asymptotes: further properties

18.51 *Lines parallel to an asymptote meet the hyperbola only once.*

Any line parallel to the asymptote $x/a + y/b = 0$ has equation

$$\frac{x}{a} + \frac{y}{b} = k.$$

It meets the hyperbola at points t for which

$$\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} = k,$$

i.e. $$(1+t)^2 = k(1-t^2),$$

i.e. $$t = -1 \quad \text{or} \quad 1+t = k(1-t).$$

Since $t = -1$ gives no point of the curve, there is a unique intersection given by $t = (k-1)/(k+1)$. A similar result holds for lines parallel to $x/a - y/b = 0$.

18.52 *The equation of the tangent at P_1 tends to the equation of an asymptote when $P_1 \to \infty$ along the curve.*

The equation $xx_1/a^2 - yy_1/b^2 = 1$ of the tangent at P_1 can be written

$$\frac{x}{a} - \frac{ay}{b^2}\frac{y_1}{x_1} = \frac{a}{x_1},$$

where $$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1, \quad \text{i.e.} \quad \frac{y_1^2}{x_1^2} = \frac{b^2}{a^2} - \frac{b^2}{x_1^2}.$$

The last equation shows that when $P_1 \to \infty$, then $y_1/x_1 \to b/a$ or $-b/a$. The limit of the equation of the tangent at P_1 is thus either

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} = 0.$$

Remark. An asymptote is sometimes defined as 'the limit of the tangent at P_1 when $P_1 \to \infty$ along the curve'. For curves whose equations are algebraic, it can be shown that the two definitions are equivalent; but for general curves a line may be an asymptote according to the definition in 18.12, but not according to that above because the equation of the tangent may have no limiting form when $P_1 \to \infty$.

18.53 *The family $x^2/a^2 - y^2/b^2 = k$ of hyperbolas has the same asymptotes for all k.*

For the equation can be written in standard form as

$$\frac{x^2}{ka^2} - \frac{y^2}{kb^2} = 1,$$

and the asymptotes have equation

$$\frac{x^2}{ka^2} - \frac{y^2}{kb^2} = 0, \quad \text{i.e.} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

which is independent of k.

Example

In the quadratic which gives the meets of $x^2/a^2 - y^2/b^2 = k$ with the line $lx + my + n = 0$, the coefficients of x^2 and x are independent of k. Hence *the sum of the roots is independent of k*. Taking $k = 1$, 0, it follows that the x-coördinate of the mid-point of any chord of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is the same as the x-coordinate of the mid-point of the same chord of the asymptotes $x^2/a^2 - y^2/b^2 = 0$. The same is true of the y-coordinates since $lx + my + n = 0$ gives y uniquely in terms of x. Consequently, *if the line meets the hyperbola at P, P' and the asymptotes at Q, Q', then PP' and QQ' have the same mid-point M.*

Since $PM = MP'$ and $QM = MQ'$, therefore $PQ = P'Q'$. In particular, when $P' \to P$ this becomes $PQ = PQ'$; i.e. *the part intercepted on a tangent by the asymptotes is bisected at the point of contact.*

18.6 The conjugate hyperbola

18.61 Definitions

In 17.15 we defined the *transverse axis* to be the intercept AA' made by the hyperbola on Ox. Although the curve does not cut Oy, it is now convenient to consider the points $B(0, b)$ and $B'(0, -b)$ on this axis of symmetry and (analogously to the ellipse) to call BB' the *conjugate axis* of the hyperbola.

Two hyperbolas are said to be *conjugate* when the transverse and conjugate axes of one are respectively the conjugate and transverse axes of the other.

It follows that the hyperbola conjugate to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{(i)}$$

is
$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = -1. \qquad \text{(ii)}$$

For, with Oy for x-axis and Ox' for y-axis, the equation of the conjugate hyperbola is $x^2/b^2-y^2/a^2 = 1$. On rotating these axes clockwise through a right-angle to positions Ox, Oy, the equation becomes $y^2/b^2-x^2/a^2 = 1$, i.e. (ii).

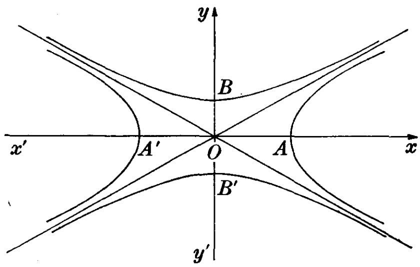

Fig. 183

The two hyperbolas have the same asymptotes, viz.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 0. \qquad \text{(iii)}$$

Remarks

(α) Equation (iii) differs from (i) by the same constant that (ii) differs from (iii). If we transform these equations by a change of axes (15.73), this relationship will be preserved.

(β) The equation of any hyperbola whose asymptotes are

$$a_1x+b_1y+c_1 = 0, \quad a_2x+b_2y+c_2 = 0$$

is
$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2) = \lambda,$$

where λ is constant; for this equation differs only by the constant λ from the equation of the asymptotes, viz.

$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2) = 0.$$

By writing $-\lambda$ for λ, we obtain the equation of the conjugate hyperbola.

(γ) In particular, any hyperbola whose asymptotes are the coordinate axes $x = 0$, $y = 0$ has equation $xy = \lambda$.

18.62 Conjugate diameters

(i) *If a pair of diameters are conjugate wo a hyperbola, they are also conjugate wo the conjugate hyperbola.*

The argument which leads to property (xi) in 18.2 can be applied to

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=k, \tag{iv}$$

and gives the same conclusion independently of k: $y = mx$ and $y = m'x$ are conjugate diameters of each of the hyperbolas (iv) if and only if $mm' = b^2/a^2$. The result stated concerns the cases $k = +1, -1$.

(ii) *Intersections of the hyperbolas with conjugate diameters.*

The diameter $y = mx$ cuts hyperbola (i) if and only if $|m| < b/a$. The conjugate diameter $y = b^2x/a^2m$ cuts (i) if and only if $|b^2/a^2m| < b/a$, i.e. $|m| > b/a$. Hence *of two conjugate diameters, only one meets the hyperbola.*

The other meets the conjugate hyperbola. For if $y = mx$ cuts (i), then $|m| < b/a$; in this case $|b^2/a^2m| > b/a$, so that $y = b^2x/a^2m$ cuts hyperbola (ii).

(iii) *Extremities of conjugate diameters.*

The points of intersection are called the *extremities* of the diameters, despite the fact that only one pair can lie on each curve.

If $P(a\sec\phi, b\tan\phi)$ is an extremity of one diameter, then $m = b\tan\phi/a\sec\phi$. Hence the conjugate diameter has gradient $m' = b^2/a^2m = b\sec\phi/a\tan\phi$, and its equation is $y = m'x$. It cuts the conjugate hyperbola where

$$\frac{x^2}{a^2}\left(1-\frac{\sec^2\phi}{\tan^2\phi}\right)=-1,$$

i.e. where $x = \pm a\tan\phi$ and hence $y = \pm b\sec\phi$. Thus $P(a\sec\phi, b\tan\phi)$ and $D(a\tan\phi, b\sec\phi)$ are extremities of conjugate semi-diameters.

Exercise 18(*b*)

1 If e, e' are the eccentricities of a hyperbola and its conjugate, prove $1/e^2+1/e'^2 = 1$.

OP, OD are conjugate semi-diameters of the hyperbola $x^2/a^2-y^2/b^2 = 1$. Prove the following.

2 $OP^2 \sim OD^2 = a^2-b^2$.

3 The mid-point of PD lies on an asymptote.

4 PD is parallel to an asymptote. 5 The area of triangle OPD is $\frac{1}{2}ab$.

6 Prove that the tangents at the extremities of a diameter are parallel to the conjugate diameter.

7 Prove that the tangents at the extremities of two conjugate diameters intersect on the asymptotes.

8 Deduce from nos. 5–7 that tangents at P, D and their diametrically opposite points P', D' form a parallelogram of constant area $4ab$, whose vertices lie on the asymptotes.

9 Deduce from no. 6 that the part intercepted on the tangent at P by the asymptotes is bisected at P.

10 Prove $SP.S'P = OD^2$. [Use Ex. 18(*a*), no. 1.]

11 If P lies on the conjugate hyperbola, prove that its chord of contact to the original hyperbola touches the conjugate at the point P' diametrically opposite to P.

12 Find the equation of the hyperbola whose asymptotes are $x-2y+1=0$ and $3x-y+2=0$ and which passes through the point $(0,1)$. [18.61, Remark (β).]

*13 Given that the equation $2x^2-5xy-3y^2+x+11y-8=0$ represents a hyperbola, find the equations of the asymptotes and the equation of the conjugate hyperbola. [Choose λ so that $2x^2-5xy-3y^2+x+11y+\lambda=0$ represents a line-pair.]

18.7 Asymptotes as (oblique) coordinate axes

18.71 $xy=c^2$

The product of the perpendiculars from a point P_1 on

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

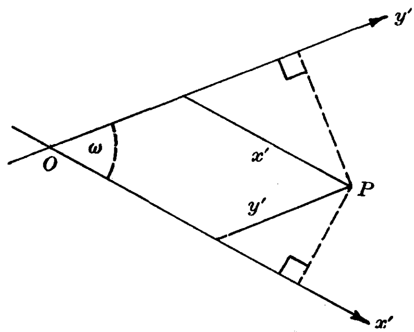

Fig. 184

to the asymptotes $x/a+y/b=0$, $x/a-y/b=0$ is

$$\frac{\dfrac{x_1}{a}+\dfrac{y_1}{b}}{\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)^{\frac{1}{2}}}\frac{\dfrac{x_1}{a}-\dfrac{y_1}{b}}{\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)^{\frac{1}{2}}}=\frac{\dfrac{x_1^2}{a^2}-\dfrac{y_1^2}{b^2}}{\dfrac{1}{a^2}+\dfrac{1}{b^2}}$$

$$=\frac{1}{\dfrac{1}{a^2}+\dfrac{1}{b^2}}$$

because $x_1^2/a^2-y_1^2/b^2=1$.

Choosing the asymptotes for axes Ox', Oy', let P_1 have coordinates (x_1', y_1'). Then the lengths of the perpendiculars are $x_1'\sin\omega$, $y_1'\sin\omega$, where ω is the angle between the asymptotes, and hence

$$x_1'y_1'\sin^2\omega=\frac{1}{1/a^2+1/b^2};$$

i.e. P_1 satisfies
$$x'y'\sin^2\omega = \frac{a^2b^2}{a^2+b^2}.$$

Since $\omega = 2\tan^{-1}(b/a)$,

$$\sin^2\omega = \left(\frac{2\tan\frac{1}{2}\omega}{1+\tan^2\frac{1}{2}\omega}\right)^2 = \left(\frac{2b/a}{1+b^2/a^2}\right)^2 = \frac{4a^2b^2}{(a^2+b^2)^2},$$

and the equation becomes
$$x'y' = \tfrac{1}{4}(a^2+b^2).$$

Omitting dashes, this can be written

$$xy = c^2,$$

where $c^2 = \frac{1}{4}(a^2+b^2)$, and is the standard equation of the hyperbola referred to its asymptotes as (oblique) coordinate axes.

Example

Any equation of the form $xy = ax+by+c$ represents a hyperbola whose asymptotes are parallel to Ox, Oy.

For the equation can be written $(x-b)(y-a) = ab+c$; i.e. $x'y' = ab+c$ after a change of origin.

Thus the locus in ex. (iii) (*b*) of 16.32 is a rectangular hyperbola, known as the *hyperbola of Apollonius* of (h, k) wo the parabola $y^2 = 4ax$. See also Ex. 17 (*c*), no. 10, and Ex. 18 (*a*), no. 19.

18.72 Parametric representation

The equation $xy = c^2$ can be written

$$\frac{x}{c} = \frac{c}{y} = t, \quad \text{say,}$$

so that $x = ct$ and $y = c/t$. For each value of t except $t = 0$ the point $(ct, c/t)$, referred to as *the point t*, lies on $xy = c^2$; and to each point of the curve corresponds just one value of t. The parametric equations can also be written
$$x:y:c = t^2:1:t.$$

The methods illustrated in 16.22 (2), (3) and 16.24 (1) do not appeal to 'gradient', and can be used in the present case of oblique axes to prove that the *chord* t_1t_2 has equation

$$x+t_1t_2y = c(t_1+t_2),$$

and hence that the *tangent at t* is

$$x+t^2y = 2ct.$$

Example

A tangent to a hyperbola and the asymptotes form a triangle of constant area.

The above tangent at $P(ct, c/t)$ meets Ox, Oy at $Q(2ct, 0)$, $R(0, 2c/t)$. Hence

$$OQ = 2ct, \; OR = 2c/t,$$

and the area of triangle OQR is

$$\tfrac{1}{2}OQ \, . \, OR \sin\omega = 2c^2 \sin\omega.$$

This work also shows that *P is the mid-point of QR*; cf. the example in 18.53.

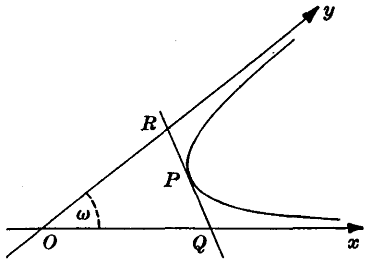

Fig. 185

18.73 The rectangular hyperbola

When $\omega = \frac{1}{2}\pi$ the asymptotes Ox, Oy are perpendicular, and the hyperbola $xy = c^2$ is *rectangular* (18.13). In addition to the results in 18.72, the usual method now shows that the *normal at t* has equation

$$t^3x - ty = c(t^4 - 1).$$

Example

The orthocentre of a triangle inscribed in a rectangular hyperbola lies on the curve.

The chord $t_1 t_2$ has gradient

$$\left(\frac{c}{t_1} - \frac{c}{t_2}\right) \Big/ (ct_1 - ct_2) = -\frac{1}{t_1 t_2};$$

similarly, chord $t_3 t_4$ has gradient $-1/t_3 t_4$. These chords are perpendicular if and only if $t_1 t_2 t_3 t_4 = -1$.

If the given triangle has vertices t_1, t_2, t_3, then the above work shows that there is a point t_4 on the curve (viz. that for which $t_1 t_2 t_3 t_4 = -1$) such that chord $t_1 t_2 \perp$ chord $t_3 t_4$. The symmetry of the relation $t_1 t_2 t_3 t_4 = -1$ shows that also $t_1 t_3 \perp t_2 t_4$ and $t_2 t_3 \perp t_1 t_4$. Hence this point $t_4 = -1/t_1 t_2 t_3$ is the orthocentre of triangle $t_1 t_2 t_3$.

It is also clear from the relation $t_1 t_2 t_3 t_4 = -1$ that the point t_1 is the orthocentre of triangle $t_2 t_3 t_4$, and so on. The set of four points t_1, t_2, t_3, t_4 on the curve may be called *orthocentric*.

Exercise 18(*c*)

In the following examples the sign $\{\omega\}$ indicates results which are true for the general hyperbola $xy = c^2$. The reader who wishes to avoid oblique axes may prove them for the rectangular case only.

1 $\{\omega\}$ The line $lx + my + n = 0$ is a tangent if and only if $4c^2lm = n^2$.

2 $\{\omega\}$ Tangents at t_1, t_2 meet at

$$\left(\frac{2ct_1 t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right).$$

3 $\{\omega\}$ The chord P_1P_2 is $c^2x+x_1x_2y = c^2(x_1+x_2)$.

4 $\{\omega\}$ The tangent at P_1 is $xy_1+x_1y = 2c^2$. Could this be written down from $xy = c^2$ by the 'rule of alternate suffixes'?

5 $\{\omega\}$ The chord of contact of tangents from P_1 is $xy_1+x_1y = 2c^2$.

*6 $\{\omega\}$ Joachimsthal's ratio quadratic is

$$(x_2y_2-c^2)\,k^2+(xy_1+x_1y-2c^2)\,kl+(x_1y_1-c^2)\,l^2 = 0.$$

*7 $\{\omega\}$ The pair of tangents from P_1 is $4(x_1y_1-c^2)(xy-c^2) = (xy_1+x_1y-2c^2)^2$.

8 The normal at P_1 is $xx_1-yy_1 = x_1^2-y_1^2$.

9 The distance quadratic is

$$r^2\sin\theta\cos\theta+(x_1\sin\theta+y_1\cos\theta)\,r+(x_1y_1-c^2) = 0.$$

10 The chord having mid-point P_1 is $xy_1+x_1y = 2x_1y_1$. [Use no. 9 or no. 3.]

11 The diameter bisecting all chords of gradient m is $y = -mx$.

12 $y = mx$, $y = m'x$ are conjugate diameters if and only if $m+m' = 0$.

13 $\{\omega\}$ The conjugate hyperbola is $xy = -c^2$. [Begin as in 18.71.]

14 Conjugate semi-diameters are equal in length and make complementary angles with the transverse axis; and the asymptotes bisect the angles between them.

15 $\{\omega\}$ Prove that lines drawn from a variable point P on a hyperbola to any two fixed points E, F on the curve intercept a constant length on either asymptote.

16 $\{\omega\}$ If the tangents at P, Q meet at T, prove that OT bisects PQ.

*17 $\{\omega\}$ A quadrilateral circumscribes a hyperbola. Prove that the line joining the mid-points of its diagonals passes through the centre of the hyperbola.

18 Prove that no two tangents to a rectangular hyperbola can be perpendicular.

19 If P, Q, R are points on a rectangular hyperbola such that PQ subtends a right-angle at R, prove that the tangent at R is perpendicular to PQ.

20 *Concyclic points.*

(i) Show that the circle $x^2+y^2+2gx+2fy+d = 0$ cuts the rectangular hyperbola $xy = c^2$ at points t for which

$$c^2t^4+2gct^3+dt^2+2fct+c^2 = 0.$$

(ii) If the points t_1, t_2, t_3, t_4 are concyclic, prove $t_1t_2t_3t_4 = 1$.

(iii) By considering the fourth intersection of the hyperbola and the circle through the points t_1, t_2, t_3, prove the converse of (ii).

21 Use no. 20 (ii) to prove that the common chords of a circle and rectangular hyperbola are equally inclined in pairs to either axis of the hyperbola.

22 If circles touch a rectangular hyperbola at a fixed point, prove that their common chords not through this point lie in a fixed direction.

*23 (i) Prove that the circle of curvature at t meets the rectangular hyperbola again at the point $1/t^3$.

(ii) Points P_1, P_2, P_3, P_4 on the rectangular hyperbola are concyclic. If Q_r is where the circle of curvature at P_r meets the curve again, prove that Q_1, Q_2, Q_3, Q_4 are also concyclic.

24 The points A, B, C, D on a rectangular hyperbola are *not concyclic*. If the circles BCD, CDA, DAB, ABC meet the curve again at A', B', C', D'

respectively, prove that the mid-points of the chords AA', BB', CC', DD' lie on another hyperbola having the same asymptotes as the given one.

25 Prove that the circumcircle of triangle $t_1 t_2 t_3$ cuts the rectangular hyperbola again at the point diametrically opposite to the orthocentre of this triangle. [Use 18.73, ex.]

26 If a circle and rectangular hyperbola meet at A, B, C, D, prove that the orthocentres of triangles BCD, CDA, DAB, ABC are concyclic. Prove also that the tangents to the hyperbola at C, D meet at a point on the diameter which is perpendicular to AB.

27 (i) Prove that the normal at t to the rectangular hyperbola meets the curve again at the point $-1/t^3$.

*(ii) Verify from no. 23 (i) that this is diametrically opposite to the point where the circle of curvature at t meets the curve again.

28 The normal at P meets the rectangular hyperbola again at Q. Prove that the mid-point of PQ lies on $4x^3y^3+c^2(x^2-y^2)^2 = 0$.

29 *Conormal points.*

(i) If the normal to the rectangular hyperbola $xy = c^2$ at the point t passes through (h, k), prove $ct^4-ht^3+kt-c = 0$.

(ii) If the normals at t_1, t_2, t_3, t_4 are concurrent, prove that $t_1t_2t_3t_4 = -1$ and $\Sigma t_1 t_2 = 0$.

30 Prove that four conormal points are also orthocentric, but that four distinct conormal points can never be concyclic.

31 The axes being rectangular, prove that the equation $y = (ax+b)/(cx+d)$, $c \neq 0$, represents a rectangular hyperbola whose asymptotes are $x = -d/c$, $y = a/c$. What are the coordinates of its centre? [Use 18.71, ex.]

32 A variable line passes through a fixed point and meets two given intersecting lines at P, Q. Prove that the locus of a point dividing PQ in a given ratio is a hyperbola whose asymptotes are parallel to the given lines. [Use oblique axes.]

33 A variable chord of $xy = c^2$ passes through the fixed point (p, q). Prove that the locus of its mid-point is another hyperbola, and give the equations of its asymptotes.

Miscellaneous Exercise 18(*d*)

1 If ω is the angle between the asymptotes of a hyperbola, prove that its eccentricity is $\sec \frac{1}{2}\omega$.

2 PN is the ordinate of a point P on $x^2/a^2-y^2/b^2 = 1$; NT is a tangent from N to the auxiliary circle $x^2+y^2 = a^2$. If P is the point ϕ, prove $N\hat{O}T = \phi$ and $NP:NT = b:a$.

3 With the notation in 17.23 and $\cos\theta = b/a$, what is the orthogonal projection of the rectangular hyperbola $x^2-y^2 = a^2$? Show that the rectangular hyperbola $xy = c^2$ projects into another rectangular hyperbola.

4 $A(a, 0)$ and $A'(-a, 0)$ are fixed points. A variable circle through A and A' cuts Oy at P, P'. If AP, $A'P'$ meet at Q, prove that the locus of Q is the rectangular hyperbola $x^2-y^2 = a^2$.

5 A variable circle of centre P touches each of two unequal intersecting circles. Prove that the locus of P is an ellipse and one branch of a hyperbola. [Use the converse bifocal property.]

6 B is a fixed point in the plane of a given circle of centre A, and P is a variable point on the circumference; BP meets the circle again at Q, and the parallel to AQ through B meets AP at R. Prove that the locus of R is an ellipse or a branch of a hyperbola according as B is inside or outside the circle.

7 The tangent at P to $x^2/a^2 - y^2/b^2 = 1$ meets the asymptotes at Q, R; VQ, VR are parallel to the axes of the hyperbola. Prove that V lies on one of the rectangular hyperbolas $xy = \pm ab$.

8 The tangents at P, Q to a hyperbola meet an asymptote at H, K. Prove that PQ passes through the mid-point of HK. [Use $xy = c^2$, oblique axes.]

9 The tangents at two points P, P' on a hyperbola meet one asymptote at Q, Q', and the other at R, R'. Prove QR', $Q'R$ are parallel.

10 For the hyperbola $x = \frac{1}{2}a(t+t^{-1})$, $y = \frac{1}{2}b(t-t^{-1})$, find the equation of the chord joining the points t given by $ut^2+vt+w = 0$.

11 Prove that tangents to $y^2 = kx$ at (kt_1^2, kt_1) and (kt_2^2, kt_2) meet at the point $\left(kt_1t_2, \frac{1}{2}k(t_1+t_2)\right)$.

The tangent to $y^2 = kx$ at any point P cuts the hyperbola $x^2-4y^2 = k^2$ at U and V. If Q, R are the points of contact of the other tangents from U, V to the parabola, prove that the chord QR touches the circle $x^2+y^2 = k^2$.

12 Prove that the chord of contact from any point (x_1, y_1) on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ to the ellipse $x^2/a^2+y^2/b^2 = 1$ touches the hyperbola, and give the coordinates of the point of contact.

13 A variable chord of $x^2/a^2 - y^2/b^2 = 1$ touches the circle $x^2+y^2 = c^2$. Prove that the mid-point of this chord lies on

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2 = c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right).$$

14 The normal at a variable point P on the rectangular hyperbola $xy = c^2$ meets the asymptotes at Q, R. Prove that the mid-point of QR lies on the curve $4x^3y^3+c^2(x^2-y^2)^2 = 0$. Explain why this is the same locus as in Ex. 18 (*c*), no. 28.

15 Prove that the tangent and normal at P on a hyperbola bisect the interior and exterior angles (respectively) between SP, $S'P$. [Either prove $SG = e.SP$ and use pure geometry (Ex. 17 (*c*), no. 3); *or* use $r'-r = \pm 2a$ and calculus (8.14, ex. (iv)).]

16 If Y, Y' are the feet of the perpendiculars from S, S' to the tangent at P to the hyperbola, prove that $SY.S'Y' = b^2$ and that triangles SPY, $S'PY'$ are similar. Hence prove that $p/r = (b^2/p)/S'P$, where $p = SY$ and $r = SP$. Deduce from the bifocal property that *the* (p, r) *equation of the hyperbola wo the focus* S *as pole is* $b^2/p^2 = 1 \pm 2a/r$, *where* $+$ *corresponds to the branch enclosing* S.

17 If an ellipse and a hyperbola have the same foci S, S' (i.e. are *confocal*), prove that they cut orthogonally at each common point P. [The tangents at P are respectively the external and internal bisectors of angle SPS'.]

18 (i) Prove that the equation $x^2/(a^2+\lambda)+y^2/(b^2+\lambda) = 1$ $(a > b)$ represents an ellipse, a hyperbola, or nothing according as $\lambda > -b^2$, $-a^2 < \lambda < -b^2$ or $\lambda < -a^2$. What happens when (*a*) $\lambda = -b^2$; (*b*) $\lambda = -a^2$?

(ii) Prove that the foci of the above conic are independent of λ. (When λ varies we obtain a *family of confocal conics*.)

*19 Prove that through any given point P in the plane (other than the common foci) pass two conics of the family in no. 18, one of which is an ellipse and the other a hyperbola. [Express the equation as a quadratic in λ, say $f(\lambda) = 0$; its discriminant reduces to $(a^2-b^2-x^2+y^2)^2+4x^2y^2$, which is positive (unless

$x^2 = a^2 - b^2$ and $y = 0$), so there are distinct roots $\lambda_1 > \lambda_2$. Since $f(\infty) > 0$, $f(-b^2) < 0$, $f(-a^2) > 0$, the roots satisfy $-a^2 < \lambda < -b^2$, $-b^2 < \lambda$ respectively.] (From 5.72, ex. (ii) or from no. 17 it follows that this ellipse and hyperbola cut orthogonally at P.)

20 Write down the condition for the lines $px^2 + 2rxy + qy^2 = 0$ to be conjugate diameters of (i) $x^2/a^2 - y^2/b^2 = 1$; (ii) the rectangular hyperbola $xy = c^2$.

21 (i) Obtain the *distance quadratic* for the hyperbola $x^2/a^2 - y^2/b^2 = k$.

(ii) Deduce that if chords P_1QR, $P_1Q'R'$ of the hyperbola in (i) are drawn through P_1 *in given directions*, then the ratio $P_1Q \, . \, P_1R : P_1Q' \, . \, P_1R'$ is independent of P_1 *and* k.

*(iii) Hence prove the analogue of Newton's theorem (17.61, ex.) for the pair of conjugate hyperbolas $x^2/a^2 - y^2/b^2 = \pm 1$.

*22 Obtain the equation $x^2/a_1^2 - y^2/b_1^2 = 1$ of the hyperbola referred to a pair of conjugate diameters OP, OD as coordinate axes, where $OP = a_1$ and $OD = b_1$. Explain why the new equation of the asymptotes is $x^2/a_1^2 - y^2/b_1^2 = 0$. [See 17.64, ex. (iv).]

Give the centre, semi-axes, eccentricity, foci, directrices, and asymptotes of the following hyperbolas.

23 $9y^2 - 4x^2 = 36$.

24 $\frac{4}{9}(x+1)^2 - \frac{1}{4}(y+2)^2 = 1$.

25 $144x^2 - 25y^2 + 50y = 169$.

19

THE GENERAL CONIC; $s = ks'$

19.1 The locus $s = 0$

19.11 Scheme of procedure

It has now been shown that the general equation of the second degree

$$s \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents either a conic (in the wide sense of 15.72), a circle, a pair of parallel lines, or nothing; cf. the last paragraph in 15.74. In this chapter it will be convenient to refer to the locus $s = 0$ as a 'conic', even when it is a circle or a parallel line-pair.

We begin the chapter by applying to the locus $s = 0$ the general methods for finding chords, tangents, etc. already illustrated individually for the parabola, ellipse, and hyperbola. The work will thus be a summary and unification of 'conics', and will contain as particular cases many results already obtained for the special standard forms of equation of these curves. Had the general case been treated first, considerable repetition would have been avoided; but dealing with the special forms separately increases the appreciation of the methods themselves. Some of our work will apply unmodified when $s = 0$ represents a line-pair. The reader should consider whether the results remain significant in this degenerate case.

The second part of the chapter is independent of the first, and is concerned with a principle which permeates the whole of coordinate geometry. The student who is pressed for time may turn at once to 19.5 *et seq.*

19.12 Notation

To simplify the writing we introduce the following notation.

(*a*) We continue to put

$$s \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

(*b*) We write

$$s_{ij} \equiv ax_i x_j + h(x_i y_j + x_j y_i) + by_i y_j + g(x_i + x_j) + f(y_i + y_j) + c.$$

Clearly $s_{ij} \equiv s_{ji}$.

(c) The result of omitting the suffix j throughout in (b) is denoted by s_i; thus

$$s_i \equiv ax_ix+h(x_iy+xy_i)+by_iy+g(x+x_i)+f(y+y_i)+c.$$

This is what would be obtained from s by applying the 'rule of alternate suffixes' (15.63, Remark).

For example,

$$s_{11} \equiv ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c,$$

$$s_{12} \equiv ax_1x_2+h(x_1y_2+x_2y_1)+by_1y_2+g(x_1+x_2)+f(y_1+y_2)+c,$$

$$s_{22} \equiv ax_2^2+2hx_2y_2+by_2^2+2gx_2+2fy_2+c,$$

$$s_1 \equiv ax_1x+h(x_1y+xy_1)+by_1y+g(x+x_1)+f(y+y_1)+c,$$

$$s_2 \equiv ax_2x+h(x_2y+xy_2)+by_2y+g(x+x_2)+f(y+y_2)+c.$$

Notice that

$$s_i \equiv (ax_i+hy_i+g)\,x+(hx_i+by_i+f)\,y+(gx_i+fy_i+c),$$

and $s_i \equiv (ax+hy+g)\,x_i+(hx+by+f)\,y_i+(gx+fy+c).$

There are similar forms for s_{ij}.

Since s_i is linear in x and y, the equation $s_i = 0$ represents a line. The geometrical interpretation of s, s_{11}, s_{12} is therefore as follows:

$s = 0$ is the equation of the conic;

$s_{11} = 0$ is the condition for P_1 to lie on this conic;

$s_{12} = 0$ is the condition for P_1 to lie on the line $s_2 = 0$,

or for P_2 to lie on the line $s_1 = 0$.

Not only does this notation make discussion of the general conic as concise as for the various special forms, but it provides a suggestive means of obtaining and memorising some of the results themselves; e.g. see 19.13. The reader should check each of the following general results with the corresponding ones obtained for the special equations used in Chs. 16–18; but first do Ex. 19 (a), nos. 1–3.

19.13 Chord P_1P_2 of $s=0$

Consider the equation

$$s_1+s_2 = s_{12}.$$

It is linear in x and y, and therefore represents a line. It is satisfied by P_1, for $s_{11} = 0$ because P_1 lies on $s = 0$; and it is satisfied by P_2 since $s_{22} = 0$. Hence *it is the equation of the chord* P_1P_2.

Letting $P_2 \to P_1$ along the conic, we obtain $2s_1 = s_{11}$ as the limit of the above equation, i.e. $2s_1 = 0$. Hence *the equation of the tangent to* $s = 0$ *at* P_1 *is* $\mathbf{s_1 = 0}$.

19.2 Joachimsthal's ratio equation

19.21 The ratio quadratic for $s=0$

The point dividing P_1P_2 in the ratio $k:l$ has coordinates

$$\left(\frac{lx_1+kx_2}{l+k}, \frac{ly_1+ky_2}{l+k}\right).$$

It will lie on $s = 0$ if and only if

$$a(lx_1+kx_2)^2+2h(lx_1+kx_2)(ly_1+ky_2)+b(ly_1+ky_2)^2 \\ +2g(lx_1+kx_2)(l+k)+2f(ly_1+ky_2)(l+k)+c(l+k)^2 = 0,$$

i.e. $$(ax_2^2+2hx_2y_2+by_2^2+2gx_2+2fy_2+c)k^2 \\ +2\{ax_1x_2+h(x_1y_2+x_2y_1)+by_1y_2+g(x_1+x_2)+f(y_1+y_2)+c\}kl \\ +(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)l^2 = 0,$$

i.e. $$\mathbf{s_{22}k^2+2s_{12}kl+s_{11}l^2 = 0.}$$

This quadratic in $k:l$ is *Joachimsthal's ratio equation for* $s = 0$. Its roots are the values of $P_1A:AP_2$ for the points of intersection A of the line P_1P_2 and the conic. Since the quadratic will have either two, one, or no roots,† *a line meets a conic in two, one, or no points.*

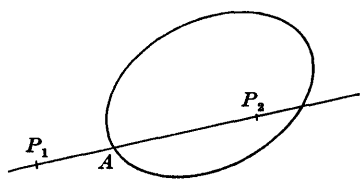

Fig. 186

The above work applies even if $s = 0$ represents a line-pair.

We now make a sequence of deductions from the ratio equation.

19.22 Sides of a conic

When s_{11} and s_{22} have opposite signs, the product of the roots is negative. Hence just one root is positive, so that the conic meets P_1P_2 internally only once. We say that P_1 and P_2 lie on *opposite sides* of the conic. Also see the Remark in 19.24.

† Unless $s_{11} = 0$, $s_{12} = 0$ and $s_{22} = 0$, in which event the quadratic is satisfied for all k and l, i.e. every point if P_1P_2 lies on $s = 0$, which is therefore degenerate.

19.23 Tangent at P_1

If P_1 is a fixed point on $s = 0$, then $s_{11} = 0$ and so one root of the quadratic for $k:l$ is 0. The second root, which corresponds to the remaining intersection of P_1P_2 and the conic, will also be zero if and only if $s_{12} = 0$. In this case P_2 will lie on the tangent at P_1; and the condition $s_{12} = 0$ shows that P_2 lies on the line $s_1 = 0$. Hence *the tangent at P_1 has equation* $\mathbf{s_1 = 0}$ (cf. 19.13).

Example*

Contact condition for $lx+my+n = 0$.

The tangent at P_1 has equation

$$(ax_1+hy_1+g)x+(hx_1+by_1+f)y+(gx_1+fy_1+c) = 0,$$

and will represent the same line as $lx+my+n = 0$ if and only if the coefficients are proportional, i.e. if for some λ we have

$$ax_1+hy_1+g-\lambda l = 0,$$
$$hx_1+by_1+f-\lambda m = 0,$$
$$gx_1+fy_1+c-\lambda n = 0,$$

and also
$$lx_1+my_1+n = 0.$$

By eliminating x_1, y_1, λ from these, we obtain (using an obvious extension of Corollary I (*b*) in 11.43)

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

When expanded this becomes (see Ex. 11 (*d*), no. 12)

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0.$$

19.24 Pair of tangents from P_1

If P_1 is a fixed point, the quadratic for $k:l$ has equal roots if and only if $s_{11}s_{22} = s_{12}^2$. This means that P_1P_2 meets the conic in only one point, i.e. is a tangent from P_1. Hence $s_{11}s_{22} = s_{12}^2$ is the condition for P_2 to lie on any tangent from P_1; but this equation shows that P_2 lies on $s_{11}s = s_1^2$. *The equation of the tangents from P_1 is therefore* $\mathbf{s_{11}s = s_1^2}$.

Remark. By writing $s_{11}s = s_1^2$ in full, it can be shown by using 15.53 (1) and 15.52 (1) that this second-degree equation will represent two distinct intersecting lines, a repeated line, or the single point P_1 according as $\Delta s_{11} \lesseqgtr 0$, where Δ is defined by equation (vii) in 15.53 (2). Hence for a non-degenerate conic (i.e. $\Delta \neq 0$), $s_{11}s = s_1^2$ represents a pair of lines if P_1 lies on that side of the conic for which $\Delta s_{11} < 0$. The *outside* of the conic $s = 0$ can therefore be defined algebraically as the set of points P_1 for which $\Delta s_{11} < 0$.

19.25 Chord of contact from P_1

Method 1. Let the tangents from P_1 (supposed outside the conic) touch $s = 0$ at P_2, P_3. The tangent at P_2 has equation $s_2 = 0$; and since it passes through P_1, we have $s_{12} = 0$. This shows that P_2 lies on the line $s_1 = 0$. Similarly $s_{13} = 0$, so that P_3 also lies on $s_1 = 0$. Hence the equation of P_2P_3 is $s_1 = 0$.

Method 2. The points of contact of tangents from P_1 satisfy both $s = 0$ and $s_{11}s = s_1^2$. Hence they also satisfy $s_1 = 0$. Since s_1 is linear, $s_1 = 0$ *is the equation of the chord of contact of tangents from* P_1.

19.26 Examples; polar of P_1 wo $s=0$

(1) *Tangents at the extremities of a variable chord through* P_1 *meet on* $s_1 = 0$.

Let the tangents at the extremities A, B of such a chord meet at P_2. Then the chord of contact from P_2 is AB, whose equation is therefore $s_2 = 0$. Since this line passes through P_1, $s_{12} = 0$; and this condition shows that P_2 lies on the line $s_1 = 0$.

(2) *Let a chord through* P_1 *cut the conic at* A *and* B. *The point* P_2 *such that* P_1 *and* P_2 *divide* AB *in the same ratio* (*one internally and the other externally*) *lies on the line* $s_1 = 0$.

From the hypothesis it follows that A and B divide P_1P_2 in the same ratio (one internally and one externally). Hence the ratio quadratic must have its roots $k:l$ equal and opposite; this is so if and only if $s_{12} = 0$, which shows that P_2 lies on $s_1 = 0$. P_2 is called the *harmonic conjugate* of P_1 wo $s = 0$.

(3) The diagrams in 15.65, with the circle replaced by any conic, show that for some positions of P_1 the locus of P_2 in (1), (2) above will be only part of the line $s_1 = 0$. We unify these results by making the following definition.

The whole line $s_1 = 0$ is called the *polar* of P_1 wo $s = 0$.

Remarks

(α) If P_1 lies outside s, the polar coincides with the chord of contact from P_1.

(β) If P_1 lies on s, the polar coincides with the tangent at P_1.

(γ) In full, the polar of P_1 is

$$s_1 \equiv (ax_1+hy_1+g)\,x+(hx_1+by_1+f)\,y+(gx_1+fy_1+c) = 0.$$

It does not exist if $ax_1+hy_1+g = 0$ and $hx_1+by_1+f = 0$; in general there is a unique point P_1 satisfying these conditions.

(δ) The argument shows (without modification) that the polar of P_1 wo a *line-pair* $s = 0$ is $s_1 = 0$. Since

$$s_1 \equiv (ax+hy+g)\,x_1+(hx+by+f)\,y_1+(gx+fy+c),$$

the polar of P_1 always passes through the vertex of the line-pair (15·53(3)).

(ϵ) Fig. 187 shows the polar of P_1 as

(i) a chord of contact TT' (Remark (α));

(ii) the harmonic locus, (2);

(iii) the locus of the meets Q of tangents at the ends of chords through P_1, (1).

(ζ) *Reciprocal property.* If the polar of P_1 passes through P_2, then the polar of P_2 passes through P_1. (This is valid for a line-pair provided P_2 is not the vertex.) For the polar of P_1 wo $s = 0$ is $s_1 = 0$, and this passes through P_2 if $s_{12} = 0$; this shows that P_1 lies on $s_2 = 0$, which is the polar of P_2 wo $s = 0$. In fig. 187, P_1Q is the polar of P_2.

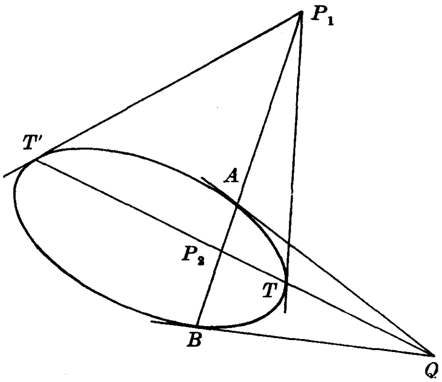

Fig. 187

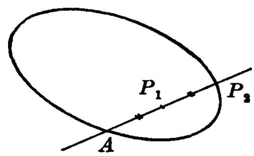

Fig. 188

19.27 Chord whose mid-point is P_1

Let the extremities of the chord be A and P_2 (fig. 188). Then $s_{22} = 0$, and the ratio quadratic has one root $l = 0$. The other root, given by

$$2ks_{12}+ls_{11} = 0,$$

corresponds to the point A; and since A divides P_1P_2 externally in the ratio $1:2$, this root must be $k/l = -\frac{1}{2}$. Hence $s_{12} = s_{11}$, which shows that P_2 lies on the locus $s_1 = s_{11}$. Since this is linear, and is also satisfied by P_1, it is the required chord. Thus *the chord of $s = 0$ whose mid-point is P_1 is* $\boldsymbol{s_1 = s_{11}}$.

Example

Show that the mid-points of all chords of $s = 0$ which pass through P_1 lie on the curve $s = s_1$.

Let P_2 be the mid-point of such a chord, which will have equation $s_2 = s_{22}$. Since the chord passes through P_1, $s_{12} = s_{22}$. This shows that P_2 lies on the locus $s_1 = s$ which, being of second degree, is also a conic; clearly it passes through P_1.

19.28 Diameters

Let P_1 be the mid-point of a chord of gradient m. Then this chord has equation

$$\frac{y-y_1}{m} = \frac{x-x_1}{1}.$$

Its equation is also $s_1 = s_{11}$, which can be written

$$ax_1(x-x_1)+h\{x_1(y-y_1)+y_1(x-x_1)\}+by_1(y-y_1)$$
$$+g(x-x_1)+f(y-y_1) = 0.$$

Hence $$ax_1+h(mx_1+y_1)+bmy_1+g+fm = 0,$$

which shows that P_1 lies on the line

$$(ax+hy+g)+m(hx+by+f) = 0.$$

By definition (16.41) *this is the equation of the diameter bisecting all chords of gradient m.*

19.29 Conjugate diameters

The diameter bisecting all chords of gradient m has gradient

$$m' = -\frac{a+hm}{h+bm},$$

so that $$a+h(m+m')+bmm' = 0.$$

The symmetry in m, m' of this relation shows that the diameter of gradient m bisects all chords of gradient m'. Two such diameters are *conjugate.*

19.3 The distance quadratic

The line through P_1 in direction θ has parametric equations

$$x = x_1+r\cos\theta, \quad y = y_1+r\sin\theta. \tag{i}$$

It meets $s = 0$ at points for which r is given by

$$a(x_1+r\cos\theta)^2+2h(x_1+r\cos\theta)(y_1+r\sin\theta)+b(y_1+r\sin\theta)^2$$
$$+2g(x_1+r\cos\theta)+2f(y_1+r\sin\theta)+c = 0,$$

i.e. $$r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)$$
$$+2r\{(ax_1+hy_1+g)\cos\theta+(hx_1+by_1+f)\sin\theta\}+s_{11} = 0. \tag{ii}$$

This quadratic in r gives the signed distances from P_1 of the points of the conic $s = 0$ which lie on the line (i) through P_1 in direction θ.

Examples

(i) *Chord whose mid-point is P_1.*

If a chord of $s = 0$ which passes through P_1 is bisected there, then the roots of equation (ii) must be equal and opposite, and hence

$$(ax_1+hy_1+g)\cos\theta+(hx_1+by_1+f)\sin\theta = 0,$$

which gives the direction $\cos\theta:\sin\theta$ of such a chord. If (x,y) is any point on this chord, then from equation (i),

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta},$$

and so $$(ax_1+hy_1+g)(x-x_1)+(hx_1+by_1+f)(y-y_1)=0.$$

This is therefore the equation of the chord of $s=0$ which has mid-point P_1; it can be arranged in the form $s_1 \doteq s_{11}$ (cf. 19.27).

(ii) *Segments of a chord.*

If the line (i) cuts the conic at Q, R, then by considering the product of the roots of (ii) we have

$$P_1Q.P_1R=r_1r_2=\frac{s_{11}}{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta}.$$

(*a*) For chords QR, $Q'R'$ through P_1 in given directions θ, θ', it follows that

$$\frac{P_1Q.P_1R}{P_1Q'.P_1R'}=\frac{a\cos^2\theta'+2h\cos\theta'\sin\theta'+b\sin^2\theta'}{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta},$$

which is independent of the coordinates of P_1. Hence the ratio of the product of the segments of two chords drawn in *given directions* through the same point is constant for all points.

(*b*) For chords P_1QR, $P_2Q'R'$ drawn through P_1, P_2 in the same direction θ, it also follows that

$$\frac{P_1Q.P_1R}{P_2Q'.P_2R'}=\frac{s_{11}}{s_{22}},$$

which is independent of θ. Hence the ratio of the products of the segments of two parallel chords drawn through *given points* P_1, P_2 is constant for all directions.

19.4 Tangent and normal as coordinate axes

Sometimes it is convenient (as for example in 8.42(1)) to choose the tangent and normal at a particular point for axes of coordinates; this point thus becomes the origin.

If $s=0$ passes through O, then $c=0$. If $s=0$ *touches* $y=0$ at $(0,0)$, the equation $ax^2+2gx+c=0$ must have both roots zero, so that also $g=0$. The equation of the conic is therefore

$$ax^2+2hxy+by^2+2fy=0.$$

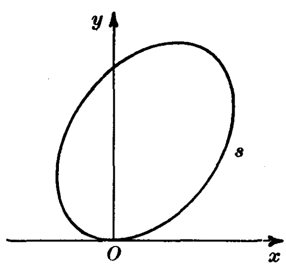

Fig. 189

Example

Frégier's point.

O is a fixed point on a non-degenerate conic $s=0$, and PQ is a chord which subtends a right-angle at O. Prove that PQ passes through a fixed point (the Frégier point of O wo s) on the normal at O, or else is parallel to this normal.

Choosing axes as above, let PQ have equation $lx+my = 1$. (This form of equation for PQ is legitimate since the hypothesis that 'PQ subtends a right-angle at O' implies that PQ does not pass through O.)

The line-pair OP, OQ has equation

$$ax^2+2hxy+by^2+2fy(lx+my) = 0, \tag{i}$$

by 15.54. Since $OP \perp OQ$, then from 15.52 (2)

$$a+(b+2fm) = 0.$$

If $a+b \neq 0$, then $m = -(a+b)/2f \neq 0$. Hence $lx+my = 1$ passes through $(0, -2f/(a+b))$, which is a fixed point on Oy, the normal at O.

If $a+b = 0$, then the perpendicularity condition becomes $fm = 0$. If $f = 0$, the conic $s = 0$ is an orthogonal line-pair—excluded by hypothesis; in such a case *every* 'chord' subtends a right-angle at O. If $m = 0$, the equation of PQ is $lx = 1$, which is a line parallel to Oy.

Exercise 19(*a*)

Write out in full the expressions for s_{11}, s_{12} *when* s *denotes*

1 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1.$ 2 $y^2-4ax.$ 3 $xy-c^2.$

*4 $(lx+my+n)(l'x+m'y+n')$; and if $u \equiv lx+my+n$, $u' \equiv l'x+m'y+n'$, show that $s_{12} = \frac{1}{2}(u_1u_2'+u_2u_1')$.

5 Interpret the equation $s_{11}+2s_{12}+s_{22} = 0$.

6 Obtain the equation of the tangent to $s = 0$ at P_1 (i) by Calculus; (ii) from $ss_{11} = s_1^2$ by taking P_1 on $s = 0$.

7 Prove that the normal to $s = 0$ at P_1 has equation

$$(y-y_1)(ax_1+hy_1+g) = (x-x_1)(hx_1+by_1+f).$$

8 Find the condition for the lines $px^2+2rxy+qy^2 = 0$ to be parallel to conjugate diameters of $s = 0$.

9 Prove that the line-pair joining O to the meets of $x^2+y^2 = r^2$ and $ax^2+2hxy+by^2 = 1$ is

$$\left(a-\frac{1}{r^2}\right)x^2+2hxy+\left(b-\frac{1}{r^2}\right)y^2 = 0.$$

Show that these will be conjugate diameters of the conic if

$$\frac{1}{r^2} = \frac{2(ab-h^2)}{a+b}.$$

Hence obtain the equation of the equi-conjugate diameters of

$$ax^2+2hxy+by^2 = 1.$$

10 Writing $u \equiv ax+hy+g$, $v \equiv hx+by+f$, $w \equiv gx+fy+c$, verify that

$$s \equiv ux+vy+w, \quad s_1 \equiv u_1x+v_1y+w_1 \equiv ux_1+vy_1+w,$$

and

$$s_{11} \equiv u_1x_1+v_1y_1+w_1.$$

Use the distance quadratic for $s = 0$ *in nos.* 11–13.

11 Obtain the equation of the tangent at P_1, and reduce it to the form $s_1 = 0$ by using no. 10.

12 (i) Prove that the equation of the pair of tangents from P_1 is

$$\{u_1(x-x_1)+v_1(y-y_1)\}^2 = \{a(x-x_1)^2+2h(x-x_1)(y-y_1)+b(y-y_1)^2\}s_{11}.$$

(ii) Using no. 10, show that this is equivalent to $(s_1-s_{11})^2 = (s-2s_1+s_{11})s_{11}$ and hence to $s_1^2 = ss_{11}$.

13 A line through O cuts $s = 0$ at P and Q. A point R is chosen on this line so that OR is (i) the arithmetic mean; (ii) the geometric mean; (iii) the harmonic mean, of OP, OQ. Find the locus of R in each case.

14 K is the point ϕ on the ellipse $x^2/a^2+y^2/b^2 = 1$. Write down the equation of the chord PQ for which KP, KQ are parallel to the axes, and find where it meets the normal at K. Deduce from the example in 19.4 that chords which subtend a right-angle at the fixed point ϕ are concurrent at

$$\left(\frac{a^2-b^2}{a^2+b^2}a\cos\phi,\ -\frac{a^2-b^2}{a^2+b^2}b\sin\phi\right).$$

15 If equation (i) in 19.4 represents lines OP, OQ equally inclined to Oy, use 15.52 (3) to prove $h+fl = 0$. Deduce that if OP, OQ are equally inclined to the normal to $s = 0$ at O, then PQ passes through a fixed point on the tangent at O, or is parallel to this tangent.

19.5 Number of conditions which a conic can satisfy

If k is any non-zero constant, the equations

$$s \equiv ax^2+2hxy+by^2+2gx+2fy+c = 0$$

and
$$ks \equiv kax^2+2khxy+kby^2+2kgx+2kfy+kc = 0$$

represent the same locus; for if P_1 satisfies one, then it also satisfies the other. Since a, b, c, f, g, h are not all zero, we can always choose k so that one coefficient in ks is 1. The remaining five coefficients (i.e. the five *ratios* $a:b:c:f:g:h$) can be chosen so as to satisfy five conditions, but in general no more. The equations which express these conditions may have more than one set of solutions but, since they are *polynomial* equations, the number of sets will be finite if the conditions are independent.

It follows that a conic can be chosen to satisfy five independent conditions, and that the number of such conics is finite.

Examples

(i) *In general a unique conic can be drawn through five given points* P_r $(r = 1, \ldots, 5)$.

For in general the five ratios $a:b:c:f:g:h$ can be determined uniquely from the five linear equations

$$ax_r^2+2hx_ry_r+by_r^2+2gx_r+2fy_r+c = 0 \quad (r = 1, \ldots, 5),$$

so that $s = 0$ is the equation of the required conic.

Alternatively, a, b, c, f, g, h can be eliminated from the above equations together with $s = 0$, giving the required equation in determinant form.

An easier way of finding the equation of a conic through five given points will be illustrated in 19.64, ex. (i).

(ii) *If three of the five points in ex.* (i) *lie on a line l, then l must be part of the conic, which therefore consists of two lines l, l'.*

For no line can meet a non-degenerate conic in three points (see the footnote on p. 666); and the only line-pair through these five points is l and the line l' which joins the other two.

If four of the points are collinear on l, the conic consists of l and *any* line l' through the fifth point. If all five points lie on l, the conic consists of l and *any* line in the plane. In these two cases of ex. (i) the conic is not unique.

19.6 The equation $s=ks'$

19.61 Number of possible intersections of two conics

The equations of the two given conics can be arranged in the form

$$s \equiv ax^2+2(hy+g)x+(by^2+2fy+c) = 0,$$

$$s' \equiv a'x^2+2(h'y+g')x+(b'y^2+2f'y+c') = 0.$$

The coordinates of any common point of these must satisfy the equation in y obtained by elimination of x. Since this equation will in general be quartic in y, there are 4, 2 or no possible values of y (apart from possible coincidences). By eliminating x^2 only, we see that to each value of y corresponds at most one value of x, given by

$$2\{(a'h-ah')y+(a'g-ag')\}x + \{(a'b-ab')y^2+2(a'f-af')y+(a'c-ac')\} = 0.$$

Hence, *ignoring coincidences, the number of possible intersections of two conics is* 4, 2 *or* 0.

19.62 $s=ks'$

This equation has already been discussed when $s = 0$, $s' = 0$ represent lines (15.41) or circles (Ex. 15 (*d*), no. 16).

(1) *If $s = 0$, $s' = 0$ are conics and k is constant, then $s = ks'$ is also the equation of a conic, which passes through the meets* (*if any*) *of s and s'.*

For if P satisfies both $s = 0$ and $s' = 0$, then it also satisfies $s = ks'$. Since k is constant, this last equation is of second degree and therefore represents a conic (or nothing).

(2) *If s, s' meet in four points A, B, C, D, then any conic σ through these points has an equation of the form $s = ks'$.*

If P_1 is on σ, k can be chosen so that $s = ks'$ passes through P_1, viz. $k = s_{11}/s'_{11}$ (assuming that P_1 is not on $s' = 0$: see the Remark below). For this k, the conics σ and $s = ks'$ have five points in common, and hence coincide (19.5, ex. (i)).

Exceptions to the last statement may arise when three of the five points are collinear. Since P_1 is an *arbitrary* point of σ, P_1 need not be collinear with any two of A, B, C, D. If A, B, C are collinear, then s, s' must be of the forms $\alpha\beta$, $\alpha\gamma$ where $\alpha = 0$ is the line ABC and $\beta = 0$, $\gamma = 0$ are lines through D (19.5, ex. (ii)). Every conic through A, B, C, D then consists of $\alpha = 0$ and a line $\beta = k\gamma$ through D. Hence

$$\sigma \equiv \alpha(\beta - k\gamma) \equiv s - ks',$$

and the theorem is still valid.

Remark. The only conic through the meets A, B, C, D of s and s' which does not have an equation of the form $s = ks'$ is s' itself. This exception can be avoided by using $\lambda s = \lambda' s'$, where λ and λ' are independent constants; but usually the equation $s = ks'$ is adequate.

For different k or $\lambda : \lambda'$, the equation $s = ks'$ or $\lambda s = \lambda' s'$ represents a family or *system* or *pencil* of conics through the intersections (if any) of s and s'.

19.63 Degenerate cases

The results in 19.62 apply when one or both of s, s' degenerate, and these cases are particularly useful.

(i) $s = k\alpha\beta$. This represents a conic through the meets (if any) of the distinct lines $\alpha = 0$, $\beta = 0$ with $s = 0$ (fig. 190).

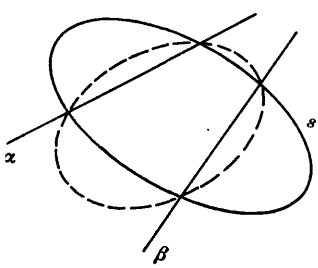

Fig. 190

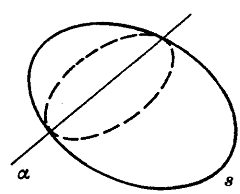

Fig. 191

(ii) $s = k\alpha^2$. When $\beta \to \alpha$ in (i), we have the equation of a conic touching s at each of its intersections (if any) with $\alpha = 0$. It is said to have *double contact* with s along $\alpha = 0$ (fig. 191).

(iii) $s = k\alpha\tau$, *where τ is a tangent to s.* When $\beta \to \tau$ in (i), we have a conic meeting s at its intersections (if any) with $\alpha = 0$, and touching it at the contact of $\tau = 0$ with s (fig. 192).

(iv) $s = k\alpha\beta$, *where α, β meet on s.* If one meet of α and s coincides with one meet of β and s in (i), we obtain a conic meeting s at a point on α and at a point on β, and touching s at the meet of α and β (fig. 193).

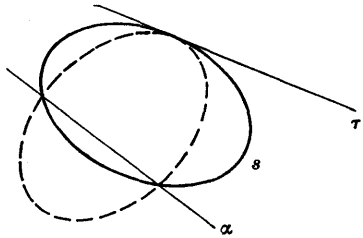

Fig. 192

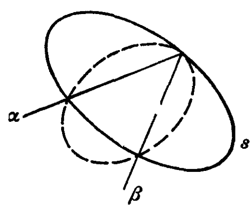

Fig. 193

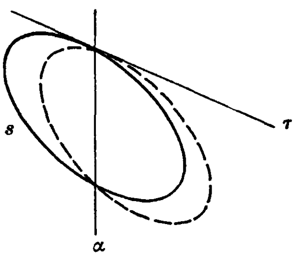

Fig. 194

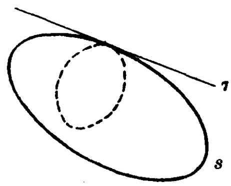

Fig. 195

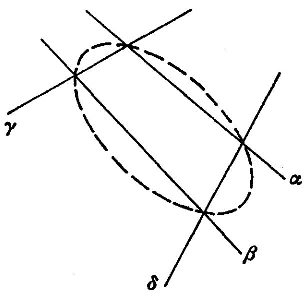

Fig. 196

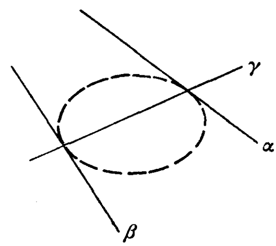

Fig. 197

(v) $s = k\alpha\tau$, *where α passes through the contact of τ.* If one meet of α and s in (iii) coincides with the contact of τ, we obtain a conic meeting s at a point on α and touching s at the meet of α and τ (fig. 194).

(vi) $s = k\tau^2$. If $\alpha \to \tau$ in (v), we get a conic touching s at its contact with τ (fig. 195).

(vii) $\alpha\beta = k\gamma\delta$ is a conic through the meets of the lines α, β, γ, δ (fig. 196).

(viii) $\alpha\beta = k\gamma^2$. If $\delta \to \gamma$ in (vii), we have a conic touching the lines α, β at their meets with γ (fig. 197).

(ix) $\alpha^2 = k\gamma^2$ represents the two lines $\alpha = \pm\sqrt{k}\gamma$, which pass through the meet of α, γ if this exists.

In the next two cases s' represents a *single* line.

(x) $s = k\alpha$ is a conic through the meets (if any) of s and α.

(xi) $s = k\tau$ is a conic touching s at its contact with τ.

19.64 Examples

The '$s = ks'$' principle has already been used for conics in Ex. 15 (f), no. 15; Ex. 16 (b), no. 1; and in the alternative method in 17.41.

(i) *Find the equation of the conic through* (3, 0), (0, −2), (5, 0), (0, 1), (15, 6).
The lines joining the first four points in pairs are as follows.

$$(3,0),\ (0,-2):\quad \frac{x}{3}-\frac{y}{2}-1=0; \qquad (5,0),\ (0,1):\quad \frac{x}{5}+y-1=0;$$

$$(0,-2),\ (0,1):\quad x=0; \qquad (3,0),\ (5,0):\quad y=0.$$

Hence any conic through the first four points is (by 19.63, (vii))

$$\left(\frac{x}{3}-\frac{y}{2}-1\right)\left(\frac{x}{5}+y-1\right) = kxy.$$

This passes through (15, 6) if $1.8 = k.15.6$, i.e. $k = \frac{4}{45}$. The required conic is therefore

$$3(2x-3y-6)(x+5y-5) = 8xy.$$

(ii) *Find the equation of the circumcircle of the triangle formed by the lines* $ax^2+2hxy+by^2 = 0$ *and* $lx+my+n = 0$.

Let the tangent at O to the required circle be $px+qy = 0$. Then by 19·63, (iii) its equation will be

$$ax^2+2hxy+by^2 = (lx+my+n)(px+qy)$$

provided we choose p and q so that

$$a-pl = b-qm \quad \text{and} \quad 2h = mp+lq.$$

If we arrange the above three equations in terms of p and q, we have

$$(lx+my+n)xp+(lx+my+n)yq-(ax^2+2hxy+by^2) = 0,$$

$$lp-mq-(a-b) = 0,$$

and
$$mp+lq-2h = 0.$$

By eliminating p, q, we obtain

$$\begin{vmatrix} (lx+my+n)x & (lx+my+n)y & ax^2+2hxy+by^2 \\ l & -m & a-b \\ m & l & 2h \end{vmatrix} = 0$$

as the equation of the required circle.

(iii) *Concyclic points on the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

The chords $\phi_1\phi_2$ and $\phi_3\phi_4$ are $\alpha = 0$, $\beta = 0$, where

$$\alpha \equiv \frac{x}{a}\cos\tfrac{1}{2}(\phi_1+\phi_2)+\frac{y}{b}\sin\tfrac{1}{2}(\phi_1+\phi_2)-\cos\tfrac{1}{2}(\phi_1-\phi_2),$$

$$\beta \equiv \frac{x}{a}\cos\tfrac{1}{2}(\phi_3+\phi_4)+\frac{y}{b}\sin\tfrac{1}{2}(\phi_3+\phi_4)-\cos\tfrac{1}{2}(\phi_3-\phi_4).$$

The points $\phi_1, \phi_2, \phi_3, \phi_4$ will be concyclic if and only if one conic of the system

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = k\alpha\beta$$

is a circle. This requires that the coefficients of x^2 and y^2 shall be equal, and the coefficient of xy zero. In general it is not possible for the single number k to satisfy these two conditions; but the coefficient of xy is zero if

$$\cos\tfrac{1}{2}(\phi_1+\phi_2)\sin\tfrac{1}{2}(\phi_3+\phi_4)+\cos\tfrac{1}{2}(\phi_3+\phi_4)\sin\tfrac{1}{2}(\phi_1+\phi_2) = 0,$$

i.e.
$$\sin\tfrac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4) = 0,$$

i.e.
$$\phi_1+\phi_2+\phi_3+\phi_4 = 2n\pi$$

for some integer n.

The coefficients of x^2 and y^2 are then equal if

$$\frac{1}{a^2}-\frac{1}{b^2} = k\left\{\frac{1}{a^2}\cos\tfrac{1}{2}(\phi_1+\phi_2)\cos\tfrac{1}{2}(\phi_3+\phi_4)-\frac{1}{b^2}\sin\tfrac{1}{2}(\phi_1+\phi_2)\sin\tfrac{1}{2}(\phi_3+\phi_4)\right\}$$

$$= k\left\{\frac{1}{a^2}\cos^2\tfrac{1}{2}(\phi_1+\phi_2)+\frac{1}{b^2}\sin^2\tfrac{1}{2}(\phi_1+\phi_2)\right\}\cos n\pi,$$

and here the coefficient of k is non-zero.

Hence if $\Sigma\phi_1 = 2n\pi$, k can be chosen uniquely to satisfy the second condition. Thus *the relation $\Sigma\phi_1 = 2n\pi$ is necessary and sufficient for the points $\phi_1, \phi_2, \phi_3, \phi_4$ of the ellipse to be concyclic.* Compare the example in 17.32.

*(iv) *Normals at the extremities of the chords $lx+my = 1$, $l'x+m'y = 1$ of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

The conic
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = k(lx+my-1)(l'x+m'y-1)$$

passes through the extremities of the chords, by 19.63, (i). *If* the normals at these extremities are concurrent, say at (x_1, y_1), then their feet lie on

$$(a^2-b^2)xy+b^2y_1x-a^2x_1y = 0$$

by Ex. 17 (*c*), no. 10. Consequently this conic must belong to the above system. Comparing coefficients of x^2, y^2, and the constant terms, we have

$$\frac{1}{a^2}-kll' = 0,\quad \frac{1}{b^2}-kmm' = 0,\quad 1+k = 0.$$

Hence *if the normals at the extremities of the chords $lx+my = 1$, $l'x+m'y = 1$ of $x^2/a^2+y^2/b^2 = 1$ are concurrent, then $a^2ll' = b^2mm' = -1$.* Also see Ex. 19 (*b*), no. 14.

*(v) *A conic passes through four given points A, B, C, D on $s = 0$ and also through the meet P_1 of tangents at A, B. Prove that it passes through the meet P_2 of tangents at C, D.*

Since the equations of the chords of contact AB, CD are respectively $s_1 = 0$, $s_2 = 0$, the conic is $s = ks_1 s_2$ by 19.63, (i). This passes through P_1, so $s_{11} = ks_{11}s_{12}$; hence $k = 1/s_{12}$, and the conic has equation $s_{12}s = s_1 s_2$. Clearly this is satisfied by P_2.

*(vi) *Pascal's theorem. If 1, 2, 3, 4, 5, 6 denote six points on a conic, then the meets of the lines*

$$12, 45; \quad 23, 56; \quad 34, 61$$

are collinear.

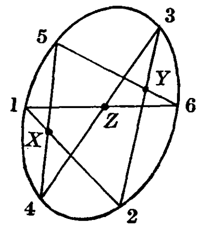

Fig. 198

For suitable k and l, the line-pair (12)(56) has equation $s = k(25)(16)$, and the line-pair (23)(45) has equation $s = l(25)(34)$. The points $X(12, 45)$ and $Y(23, 56)$ lie on both pairs, but do not lie on $s = 0$ or on (25). Hence they lie on the line $k(16) = l(34)$, which passes through $Z(34, 61)$.

The argument holds even when $s = 0$ is a line-pair provided that the sets 1, 3, 5 and 2, 4, 6 lie on different lines; the result is then known as *Pappus's theorem.*

19.65 Contact of two conics

Although in general two distinct conics intersect in at most four points A, B, C, D, there may be coincidences of the following types:

$AACD$	2-point contact at A	(figs. 192, 193),
$AACC$	double contact along AC	(fig. 191),
$AAAD$	3-point contact at A	(fig. 194),
$AAAA$	4-point contact at A	(fig. 195).

Since a circle can be determined to pass through only three general points, we may expect that *in general* a circle cannot have more than 3-point contact with a conic at A.† By regarding such a circle as the limit when $P \to A$ along the conic of a circle touching the conic at A and cutting it at P, we see from 8.42 (2) that *the circle having 3-point contact at A is the circle of curvature of the conic at A.*

A conic having 3-point contact with $s = 0$ at A is $s = k\alpha\tau$, where $\tau = 0$ is the tangent at A to s and $\alpha = 0$ is any chord through A. If k and α are chosen so that this conic is a *circle*, it will be the circle of curvature at A. It is known (e.g. Ex. 16 (*b*), no. 16 (i); Ex. 17 (*a*), no. 13) that α and τ must be equally inclined to the axes.

† For examples of exceptions, see Ex. 19 (*b*), nos. 9, 10.

Example

Find the circle of curvature of $y^2 = 4ax$ at the point t.

The tangent at the point t is $\tau \equiv x-ty+at^2 = 0$, and has gradient $1/t$. The chord through $(at^2, 2at)$ with gradient $-1/t$ is

$$y-2at = -\frac{1}{t}(x-at^2),$$

i.e.
$$\alpha \equiv x+ty-3at^2 = 0.$$

The required circle is

$$y^2-4ax = k(x+ty-3at^2)(x-ty+at^2),$$

where k must be chosen so that the coefficients of x^2 and y^2 are equal (the xy-term is already absent). This gives $k = -kt^2-1$, i.e. $k = -1/(1+t^2)$, so that the circle is
$$x^2+y^2-2a(3t^2+2)x+4at^3y-3a^2t^4 = 0.$$

The *centre of curvature* at t is therefore $(2a+3at^2, -2at^3)$; and the *radius of curvature* is found to be $2a(1+t^2)^{\frac{3}{2}}$. These results can of course be obtained by the methods of Ch. 8.

Remark. If $f(x)$ and $g(x)$ are polynomials, then Remark (α) in 6.72 shows that when the curves $y = f(x)$, $y = g(x)$ have mth-order contact at $x = a$, they have $(m+1)$-point contact in the sense of the present section, and conversely. For general curves the concepts are not equivalent.

19.7 Equations of a type more general than $s=ks'$

If $f = 0$, $g = 0$ represent any two curves, their intersections (if any) satisfy every equation which can be deduced *algebraically* from $f = 0$ and $g = 0$. The new equation need not be of the form $f = kg$; and even when it is, k may not be constant. The problem in 15.54 is of this type.

Sometimes the equations are combined in such a way that the deductions do not hold for *all* the common points.

Examples

(i) *Circle through the feet of conormal points on a parabola.*

The points in question satisfy the equations

$$y^2 = 4ax \quad \text{and} \quad xy+(2a-h)y-2ak = 0,$$

where (h, k) is the point of concurrence of the normals (see 16.32, ex. (iii) (*b*)). Hence they also satisfy

$$xy^2+(2a-h)y^2 = 2aky,$$
$$x(4ax)+(2a-h)y^2 = 2aky,$$
$$4a(x^2+y^2)-(2a+h)4ax = 2aky,$$

and
$$x^2+y^2-(2a+h)x = \tfrac{1}{2}ky.$$

This is the required circle; cf. 16.32, ex. (ii).

(ii) *Find the equation of conics through the meets, other than the origin, of*

$$s \equiv ax^2+2hxy+by^2+2gx+2fy = 0$$

and $$s' \equiv a'x^2+2h'xy+b'y^2+2g'x+2f'y = 0,$$

where $fg' \neq f'g$.

The given equations can be written

$$x(ax+2hy+2g) = -y(by+2f),$$

$$x(a'x+2h'y+2g') = -y(b'y+2f').$$

Coordinates other than (0, 0) which satisfy both these equations also satisfy

$$\sigma \equiv (ax+2hy+2g)(b'y+2f')-(a'x+2h'y+2g')(by+2f) = 0.$$

This is therefore one conic through the meets of s, s', and does not pass through O. Clearly

$$\sigma+ks+k's' = 0 \qquad \text{(i)}$$

also passes through the same points for all k, k'.

If s, s' meet in three points other than O, then k, k' can be chosen to make the conic (i) pass through two other general points. Hence *any* conic through the three points has an equation of this form. The system (i) is called a *net* of conics.

Exercise 19(*b*)

Find the equation of the conic through

1 (0, 0), (0, 1), (3, 0), (2, 1), (2, −3).

2 (2, 3), (−1, 1), (4, 0), (3, −2), (5, 3).

3 Assuming that $ax^2+2hxy+by^2 = c$ and $a'x^2+2h'xy+b'y^2 = 1$ intersect in four points, find the condition for these points to be concyclic.

4 Find the common chords of $x^2+y^2 = 25$ and $x^2+xy+y^2 = 36$.

5 Two chords are drawn through a focus of an ellipse, and a conic is drawn through their extremities and the centre of the ellipse. Prove that this conic cuts the major axis in another fixed point.

6 (i) Prove that, for any c,

$$(1+t^2)(y^2-4ax)+(x-ty+at^2)(x+ty+c) = 0$$

is the equation of a circle which touches $y^2 = 4ax$ at the point t.

(ii) PSQ is a focal chord of a parabola. Circles are drawn through the focus S to touch the parabola at P, Q respectively. Prove that these circles cut orthogonally.

7 A circle has double contact with $ax^2+by^2+c = 0$. Prove that the chord of contact is parallel to either $x = 0$ or $y = 0$. [Let the chord be $lx+my+n = 0$.]

8 Find the equation of the circle touching the ellipse $x^2/a^2+y^2/b^2 = 1$ at each end of a latus rectum.

9 Find the equation of the circle which has 4-point contact with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \quad \text{at } (a, 0).$$

[The circle has equation $s = k\tau^2$, where $\tau \equiv x-a$.]

10 Show that a circle can have 4-point contact with a parabola only at the vertex.

11 Write down the equation of the tangent at the point ϕ of $x^2/a^2 + y^2/b^2 = 1$, and find the equation of the chord through ϕ which has the same inclination to Ox as this tangent. Hence show that the circle of curvature at ϕ is

$$(b^2\cos^2\phi + a^2\sin^2\phi)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$
$$+ (a^2 - b^2)\left(\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1\right)\left(\frac{x}{a}\cos\phi - \frac{y}{b}\sin\phi - \cos 2\phi\right) = 0,$$

and find its centre and radius.

12 Find the circle having 3-point contact with $xy = c^2$ at the point t, and deduce the centre and radius of curvature at this point.

13 Prove that the circle of curvature of $ax^2 + 2hxy + by^2 = 2y$ at O is $a(x^2 + y^2) = 2y$.

*14 Use example (iv) in 19.64 to prove that *if* normals at the ends of the chords $\phi_1\phi_2$ and $\phi_3\phi_4$ of $x^2/a^2 + y^2/b^2 = 1$ are concurrent, then

$$\cos\tfrac{1}{2}(\phi_1 + \phi_2)\cos\tfrac{1}{2}(\phi_3 + \phi_4) + \cos\tfrac{1}{2}(\phi_1 - \phi_2)\cos\tfrac{1}{2}(\phi_3 - \phi_4) = 0$$

and $$\sin\tfrac{1}{2}(\phi_1 + \phi_2)\sin\tfrac{1}{2}(\phi_3 + \phi_4) + \cos\tfrac{1}{2}(\phi_1 - \phi_2)\cos\tfrac{1}{2}(\phi_3 - \phi_4) = 0.$$

By subtraction, deduce that $\Sigma\phi_1 = (2n+1)\pi$ is a necessary (but not sufficient) condition for the points $\phi_1, \phi_2, \phi_3, \phi_4$ to be conormal. (Cf. 17.52, ex.)

*15 (i) Write down the equation of a conic circumscribing the quadrilateral whose opposite sides are the lines $\alpha, \beta; \gamma, \delta$.

(ii) If p is the length of the perpendicular from P_0 to the line $\alpha = 0$, prove that $p = \text{constant} \times \alpha_0$.

(iii) If a conic circumscribes a quadrilateral, prove that the ratio of the product of perpendiculars from any point P_0 of the conic onto two opposite sides, to the product of perpendiculars from P_0 onto the other two sides, is constant (a result due to Pappus).

*16 Show that the feet of normals from (h, k) to $y^2 = 4ax$ lie on the parabola $x^2 + (2a - h)x = \frac{1}{2}ky$. [See 19.7, ex. (i).]

*17 Explain why the conic $\Sigma k_1(x - t_2 y + at_2^2)(x - t_3 y + at_3^2) = 0$ passes through the vertices of the triangle formed by the tangents to $y^2 = 4ax$ at t_1, t_2, t_3.

Find the ratios $k_1 : k_2 : k_3$ for which this conic is a circle, and verify that the circle passes through the focus. State this result as a geometrical theorem.

Miscellaneous Exercise 19(*c*)

1 A pair of tangents to $ax^2 + by^2 = 1$ intercepts a constant length $2c$ on Ox. Prove that the point from which the tangents are drawn lies on the curve

$$by^2(ax^2 + by^2 - 1) = ac^2(by^2 - 1)^2.$$

2 Prove that chords of $y^2 = 4ax$ which subtend a right-angle at $(at_0^2, 2at_0)$ are concurrent at $\left(a(t_0^2 + 4), -2at_0\right)$. [Use 19.4, ex.]

3 Through a fixed point O a line is drawn to meet a conic at P and P'. A point Q on OPP' is chosen so that

$$\frac{1}{OQ^2} = \frac{1}{OP^2} + \frac{1}{OP'^2}.$$

Prove that Q lies on a conic. [Use the distance quadratic.]

4 If $ax^2+2hxy+by^2+2gx+2fy+c = 0$ represents a line-pair, prove that the pair (other than the coordinate axes) passing through the meets of these with the axes has equation $ax^2+2(2fg/c-h)\,xy+by^2+2gx+2fy+c = 0$.

5 The lines $bx^2+ay^2 = 0$ $(a+b \neq 0)$ meet the conic $s = 0$ in four points A, B, C, D. If the diagonals not through O of the quadrilateral $ABCD$ are perpendicular, prove that

$$\begin{vmatrix} a-b & h & g \\ h & b-a & f \\ g & f & c \end{vmatrix} = 0.$$

6 $ABCD$ is cyclic; AB, DC produced meet at E; BC, AD produced meet at F. Prove that the bisectors of angles CEB, CFD are perpendicular. [With A for origin, let the pair AB, DC be $t \equiv px^2+2rxy+qy^2 = 0$, and BC, AD be $s = 0$. Express the condition for $t = ks$ to be a circle, and use 15.52 (3).]

7 Prove that the points of intersection of $x^2-2xy+3y^2+9x-16y+24 = 0$ and $5x^2+6xy-y^2-43x+60y-62 = 0$ are concyclic. [Eliminate xy.]

8 Find the equation of the rectangular hyperbola through the meets of the ellipse $x^2/a^2+y^2/b^2 = 1$ and the line-pair $y^2 = m^2x^2$. Show that when $m = b/a$, this rectangular hyperbola cuts the ellipse orthogonally.

9 A circle through the origin touches the rectangular hyperbola $xy = c^2$ and meets it again at P, Q. Prove that the foot of the perpendicular from the origin to PQ lies on $4xy = c^2$.

*10 Obtain the general equation of conics through $(3,2)$, $(-3,2)$, $(2,-3)$, $(-2,-3)$. Show that this family contains a parabola, and find its equation. [Use Ex. 16 (*e*), no. 26 (i).]

*11 The normal to $y^2 = 4ax$ at the point $P(t)$ meets the parabola again at $Q(\theta)$, and the normal at Q meets the curve again at R. Prove that the equation of the parabola which passes through P, R and touches the given one at Q can be written

$$\lambda(y^2-4ax)+(tx+y-at^3-2at)(\theta x+y-a\theta^3-2a\theta) = 0,$$

where $\lambda = (\theta-t)^2/4\theta t$. Prove that its axis is parallel to SN, where S is the focus of $y^2 = 4ax$ and N is the point where the normal at P to $y^2 = 4ax$ cuts Oy. [In the last part use $\theta = -t-2/t$.]

*12 Prove that the tangent at P_1 to the conic obtained in ex. (v) of 19.64 is the line joining P_1 to the meet of AB and CD. [Use Ex 19 (*a*), no. 4.]

20

POLAR EQUATION OF A CONIC

20.1 The straight line

Polar coordinates and equations have been used in various parts of the book, particularly in calculus applications. The purpose of this chapter is to obtain and use the polar equation of a conic to prove geometrical properties, especially focal ones. We begin with short sections on the straight line and circle, in which we assemble the material required.

20.11 Distance formula

Given two points $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$, the length of AB can be found by applying the cosine rule to triangle AOB (fig. 199):

$$AB^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2).$$

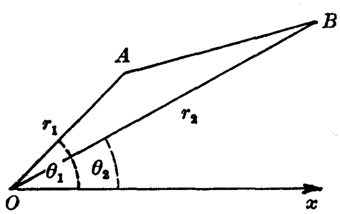

Fig. 199

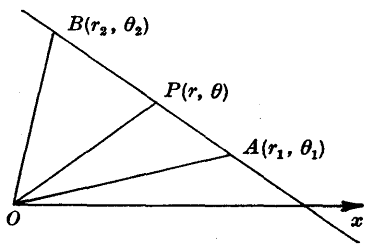

Fig. 200

20.12 Line joining two points

Let $P(r, \theta)$ be any point on AB. If P lies between A and B, then (fig. 200)

$$\triangle AOB = \triangle AOP + \triangle POB,$$

i.e. $$\tfrac{1}{2}r_1r_2\sin(\theta_2-\theta_1) = \tfrac{1}{2}r_1r\sin(\theta-\theta_1) + \tfrac{1}{2}rr_2\sin(\theta_2-\theta),$$

i.e. $$\frac{\sin(\theta_2-\theta_1)}{r} = \frac{\sin(\theta-\theta_1)}{r_2} + \frac{\sin(\theta_2-\theta)}{r_1}. \qquad \text{(i)}$$

If P lies on AB produced or on BA produced, the reader should show similarly by considering areas that the same result holds.

Accordingly, since this equation is satisfied by any point (r, θ) of the line AB, it is the equation of the line.

Remark. The line joining O to the point (p, α) has equation $\theta = \alpha$. This fact, already noticed in 1.63 (*c*), is obvious from first principles.

20.13 Line in 'perpendicular form'

Let the perpendicular from O to the given line have length p and make angle α with Ox. If $P(r, \theta)$ is any point on the line, then from triangle ONP we have

$$p = r\cos(\theta - \alpha). \tag{ii}$$

This (cf. 15.28) is the equation of the line in 'perpendicular form'.

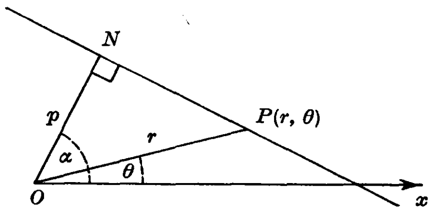

Fig. 201

Remark. The foot N of the perpendicular from O has polar co-ordinates (p, α).

20.14 General equation of a line

When converted to polar coordinates, the general linear equation $Ax + By = C$ becomes

$$A\cos\theta + B\sin\theta = \frac{C}{r}. \tag{iii}$$

Example

Any line perpendicular to (iii) *has an equation of the form*

$$A\cos(\theta + \tfrac{1}{2}\pi) + B\sin(\theta + \tfrac{1}{2}\pi) = \frac{C'}{r}.$$

For any line perpendicular to $Ax + By = C$ has equation

$$-Bx + Ay + C' = 0,$$

i.e.
$$-A\sin\theta + B\cos\theta = \frac{C'}{r},$$

i.e.
$$A\cos(\theta + \tfrac{1}{2}\pi) + B\sin(\theta + \tfrac{1}{2}\pi) = \frac{C'}{r}.$$

20.2 The circle

20.21 Polar equation

Let $P(r, \theta)$ be any point on the circle with centre $C(\rho, \alpha)$ and radius a (fig. 202). From the distance formula (20.11),

$$a^2 = CP^2 = \rho^2 + r^2 - 2\rho r \cos(\theta - \alpha).$$

Hence the circle has equation

$$r^2 - 2\rho r \cos(\theta - \alpha) = a^2 - \rho^2.$$

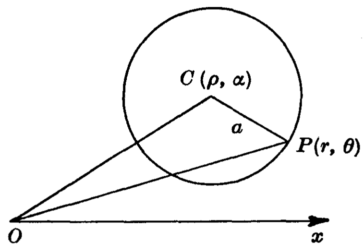

Fig. 202

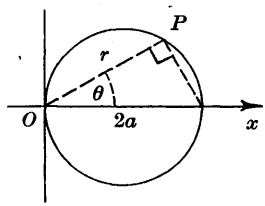

Fig. 203

We notice the following particular cases.

(*a*) If C lies on Ox, then $\alpha = 0$; the equation is

$$r^2 - 2\rho r \cos\theta = a^2 - \rho^2.$$

(*b*) If O lies on the circle, then $\rho = a$; the equation becomes

$$r^2 - 2ar \cos(\theta - \alpha) = 0,$$

so that $\quad r = 2a\cos(\theta - \alpha)$ *represents a circle through O.*

(*c*) If C lies on Ox and O lies on the circle, then $\alpha = 0$ and $\rho = a$; the equation is then $r = 2a\cos\theta$, which is easily written down from fig. 203.

20.22 Chord P_1P_2 of $r = 2a\cos\theta$; tangent at P_1

Let the required chord have equation $p = r\cos(\theta - \alpha)$. This is therefore satisfied by $(2a\cos\theta_1, \theta_1)$ and $(2a\cos\theta_2, \theta_2)$, so that

$$2a\cos\theta_1 . \cos(\theta_1 - \alpha) = p = 2a\cos\theta_2 . \cos(\theta_2 - \alpha), \tag{i}$$

$$\cos(2\theta_1 - \alpha) + \cos\alpha = \cos(2\theta_2 - \alpha) + \cos\alpha,$$

and hence† $$2\theta_1-\alpha = -(2\theta_2-\alpha),$$

i.e. $$\alpha = \theta_1+\theta_2.$$

From (i) we now have $p = 2a\cos\theta_1\cos\theta_2$, and the equation of the chord becomes

$$r\cos(\theta-\theta_1-\theta_2) = 2a\cos\theta_1\cos\theta_2. \qquad \text{(ii)}$$

By letting $\theta_2 \to \theta_1$ we obtain the equation of the *tangent at* P_1, viz.

$$r\cos(\theta-2\theta_1) = 2a\cos^2\theta_1. \qquad \text{(iii)}$$

20.23 Examples

(i) *Simson's line.*‡ *From any point O on the circumcircle of triangle ABC, perpendiculars are drawn to the sides. Prove that their feet are collinear.*

Choose O for pole, and the diameter through O for initial line. The equation of the circumcircle is then $r = 2a\cos\theta$.

Let A, B, C correspond to the values $\theta = \alpha, \beta, \gamma$. Then the chord BC has equation

$$2a\cos\beta\cos\gamma = r\cos(\theta-\beta-\gamma),$$

and hence by the Remark in 20.13, the foot of the perpendicular from O to BC has polar coordinates $(2a\cos\beta\cos\gamma, \beta+\gamma)$. Similarly, the other feet are

$$(2a\cos\gamma\cos\alpha, \gamma+\alpha), \quad (2a\cos\alpha\cos\beta, \alpha+\beta).$$

These three points clearly lie on the line whose polar equation is

$$2a\cos\alpha\cos\beta\cos\gamma = r\cos(\theta-\alpha-\beta-\gamma). \qquad \text{(iv)}$$

This is known as the *Simson line* of O wo triangle ABC.

(ii) *Extension of ex.* (i). If D is another point on the circumcircle of triangle ABC, then A, B, C, D can be selected three at a time in 4 ways, and hence there are 4 Simson lines wo O corresponding to the 4 possible triangles.

By the Remark in 20.13, the foot of the perpendicular from O to (iv) is $(2a\cos\alpha\cos\beta\cos\gamma, \alpha+\beta+\gamma)$. Similarly, if D corresponds to the value $\theta = \delta$, the feet of the perpendiculars from O to the other Simson lines are

$$(2a\cos\beta\cos\gamma\cos\delta, \beta+\gamma+\delta), \quad (2a\cos\gamma\cos\delta\cos\alpha, \gamma+\delta+\alpha),$$

$$(2a\cos\delta\cos\alpha\cos\beta, \delta+\alpha+\beta).$$

Clearly these four points lie on the line

$$2a\cos\alpha\cos\beta\cos\gamma\cos\delta = r\cos(\theta-\alpha-\beta-\gamma-\delta).$$

This result is likewise capable of extension.

† The general solution (1.52(3), ex. (iv)) is $2\theta_1-\alpha = 2n\pi \pm (2\theta_2-\alpha)$. The sign $+$ is inadmissible since we are assuming P_1, P_2 to be *distinct*. The sign $-$ gives $\alpha = \theta_1+\theta_2-n\pi$. Using this value of α and the fact that $\cos n\pi = (-1)^n$, we find that $p = 2a(-1)^n\cos\theta_1\cos\theta_2$ and $\cos(\theta-\alpha) = (-1)^n\cos(\theta-\theta_1-\theta_2)$, so that the result (ii) is unchanged.

‡ Robert Simson (1687–1768).

Exercise 20(*a*)

1 Sketch the figures for the cases in 20.12 when P lies on (*a*) AB produced; (*b*) BA produced. Verify for each that the equation (i) still holds.

2 Show that the area of the triangle whose vertices are $A(r_1, \theta_1)$, $B(r_2, \theta_2)$, $C(r_3, \theta_3)$ is $\frac{1}{2}\{r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_1 - \theta_3) + r_1 r_2 \sin(\theta_2 - \theta_1)\}$. Deduce the condition for A, B, C to be collinear.

3 Prove that the perpendicular from (r_1, θ_1) to the line $r\cos(\theta - \alpha) = p$ has length $\pm\{p - r_1 \cos(\theta_1 - \alpha)\}$.

4 A variable line through a fixed point O meets three given lines at points P_1, P_2, P_3. On this line is taken a point P such that

$$\frac{1}{OP} = \frac{1}{OP_1} + \frac{1}{OP_2} + \frac{1}{OP_3}.$$

Prove that the locus of P is another straight line.

5 (i) Write down the polar equation of (*a*) the circle whose centre is (ρ, α) and which touches the initial line; (*b*) the circles of radius a which touch the initial line at the pole.

(ii) Explain why the circles $r = a\cos(\theta - \alpha)$, $r = b\sin(\theta - \alpha)$ cut orthogonally.

6 Find the centre of the circle $r = a\cos\theta + b\sin\theta$.

7 If OPQ is a chord of the circle $r^2 - 2\rho r\cos(\theta - \alpha) + \rho^2 - a^2 = 0$, prove that $OP.OQ = \rho^2 - a^2$. Deduce the 'product property' that, for chords OPQ, ORS through O, $OP.OQ = OR.OS$.

8 Let ON be the perpendicular from O to the tangent at $A(r_1, \theta_1)$ on the circle $r = 2a\cos\theta$, and $P(r, \theta)$ be any point on this tangent. By expressing ON in two ways as $r\cos(\theta - 2\theta_1)$ and $r_1 \cos\theta_1$, obtain equation (iii) of 20.22.

*9 Prove that the normal to $r = 2a\cos\theta$ at $\theta = \alpha$ has equation

$$r\sin(2\alpha - \theta) = a\sin 2\alpha.$$

10 Find the equation of the chord joining the points of the circle

$$r = 2a\cos(\theta - \alpha)$$

for which $\theta = \theta_1, \theta_2$. Deduce the equation of the tangent at θ_1.

11 Two circles meet at O, and a line through O meets them again at P, Q respectively. Find the locus of the mid-point of PQ.

12 Find the condition for the line $1/r = a\cos\theta + b\sin\theta$ to touch the circle $r = 2c\cos\theta$.

*13 Prove the Simson line property (20.23, ex. (i)) by pure geometry.

20.3 Conics: pole at a focus

20.31 Polar equation of all non-degenerate conics

Choose a focus S of the conic for pole, and the perpendicular from S to the corresponding directrix for initial line. Let the latus rectum LL' have length $2l$, and let $P(r, \theta)$ be any point on the conic.

In figs. 204 and 205, P lies on the same side of the directrix as S. Fig. 206, where P and S are on opposite sides, will arise when P is on that branch of a hyperbola which is remote from S. We have

$$r = SP = e.PM = e.ND$$

$$= \begin{cases} e(SD-SN) \\ \quad \text{in 204} \\ e(SD+SN) \\ \quad \text{in 205} \\ e(SN-SD) \\ \quad \text{in 206} \end{cases} = \begin{cases} e(SD-r\cos\theta) \\ \quad \text{in 204, 205} \\ e(r\cos\theta-SD) \\ \quad \text{in 206} \end{cases} = \begin{cases} l-er\cos\theta \\ \quad \text{in 204, 205,} \\ er\cos\theta-l \\ \quad \text{in 206,} \end{cases}$$

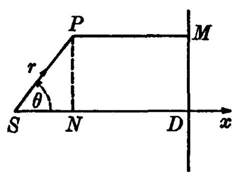

Fig. 204

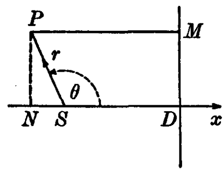

Fig. 205

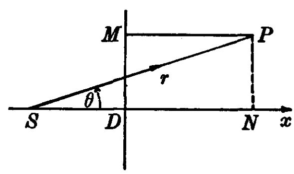

Fig. 206

since $l = SL = e.SD$. Hence

$$\text{in figs. 204, 205} \qquad \frac{l}{r} = 1 + e\cos\theta;$$

$$\text{in fig. 206} \qquad \frac{l}{r} = -1 + e\cos\theta.$$

The last equation represents the branch of a hyperbola remote from S.

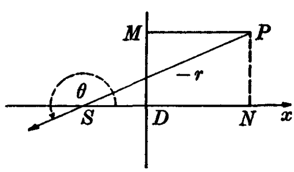

Fig. 207

If we replace r by $-r$ and θ by $\theta-\pi$ (i.e. if we use the notation of fig. 207), then the equation $l/r = -1+e\cos\theta$ becomes

$$-\frac{l}{r} = -1 + e\cos(\theta-\pi) = -1 - e\cos\theta,$$

i.e.
$$\frac{l}{r} = 1 + e\cos\theta$$

as for figs. 204 and 205. Hence the equation $l/r = 1+e\cos\theta$ will represent the branch of a hyperbola remote from S if r is allowed to take negative values.

It follows that, *if negative values of r are allowed, then every proper* (i.e. non-degenerate) *conic is completely represented by* $l/r = 1+e\cos\theta$.

Remarks

(α) If the line DS (in the sense from D towards S) is taken as initial line, this is equivalent to replacing θ by $\pi-\theta$ in the above discussion. The equation becomes $l/r = 1-e\cos\theta$; it will completely represent all proper conics if r is allowed to take negative values.

(β) The equation $l/r = -1+e\cos\theta$, obtained from fig. 206, also completely represents all proper conics (with the same proviso about negative values of r). Indeed, *the two equations* $l/r = -1+e\cos\theta$ *and* $l/r = 1+e\cos\theta$ *represent the same conic*, but any chosen point of the conic will of course correspond to different pairs of values of r, θ in the two representations. This fact is used in 20.33, ex. (iii).

(γ) The equation of a conic whose major axis is inclined at angle α to the initial line is $l/r = 1+e\cos(\theta-\alpha)$. For, wo its axis SD as initial line, the equation is $l/r = 1+e\cos\theta$; rotation of the initial line clockwise through angle α gives the new equation.

20.32 Tracing of the curve $l/r = 1+e\cos\theta$

(1) *When* $e = 1$, the conic is a parabola whose equation can be written

$$\frac{l}{r} = 1+\cos\theta = 2\cos^2\tfrac{1}{2}\theta.$$

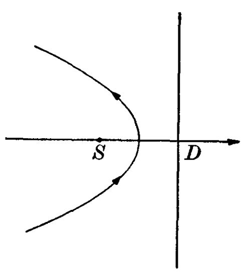

Fig. 208

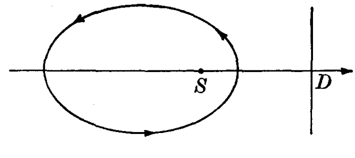

Fig. 209

It is represented completely by

$$-\pi < \theta < \pi \quad \text{and} \quad r > 0.$$

(2) *When* $e < 1$, the conic is an ellipse which is completely represented by

$$-\pi < \theta \leqslant \pi \quad \text{and} \quad r > 0.$$

(3) *When* $e > 1$, the conic is a hyperbola. The branch enclosing S is given by

$$-\pi + \sec^{-1} e < \theta < \pi - \sec^{-1} e \quad \text{and} \quad r > 0;$$

and the branch remote from S by

$$\pi - \sec^{-1} e < \theta < \pi + \sec^{-1} e \quad \text{and} \quad r < 0.$$

The reader should check the above statements.

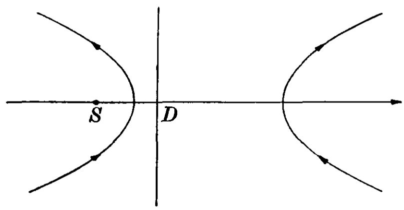

Fig. 210

20.33 Examples

(i) *Equations of the directrices.*

(*a*) From the focus-directrix definition, $SL = e.SD$, so that $SD = l/e$. If (r, θ) is any point on the directrix corresponding to S, then

$$r\cos\theta = SD = l/e.$$

Hence this directrix has equation

$$\frac{l}{r} = e\cos\theta.$$

*(*b*) For central conics we obtain the equation of the directrix remote from S as follows.

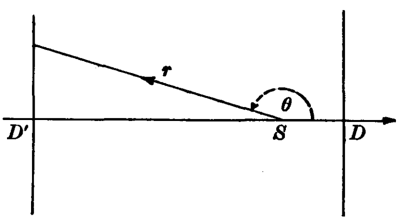

Fig. 211

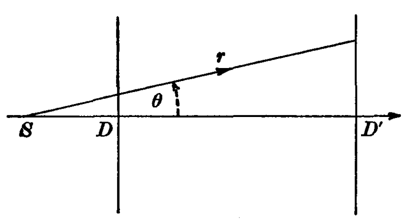

Fig. 212

If $e < 1$ (fig. 211), then

$$r\cos(\pi - \theta) = SD' = DD' - SD$$

$$= \frac{2a}{e} - \frac{l}{e} = \frac{l}{e}\left(\frac{2a}{l} - 1\right);$$

and since $l/a = 1-e^2$ (see Ex. 20 (b), no. 1), this becomes

$$-r\cos\theta = \frac{l}{e}\left(\frac{2}{1-e^2}-1\right) = \frac{l}{e}\frac{1+e^2}{1-e^2}.$$

Hence the equation is $$r\cos\theta = \frac{l}{e}\frac{e^2+1}{e^2-1}.$$

If $e > 1$ (fig. 212),

$$\begin{aligned} r\cos\theta = SD' &= SD+DD' \\ &= \frac{l}{e}+\frac{2a}{e} = \frac{l}{e}\left(1+\frac{2a}{l}\right) \\ &= \frac{l}{e}\left(1+\frac{2}{e^2-1}\right) = \frac{l}{e}\frac{e^2+1}{e^2-1}, \end{aligned}$$

since $l/a = e^2-1$. The equation is thus the same for each case.

*(ii) *Equations of the asymptotes of a hyperbola.*

The asymptote whose cartesian equation is $y = bx/a$ is inclined to Ox at angle λ, where $\tan\lambda = b/a$ and hence $\cos\lambda = a/\sqrt{(a^2+b^2)} = 1/e$. Therefore the perpendicular from S to this asymptote makes angle $2\pi - \sin^{-1}(1/e)$ with Ox; its length is

$$OS\sin\lambda = ae\frac{b}{\sqrt{(a^2+b^2)}} = ae\frac{b}{ae} = b,$$

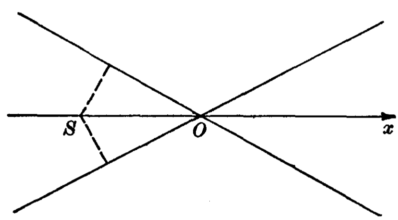

Fig. 213

and so its equation is (20.13)

$$b = r\cos\left(\theta - 2\pi + \sin^{-1}\frac{1}{e}\right).$$

Since $b = l/\sqrt{(e^2-1)}$, this can be written

$$r\cos\left(\theta+\sin^{-1}\frac{1}{e}\right) = \frac{l}{\sqrt{(e^2-1)}}.$$

Similarly, the asymptote $y = -bx/a$ has polar equation

$$b = r\cos\left(\theta-\sin^{-1}\frac{1}{e}\right), \quad \text{i.e.} \quad r\cos\left(\theta-\sin^{-1}\frac{1}{e}\right) = \frac{l}{\sqrt{(e^2-1)}}.$$

(iii) *Two conics have a common focus. Prove that two of their common chords pass through the intersection of the directrices which correspond to that focus.*

The conics can be taken as

$$\frac{l}{r} = 1+e\cos\theta \quad \text{and} \quad \frac{l'}{r} = 1+e'\cos(\theta-\alpha).$$

If we subtract these equations, we obtain

$$\frac{l}{r}-e\cos\theta = \frac{l'}{r}-e'\cos(\theta-\alpha),$$

which is the equation of a line. Since it is satisfied by the points (r,θ) which satisfy the equations of both conics, it represents a common chord. Clearly it is also satisfied by the points (r,θ) which satisfy both the equations

$$\frac{l}{r}-e\cos\theta = 0, \quad \frac{l'}{r}-e'\cos(\theta-\alpha) = 0,$$

and these represent the directrices corresponding to S, by ex. (i). This common chord therefore passes through the meet of these directrices.

The first conic is also given by

$$\frac{l}{r} = -1+e\cos\theta$$

(see Remark (β) in 20.31). Proceeding similarly, another common chord is

$$\frac{l}{r}-e\cos\theta+\frac{l'}{r}-e'\cos(\theta-\alpha) = 0,$$

which also passes through the meet of the same directrices.

20.34 Chord and tangent

Let P, Q be the points of the conic $l/r = 1+e\cos\theta$ corresponding to $\theta = \alpha$, $\theta = \beta$. Since by 20.14 the polar equation of any line can be written

$$A\cos\theta+B\sin\theta = \frac{C}{r},$$

let the required chord have equation

$$p\cos\theta+q\sin\theta = \frac{l}{r}.$$

Then $$p\cos\alpha+q\sin\alpha = 1+e\cos\alpha$$

and $$p\cos\beta+q\sin\beta = 1+e\cos\beta;$$

i.e. $$(p-e)\cos\alpha+q\sin\alpha = 1, \quad (p-e)\cos\beta+q\sin\beta = 1.$$

Solving these,

$$p-e = \frac{\sin\alpha-\sin\beta}{\sin(\alpha-\beta)} = \cos\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta)$$

and $$q = \frac{\cos\beta-\cos\alpha}{\sin(\alpha-\beta)} = \sin\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta).$$

The chord PQ is therefore

$$\frac{l}{r} = e\cos\theta+\sec\tfrac{1}{2}(\alpha-\beta)\{\cos\theta\cos\tfrac{1}{2}(\alpha+\beta)+\sin\theta\sin\tfrac{1}{2}(\alpha+\beta)\},$$

i.e. $$\frac{l}{r} = e\cos\theta + \sec\tfrac{1}{2}(\alpha-\beta)\cos\left(\theta - \frac{\alpha+\beta}{2}\right).$$

For an alternative method, see Ex. 20 (*b*), no. 7.

By letting $\beta \to \alpha$ we find that the *tangent at the point* $\theta = \alpha$ is

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha).$$

For a calculus method, see Ex. 20 (*b*), no. 24.

Examples

(i) *The tangents at the points P, Q corresponding to $\theta = \alpha, \beta$ meet at the point $T(r, \theta)$, where $\theta = \frac{1}{2}(\alpha+\beta)$.*

For the coordinates of T satisfy

$$e\cos\theta + \cos(\theta-\alpha) = \frac{l}{r} = e\cos\theta + \cos(\theta-\beta),$$

i.e. $$\cos(\theta-\alpha) = \cos(\theta-\beta),$$

from which $$\theta = \tfrac{1}{2}(\alpha+\beta).$$

Remark. The general solution is $\theta-\alpha = 2n\pi \pm (\theta-\beta)$. The sign $+$ is inadmissible since P, Q are assumed to be distinct; the sign $-$ gives

$$\theta = n\pi + \tfrac{1}{2}(\alpha+\beta).$$

Since $-\pi < \alpha \leqslant \pi$ and $-\pi < \beta \leqslant \pi$, also $-\pi < \frac{1}{2}(\alpha+\beta) \leqslant \pi$, and so there is no loss of generality by taking $n = 0$ because any point in the plane can be specified by an angular coordinate which lies between $\pm\pi$.

(ii) With the notation of ex. (i), *ST bisects $P\hat{S}Q$.*

If P, Q are not on different branches of a hyperbola (fig. 214),

$$P\hat{S}T = \tfrac{1}{2}(\alpha+\beta) - \alpha = \tfrac{1}{2}(\beta-\alpha)$$

and $$T\hat{S}Q = \beta - \tfrac{1}{2}(\alpha+\beta) = \tfrac{1}{2}(\beta-\alpha).$$

Hence ST bisects $P\hat{S}Q$ (internally).

If P, Q are on different branches of a hyperbola, let Q be on the branch remote from S. Then β is the angle xSQ', and $\frac{1}{2}(\alpha+\beta)$ is angle xST', as shown in fig. 215.

$$P\hat{S}T' = \tfrac{1}{2}(\alpha+\beta) - \alpha = \tfrac{1}{2}(\beta-\alpha)$$

and $$T'\hat{S}Q' = \beta - \tfrac{1}{2}(\alpha+\beta) = \tfrac{1}{2}(\beta-\alpha),$$

so that ST bisects $P\hat{S}Q$ externally.

(iii) *Find the equation of the circumcircle of the triangle formed by three tangents to a parabola, and verify that this circle passes through the focus.*

Taking the parabola as $l/r = 1 + \cos\theta$, the tangents at the points A, B, C corresponding to $\theta = \alpha, \beta, \gamma$ are

$$\frac{l}{r} = \cos\theta + \cos(\theta-\alpha), \quad \text{etc.}$$

Tangents at A and B meet where $\theta = \frac{1}{2}(\alpha+\beta)$, by ex. (i), and hence

$$\frac{l}{r} = \cos\tfrac{1}{2}(\alpha+\beta)+\cos\tfrac{1}{2}(\beta-\alpha) = 2\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta,$$

i.e. at the point $\left(\frac{1}{2}l\sec\frac{1}{2}\alpha\sec\frac{1}{2}\beta,\ \frac{1}{2}(\alpha+\beta)\right)$. Similarly the other meets are

$$\left(\tfrac{1}{2}l\sec\tfrac{1}{2}\beta\sec\tfrac{1}{2}\gamma,\ \tfrac{1}{2}(\beta+\gamma)\right),\quad \left(\tfrac{1}{2}l\sec\tfrac{1}{2}\gamma\sec\tfrac{1}{2}\alpha,\ \tfrac{1}{2}(\gamma+\alpha)\right).$$

These three points satisfy the equation

$$r = \tfrac{1}{2}l\sec\tfrac{1}{2}\alpha\sec\tfrac{1}{2}\beta\sec\tfrac{1}{2}\gamma\cos\left(\theta-\frac{\alpha+\beta+\gamma}{2}\right),$$

which by 20.21 (b) represents a circle of radius $R = \frac{1}{4}l\sec\frac{1}{2}\alpha\sec\frac{1}{2}\beta\sec\frac{1}{2}\gamma$, centre $\left(R, \frac{1}{2}(\alpha+\beta+\gamma)\right)$, and passing through the pole S.

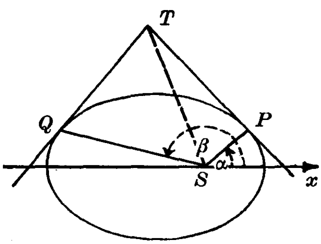

Fig. 214

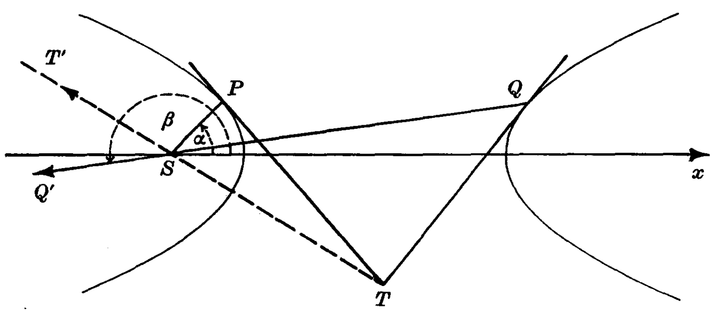

Fig. 215

(iv) *If a variable chord of a conic subtends a constant angle at the focus, then tangents at the extremities of the chord meet on a fixed conic, and the chord itself touches another fixed conic.*

Let the angle subtended at the focus be 2β, and take the angular coordinate of one extremity to be $\alpha-\beta$; then that of the other is $\alpha+\beta$. The chord thus has equation

$$\frac{l}{r} = e\cos\theta+\sec\beta\cos(\theta-\alpha),$$

i.e.

$$\frac{l\cos\beta}{r} = e\cos\beta.\cos\theta+\cos(\theta-\alpha).$$

This is the equation of the tangent at α to the conic

$$\frac{l\cos\beta}{r} = 1+e\cos\beta\cos\theta, \tag{i}$$

whose eccentricity is $e\cos\beta$ and latus rectum $2l\cos\beta$.

The tangents at the extremities of the chord are

$$\frac{l}{r} = e\cos\theta+\cos(\theta-\alpha+\beta), \quad \frac{l}{r} = e\cos\theta+\cos(\theta-\alpha-\beta),$$

and these meet where (by ex. (i)) $\theta = \alpha$ and $l/r = e\cos\alpha+\cos\beta$. This point lies on the conic

$$\frac{l}{r} = e\cos\theta+\cos\beta,$$

i.e.

$$\frac{l\sec\beta}{r} = 1+e\sec\beta\cos\theta, \tag{ii}$$

which has eccentricity $e\sec\beta$ and latus rectum $2l\sec\beta$.

The conics (i), (ii) have the same focus and directrix as the given one.

Exercise 20(*b*)

1 If a, b are the semi-axes of $l/r = 1+e\cos\theta$ ($e \neq 1$), prove that when $e < 1$,

$$a = l/(1-e^2) \quad \text{and} \quad b = l/\sqrt{(1-e^2)}.$$

Give the corresponding results when $e > 1$.

2 What locus is represented by $a = r\sin^2\frac{1}{2}\theta$? Sketch this curve.

3 What is represented by $k/r = a+b\cos\theta+c\sin\theta$?

4 If $P(r_1, \alpha)$ is an extremity of a focal chord of $l/r = 1+e\cos\theta$, show that the other extremity Q has coordinates of the form $(r_2, \pi+\alpha)$. Prove that

$$\frac{1}{SP}+\frac{1}{SQ} = \frac{2}{l},$$

i.e. that *half the latus rectum is the harmonic mean of the segments of any focal chord.*

5 Prove that the sum of the reciprocals of the lengths of two perpendicular focal chords of a conic is constant. Express this constant in terms of e and l.

6 Prove that mutually perpendicular focal chords of a rectangular hyperbola are equal in length.

7 Obtain the equation of the chord joining the points of $l/r = 1+e\cos\theta$ which correspond to $\theta = \alpha, \beta$ from the result in 20.12.

8 Find the equation of the chord of $l/r = 1+\cos(\theta-\gamma)$ which joins the points $\theta = \alpha, \beta$. Deduce that the tangent at α is $l/r = e\cos(\theta-\gamma)+\cos(\theta-\alpha)$.

9 Find the common points of the conics

$$l\sqrt{3} = r(\sqrt{3}+\cos\theta) \quad \text{and} \quad l\sqrt{3} = 2r\{\sqrt{3}+\cos(\theta+\tfrac{1}{3}\pi)\},$$

and show that they touch there.

10 Prove that the tangents at the extremities of a focal chord meet on the corresponding directrix.

11 If the tangent at P meets the directrix at Z, prove that $P\hat{S}Z$ is a right-angle.

12 A chord of $l/r = 1+e\cos\theta$ subtends an angle 2α at the focus. If the bisector of this angle meets the chord at M, prove that M lies on the conic

$$\frac{l\cos\alpha}{r} = 1+e\cos\alpha.\cos\theta.$$

13 P is any point on a conic with focus S. A line through S making a given angle with SP meets the tangent at P in T. Prove that T lies on a conic which has the same focus S and corresponding directrix as the given conic.

14 Two conics have a common focus, about which one of them rotates. Prove that the common chord touches a conic which has the same focus as the given conics, and whose eccentricity is the ratio of the eccentricity of the fixed conic to that of the rotating one.

15 An ellipse of eccentricity e and a parabola intersect at the ends L, L' of a common latus rectum, and S is their common focus. If a common tangent touches them at P, Q respectively, prove that SP, SQ are each inclined to LL' at an angle $\sin^{-1}\{\frac{1}{2}(1\pm e)\}$. Explain geometrically how the two cases arise.

16. P, Q are the points of the conic $l/r = 1+e\cos\theta$ corresponding to $\theta = \alpha, \beta$. If the tangents at P, Q meet at T, prove that T has coordinates

$$\left(\frac{l}{\cos\frac{1}{2}(\alpha-\beta)+e\cos\frac{1}{2}(\alpha+\beta)}, \ \tfrac{1}{2}(\alpha+\beta)\right).$$

17. If P, Q are points on a parabola with focus S, and the tangents at P, Q meet at T, prove that $SP.SQ = ST^2$.

18 (i) P, Q are the points $\theta = \alpha, \beta$ of the parabola $l/r = 1+\cos\theta$. Prove that the tangents at P, Q contain an angle $\frac{1}{2}|\alpha-\beta|$.

(ii) Tangents to a parabola with latus rectum $2l$ contain the constant angle ϕ. Prove that they meet at a point on a hyperbola having the same focus as the parabola, and give the latus rectum and eccentricity. [Use no. 16.]

*19 Prove that the chord of contact of tangents from (r_1, θ_1) to $l/r = 1+e\cos\theta$ has equation

$$\left(\frac{l}{r_1}-e\cos\theta_1\right)\left(\frac{l}{r}-e\cos\theta\right) = \cos(\theta-\theta_1).$$

[If its extremities are given by $\theta = \alpha, \beta$, the chord is

$$\cos\tfrac{1}{2}(\alpha-\beta)\left(\frac{l}{r}-e\cos\theta\right) = \cos\left(\theta-\frac{\alpha+\beta}{2}\right).$$

Use no. 16.]

*20 Show that the tangents at the points $\theta = \alpha, \beta$ of $l/r = 1+e\cos\theta$ are perpendicular if and only if $e^2+(\cos\alpha+\cos\beta)e+\cos(\alpha-\beta) = 0$. [Change the equation of the tangent into cartesian coordinates.]

*21 Prove that the meet of perpendicular tangents to $l/r = 1+e\cos\theta$ lies on the locus $2l(l/r-e\cos\theta) = r(1-e^2)$. If $e \neq 1$, show that this locus is a circle concentric with the given conic (the *director circle*). What is the locus when $e = 1$?

*22 Tangents are drawn to the parabola $l/r = 1+\cos\theta$ at $\theta = \alpha, \beta, \gamma, \delta$. Prove that the circumcentres of the four triangles so formed lie on the circle

$$r = \tfrac{1}{4}l\sec\tfrac{1}{2}\alpha\sec\tfrac{1}{2}\beta\sec\tfrac{1}{2}\gamma\sec\tfrac{1}{2}\delta\cos\left(\theta-\frac{\alpha+\beta+\gamma+\delta}{2}\right),$$

which passes through the focus. [Use 20.34, ex. (iii).]

*23 Show that the equation of any line perpendicular to the tangent at α to $l/r = 1+e\cos\theta$ has an equation of the form

$$\frac{kl}{r} = e\cos(\theta+\tfrac{1}{2}\pi)+\cos(\theta+\tfrac{1}{2}\pi-\alpha).$$

[See 20.14, ex.] If this line passes through the point α, prove that

$$k = -e\sin\alpha/(1+e\cos\alpha).$$

Deduce that *the normal at α has equation*

$$\frac{e\sin\alpha}{1+e\cos\alpha}\frac{l}{r} = e\sin\theta+\sin(\theta-\alpha).$$

*24 Prove that the tangent at $A(\rho,\alpha)$ to the curve $r=f(\theta)$ has equation

$$\frac{1}{\rho}\cos(\theta-\alpha)-\frac{1}{\rho^2}\left(\frac{dr}{d\theta}\right)_\alpha\sin(\theta-\alpha) = \frac{1}{r}.$$

[If $P(r,\theta)$ is any point on the tangent at A, apply the sine rule to triangle OAP, and use $\cot\phi = dr/rd\theta$ (8.14).] Deduce the equation of the tangent to

$$l/r = 1+e\cos\theta \quad \text{at} \quad \theta=\alpha.$$

*25 Obtain similarly the equation of the normal at (ρ,α) to $r=f(\theta)$, and use it to obtain the result of no. 23.

Miscellaneous Exercise 20(*c*)

1 A, B are the points (r_1,θ_1), (r_2,θ_2), and $M(r,\theta)$ is the mid-point of AB. Prove that

$$r = \tfrac{1}{2}\sqrt{\{r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)\}} \quad \text{and} \quad \tan\theta = \frac{r_1\sin\theta_1+r_2\sin\theta_2}{r_1\cos\theta_1+r_2\cos\theta_2}.$$

2 A variable line meets n given lines through O at $P_1, P_2, \ldots, P_n$ respectively. If $\sum_{r=1}^{n}(1/OP_r)$ is constant, prove that it passes through a fixed point.

3 Find the equation of the common chord and of the line of centres of the circles $r = 2a\cos(\theta-\alpha)$, $r = 2b\cos(\theta-\beta)$.

4 Obtain the polar equation of the circle whose diameter is the join of (a,α) and (b,β).

5 Prove that the chord of contact from (r_1,θ_1) to the circle $r = 2a\cos\theta$ is

$$\frac{\cos\theta}{r_1}+\frac{\cos\theta_1}{r} = \frac{\cos(\theta-\theta_1)}{a}.$$

[Use the cartesian equation of the circle and chord.]

6 If PSP' and QSQ' are mutually perpendicular focal chords of a conic, prove that $1/PS.SP'+1/QS.SQ'$ is constant, and express this constant in terms of l and e.

7 (i) Prove that the locus of the mid-points of chords through the focus S of $l/r = 1+e\cos\theta$ is $r(e^2\cos^2\theta-1) = el\cos\theta$.

(ii) By changing this equation to cartesian coordinates, show that the locus is a conic with the same eccentricity as the original one.

8 (i) If the tangent to the parabola $a = r\cos^2\frac{1}{2}\theta$ at the point P given by $\theta = \alpha$ meets the initial line at T, prove $ST = a\sec^2\frac{1}{2}\alpha = SP$.

(ii) Find the equation of the circle SPT.

9 Find the value of α for which the tangent at α to $l/r = 1+e\cos\theta$ is parallel to the initial line. Hence find the locus of the ends of the minor axis of this ellipse as e increases from 0 to 1.

10 Prove that the line $l/r = a\cos\theta + b\sin\theta$ touches the conic $l/r = 1+e\cos\theta$ if and only if $(a-e)^2+b^2 = 1$.

11 Two parabolas have a common focus and their axes perpendicular. Prove that the directrix of either passes through the point of contact with the other of their common tangent.

12 Find the locus of the foot of the perpendicular from the pole to the line $l/r = e\cos\theta + \cos(\theta-\alpha)$ when α varies. Interpret the result geometrically.

13 The perpendicular from the focus S to the directrix meets the conic at A. Prove that *if* the foot of the perpendicular from S to any tangent lies on the tangent at A, then the conic is a parabola. [Do *not* prove the converse result.]

14 A variable chord PQ of the ellipse $l/r = 1+e\cos\theta$ with focus S is parallel to the major axis. If its extremities are the points $\theta = 2\alpha,\ 2\beta$, prove that $\cos(\alpha+\beta)+e\cos(\alpha-\beta) = 0$.

If the internal bisector of $P\hat{S}Q$ meets PQ at K, prove that

$$SK = \frac{2SP.SQ}{SP+SQ}\cos\tfrac{1}{2}P\hat{S}Q,$$

and deduce that K lies on a parabola whose vertex is S.

*15 PQ is a focal chord of a conic, and its mid-point is M. The normals at P, Q meet at N. Prove that MN is parallel to the (major) axis. [Use no. 7 (i) and Ex. 20 (b), no. 23 to prove that $r\sin\theta$ is the same for M and N.]

21

COORDINATE GEOMETRY IN SPACE: THE PLANE AND LINE

21.1 Coordinates in space

21.11 Rectangular cartesian coordinates

The method of locating points in a plane by coordinates x, y referred to two intersecting lines Ox, Oy (15.11) can be extended to points in space. Clearly three axes of reference will be required, and for simplicity we shall choose them to be mutually perpendicular.

Take any two intersecting perpendicular lines; let them meet at O.

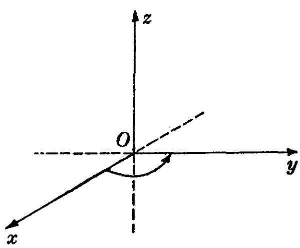

Fig. 216

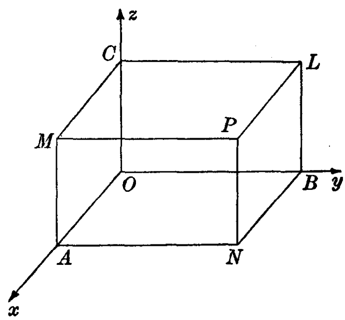

Fig. 217

On each select a positive sense, indicated by an arrow, and label these directions Ox, Oy. Through O draw a line perpendicular to both Ox and Oy (and therefore to the plane xOy), and choose the positive sense along it to be the direction of advance of a right-handed screw turned in the sense from Ox towards Oy; label this direction Oz. Then we have constructed a *right-handed system of rectangular axes* Ox, Oy, Oz.† The planes xOy, yOz, zOx determined by these axes are also mutually perpendicular, and are called the *coordinate planes*.

Given a point P in space, we can draw the perpendiculars PL, PM, PN to the planes yOz, zOx, xOy. The rectangular box having OP for diagonal and PL, PM, PN for edges can now be completed; let its

† If the direction of any one axis is reversed, we should obtain a *left-handed* system. We use only right-handed axes in this book.

edges through O be OA, OB, OC along Ox, Oy, Oz respectively. The distances OA, OB, OC can be specified in magnitude and sense by signed numbers x, y, z, and the ordered triplet of numbers (x, y, z) are the *rectangular cartesian coordinates* of P wo the axes Ox, Oy, Oz. In particular, observe that L has coordinates $(0, y, z)$, $M(x, 0, z)$, $N(x, y, 0)$; and that A is $(x, 0, 0)$, $B(0, y, 0)$, $C(0, 0, z)$.

Conversely, given three signed numbers x, y, z, the points A, B, C on the axes can be determined, and the box then completed. The corresponding point P is the extremity of the diagonal through O.

It follows that *to each point P in space there corresponds a unique set of coordinates x, y, z wo the axes Ox, Oy, Oz; and to each ordered set of numbers x, y, z corresponds a unique point P in space.*

The three coordinate planes divide space into eight regions called *octants*; to each corresponds a distribution of + and − signs in the coordinates (see Ex. 21 (*a*), no. 1), the number of such possible sign-distributions being $2^3 = 8$. In fig. 217 all coordinates of P are positive.

Finally, observe that A in fig. 217 is the point in which Ox is met by the plane through P drawn parallel to the plane yOz. Hence PA is perpendicular to Ox, so that A is the orthogonal projection of P on Ox. Similarly, B and C are the projections of P on Oy, Oz respectively. *The coordinates of a point P are therefore the (signed) lengths of the orthogonal projections of OP on the axes.*

Notation. In the remainder of this book P_1 is the point whose coordinates are (x_1, y_1, z_1), and so on.

21.12 Other coordinate systems

(1) *Cylindrical polar coordinates.* We may replace the coordinates (x, y) in the plane xOy by polar coordinates (ρ, ϕ), where $\rho = ON$ and ϕ is the angle

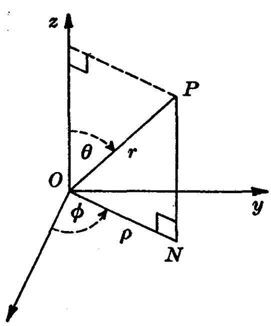

Fig. 218

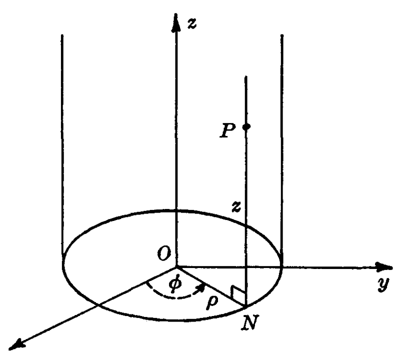

Fig. 219

xON, measured positively in the sense from Ox towards Oy. P is then determined by the triplet of numbers (ρ, ϕ, z), called the *cylindrical coordinates* of P.

(2) *Spherical polar coordinates*. Further, in the plane $zONP$ the coordinates ρ and z can be replaced by polar coordinates (r, θ), where $r = OP$ and θ is the angle zOP, measured positively from Oz towards ON. P is then determined by (r, θ, ϕ), the *spherical coordinates* of P.

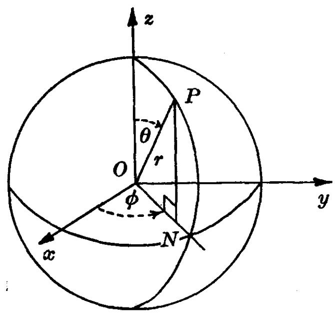

Fig. 220

If the sphere in fig. 220 represents the Earth and xOy, xOz are the planes of the equator and Greenwich meridian, then θ is the *colatitude* and ϕ is the *longitude* of P.

21.2 Fundamental formulae

21.21 Distance formula

Given two points P_1 and P_2, let the perpendiculars from them to the plane xOy be P_1N_1, P_2N_2. The coordinates of N_1 are $(x_1, y_1, 0)$, and of N_2 are $(x_2, y_2, 0)$. Regarded as points of the plane xOy, their cartesian coordinates wo axes Ox, Oy are (x_1, y_1), (x_2, y_2), so that N_1N_2 is given by

$$N_1N_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

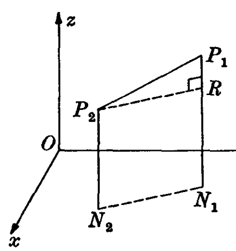

Fig. 221

Through P_2 draw P_2R parallel to N_2N_1; since P_2R lies in the plane of P_1N_1 and P_2N_2, it meets P_1N_1, say at R. From the right-angled triangle P_1P_2R,

$$P_1P_2^2 = P_1R^2 + P_2R^2 = (z_1 - z_2)^2 + N_2N_1^2$$
$$= (z_1 - z_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

and hence $$P_1P_2 = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\}}.$$

In particular, the distance of $P(x, y, z)$ from O is $\sqrt{(x^2 + y^2 + z^2)}$

21.22 Section formulae

Given two points P_1, P_2, let $P(x, y, z)$ be the point dividing P_1P_2 in the ratio $k:l$; then $P_1P/PP_2 = k/l$.

Draw P_1N_1, P_2N_2, PN perpendicular to plane xOy. Through P draw PR parallel to N_1N_2 to meet P_2N_2 at R; and through P_1 draw P_1Q parallel to N_1N_2 to meet PN at Q.

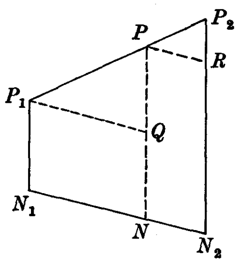

Fig. 222

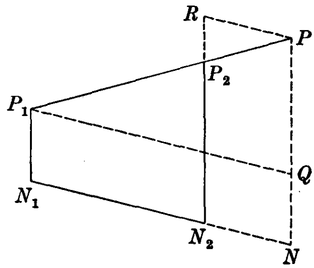

Fig. 223

From the similar triangles P_1PQ, PP_2R we have $PQ/P_2R = P_1P/PP_2$.

(i) If the division is *internal* (fig. 222), it follows that

$$\frac{z-z_1}{z_2-z} = \frac{k}{l},$$

from which
$$z = \frac{lz_1+kz_2}{l+k}.$$

Similarly, by dropping perpendiculars from P_1, P_2, P to plane yOz, we find that $x = (lx_1+kx_2)/(l+k)$; and by perpendiculars to plane zOx, that $y = (ly_1+ky_2)/(l+k)$.

Alternatively, since N divides N_1N_2 internally in the ratio $k:l$ and has the same x- and y-coordinates as P, the last two results follow from the section formulae in the plane xOy (15.14).

(ii) If the division is *external* (fig. 223) we have

$$\frac{z-z_1}{z-z_2} = \frac{k}{l},$$

from which
$$z = \frac{lz_1-kz_2}{l-k}.$$

There are similar expressions for the x- and y-coordinates of P.

Hence, with the convention in 15.14 (3), *the point dividing P_1P_2 in the (signed) ratio $k:l$ has coordinates*

$$\left(\frac{lx_1+kx_2}{l+k}, \frac{ly_1+ky_2}{l+k}, \frac{lz_1+kz_2}{l+k}\right).$$

Examples

(i) *The line joining $A(1,2,-3)$ and $B(3,0,2)$ meets the coordinate planes xOy, yOz, zOx at P, Q, R. Find the coordinates of these points, and the ratios in which they divide AB.*

The point dividing AB in the ratio $k:l$ has coordinates

$$\left(\frac{l+3k}{l+k}, \frac{2l}{l+k}, \frac{-3l+2k}{l+k}\right).$$

It lies in the plane xOy if and only if its z-coordinate is zero, i.e. $2k = 3l$. Hence P has coordinates $(\frac{11}{5}, \frac{4}{5}, 0)$, and divides AB internally in the ratio $3:2$.

The point lies in yOz if and only if its x-coordinate is zero, i.e. $l = -3k$. Hence Q is $(0, 3, -5\frac{1}{2})$, and divides AB externally in the ratio $1:3$.

The point lies in plane zOx if its y-coordinate is zero, i.e. $l = 0$. Thus R is $(3, 0, 1)$ and coincides with B. This is evident since B already lies in zOx.

(ii) *The four lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent at a point one-quarter the way up each line from the corresponding face.*

If the vertices are P_1, P_2, P_3, P_4, let G_4 be the centroid of triangle $P_1P_2P_3$, etc. The mid-point M of P_1P_2 has coordinates

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right),$$

and G_4 is the point dividing MP_3 in the ratio $1:2$ (cf. Ex. 15 (*a*), no. 1). Hence G_4 has coordinates

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right).$$

Consider the point G dividing G_4P_4 in the ratio $1:3$; its coordinates are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right),$$

which are symmetrical in those of P_1, P_2, P_3, P_4. Therefore G is also the point dividing each of the lines G_1P_1, G_2P_2, G_3P_3 in the ratio $1:3$, and the results follow.

The point G of concurrence is called the *centroid* of the tetrahedron $P_1P_2P_3P_4$, and the lines $P_1G_1, \ldots, P_4G_4$ are called its *medians*.

Exercise 21 (*a*)

1 If the negative x-axis is labelled Ox', and similarly for Oy', Oz', give the signs of the coordinates of points in each of the octants

$$Oxyz,\quad Ox'yz,\quad Ox'y'z,\quad Oxy'z,\quad Oxyz',\quad Ox'yz',\quad Ox'y'z',\quad Oxy'z'.$$

2 What points have

(i) $x = 0$; (ii) $y = 0$; (iii) $z = 0$;

(iv) $y = 0 = z$; (v) $z = 0 = x$; (vi) $x = 0 = y$?

3 In fig. 217 prove that $PA = \sqrt{(y^2+z^2)}$, and give expressions for PB, PC.

4 Prove that the points $(2, 3, 7)$, $(5, -1, -5)$, $(-10, 8, 7)$ are the vertices of an isosceles triangle, and find the length of the base.

5 Show that $(1, 2, -1)$ is the centre of the sphere through the points $(4, 6, 11)$, $(13, -1, 3)$, $(6, 14, -1)$, $(-3, -10, 2)$.

6 For $A(3, -1, 2)$, $B(5, 3, -6)$, $C(-1, 1, 7)$, $D(9, 1, -11)$, prove that AB and CD bisect each other.

7 Prove that the points $(1, 3, -4)$, $(2, -1, -3)$, $(7, 1, 8)$, $(6, 5, 7)$ are the vertices of a parallelogram, and find the lengths of the diagonals.

8 Find the ratios in which the coordinate planes divide the join of the points $(3, -5, 2)$ and $(-4, 7, 1)$.

9 Show that the point $(6, 2, 0)$ lies on the line joining $(-4, 6, 2)$ and $(-9, 8, 3)$, and also on the line joining $(7, -5, 1)$, $(4, 16, -2)$.

10 Prove that the lines joining the mid-points of opposite edges of a tetrahedron are concurrent at the centroid of the tetrahedron. [With the notation of 21.22, ex. (ii), P_1P_2 and P_3P_4 are a pair of 'opposite edges'.]

11 *Change of origin.* If new axes P_1x', P_1y', P_1z' are taken through P_1 parallel to Ox, Oy, Oz respectively, show that the new coordinates of $P(x, y, z)$ are (x', y', z') where $x' = x-x_1$, $y' = y-y_1$, $z' = z-z_1$.

12 (i) By using triangle ONP in fig. 217, prove $OP^2 = x^2+y^2+z^2$.

(ii) Obtain the formula for the distance P_1P_2 by first changing the origin to P_1.

The notation of 21.12 *is used in the following examples.*

*13 Express x, y in terms of ρ and ϕ, and conversely.

*14 If k and α are constants, what points have (i) $\rho = k$; (ii) $\phi = \alpha$; (iii) $\rho = k$ and $\phi = \alpha$?

*15 Express x, y, z in terms of r, θ and ϕ, and conversely.

*16 If k, α, β are constants, what points have

(i) $r = k$; (ii) $\theta = \alpha$; (iii) $\phi = \beta$;

(iv) $\theta = \alpha$ and $\phi = \beta$; (v) $r = k$ and $\phi = \beta$; (vi) $r = k$ and $\theta = \alpha$?

21.3 Direction cosines and direction ratios of a line

21.31 Direction cosines

We now consider how to specify the direction of a line, i.e. we seek the space analogue of 'gradient'. Whereas the latter is defined to be the *tangent* of an angle, we shall find that in space the *cosine* is the important trigonometrical function.

Let PQ be a line in space, described in the sense from P towards Q. Through O draw a line OA of unit length parallel to and in the same sense as PQ (fig. 224). We call $\mathbf{OA}$ the *unit ray* corresponding to the (sensed) line PQ.

The direction of $\mathbf{OA}$, and therefore that of PQ, will be determined by the coordinates of A. Since these coordinates are the orthogonal projections of $\mathbf{OA}$ on the axes Ox, Oy, Oz, hence A is the point $(\cos\alpha, \cos\beta, \cos\gamma)$, where $\alpha = x\hat{O}A$, $\beta = y\hat{O}A$ and $\gamma = z\hat{O}A$. It is customary to write

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma,$$

and call $\{l, m, n\}$ the *direction cosines* of PQ (and therefore of any line parallel to and in the same sense as PQ).

Since $\mathbf{OA}$ has unit length, the distance formula shows that

$$l^2+m^2+n^2 = 1. \tag{i}$$

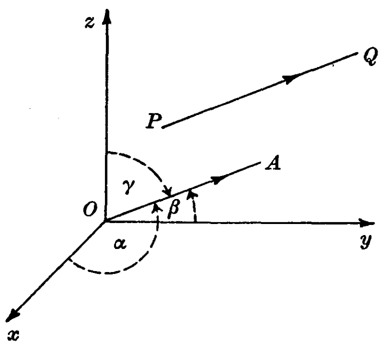

Fig. 224

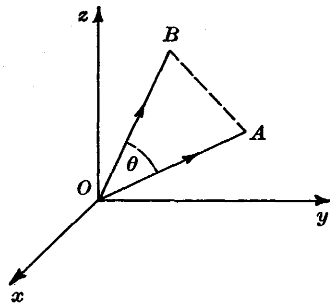

Fig. 225

Remarks

(α) If the line is described in the sense from Q towards P, then the corresponding unit ray through O is $\mathbf{OA'}$, where A' is the point

$$\big(\cos(\pi-\alpha), \cos(\pi-\beta), \cos(\pi-\gamma)\big),$$

i.e. $\quad (-\cos\alpha, -\cos\beta, -\cos\gamma), \quad$ i.e. $\quad (-l, -m, -n)$.

(β) A line parallel to the plane yOz has $\alpha = \frac{1}{2}\pi$, so that $l = 0$. Similarly, a line parallel to zOx has $m = 0$, and one parallel to xOy has $n = 0$.

(γ) In particular, the coordinate axes described in the positive senses Ox, Oy, Oz have direction cosines $\{1, 0, 0\}$, $\{0, 1, 0\}$, $\{0, 0, 1\}$.

21.32 Angle between two lines

If RS is another line, described in the sense from R towards S, let its direction cosines be $\{l', m', n'\}$. If $\mathbf{OB}$ is the corresponding unit ray through O, then B is the point (l', m', n').

The *angle between the (sensed) lines PQ, RS* (intersecting or not) is defined to be the angle AOB for which $0 \leqslant A\hat{O}B \leqslant \pi$ (fig. 225).

Let $A\hat{O}B = \theta$; then from triangle AOB, in which $OA = 1 = OB$, we have by the cosine rule that

$$AB^2 = 2 - 2\cos\theta.$$

Using the distance formula for AB, we also have

$$\begin{aligned} AB^2 &= (l-l')^2 + (m-m')^2 + (n-n')^2 \\ &= l^2 + m^2 + n^2 + l'^2 + m'^2 + n'^2 - 2ll' - 2mm' - 2nn' \\ &= 2 - 2(ll' + mm' + nn'), \end{aligned}$$

since by (i), $l^2 + m^2 + n^2 = 1$ and similarly $l'^2 + m'^2 + n'^2 = 1$. Comparing the two expressions for AB^2, we find

$$\cos\theta = ll' + mm' + nn'. \tag{ii}$$

In particular, *the directions* $\{l, m, n\}$, $\{l', m', n'\}$ *are perpendicular if and only if* $ll' + mm' + nn' = 0$. They are parallel (in the same sense) if and only if $l = l'$, $m = m'$, $n = n'$; and 'antiparallel' if and only if $l = -l'$, $m = -m'$, $n = -n'$.

21.33 Direction ratios

It is often convenient to specify the direction of a line by a set of three numbers which are proportional to the direction cosines l, m, n. Any such three numbers are called *direction ratios* of the line. For example, the line in the plane xOy which bisects the angle xOy has $\alpha = \frac{1}{4}\pi$, $\beta = \frac{1}{4}\pi$, $\gamma = \frac{1}{2}\pi$, so that its direction cosines are $\{1/\sqrt{2}, 1/\sqrt{2}, 0\}$; a convenient set of direction ratios for this bisector would be $1:1:0$.

In general, suppose $l:m:n = p:q:r$;† then $l = \lambda p$, $m = \lambda q$, $n = \lambda r$ where λ is determined from equation (i) by

$$\lambda^2(p^2 + q^2 + r^2) = 1.$$

Hence when the direction ratios $p:q:r$ are given, the direction cosines can be written down as

$$\{p/\sqrt{(p^2+q^2+r^2)},\ q/\sqrt{(p^2+q^2+r^2)},\ r/\sqrt{(p^2+q^2+r^2)}\}.$$

In any problem it is important to observe whether the numbers

† An equation $a:b = a':b'$ in which $a' = 0$ is understood to mean that $a = 0$; if $b' = 0$, the equation means that $b = 0$. An equation $a:b:c = a':b':c'$ in which (for example) $c' = 0$ means that $a:b = a':b'$ and $c = 0$, and so on. Cf. 11.43 (2).

p, q, r specifying a direction are meant to be direction cosines or direction ratios. We remark that direction ratios do not distinguish the two senses of a line.

Examples

(i) If θ is the acute angle between two lines having direction ratios $p:q:r$, $p':q':r'$, then by equation (ii)

$$\cos\theta = \frac{\pm(pp'+qq'+rr')}{\sqrt{(p^2+q^2+r^2)}\sqrt{(p'^2+q'^2+r'^2)}},$$

where the sign is chosen so that the expression is positive.

In particular, *the lines will be perpendicular if and only if*

$$pp'+qq'+rr' = 0$$

(the 'perpendicularity condition').

They will be parallel or 'antiparallel' if and only if $p:q:r = p':q':r'$.

(ii) If P is (x, y, z), the direction *cosines* of OP are $\{x/OP, y/OP, z/OP\}$; for $\cos\alpha = x/OP$, etc. A convenient set of direction *ratios* for OP is $x:y:z$.

(iii) The projections of P_1P_2 on Ox, Oy, Oz are x_2-x_1, y_2-y_1, z_2-z_1. Thus *a set of direction ratios for P_1P_2 is $x_2-x_1:y_2-y_1:z_2-z_1$.*

(iv) *The lines OP, OQ have direction ratios $l:m:n$, $l':m':n'$. Find direction ratios for the line OR which is perpendicular to both OP and OQ.*

Let OR have direction ratios $p:q:r$. Then by the perpendicularity condition in ex. (i),

$$lp+mq+nr = 0 \quad \text{and} \quad l'p+m'q+n'r = 0.$$

Solving these for $p:q:r$ (see 11.43 (2)), we have

$$p:q:r = mn'-m'n:nl'-n'l:lm'-l'm.$$

Exercise 21(*b*)

1 For $A(3, 2, 5)$, $B(-1, 6, 4)$, $C(5, -3, 7)$, $D(-3, 5, 5)$, prove $AB \parallel CD$.

2 For $A(2, -4, 3)$, $B(3, 2, -1)$, $C(-4, -3, 0)$, $D(6, -2, 4)$, prove $AB \perp CD$.

3 Prove that the points $(-5, 2, 7)$, $(-9, 3, 6)$, $(3, 0, 9)$ are collinear.

4 Prove that $(2, 3, 0)$, $(1, 5, 2)$, $(3, 7, 1)$, $(4, 5, -1)$ are the vertices of a square.

5 Find the direction ratios of the lines through O which make equal angles with the coordinate axes.

6 Calculate the angle between any two diagonals of a cube.

7 Find the angles between the diagonals of a rectangular box whose edges are a, b, c.

8 Calculate the angles of the triangle whose vertices are $A(2, 1, 3)$, $B(6, -2, -9)$, $C(5, 1, -1)$.

9 From $P(x, y, z)$ a perpendicular is drawn to meet the line through O with direction cosines $\{l, m, n\}$ at N. Prove that N has coordinates (pl, pm, pn) where $p = ON$, and write down direction ratios for PN. Hence prove that $p = lx+my+nz$.

10 If M is the foot of the perpendicular from N to OP in no. 9, find the coordinates of M.

*11 If the directions $p_1:q_1:r_1, p_2:q_2:r_2, p_3:q_3:r_3$ are parallel to the same plane, prove that

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0.$$

[They are perpendicular to the normal to the plane.]

*12 Prove that the converse of no. 11 is true. [Use Theorem II of 11.43.]

*13 If Ox, Oy, Oz and Ox', Oy', Oz' are two systems $\mathscr{S}, \mathscr{S}'$ of rectangular axes with the common origin O, let Ox', Oy', Oz' have direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) wo $\mathscr{S}$.

(i) Prove that

$$l_1^2+m_1^2+n_1^2 = l_2^2+m_2^2+n_2^2 = l_3^2+m_3^2+n_3^2 = 1$$

and

$$l_2 l_3+m_2 m_3+n_2 n_3 = l_3 l_1+m_3 m_1+n_3 n_1$$
$$= l_1 l_2+m_1 m_2+n_1 n_2 = 0.$$

(ii) Verify that (l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3) are the direction cosines of Ox, Oy, Oz wo $\mathscr{S}'$, and hence that

$$l_1^2+l_2^2+l_3^2 = m_1^2+m_2^2+m_3^2 = n_1^2+n_2^2+n_3^2 = 1$$

and $$m_1 n_1+m_2 n_2+m_3 n_3 = n_1 l_1+n_2 l_2+n_3 l_3 = l_1 m_1+l_2 m_2+l_3 m_3 = 0.$$

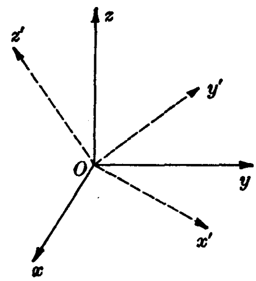

Fig. 226

*14 *Rotation of axes.*

(i) If P has coordinates (x, y, z) wo $\mathscr{S}$ and (x', y', z') wo $\mathscr{S}'$, prove that

$$x' = l_1 x+m_1 y+n_1 z, \quad y' = l_2 x+m_2 y+n_2 z, \quad z' = l_3 x+m_3 y+n_3 z.$$

[x' = projection of OP on Ox'; use no. 9.]

(ii) Also prove that

$$x = l_1 x'+l_2 y'+l_3 z', \quad y = m_1 x'+m_2 y'+m_3 z', \quad z = n_1 x'+n_2 y'+n_3 z'.$$

[Symmetry; *or* multiply x', y', z' by l_1, l_2, l_3 in (i), add, and use no. 13 (ii).]

*15 Explain geometrically why $x^2+y^2+z^2 = x'^2+y'^2+z'^2$, and verify this algebraically.

21.4 The plane

In this section we obtain the equations of planes determined by various conditions, and show that the general linear equation in x, y, z is the equation of some plane in space. The reader should note resemblances to some of the equations obtained in 15.2.

21.41 Equation of a plane in 'perpendicular form'

Let the perpendicular from O to the given plane have foot N. If ON has length p and direction cosines $\{l, m, n\}$, then N is the point (pl, pm, pn).

If $P(x, y, z)$ is any point of the plane, then $PN \perp ON$. Hence the direction given by $x-pl:y-pm:z-pn$ is perpendicular to that given by $\{l, m, n\}$, and so

$$(x-pl)\,l+(y-pm)\,m+(z-pn)\,n = 0,$$

i.e. $lx+my+nz = p(l^2+m^2+n^2) = p.$

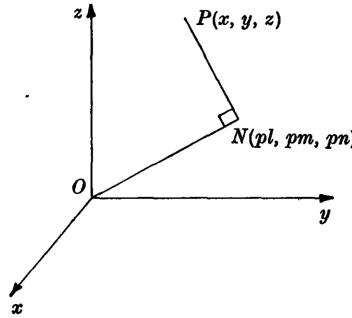

Fig. 227

The equation

$$lx+my+nz = p,$$

which is satisfied by the coordinates of any point of the plane, is consequently the equation of the plane. We observe that it is linear in x, y, z.

21.42 General linear equation $Ax+By+Cz+D=0$

(1) Let P_1 and P_2 be two distinct points which satisfy the above equation; then

$$Ax_1+By_1+Cz_1+D = 0, \quad Ax_2+By_2+Cz_2+D = 0. \qquad \text{(i)}$$

Multiply the first of these by $l/(l+k)$, the second by $k/(l+k)$, and add; we get

$$A\left(\frac{lx_1+kx_2}{l+k}\right)+B\left(\frac{ly_1+ky_2}{l+k}\right)+C\left(\frac{lz_1+kz_2}{l+k}\right)+D = 0,$$

which shows that the point dividing P_1P_2 in the ratio $k:l$ also satisfies

$$Ax+By+Cz+D = 0. \qquad \text{(ii)}$$

Since $k:l$ is arbitrary, this means that all points of the line P_1P_2 satisfy (ii).

By definition a plane is a locus in space (or *surface*) such that the line joining any two points of it lies entirely on the locus; hence *equation* (ii) *represents some plane.*

(2) *The normal to this plane has direction ratios $A:B:C$.* For if P_1 and P_2 are any two points in the plane, then by subtraction of equations (i) we have

$$A(x_1-x_2)+B(y_1-y_2)+C(z_1-z_2) = 0.$$

This shows that the direction $A:B:C$ is perpendicular to the direction $x_1-x_2:y_1-y_2:z_1-z_2$, i.e. to P_1P_2, for *any* line P_1P_2 in the plane. Hence the direction $A:B:C$ is perpendicular to the plane.

(3) At least one of the coefficients in (ii) is non-zero. On dividing by it we obtain an equation containing three arbitrary coefficients. This confirms the geometrical fact that a plane is determined by three independent conditions.

21.43 Conditions for $Ax+By+Cz+D=0, A'x+B'y+C'z+D'=0$ to represent the same plane

The planes are parallel if and only if their normals are parallel, i.e.

$$\frac{A'}{A}=\frac{B'}{B}=\frac{C'}{C}=k \quad \text{(say)}.$$

The equation of the second plane can therefore be written

$$k(Ax+By+Cz)+D'=0.$$

These parallel planes will be the *same* plane if and only if they have a point in common, say (x_0, y_0, z_0); and then

$$Ax_0+By_0+Cz_0+D=0, \quad k(Ax_0+By_0+Cz_0)+D'=0,$$

so that $$k(-D)+D'=0, \quad \text{i.e.} \quad k=D'/D.$$

Hence the planes are identical if and only if

$$\frac{A'}{A}=\frac{B'}{B}=\frac{C'}{C}=\frac{D'}{D},$$

i.e. if corresponding coefficients are *proportional*; in general the latter will *not* be equal.

21.44 Plane having normal $l:m:n$ and passing through P_1

By 21.41 the plane with normal in direction $l:m:n$ has an equation of the *form*

$$lx+my+nz=p,$$

even when l, m, n are direction ratios and not direction cosines. Since the plane passes through (x_1, y_1, z_1),

$$lx_1+my_1+nz_1=p.$$

By subtraction,

$$l(x-x_1)+m(y-y_1)+n(z-z_1)=0. \qquad \text{(iii)}$$

This relation holds for any point (x, y, z) of the required plane, and therefore is the equation of the plane.

Alternatively, consider the equation (iii). It is linear in x, y, z and therefore represents some plane (21.42 (1)), whose normal has direction $l:m:n$ (21.42 (2)). It is satisfied by (x_1, y_1, z_1). Hence it is the required equation.

21.45 Plane $P_1P_2P_3$

Method 1. Suppose this plane has normal $l:m:n$; then by 21.44 its equation is

$$l(x-x_1)+m(y-y_1)+n(z-z_1) = 0.$$

The points P_2, P_3 must satisfy this, so

$$l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1) = 0$$

and

$$l(x_3-x_1)+m(y_3-y_1)+n(z_3-z_1) = 0.$$

Elimination of l, m, n from this homogeneous system of three equations gives

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

as the equation of the plane $P_1P_2P_3$.

Method 2. Let the required equation be

$$Ax+By+Cz+D = 0.$$

This is satisfied by P_1, P_2 and P_3, so

$$Ax_1+By_1+Cz_1+D = 0,$$

$$Ax_2+By_2+Cz_2+D = 0,$$

$$Ax_3+By_3+Cz_3+D = 0.$$

Elimination of A, B, C, D from this homogeneous system of four equations gives

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

as the required equation. It is easy to reduce this fourth-order determinant to the third-order one just obtained, and vice versa.

21.46 Intercept form: plane making intercepts *a, b, c* on the coordinate axes

The required plane passes through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. Any plane through $(a, 0, 0)$ has an equation of the form

$$l(x-a)+my+nz = 0.$$

This is satisfied by $(0, b, 0)$, $(0, 0, c)$ if

$$-al+bm = 0, \quad -al+cn = 0.$$

Hence
$$al = bm = cn,$$

so that
$$l:m:n = \frac{1}{a}:\frac{1}{b}:\frac{1}{c},$$

and the required equation is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1.$$

21.47 Angle between two planes

The angle between two intersecting planes (a *dihedral angle*) is by definition the acute angle between the perpendiculars drawn in each to the common line, and is clearly equal to that between their normals.

Given two planes

$$Ax+By+Cz+D = 0, \quad A'x+B'y+C'z+D' = 0,$$

their normals have directions $A:B:C$, $A':B':C'$. The angle θ between the planes is therefore given by (see 21.33, ex. (i))

$$\cos\theta = \frac{\pm(AA'+BB'+CC')}{\sqrt{(A^2+B^2+C^2)}\sqrt{(A'^2+B'^2+C'^2)}}.$$

In particular, *the planes are perpendicular if and only if*

$$AA'+BB'+CC' = 0.$$

They are parallel if and only if their normals are parallel, i.e.

$$A:B:C = A':B':C'.$$

Exercise 21(*c*)

1 Find the locus of points equidistant from (2, −1, 3) and (1, 3, −2).

2 Find the equation of the plane which bisects AB and is perpendicular to CD, where $A(3, -2, 1)$, $B(5, 4, -2)$, $C(2, 4, -3)$, $D(-1, 3, 2)$.

3 Find the equation of the plane through (−2, 4, 0) and (−3, 2, −9) which is parallel to the line joining (3, 5, −1) and (6, 3, 2).

4 Obtain the equation of the plane through (2, 2, −1), (7, 0, 6), (3, 4, 2).

5 Find the equations of the two planes through (0, 4, −3) and (6, −4, 3), other than the plane through O, which make intercepts on the axes whose sum is zero.

6 If $A(3, 0, 0)$, $B(2, 3, 0)$, $C(1, 1, 1)$, find (i) the angle between planes OBC, OAB; (ii) the angle between line BC and plane OAB.

7 At three points O, X, Y on a horizontal plane, where $OX = c = OY$ and $X\hat{O}Y = \frac{1}{2}\pi$, vertical shafts are sunk to meet a seam of coal at depths p, q, r respectively. Assuming that the coal face is plane, find the angle between this plane and the horizontal, and the distances from O of the points where this plane meets OX and OY.

8 If $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$, $P(a, b, c)$, prove that the angle between OP and plane ABC is $\sin^{-1}(3q/p)$, where $p^2 = a^2+b^2+c^2$ and $q^{-2} = a^{-2}+b^{-2}+c^{-2}$. Also prove that the distance between the feet of the perpendiculars from O and P to ABC is $\sqrt{(p^2-9q^2)}$, and find the angle between planes PBC, PCA.

9 Find the equation of the plane bisecting P_1P_2 at right-angles.

*10 Show that the two determinantal equations in 21.45 are equivalent.

11 Prove that the plane through P_1 and P_2 which is parallel to the direction $p:q:r$ has equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ p & q & r \end{vmatrix} = 0.$$

12 Prove that the plane through P_1 which is parallel to each of the directions $p:q:r$, $p':q':r'$ has equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ p & q & r \\ p' & q' & r' \end{vmatrix} = 0.$$

13 *Sides of a plane.* Find the value of $k:l$ if the point dividing P_1P_2 in the ratio $k:l$ lies in the plane $Ax+By+Cz+D = 0$. Deduce that P_1, P_2 are on the same or opposite sides of this plane according as $Ax_1+By_1+Cz_1+D$, $Ax_2+By_2+Cz_2+D$ have the same or opposite signs.

21.5 The line

A straight line in space can be determined by (*a*) the meet of two planes; (*b*) the join of two points; (*c*) a single point together with a set of direction ratios.

We begin most conveniently with case (*c*).

21.51 Line through P_1 in direction $l:m:n$

Let $P(x, y, z)$ be any point on the line; then a set of direction ratios for the line is $x-x_1:y-y_1:z-z_1$. Since $l:m:n$ is also such a set of direction ratios, hence

$$x-x_1:y-y_1:z-z_1 = l:m:n.$$

This can be expressed in the form†

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}. \tag{i}$$

These equations are satisfied by any point of the line, and are therefore the equations of the line.

Notice that *two* equations are necessary to determine a line in space: a single linear equation represents a plane. The two equations (i) are in fact the equations of two planes, each of which contains the line; e.g.

$$\frac{x-x_1}{l} = \frac{y-y_1}{m}, \quad \frac{x-x_1}{l} = \frac{z-z_1}{n}$$

represent two such planes.

We refer to (i) as the *symmetrical equations* of a line.

Example

Equation of the line P_1P_2 (case (*b*) above).

Direction ratios of the line P_1P_2 are $x_1-x_2:y_1-y_2:z_1-z_2$. Hence the line has equations

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2}.$$

21.52 Parametric equations of a line

Putting each of the ratios in (i) above equal to λ, we have

$$x = x_1+\lambda l, \quad y = y_1+\lambda m, \quad z = z_1+\lambda n. \tag{ii}$$

These are the parametric equations of the line through P_1 in direction $l:m:n$, λ being the parameter (cf. 15.26‡). They are particularly useful when dealing with intersections of the line with a line, plane, or other surface.

† If the line is parallel to a coordinate plane, say to yOz, then $n = 0$. The equation

$$x-x_1:y-y_1:z-z_1 = l:m:0 \tag{i)$'$}$$

then means (see the footnote on p. 707) that $x-x_1:y-y_1 = l:m$ and $z-z_1 = 0$. This is consistent with the geometrical situation, since every point of the line has its z-coordinate constant, say $z = z_1$: the line lies in the plane whose equation is $z = z_1$. We continue to write $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{0}$ as alternative to (i)′, with a similar understanding when two of l, m, n are zero.

‡ Unless l, m, n are direction cosines, λ is not the distance of the point $(x_1+\lambda l, y_1+\lambda m, z_1+\lambda n)$ from P_1; but it is proportional to this distance.

Examples

(i) *Find the length of the perpendicular from* $A(3, 7, -2)$ *to the line*

$$\frac{x-10}{3} = \frac{y+8}{-4} = \frac{z+5}{1},$$

and find the coordinates of its foot N.

The parametric equations of the line, obtained by putting the ratios in its equations equal to λ, are

$$x = 3\lambda+10, \quad y = -4\lambda-8, \quad z = \lambda-5.$$

Calling this general point P, the direction ratios of AP are

$$3\lambda+7:-4\lambda-15:\lambda-3.$$

AP will be perpendicular to the given line (whose direction ratios are $3:-4:1$) if and only if

$$3(3\lambda+7)-4(-4\lambda-15)+1(\lambda-3) = 0,$$

from which $\lambda = -3$. Hence the foot N of the perpendicular from A to the line is obtained by putting $\lambda = -3$ in the coordinates of the general point P; this gives $N(1, 4, -8)$.

It now follows that

$$AN^2 = 2^2+3^2+6^2 = 49, \quad \therefore\ AN = 7.$$

(ii) *Find the image of the point* $P(3, -5, 2)$ *in the plane* $2x+7y-3z = 27$.

By definition, the *image* P' of P in the plane is such that PP' is perpendicular to the plane and is bisected by it.

Since the normal to the plane has direction ratios $2:7:-3$, the equation of the normal through P is

$$\frac{x-3}{2} = \frac{y+5}{7} = \frac{z-2}{-3},$$

or parametrically

$$x = 2\lambda+3, \quad y = 7\lambda-5, \quad z = -3\lambda+2.$$

This point will be P', the image of P, provided that the mid-point of $P'P$ lies on the plane, i.e.

$$2(\lambda+3)+7(\tfrac{7}{2}\lambda-5)-3(-\tfrac{3}{2}\lambda+2) = 27,$$

from which $\lambda = 2$. Hence P' is the point $(7, 9, -4)$.

(iii) *Find the image of the line*

$$\frac{x-2}{3} = \frac{y-1}{2} = \frac{z+3}{5}$$

in the plane $x-2y+3z = 5$.

We find (a) the point Q where the line and plane meet; (b) the image P' of the point $P(2, 1, -3)$ of the line; (c) the required line as the join of P', Q.

(a) The parametric equations of the given line are

$$x = 3\lambda+2, \quad y = 2\lambda+1, \quad z = 5\lambda-3.$$

The line cuts the given plane at the point Q for which

$$(3\lambda+2)-2(2\lambda+1)+3(5\lambda-3) = 5,$$

i.e. $\lambda = 1$; hence Q is $(5, 3, 2)$.

(*b*) The method of ex. (ii) shows that the image of P in the given plane is $P'(4, -3, 3)$.

(*c*) The line $P'Q$ has direction ratios $1:6:-1$, so its equations can be written

$$\frac{x-5}{1} = \frac{y-3}{6} = \frac{z-2}{-1}.$$

(iv) *Condition for two lines to be coplanar.*

Suppose that the lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \quad \frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'}$$

are coplanar. Then *either* they are parallel, *or* they intersect.

If they are parallel, then $l:m:n = l':m':n'$.

If the lines intersect, then the common point has coordinates

$$(a+\lambda l, b+\lambda m, c+\lambda n) \quad \text{for some } \lambda,$$

and also $$(a'+\mu l', b'+\mu m', c'+\mu n') \quad \text{for some } \mu.$$

Hence there exist values of λ and μ for which

$$\left.\begin{aligned} a-a'+\lambda l-\mu l' &= 0,\\ b-b'+\lambda m-\mu m' &= 0,\\ c-c'+\lambda n-\mu n' &= 0.\end{aligned}\right\} \qquad \text{(iii)}$$

Elimination of λ, μ (11.43, Corollary I (*b*)) gives the condition

$$\begin{vmatrix} a-a' & l & l' \\ b-b' & m & m' \\ c-c' & n & n' \end{vmatrix} = 0.$$

The condition is also satisfied when the lines are parallel, for in this case the second and third columns of the determinant are proportional.

This *necessary* condition for two lines to be coplanar is also *sufficient*. For when it is satisfied, then by 11.43, Theorem II, there exist numbers α, β, γ not all zero such that

$$\alpha(a-a')+\beta l+\gamma l' = 0,$$

$$\alpha(b-b')+\beta m+\gamma m' = 0,$$

and $$\alpha(c-c')+\beta n+\gamma n' = 0.$$

If $\alpha = 0$, then $l:m:n = l':m':n'$, so that the lines are parallel, and hence coplanar.

If $\alpha \neq 0$, the above equations can be written in the form (iii), which show that, for some pair of values of λ and μ, the point $(a+\lambda l, b+\lambda m, c+\lambda n)$ of the first line coincides with the point $(a'+\mu l', b'+\mu m', c'+\mu n')$ of the second. The lines therefore intersect, and hence are coplanar.

21.53 Line of intersection of two planes

Given two planes

$$ax+by+cz+d = 0, \quad a'x+b'y+c'z+d' = 0$$

which intersect, we require to express the equations of their common line† in the symmetrical form.

The planes will not be parallel if and only if *not all* of

$$bc'-b'c, \quad ca'-c'a, \quad ab'-a'b$$

are zero. When this is so, let the common line have direction ratios $l:m:n$. Then since this line lies in each plane, it is perpendicular to the normal to each plane, i.e. to the directions $a:b:c$, $a':b':c'$. Hence

$$al+bm+cn = 0, \quad a'l+b'm+c'n = 0,$$

from which (11.43 (2))

$$\frac{l}{bc'-b'c} = \frac{m}{ca'-c'a} = \frac{n}{ab'-a'b}.$$

We now require the coordinates of one point on the line of intersection. This can be chosen in infinitely many ways; e.g. we may find the point where the line meets any one of the coordinate planes.

Example

Find symmetrical equations for the line of intersection of the planes

$$2x-3y+z = 6, \quad 5x+4y-3z = 7.$$

If the required line has direction ratios $l:m:n$, then

$$2l-3m+n = 0, \quad 5l+4m-3n = 0,$$

and hence $$l:m:n = 5:11:23.$$

Let us find the point where the line cuts the plane $x = 0$. The y- and z-coordinates are given by

$$-3y+z = 6, \quad 4y-3z = 7;$$

hence $y = -5$, $z = -9$, and a point on the line is $(0, -5, -9)$. Symmetrical equations of the line are therefore

$$\frac{x}{5} = \frac{y+5}{11} = \frac{z+9}{23}.$$

21.54 Distance of a point from a plane

Given a point P_1 and a plane $ax+by+cz+d = 0$, let N be the foot of the perpendicular from P_1 to this plane. We require to find the length of P_1N.

The normal through P_1 to the plane has equations

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c},$$

† The two equations just written are equations of the common line (case (a)).

or parametrically

$$x = x_1 + \lambda a, \quad y = y_1 + \lambda b, \quad z = z_1 + \lambda c.$$

These expressions will be the coordinates of N if they satisfy the equation of the plane, i.e. if

$$a(x_1 + \lambda a) + b(y_1 + \lambda b) + c(z_1 + \lambda c) + d = 0,$$

from which
$$\lambda = -\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}.$$

With this value of λ we have, by the distance formula,

$$P_1N^2 = (\lambda a)^2 + (\lambda b)^2 + (\lambda c)^2 = \lambda^2(a^2 + b^2 + c^2)$$
$$= \frac{(ax_1 + by_1 + cz_1 + d)^2}{a^2 + b^2 + c^2}.$$

Hence
$$P_1N = \pm \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}},$$

where the sign is chosen so that the expression is positive.

Example

Planes bisecting the angles between two given planes.

If $P(x, y, z)$ is any point on a plane which bisects an angle between the planes

$$ax + by + cz + d = 0, \quad a'x + b'y + c'z + d' = 0,$$

then the perpendiculars PN, PN' to these two planes are equal. Hence

$$\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a'x + b'y + c'z + d'}{\sqrt{(a'^2 + b'^2 + c'^2)}},$$

and these are the equations of the two bisector planes.

21.55 Areas and volumes

(1) *Area of triangle* $P_1P_2P_3$. Let α, β, γ be the angles between the plane $P_1P_2P_3$ and the coordinate planes yOz, zOx, xOy. Then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal to this plane, and hence

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \qquad \text{(iv)}$$

Let Δ be the area of the triangle $P_1P_2P_3$, and let the areas of its orthogonal projections on the planes yOz, zOx, xOy be Δ_1, Δ_2, Δ_3. Then by 17.24 (4),

$$\Delta_1 = \Delta\cos\alpha, \quad \Delta_2 = \Delta\cos\beta, \quad \Delta_3 = \Delta\cos\gamma,$$

and hence by (iv),
$$\Delta^2 = \Delta_1^2 + \Delta_2^2 + \Delta_3^2. \qquad \text{(v)}$$

The projections of P_1, P_2, P_3 on plane yOz are the points $(0, y_1, z_1)$, $(0, y_2, z_2)$,

$(0, y_3, z_3)$ of space. Regarded as points of the plane yOz referred to axes Oy, Oz, they have coordinates (y_1, z_1), (y_2, z_2), (y_3, z_3). Hence by 15.16 (2),

$$\Delta_1 = \pm\tfrac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}.$$

Similarly
$$\Delta_2 = \pm\tfrac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \quad \Delta_3 = \pm\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

On substituting these expressions in (v), we obtain a formula for Δ in terms of the coordinates of the vertices of the triangle.

(2) *Volume of tetrahedron* $P_0P_1P_2P_3$. The volume is given by

$\tfrac{1}{3}\times$ (area of one face) $\times$ (perpendicular to that face from the remaining vertex).

If V is the required volume, Δ the area of triangle $P_1P_2P_3$, and h the perpendicular from P_0 to the plane $P_1P_2P_3$, then

$$V = \tfrac{1}{3}\Delta h. \qquad \text{(vi)}$$

The plane $P_1P_2P_3$ has equation (see 21.45, Method 2)

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

If we expand this determinant by the first row, we obtain (11.7)

$$x\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} - y\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} + z\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

The coefficients of x, y, z are numerically equal to the expressions $2\Delta_1$, $2\Delta_2$, $2\Delta_3$ in (1).

The perpendicular distance h of P_0 from the plane $P_1P_2P_3$ is obtained (see 21.54) by writing x_0, y_0, z_0 for x, y, z in the left-hand side of this equation, and then dividing by the square root of the sum of the squares of the coefficients of x, y, z:

$$h = \pm\begin{vmatrix} x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} \div 2\sqrt{(\Delta_1^2+\Delta_2^2+\Delta_3^2)}.$$

Using (vi) and (v), we now have

$$V = \pm\tfrac{1}{6}\begin{vmatrix} x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix},$$

where the sign is chosen to make the result positive.

Example*

With the notation of Ex. 21 (*b*), no. 13, let I, J, K be the points which have coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ wo $Ox'y'z'$. Then triangle OJK has area $\frac{1}{2}$, and tetrahedron $OIJK$ has volume $\frac{1}{6}$.

Referred to axes $Oxyz$, the points are $I(l_1, m_1, n_1)$, $J(l_2, m_2, n_2)$, $K(l_3, m_3, n_3)$, and so

$$\text{volume } OIJK = \pm\tfrac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ l_1 & m_1 & n_1 & 1 \\ l_2 & m_2 & n_2 & 1 \\ l_3 & m_3 & n_3 & 1 \end{vmatrix}.$$

Hence
$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1.$$

Exercise 21(*d*)

1 Find the point where the line

$$\frac{x-1}{2} = \frac{y+3}{5} = \frac{z-4}{3}$$

cuts the plane $3x-2y+z = 8$. Also find the distance between $(1, -3, 4)$ and this point.

2 Find the points where the line through (a, b, c) in direction $l:m:n$ cuts the coordinate planes.

3 G is the centroid of the triangle whose vertices are the points where the plane $lx+my+nz = p$ cuts the coordinate axes, and $l^2+m^2+n^2 = 1$. The perpendicular at G to this plane meets the coordinate planes at A, B, C. Prove

$$\frac{1}{GA}+\frac{1}{GB}+\frac{1}{GC} = \frac{3}{p}.$$

4 Find the perpendicular distance of $(5, 4, 1)$ from the line

$$\frac{x-6}{5} = \frac{y+15}{1} = \frac{z-14}{8}.$$

5 Obtain the equations of the line through $(1, 2, 3)$ which meets the line

$$\frac{x+1}{2} = \frac{y-2}{1} = \frac{z+4}{2}$$

and is parallel to the plane $x+5y+z = 7$. Give the point of intersection.

6 Prove that the line $x-1 = -9y+18 = -3z-9$ is parallel to the plane $3x-3y+10z = 26$. Find the image of the line in this plane.

7 Find the image of the line

$$\frac{x+1}{2} = \frac{y-2}{1} = \frac{z-3}{-6}$$

in the plane $3x+2y-5z = 24$.

8 Obtain the equation of the plane which contains the parallel lines

$$\frac{x-1}{2}=\frac{y+2}{-3}=\frac{z-4}{4}, \quad \frac{x-4}{2}=\frac{y-3}{-3}=\frac{z+1}{4}.$$

9 Prove that the lines

$$\frac{x-2}{2}=\frac{y-3}{-1}=\frac{z+4}{3}, \quad \frac{x-3}{1}=\frac{y+1}{3}=\frac{z-1}{-2}$$

are coplanar. Find their common point, and the equation of the plane containing them.

10 Obtain the condition in ex. (iv) of 21.52 by using Ex. 21 (c), no. 12.

11 Prove that the line $3x+2y+z=4$, $x+y-2z=1$ is perpendicular to the line $2x-y-z=16$, $7x+10y-8z=2$.

12 Find the equation of the plane through O which is perpendicular to the common line of the planes $x+y+z=2$, $3x-y+2z=1$.

13 Q is the point $(\frac{2}{3}, \frac{4}{3}, \frac{4}{3})$, and a line l through Q has direction ratios $-4:1:1$. The line OQ meets the line l' at P, where l' has equations $3x-3y+4=0$, $x+2y+2z=12$. Prove that OPQ is perpendicular to both l and l'.

If R is a point on l' and S is a point on l such that $PR=QS$, prove that OR^2-OS^2 is constant, and state its value.

14 A plane makes angle 60° with the line $x=y=z$ and 45° with the line $x=0=y-z$. Find the angle which it makes with the plane $x=0$.

15 Find the equations of the two lines through O which meet the line

$$\frac{x-3}{2}=y-3=z$$

at angles of 60°.

16 Find the equations of the lines which meet the line

$$\frac{x+5}{2}=\frac{y+6}{2}=\frac{z+7}{1}$$

at 60° and which lie in the plane $x+y+2z+1=0$.

17 Calculate the distance of the point $(3, -1, 2)$ from the plane

$$5x-6y-30z=23.$$

18 Find the equations of the planes which bisect the angles between the planes $2x-3y+6z=1$, $4x+3y-12z+2=0$.

*19 Find the area of the triangle formed by the lines in which the plane $lx+my+nz=p$ cuts the coordinate planes ($lmn \neq 0$).

*20 Interpret geometrically the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

21.6 Planes in space

21.61 Planes through a common line

Consider the equation $\alpha + k\alpha' = 0$, where k is constant and

$$\alpha \equiv ax+by+cz+d, \quad \alpha' \equiv a'x+b'y+c'z+d'.$$

It is linear in x, y, z, and therefore represents some plane. It is satisfied by the coordinates of any point P which satisfies both $\alpha = 0$ and $\alpha' = 0$, i.e. by any point on the common line of these planes.

Hence $\alpha + k\alpha' = 0$ *represents a plane through the line of intersection of the planes* $\alpha = 0$, $\alpha' = 0$, provided that such a line exists. If $\alpha = 0$, $\alpha' = 0$ are parallel, then clearly $\alpha + k\alpha' = 0$ is a plane parallel to both.

Conversely, every plane through the common line of $\alpha = 0$, $\alpha' = 0$ can be represented by an equation $\alpha + k\alpha' = 0$ for some value of k (except the plane $\alpha' = 0$ itself†). For if P_1 is a point (not on the common line) of such a plane, then $\alpha_1 + k\alpha_1' = 0$ gives a unique value of k, viz. $-\alpha_1/\alpha_1'$, and the required equation is $\alpha\alpha_1' = \alpha'\alpha_1$.

A case of $\alpha + k\alpha' = 0$ arose in 21.54, example.

Examples

(i) *Find the equations of the orthogonal projection of the line*

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$$

on the plane $x+2y+z = 8$.

Two planes through the given line are

$$\frac{x-1}{2} = \frac{y+1}{-1} \quad \text{and} \quad \frac{x-1}{2} = \frac{z-3}{4},$$

i.e. $$x+2y+1 = 0 \quad \text{and} \quad 2x-z+1 = 0.$$

Hence any plane through the line has an equation of the form

$$(2x-z+1)+k(x+2y+1) = 0,$$

i.e. $$(2+k)x+2ky-z+(1+k) = 0.$$

The orthogonal projection of the line is the meet of the plane $x+2y+z = 8$ with the plane through the line and perpendicular to this given plane. We therefore choose k so that the directions $1:2:1$ and $2+k:2k:-1$ are perpendicular:

$$(2+k)+4k-1 = 0,$$

so that $k = -\frac{1}{5}$. The required plane through the line is therefore

$$2x-z+1-\tfrac{1}{5}(x+2y+1) = 0,$$

i.e. $$9x-2y-5z = -4.$$

This and $x+2y+z = 8$ are the equations of the projection.

† This exception can be avoided, if necessary, by using the equation $\lambda\alpha + \lambda'\alpha' = 0$.

(ii) *If the direction ratios of a vertical line are* $1:0:-2$, *find direction ratios of a line of greatest slope of the plane* $2x-y-3z = 4$.

Let the vertical through the origin O cut the given plane at P, and let PQ be the line of greatest slope through P. Then the plane OPQ must be perpendicular to the given plane.

The equations of OP are

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{-2},$$

which are equivalent to $\quad y = 0, \quad 2x+z = 0.$

Hence any plane through OP has an equation of the form

$$2x+z+ky = 0.$$

This will be the plane OPQ if k is chosen so that the direction $2:k:1$ is perpendicular to $2:-1:-3$ (the normal to the given plane):

$$4-k-3 = 0, \quad \text{i.e.} \quad k = 1.$$

Hence the plane OPQ is $2x+y+z = 0$, and this equation together with $2x-y-3z = 4$ determines the line of greatest slope through P. Its direction is $l:m:n$, where

$$2l+m+n = 0 \quad \text{and} \quad 2l-m-3n = 0;$$

hence $l:m:n = 1:-4:2$.

(iii) *Show that the planes* $3x+y+3z = 4$, $x-y-5z = 2$, *and* $6x+4y+15z = 7$ *pass through a common line, and find direction ratios for this line.*

Any plane through the line of intersection of the first two planes has an equation of the form

$$3x+y+3z-4+k(x-y-5z-2) = 0,$$

i.e. $$(k+3)x+(1-k)y+(3-5k)z = 2k+4.$$

This will represent the same plane as $6x+4y+15z = 7$ if and only if (21.43)

$$\frac{k+3}{6} = \frac{1-k}{4} = \frac{3-5k}{15} = \frac{2k+4}{7}.$$

The first of these equations gives $k = -\frac{3}{5}$; and this value is found to satisfy the second and third. Hence the three given planes possess a common line, whose direction ratios $l:m:n$ are given by any two of the equations

$$3l+m+3n = 0, \quad l-m-5n = 0, \quad 6l+4m+15n = 0.$$

Thus $$l:m:n = -2:18:-4 = 1:-9:2.$$

21.62 Incidence of three planes

Let
$$\alpha_1 \equiv a_1x+b_1y+c_1z+d_1 = 0,$$
$$\alpha_2 \equiv a_2x+b_2y+c_2z+d_2 = 0,$$
$$\alpha_3 \equiv a_3x+b_3y+c_3z+d_3 = 0,$$

be the equations of three *distinct* planes.

If (x, y, z) satisfies all three equations, then as in 11.41 we have

$$\Delta x = \Delta^{(1)}, \quad \Delta y = \Delta^{(2)}, \quad \Delta z = \Delta^{(3)}, \tag{i}$$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad \Delta^{(1)} = \begin{vmatrix} -d_1 & b_1 & c_1 \\ -d_2 & b_2 & c_2 \\ -d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta^{(2)} = \begin{vmatrix} a_1 & -d_1 & c_1 \\ a_2 & -d_2 & c_2 \\ a_3 & -d_3 & c_3 \end{vmatrix}, \qquad \Delta^{(3)} = \begin{vmatrix} a_1 & b_1 & -d_1 \\ a_2 & b_2 & -d_2 \\ a_3 & b_3 & -d_3 \end{vmatrix}.$$

If $\Delta \neq 0$, the equations possess a unique solution (x, y, z), i.e. *the planes* $\alpha_1, \alpha_2, \alpha_3$ *have a unique common point.*

If $\Delta = 0$, then by Theorem II of 11.43 there exist numbers l, m, n not all zero such that

$$a_1 l + b_1 m + c_1 n = 0,$$
$$a_2 l + b_2 m + c_2 n = 0,$$
$$a_3 l + b_3 m + c_3 n = 0.$$

Hence the line with direction ratios $l:m:n$ is perpendicular to each of the directions $a_1:b_1:c_1$, $a_2:b_2:c_2$, $a_3:b_3:c_3$. Since these are the directions of the normals to $\alpha_1, \alpha_2, \alpha_3$, these planes must all be parallel to some line.

Conversely, if the planes are all parallel to a line having direction ratios $l:m:n$, then the above three homogeneous equations in three unknowns l, m, n hold, and so $\Delta = 0$ by 11.43, Theorem I.

Hence $\alpha_1, \alpha_2, \alpha_3$ *are parallel to some line if and only if* $\Delta = 0$. There are various possibilities; to illustrate them, let a plane perpendicular to this line cut $\alpha_1, \alpha_2, \alpha_3$ in lines $\lambda_1, \lambda_2, \lambda_3$.

(i) *All of* $\alpha_1, \alpha_2, \alpha_3$ *are parallel* (fig. 228). This is so if and only if

$$a_1:b_1:c_1 = a_2:b_2:c_2 = a_3:b_3:c_3.$$

(ii) *Two of* $\alpha_1, \alpha_2, \alpha_3$ *are parallel, and the third intersects them* (fig. 229). This happens if and only if just two of the ratio-sets $a_1:b_1:c_1$, $a_2:b_2:c_2$, $a_3:b_3:c_3$ are equal.

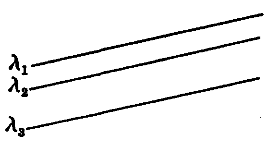

Fig. 228

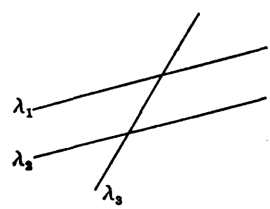

Fig. 229

In both of these cases we certainly have $\Delta = 0$ since at least two columns are proportional.

(iii) *No two of* $\alpha_1, \alpha_2, \alpha_3$ *are parallel.* This occurs when no two of the above ratio-sets are equal. The planes may

(*a*) meet in pairs in three parallel lines (fig. 230), or

(*b*) meet in a common line (fig. 231).

Unless $\Delta^{(1)} = \Delta^{(2)} = \Delta^{(3)} = 0$, the three planes cannot possess any point in common; this follows from equations (i) (and the general hypothesis that $\Delta = 0$).

Since (iii)(*a*) and (iii)(*b*) are mutually exclusive, (iii)(*a*) occurs when and only when $\Delta = 0$ but at least one of $\Delta^{(1)}$, $\Delta^{(2)}$, $\Delta^{(3)}$ is non-zero; and the planes have a common line (case (iii)(*b*)) if and only if $\Delta = \Delta^{(1)} = \Delta^{(2)} = \Delta^{(3)} = 0$.

The algebra in 11.42(1) corresponds to the case (iii)(*b*), but includes the possibility that two of the planes may coincide; that of 11.42(2) corresponds to the possibility of all the planes coinciding.

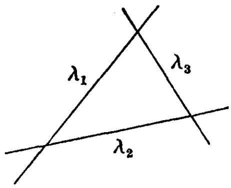

Fig. 230

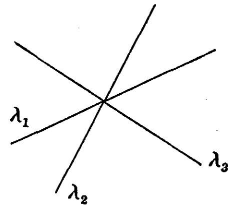

Fig. 231

Exercise 21(*e*)

1 P is the point $(-2, -3, -2)$, and l is the line

$$\frac{x-2}{3} = \frac{y-1}{4} = \frac{z+3}{2}.$$

Find the equation of the plane determined by P and l.

2 Find the equation of the plane through the line

$$\frac{x-1}{2} = \frac{y}{-1} = \frac{z+2}{1}$$

and parallel to the line $2x+5y+3z = 8$, $x-y-5z = 3$.

3 Find the plane through

$$x = \frac{y-3}{2} = \frac{z-5}{3}$$

and perpendicular to the plane $2x+7y-3z = 4$.

4 Find direction ratios for the projection of

$$\frac{x-1}{2} = \frac{y}{-1} = \frac{z+2}{1}$$

on the plane $2x+y-3z = 9$.

5 Find the equations of the projection of the line

$$3x-y+2z = 1, \quad x+2y-z = 2$$

on the plane $3x+2y+z = 0$. Verify that $(-1, 1, 1)$ lies on the projection, and obtain its equations in symmetrical form.

6 Taking Oz to be vertical, find direction ratios for a line of greatest slope of the plane through $(0, 0, 0)$, $(3, 5, -2)$, $(4, 1, 1)$.

7 Find the equations of the planes through the line $3x = 2y = 3z$ which make angle 30° with the plane $z = 0$. If Oz is vertical, prove that the lines of greatest slope of these planes make an angle $\tan^{-1}\frac{1}{4}$ with the given line.

8 Obtain the equations of the projection l' of the line l whose equations are $2x+y-4 = 0$, $y+2z-8 = 0$ on the plane α whose equation is $2x-y-z+3 = 0$. If l' is a line of greatest slope in α and the angle between l' and the vertical is 60°, find direction ratios for either possible vertical.

9 Find the equations of the three planes which pass through the line of intersection of two of the planes

$$x+y+z+3 = 0, \quad 2x+y+2z+5 = 0, \quad x+3y+2z+6 = 0$$

and are perpendicular to the third.

Prove that the planes so obtained have a common line, and that the plane through O perpendicular to this line is $7x+5y-2z = 0$.

10 Show that the planes $x+2y-z = 0$, $3x-4y+z = 3$, and $4x+3y-2z = 24$ form a triangular prism, and that the lengths of the sides of a normal cross-section are in the ratio $\sqrt{13}:\sqrt{58}:5\sqrt{3}$. [Use the sine rule.]

21.7 Skew lines

21.71 Geometrical introduction

Two lines in space may (i) intersect, (ii) be parallel, or (iii) neither intersect nor be parallel. In cases (i) and (ii) the lines are coplanar; in (iii) the lines are said to be *skew*.

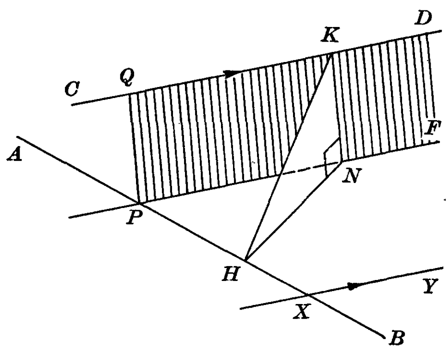

Fig. 232

Two skew lines have infinitely many common transversals; for any point on one can be joined to any point on the other. Cf. the example below. We now prove that there is exactly one transversal which cuts both lines at right-angles, and that the distance between the intersections is the shortest distance between the two lines.

Let AB, CD be the given skew lines. Through any point X on AB draw XY parallel to CD. Then the plane AXY contains AB, and is parallel to CD; for if CD met plane AXY, say at E, we could draw a line through E parallel to XY, i.e. parallel to CD; and this is impossible.

Let the orthogonal projection of CD on plane AXY be the line PF, meeting AB at P. (If PF did not cut AB, then $PF \parallel AB$; and since CD is parallel to plane AXY, therefore $CD \parallel PF$. Hence we should have $CD \parallel AB$, contradicting the hypothesis of skewness.) The point P is the projection of some point of CD, say Q.

We prove that PQ *is the unique common perpendicular of* AB *and* CD. For since PQ is perpendicular to plane AXY, hence PQ is perpendicular to AB and to PF. Also, since $CD \parallel PF$, PQ is perpendicular to CD. The uniqueness follows from the steps of the construction for PQ.

If H, K *are any two points of* AB, CD *respectively, then* $HK > PQ$. To prove this, let the perpendicular from K to plane AXY have foot N; then N lies on PF, and $KN \perp NH$, so that $HK > KN$ (the hypotenuse being the longest side of triangle KNH). Since both PQ and NK are perpendicular to PF and coplanar, hence $PQ \parallel NK$; and since $QK \parallel PN$, $PQKN$ is a rectangle, so that $PQ = KN$. Thus $HK > PQ$.

It follows that PQ is the shortest distance between the two skew lines. It is equal to the perpendicular distance from any point K on CD to the plane AXY.

Example

Prove that through a given point there is a unique common transversal of two skew lines l, l'.

Let the given point be P, not on either line l, l'. If a line through P meets l, it must lie in the plane (P, l). If a line through P meets l', it must lie in the plane (P, l'). Hence a line through P meeting both l and l' would have to lie in both these planes. Now these planes certainly intersect, because P lies in both; hence they have a unique common line which passes through P, and this is the common transversal of l, l' through P.

21.72 Length of the common perpendicular

Let the lines AB, CD in 21.71 have equations

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \quad \frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'}.$$

Then the plane AXY, which is parallel to CD and contains AB, has an equation of the form

$$p(x-a)+q(y-b)+r(z-c) = 0,$$

where
$$pl+qm+rn = 0$$
and
$$pl'+qm'+rn' = 0$$

since its normal is perpendicular to both AB and CD. Elimination of p, q, r gives

$$\begin{vmatrix} x-a & y-b & z-c \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$$

as the equation of plane AXY. Its normal therefore has direction ratios

$$mn'-m'n : nl'-n'l : lm'-l'm.$$

The length of PQ is equal to the perpendicular distance of any point K of CD from this plane. Since (a', b', c') is a point on CD, we have

$$PQ = \frac{\pm \begin{vmatrix} a'-a & b'-b & c'-c \\ l & m & n \\ l' & m' & n' \end{vmatrix}}{\sqrt{\{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2\}}},$$

where the sign is to be chosen so that the result is positive.

21.73 Equations of the common perpendicular

The line PQ is the line of intersection of the planes QAB, PCD. The plane QAB has an equation of the form

$$f(x-a)+g(y-b)+h(z-c) = 0,$$

where

$$fl+gm+hn = 0$$

and

$$f(mn'-m'n)+g(nl'-n'l)+h(lm'-l'm) = 0,$$

since its normal is perpendicular to both AB and PQ. Elimination of f, g, h gives

$$\begin{vmatrix} x-a & y-b & z-c \\ l & m & n \\ mn'-m'n & nl'-n'l & lm'-l'm \end{vmatrix} = 0$$

as the equation of QAB. Similarly, plane PCD has equation

$$\begin{vmatrix} x-a' & y-b' & z-c' \\ l' & m' & n' \\ mn'-m'n & nl'-n'l & lm'-l'm \end{vmatrix} = 0.$$

These are the two equations of the line PQ.

21.74 Alternative method

Although the methods in 21.72, 21.73 give the results for two general lines, it is often more convenient in a given particular case to find the coordinates of P, Q, and from these deduce the length and equations of PQ, as follows.

Since P lies on AB, its coordinates are of the form

$$(a+\lambda l,\, b+\lambda m,\, c+\lambda n).$$

Similarly, Q on CD has coordinates $(a'+\mu l', b'+\mu m', c'+\mu n')$. Direction ratios for PQ are therefore

$$a-a'+\lambda l-\mu l' : b-b'+\lambda m-\mu m' : c-c'+\lambda n-\mu n'.$$

As PQ is perpendicular to AB (whose direction ratios are $l:m:n$) and to CD ($l':m':n'$), hence

$$l(a-a')+m(b-b')+n(c-c') \\ +\lambda(l^2+m^2+n^2)-\mu(ll'+mm'+nn') = 0$$

and
$$l'(a-a')+m'(b-b')+n'(c-c') \\ +\lambda(ll'+mm'+nn')-\mu(l'^2+m'^2+n'^2) = 0.$$

From these λ, μ can be calculated, and then the coordinates of P and Q found.

Example

Find the shortest distance between the lines

$$\frac{x}{2}=\frac{y}{-3}=\frac{z}{1} \quad \textit{and} \quad \frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2},$$

and find equations for the line along which it lies.

Any point P on the first line is $(2\lambda, -3\lambda, \lambda)$, and any point Q on the second is $(2+3\mu, 1-5\mu, -2+2\mu)$. Hence PQ has direction ratios

$$2\lambda-3\mu-2 : -3\lambda+5\mu-1 : \lambda-2\mu+2.$$

If PQ is perpendicular to the first line,

$$2(2\lambda-3\mu-2)-3(-3\lambda+5\mu-1)+(\lambda-2\mu+2) = 0,$$

i.e.
$$14\lambda-23\mu+1 = 0.$$

If PQ is perpendicular to the second line,

$$3(2\lambda-3\mu-2)-5(-3\lambda+5\mu-1)+2(\lambda-2\mu+2) = 0,$$

i.e.
$$23\lambda-38\mu+3 = 0.$$

On solving we find $\lambda = \frac{31}{3}$, $\mu = \frac{19}{3}$. Hence P is $(\frac{62}{3}, -31, \frac{31}{3})$, and Q is $(21, -\frac{92}{3}, \frac{32}{3})$. By the distance formula,

$$PQ^2 = (\tfrac{1}{3})^2+(\tfrac{1}{3})^2+(\tfrac{1}{3})^2,$$

and so $PQ = \frac{1}{3}\sqrt{3}$.

Equations of PQ are
$$\frac{x-21}{\frac{1}{3}}=\frac{y+\frac{92}{3}}{\frac{1}{3}}=\frac{z-\frac{32}{3}}{\frac{1}{3}},$$

i.e.
$$x-21 = y+\tfrac{92}{3} = z-\tfrac{32}{3}.$$

21.75 Standard form for the equations of two skew lines

Calculations for proving properties of skew lines can be simplified by a suitable choice of coordinate axes as follows.

If AB, CD are the skew lines, let their common perpendicular PQ be chosen for z-axis. To introduce as much symmetry as possible, choose O at the mid-point of PQ, and through O draw lines OB', OD' parallel to AB, CD respectively. For Ox, Oy choose (again with a view to symmetry) the bisectors of the angles between OB', OD', and label them in such a way that $Oxyz$ is a right-handed system.

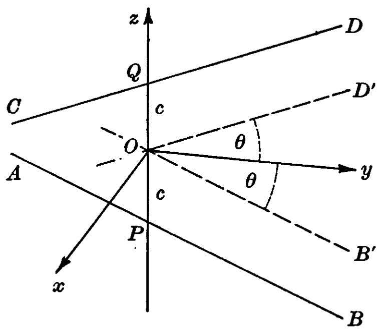

Fig. 233

From the construction, the plane xOy is parallel to both AB and CD, and hence is perpendicular to PQ, i.e. to Oz; also $x\hat{O}y$ is a right-angle. The coordinate axes are thus mutually perpendicular.

Let angle $B'OD'$ be 2θ; by definition this is the angle between AB and CD (21.32). Then OB' makes angles $\frac{1}{2}\pi-\theta$, θ, $\frac{1}{2}\pi$ with Ox, Oy, Oz, and therefore has direction cosines $\{\sin\theta, \cos\theta, 0\}$; these are the direction cosines of the parallel line AB. Also OD' makes angles $\frac{1}{2}\pi+\theta$, θ, $\frac{1}{2}\pi$ with the axes, and its direction cosines are

$$\{-\sin\theta, \cos\theta, 0\},$$

which are also those of CD.

If $PQ = 2c$, then P is $(0, 0, -c)$ and Q is $(0, 0, c)$. The equations of AB, CD are therefore

$$\frac{x}{\sin\theta} = \frac{y}{\cos\theta} = \frac{z+c}{0}, \quad \frac{x}{-\sin\theta} = \frac{y}{\cos\theta} = \frac{z-c}{0}.$$

Putting $m = \cot\theta$, these can be written in the *standard forms*

$$y-mx = 0 = z+c, \quad y+mx = 0 = z-c. \tag{i}$$

Example

A line is drawn to meet the lines (i) *and to make a constant angle with Oz. Prove that the locus of the mid-point of the intercept is an ellipse.*

The point P on the first line having x-coordinate λ has $y = m\lambda$, $z = -c$, and so is $P(\lambda, m\lambda, -c)$. Similarly, a point Q on the second line is $Q(\mu, -m\mu, c)$. If $M(x, y, z)$ is the mid-point of PQ, then

$$x = \tfrac{1}{2}(\lambda+\mu), \quad y = \tfrac{1}{2}m(\lambda-\mu), \quad z = 0.$$

Direction ratios for PQ are $\lambda-\mu : m(\lambda+\mu) : -2c$. If PQ makes angle α with Oz, which has direction cosines $\{0, 0, 1\}$, then

$$\cos\alpha = \frac{-2c}{\sqrt{\{(\lambda-\mu)^2+m^2(\lambda+\mu)^2+4c^2\}}}.$$

On squaring and rearranging, and using the above equations to eliminate λ and μ,

$$\left(\frac{2y}{m}\right)^2 + m^2(2x)^2 + 4c^2 = 4c^2\sec^2\alpha,$$

i.e.

$$\frac{x^2}{k^2/m^2} + \frac{y^2}{k^2m^2} = 1,$$

where $k = c\tan\alpha$. The fact that the z-coordinate of M is zero shows that M lies in the plane xOy; and the above equation in (x, y) represents an ellipse in this plane.

Exercise 21(*f*)

1 Obtain direction ratios of the line through O which meets each of the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}, \quad \frac{x+2}{1} = \frac{y-3}{3} = \frac{z-4}{2}.$$

[Method of 21.71, ex.]

2 Find the equations of the line through $(-6, -4, -6)$ which meets each of the lines

$$\frac{x}{2} = y = \frac{z}{3}, \quad -x-2 = \frac{y-1}{2} = -z-1.$$

Also find the points of intersection.

3 Prove that there are two lines which meet Ox at right-angles and also meet the lines

$$x-4 = \frac{y-7}{5} = \frac{z-13}{8}, \quad \frac{x+1}{2} = \frac{y+2}{4} = \frac{z-9}{3},$$

and find the points where they meet Ox. [Use the parametric equations of the lines.]

4 Prove geometrically that three given skew lines have infinitely many common transversals, *no two of which intersect.*

Find the shortest distance between the following pairs of skew lines.

5 $\dfrac{x+2}{3} = \dfrac{y}{2} = \dfrac{z+2}{-1}$ and $\dfrac{x+5}{4} = \dfrac{y+6}{3} = \dfrac{z-1}{-1}$.

6 $y = 0 = x+2z-17$ and $\dfrac{x+2}{4} = \dfrac{y-5}{3} = \dfrac{z-2}{-3}$.

7 $\frac{x+7}{-8} = \frac{y-5}{3} = \frac{z-4}{1}$ and $\frac{x+4}{4} = \frac{y}{3} = \frac{z-19}{-2}$ [use the formula in 21.72]

and find equations of the line along which it lies.

8 A fixed line APB passes through $A(2,2,2)$ and $B(-1,-1,-3)$, and a variable line CQD passes through $C(2,3,1)$ in such a way that the common perpendicular PQ has direction ratios $0:5:-3$. Find the equations of the locus of Q and the length of PQ.

9 From the result of 21.72 deduce the condition for the lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \quad \frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'}$$

to intersect.

Two skew lines l, l' are met by their common perpendicular at A on l and B on l'; P, Q are variable points on l, l' respectively, and M is the mid-point of PQ (nos. 10–13).

10 If $AQ \perp BP$ and l, l' are not perpendicular, prove that M lies on a hyperbola whose asymptotes are parallel to l, l'. [Choose axes as in 21.75.]

11 If AP^2+BQ^2 is constant, prove that the locus of M is an ellipse.

12 If $PQ = AP-BQ$ and 2α is the angle between l, l', prove that (with axes as in 21.75) PQ makes an angle $\frac{1}{2}\pi-\alpha$ with Ox, and that the meet of PQ with the plane $x = 0$ lies on the circle whose diameter is AB.

13 Prove that $AP.BQ$ is constant when (*a*) the lines AQ, BP are perpendicular, or (*b*) the planes APQ, BQP are perpendicular.

14 Prove that the coordinates of any point equidistant from the lines $y-mx = 0 = z+c$, $y+mx = 0 = z-c$ satisfy the equation $mxy = (1+m^2)cz$.

Miscellaneous Exercise 21(*g*)

1 If $AB^2+CD^2 = AC^2+BD^2$, prove $BC \perp AD$.

2 Two edges AB, CD of the tetrahedron $ABCD$ are perpendicular. Prove that the distance between the mid-points of AC, BD is equal to the distance between the mid-points of AD, BC.

If also the edges AC, BD are perpendicular, prove that the third pair BC, AD are perpendicular.

3 Find the ratio in which N, the foot of the perpendicular from O, divides the join of $A(4,6,0)$ and $B(1,2,-1)$; and prove that N does not lie between A and B.

4 P moves with uniform speed in a straight line from $A(0,0,12)$ to $B(3,4,0)$ in 13 seconds. Find the coordinates of P after it has been moving for t seconds. For what value of t is P nearest to O?

If C is $(5,7,2)$ and CP meets the plane $z = 0$ at Q, find the speed of Q at the instant when P leaves A.

5 If a moving rod has direction cosines

$$\{l, m, n\} \text{ at time } t \quad \text{and} \quad \{l+\delta l, m+\delta m, n+\delta n\} \text{ at time } t+\delta t,$$

prove that the angle $\delta\theta$ turned through in time δt is given approximately by $(\delta\theta)^2 \doteqdot (\delta l)^2+(\delta m)^2+(\delta n)^2$. [Use $\cos x = 1-\frac{1}{2}x^2+O(x^4)$: see Ex. 6(*b*), no. 21.]

6 Find the equations of the line through O which intersects and is perpendicular to the line $x+2y+3z+4 = 0$, $2x+3y+4z+5 = 0$, and find the point of intersection.

7 Find the angle between the line $6x+4y-5z = 4$, $x-5y+2z = 12$ and the line

$$\frac{x-9}{2} = \frac{y+4}{-1} = \frac{z-5}{1}.$$

Prove that these lines intersect, and obtain the equation of their plane.

8 Prove that any point equidistant from the lines through O which have direction cosines $\{l, m, n\}$, $\{l', m', n'\}$ lies in one of the planes

$$(l+l')x+(m+m')y+(n+n')z = 0, \quad (l-l')x+(m-m')y+(n-n')z = 0.$$

Hence find the equations of the lines which bisect the angles between the given lines.

9 Find the coordinates of the point P on the intersection of the planes $2x+2y-z = 0$, $x+y-3z = 0$ which has least distance from the line l joining the points $(-2, 3, 2)$, $(-5, 5, -3)$. Find the point Q on l which is nearest to P.

10 The plane $4x+7y+4z+81 = 0$ is rotated through a right-angle about its line of intersection with the plane $5x+3y+10z = 25$. Find the equation of the plane in its new position. Also find the distance between the feet of the perpendiculars from O to the plane in its two positions.

11 Obtain the equation of the plane through the origin and the line

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}.$$

Find the equations of the line meeting Ox at right-angles whose projection on this plane coincides with the given line.

12 Write down the equation of the plane π_{AB} which bisects the line AB at right-angles, where A is (a, b, c) and B is (d, e, f). Prove that, for any points A, B, C, the planes π_{AB}, π_{BC}, π_{CA} have in general a common line λ. If $A(1, 0, 1)$, $B(0, 1, 1)$, $C(2, 2, 0)$, show that the distance of O from λ is $\frac{3}{11}\sqrt{22}$.

13 Prove that the planes $\nu y-\mu z = p$, $\lambda z-\nu x = q$, $\mu x-\lambda y = r$ possess a common line if and only if $\lambda p+\mu q+\nu r = 0$, and show that this line lies in the plane $px+qy+rz = 0$.

14 A line meets the planes $\alpha = 0$, $\alpha' = 0$ at P, P', and meets the planes $\alpha \pm \lambda\alpha' = 0$ at Q, Q'. Prove that Q, Q' divide PP' internally and externally in the same ratio.

15 Prove that the two lines through (a, a, a) which lie in the plane $x+y+z = 3a$ and are inclined at 30° to the plane $x = 0$ have equations

$$\frac{x-a}{2} = \frac{y-a}{1+\sqrt{5}} = \frac{z-a}{1-\sqrt{5}}, \quad \frac{x-a}{2} = \frac{y-a}{1-\sqrt{5}} = \frac{z-a}{1+\sqrt{5}}.$$

Find the angle between these lines, and also the angle between their projections on the plane $x = 0$.

16 Show that the line of shortest distance between Oz and the join of the point $P(p, p, c)$ on the line $x = y$, $z = c$, to $Q(q, -q, -c)$ on the line $x = -y$, $z = -c$, divides PQ in the ratio $p^2 : q^2$. Prove also that this line makes angle $\tan^{-1}(p/q)$ with the first line and angle $\tan^{-1}(q/p)$ with the second.

17 Verify that the line $x = p$, $cy = mpz$ cuts each of the three lines

$$y = 0 = z, \quad y - mx = 0 = z - c, \quad y + mx = 0 = z + c.$$

As p varies, prove that all such lines lie on the surface whose equation is $cy = mxz$.

18 If r is any transversal of the three lines

$$y - mx = 0 = z - c, \quad y = 0 = z, \quad y + mx = 0 = z + c,$$

and s is any transversal of the three lines

$$y - mz = 0 = x - c, \quad y = 0 = x, \quad y + mz = 0 = x + c,$$

prove that r and s intersect.

19 Points P, P' are taken on the lines $y = x$, $z = 1$; $y = -x$, $z = -1$ respectively, such that $OP = 3OP'$. Prove that the meet of PP' with the plane $z = 0$ lies on the curve $2x^2 - 5xy + 2y^2 + 1 = 0$, $z = 0$.

20 The common perpendicular to two skew lines $y - mx = 0 = z + c$, $y + mx = 0 = z - c$ meets them at A, B respectively. Points P on the first and Q on the second line are such that $AP = BQ$. Prove that PQ lies on one of the surfaces $mxz + cy = 0$, $yz + cmx = 0$.

21 Discuss the intersections of the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

and the plane $ax + by + cz + d = 0$ (*a*) algebraically, (*b*) geometrically, showing that:

(i) if $al + bm + cn \neq 0$, there is a unique point of intersection;
(ii) if $al + bm + cn = 0$, the line and plane are parallel *unless also*
(iii) $ax_1 + by_1 + cz_1 + d = 0$, in which case the line lies in the plane.

22

THE SPHERE; SPHERICAL TRIGONOMETRY

22.1 Coordinate geometry of the sphere

22.11 Equation of a sphere

The distance formula shows that the sphere of centre $C(a, b, c)$ and radius r (defined as the locus of points P in space such that $CP = r$) has equation

$$(x-a)^2+(y-b)^2+(z-c)^2 = r^2. \tag{i}$$

When expanded, this takes the form

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0. \tag{ii}$$

Conversely, (ii) can be written in the form (i) by separately completing the square for the terms involving x, y, and z:

$$(x+u)^2+(y+v)^2+(z+w)^2 = u^2+v^2+w^2-d.$$

If $u^2+v^2+w^2 > d$, (ii) is therefore the equation of the sphere having *centre* $(-u, -v, -w)$ *and radius* $\sqrt{(u^2+v^2+w^2-d)}$.

If $u^2+v^2+w^2 = d$, (ii) represents the single point $(-u, -v, -w)$. We may call this a point-sphere.

When $u^2+v^2+w^2 < d$, the equation does not represent any locus.

The slightly more general equation

$$cx^2+cy^2+cz^2+2ux+2vy+2wz+d = 0,$$

where $c \neq 0$, can be written in the 'normalised' form (ii) on dividing by c, and hence in general this also represents a sphere. We therefore take (ii) as the standard general equation of a sphere, and observe that it involves four arbitrary coefficients u, v, w, d. Consequently a sphere can be determined by four independent conditions.

22.12 Some definitions and results from pure geometry

A line and sphere, a plane and sphere, or two spheres are said to

(*a*) *intersect* when they have more than one point in common;

(*b*) *touch* when they have exactly one point in common.

The following results are proved in elementary solid geometry.

(i) If a plane and sphere intersect, they do so in all points of a circle (whose centre is the foot of the perpendicular from the centre C of the sphere to this plane).

If the plane already passes through C, the circle cut on the sphere is called a *great circle* of the sphere; otherwise it is a *small circle*.

(ii) If two spheres intersect, they do so in all points of a circle (whose plane is perpendicular to the line of centres and whose centre lies on this line).

(iii) If a line touches a sphere at P, it is perpendicular to the radius CP. Conversely, all lines through P which are perpendicular to CP will touch the sphere at P. All such lines therefore lie in a plane through P which has PC for normal.

(iv) If a plane touches a sphere at P, it is perpendicular to the radius CP. Conversely, a plane through P and perpendicular to CP touches the sphere at P.

It follows that just one plane can touch a sphere at a given point P, and that all tangent lines at P lie in this tangent plane.

Example

Sphere on diameter P_1P_2.

If $P(x, y, z)$ is any point on the required sphere, then the plane PP_1P_2 cuts the sphere in a great circle having diameter P_1P_2, and hence (by 'angle in a semi-circle') $PP_1 \perp PP_2$. Since PP_1, PP_2 have direction ratios

$$x-x_1 : y-y_1 : z-z_1, \quad x-x_2 : y-y_2 : z-z_2,$$

hence by the perpendicularity condition,

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2) = 0.$$

This relation, satisfied by *any* point P of the sphere on diameter P_1P_2, is the required equation.

22.13 Tangent plane at P_1

We employ some of the results in 22.12. Algebraical treatments using the ratio quadratic or the 'distance quadratic' are indicated in Ex. 22 (a), nos. 26–30, 23–25; these are applicable to surfaces more general than the sphere, whereas the present is not.

For simplicity we first discuss the sphere of radius r and centre the origin O; its equation is $\quad x^2+y^2+z^2 = r^2.$

Given a point P_1 on this sphere, the direction ratios of OP_1 are $x_1 : y_1 : z_1$. The plane through P_1 and perpendicular to OP_1 therefore has equation

$$x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1) = 0,$$

i.e. $$xx_1+yy_1+zz_1 = x_1^2+y_1^2+z_1^2.$$

Since P_1 lies on the sphere, we have $x_1^2+y_1^2+z_1^2 = r^2$, and the above equation becomes

$$xx_1+yy_1+zz_1 = r^2. \tag{iii}$$

By 22.12, (iv) this represents the tangent plane at P_1.

If P_1 is a point on the general sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0,$$

whose centre is $C(-u, -v, -w)$, then direction ratios for CP_1 are

$$x_1+u : y_1+v : z_1+w.$$

The plane through P_1 and perpendicular to this direction has equation

$$(x_1+u)(x-x_1)+(y_1+v)(y-y_1)+(z_1+w)(z-z_1) = 0,$$

i.e. $(x_1+u)x+(y_1+v)y+(z_1+w)z = x_1^2+y_1^2+z_1^2+ux_1+vy_1+wz_1.$

Since P_1 lies on the sphere,

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d = 0.$$

By adding the last two equations,

$$(x_1+u)x+(y_1+v)y+(z_1+w)z+ux_1+vy_1+wz_1+d = 0, \tag{iv}$$

which is the equation of the tangent plane at P_1 to the general sphere.

Since (iv) can be arranged as

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d = 0,$$

we see that the 'rule of alternate suffixes' (15.63, Remark) is still valid as a mnemonic for writing down the equation of a tangent plane to a sphere.

22.14 Examples

(i) *Condition for the plane $lx+my+nz = p$ to touch the sphere $x^2+y^2+z^2 = r^2$.*

The plane and sphere will touch if and only if (22.12, (iv)) the perpendicular distance of the centre from the plane is equal to the radius, i.e.

$$\pm p/\sqrt{(l^2+m^2+n^2)} = r,$$

i.e.

$$p^2 = r^2(l^2+m^2+n^2).$$

Also see Ex. 22 (*a*), no. 17.

(ii) *Find the equations of the tangent planes to the sphere*

$$x^2+y^2+z^2-6x+6y+2z-2 = 0$$

which pass through the line $x-2z+4 = 0$, $y = 0$.

The sphere has centre $(3, -3, -1)$ and radius $\sqrt{\{(-3)^2+3^2+1^2-(-2)\}} = \sqrt{21}$. Any plane through the given line is (21.61)

$$(x-2z+4)+ky = 0,$$

and this will touch the sphere if and only if (as in ex. (i))

$$\pm\frac{3-3k+2+4}{\sqrt{\{1^2+k^2+(-2)^2\}}} = \sqrt{21},$$

i.e.

$$(9-3k)^2 = 21(5+k^2),$$

from which $2k^2+9k+4 = 0$ and so $k = -4$ or $-\frac{1}{2}$. The required tangent planes are therefore

$$x-4y-2z+4 = 0, \quad 2x-y-4z+8 = 0.$$

(iii) *Plane of contact from P_1 to the sphere $x^2+y^2+z^2 = r^2$.*

Let a line through P_1 touch the sphere at P_2. Then the tangent plane at P_2 is

$$xx_2+yy_2+zz_2 = r^2$$

and (since this contains the tangent line P_2P_1) it passes through P_1. Hence

$$x_1x_2+y_1y_2+z_1z_2 = r^2,$$

which shows that the coordinates of P_2 satisfy the linear equation

$$x_1x+y_1y+z_1z = r^2. \qquad \text{(v)}$$

Therefore the points of contact of all tangent lines from P_1 lie in the plane (v), which is called the *plane of contact* from P_1. Its normal has direction ratios $x_1:y_1:z_1$, so that the plane is perpendicular to OP_1.

This plane cuts the sphere in a circle, and hence the points of contact of tangent lines from P_1 lie on this circle. The tangent lines themselves lie on a right circular cone with vertex P_1, called the *tangent cone* from P_1.

(iv) *Condition for the line*

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

to touch the sphere $x^2+y^2+z^2 = r^2$; tangent cone from P_1.

The point $(x_1+\lambda l, y_1+\lambda m, z_1+\lambda n)$ of the line also lies on the sphere if and only if

$$(x_1+\lambda l)^2+(y_1+\lambda m)^2+(z_1+\lambda n)^2 = r^2,$$

i.e. if λ satisfies

$$\lambda^2(l^2+m^2+n^2)+2\lambda(lx_1+my_1+nz_1)+(x_1^2+y_1^2+z_1^2-r^2) = 0. \qquad \text{(vi)}$$

In general this quadratic† gives two values of λ (confirming algebraically that a line can meet a sphere in at most two points); but the line and sphere will touch when there is a repeated root, i.e.

$$(lx_1+my_1+nz_1)^2 = (l^2+m^2+n^2)(x_1^2+y_1^2+z_1^2-r^2).$$

When this condition is satisfied, the line lies on the tangent cone from P_1. Elimination of $l:m:n$ from the condition and the equations of the line gives

$$\{\Sigma x_1(x-x_1)\}^2 = \{\Sigma(x-x_1)^2\}(x_1^2+y_1^2+z_1^2-r^2).$$

This equation is satisfied by the coordinates of any point P on *any* tangent line through P_1, and therefore represents the tangent cone. A more convenient form of the equation is given in Ex. 22 (a), no. 28.

† The '*distance quadratic*': see footnote‡ on p. 715.

(v) *Orthogonal spheres.*

The *angle of intersection* of two spheres at a common point P is defined to be the angle between the tangent planes at P. If C, C' are the centres, then this angle is equal to that between the radii $C'P$, CP because these are normal to the respective tangent planes. When the angle is a right-angle, the spheres are *orthogonal*; in this case triangle CPC' is right-angled at P, and so

$$C'C^2 = CP^2 + C'P^2.$$

If the spheres are

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0,$$
$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d' = 0,$$

then $C(-u, -v, -w)$, $C'(-u', -v', -w')$, and the orthogonality condition becomes

$$(u-u')^2+(v-v')^2+(w-w')^2 = (u^2+v^2+w^2-d)+(u'^2+v'^2+w'^2-d'),$$

which reduces to

$$2uu'+2vv'+2ww' = d+d'. \qquad \text{(vii)}$$

Conversely, if this condition is satisfied, then by adding the expression $u^2+v^2+w^2+u'^2+v'^2+w'^2$ to both sides we obtain the previous equation, which is equivalent to $C'C^2 = CP^2+C'P^2$. The converse of the theorem of Pythagoras then shows that $CP \perp C'P$, so that the spheres are orthogonal.†

Exercise 22(*a*)

1 A point moves so that the square of its distance from $(-1, 2, 1)$ is equal to its distance from the plane $2x-3y+6z = 8$. Show that it lies on a sphere, and state the centre.

2 A, B are fixed points, and P varies so that $AP:PB = \lambda:1$, where λ is constant. Find the locus of P. [Choose axes so that A is $(a, 0, 0)$, $B(-a, 0, 0)$.]

Find the equation of the sphere through

3 $(0, 1, 3)$, $(1, 2, 4)$, $(3, 0, 2)$, $(2, 3, 1)$.

4 $(1, 2, 1)$, $(0, 3, 1)$, $(-1, 1, 2)$ and having its centre in the plane $3y+2z = 1$.

5 $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ which has its radius as small as possible.

6 O and the points where the plane $2x+3y+z = 6$ cuts the coordinate axes, and find its diameter.

7 Find the equation of the sphere which circumscribes the tetrahedron whose faces lie in the planes $x = 0$, $y = 0$, $z = 0$, $x+2y+3z = 4$.

8 A sphere of radius r passes through O. Show that the ends of the diameter parallel to Oz lie one on each of the spheres $x^2+y^2+z^2 \pm 2rz = 0$.

9 Spheres are drawn through $(2, 0, 0)$ and $(8, 0, 0)$ to touch Oy and Oz. Prove that there are four such spheres, and give their equations.

10 Find the centres of the two spheres which touch the plane $3x+4z = 47$ at $(5, 4, 8)$ and which touch the sphere $x^2+y^2+z^2 = 1$.

11 Find the centre and radius of the sphere which touches the plane

$$3x+2y-z+2 = 0$$

at $(1, -2, 1)$ and cuts orthogonally the sphere $x^2+y^2+z^2-4x+6y+4 = 0$.

† When (vii) is satisfied, the spheres certainly do intersect; for it implies that the sum of the radii is greater than the distance between the centres.

12 Find the centres and radii of the two spheres whose centres lie in the octant for which all of x, y, z are positive, and which touch the planes $x = 0$, $y = 0$, $z = 0$, $2x+2y+z = 4$.

13 A sphere is inscribed in the tetrahedron whose faces lie in the planes $x = 0$, $y = 0$, $z = 0$, $2x+6y+3z = 14$. Find its equation.

14 Three spheres have centres $(0,0,0)$, $(3a,0,0)$, $(0,4a,0)$ and radii a, $2a$, $3a$ respectively. Two planes, including an acute angle ϕ, are such that every one of the spheres touches both planes. Prove that $\cos\phi = \frac{5}{18}$.

15 Spheres s_1, s_2, s_3 have centres $(0,0,0)$, $(3,0,0)$, $(0,30,0)$ and radii 1, 1, 19 respectively. Find the equations of all common tangent planes π of the spheres such that s_1 and s_3 lie on opposite sides of π, and s_2, s_3 lie on the same side. Show that two such planes exist, and that the acute angle ϕ between them satisfies $9\cos\phi = 7$. [Use Ex. 21 (c), no. 13.]

16 Find the tangent planes to the sphere $x^2+y^2+z^2-2x+4y-6z+10 = 0$ which pass through the line

$$\frac{x+3}{14} = \frac{y+1}{-3} = \frac{z-5}{4}.$$

17 Obtain the result of 22.14, ex. (i) by comparing the equations

$$lx+my+nz = p \quad \text{and} \quad xx_1+yy_1+zz_1 = r^2.$$

Hence obtain the point of contact when the condition is satisfied.

18 Find the condition for the plane $lx+my+nz = p$ to touch the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0.$$

19 Find the equation of the sphere having centre $(5, -2, 3)$ and touching the line

$$\frac{x-1}{6} = \frac{y+1}{2} = \frac{z-12}{-3}.$$

Also find the equation of the tangent plane which contains this tangent line.

20 Prove that a tangent line from the point P_1 outside the sphere

$$s \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0$$

has length $\sqrt{s_{11}}$, where $s_{11} = x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$.

21 Show that the points from which equal tangents can be drawn to the three spheres

$$x^2+y^2+z^2 = 1, \quad x^2+y^2+z^2+2x-2y+2z-1 = 0,$$

and

$$x^2+y^2+z^2-x+4y-6z-2 = 0$$

lie on the line

$$\frac{x-1}{2} = \frac{y-2}{5} = \frac{z-1}{3}.$$

Also find the point of this line from which the length of the tangents is least.

22 (i) Prove that the points from which tangents to the spheres

$$s \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0,$$

$$s' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d' = 0$$

are of equal length lie in the plane $s-s' = 0$.

(ii) If the spheres s, s' intersect, explain why the plane $s-s' = 0$ contains their common circle.

*23 By taking P_1 on the sphere in 22.14, ex. (iv), use the 'distance quadratic' (vi) to show that the line through P_1 in direction $l:m:n$ will have no point other than P_1 in common with the sphere (i.e. will be a *tangent line* at P_1) if and only if $lx_1+my_1+nz_1 = 0$. (This confirms that the tangent line is perpendicular to OP_1: see 22.12, (iii).)

*24 If $P_2(x_2, y_2, z_2)$ is any point on the tangent line at P_1 in direction $l:m:n$, prove that $(x_2-x_1)x_1+(y_2-y_1)y_1+(z_2-z_1)z_1 = 0$. [Eliminate $l:m:n$ from the result of no. 23 and the equations $(x_2-x_1)/l = (y_2-y_1)/m = (z_2-z_1)/n$.]

Deduce that all tangent lines at P_1 lie in the plane $xx_1+yy_1+zz_1 = r^2$ (the *tangent plane* at P_1).

*25 Use the 'distance quadratic' (vi) to show that the mid-points of all chords in direction $l:m:n$ lie in the plane $lx+my+nz = 0$ (which is clearly normal to this direction).

*26 *The ratio quadratic.* Show that the point dividing P_1P_2 in the ratio $k:l$ lies on the sphere $s = 0$ if and only if

$$s_{22}k^2+2s_{12}kl+s_{11}l^2 = 0.$$

(Here s may denote $x^2+y^2+z^2-r^2$ or the general expression in no. 20; the notation of 19.12 is used, extended to three variables in the obvious way.)

*27 Taking P_1 on $s = 0$, deduce from no. 26 that P_2 lies on a tangent line at P_1 if and only if $s_{12} = 0$. Hence show that the *tangent plane* at P_1 is $s_1 = 0$.

*28 By expressing the condition for equal roots, deduce from no. 26 that the *tangent cone* from P_1 has equation $ss_{11} = s_1^2$.

*29 Prove that the *plane of contact* of tangents from P_1 has equation $s_1 = 0$.

*30 A line through the fixed point P_1 meets s at A, B, and P_2 is chosen so that P_1, P_2 divide AB internally and externally in the same ratio. When the line through P_1 varies, prove that P_2 moves in the plane $s_1 = 0$.

22.2 $s = ks'$

22.21 The general principle

If $s = 0$, $s' = 0$ represent any two loci in space or in a plane, then $s = ks'$ represents some locus which passes through their common points (if any). For, points whose coordinates satisfy $s = 0$ and $s' = 0$ will also satisfy $s = ks'$.

In this section we consider the equation $s = ks'$ when k is constant (i.e. independent of x, y, z) and one of $s = 0$, $s' = 0$ represents a sphere, while the other represents either a plane or a sphere.

22.22 Spheres through a given circle

(1) If $$s \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0$$

represents a sphere, and

$$s' \equiv Ax+By+Cz+D = 0,$$

then provided that these loci intersect, they do so in a circle. The pair of equations $s = 0$, $s' = 0$ are therefore sufficient to determine a circle in space, and can be referred to as the *equations of the circle*. As for the straight line, more than one equation is needed; a circle cannot be specified more simply than by one linear and one second-degree equation, although the latter may not be the equation of a *sphere*: e.g. see ex. (iii) below, where it represents a *cylinder*.

The equation $s-ks'=0$ (k constant) will represent a sphere, a single point, or nothing, since it has the form appropriate for the general equation of a sphere (22.11). If s, s' intersect, then the locus is a sphere through the circle determined by $s=0$, $s'=0$. If s, s' touch, then $s=ks'$ represents a sphere through their common point, i.e. touching s at its contact with s'. In both cases the constant k can be chosen to make the sphere satisfy one further condition.

(2) If $$s \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$$
and $$s' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$$
both represent spheres, then $s=ks'$ (k constant) will represent a sphere, a single point, or nothing, provided that $k \neq 1$; if $k=1$, it represents a plane.

If s, s' intersect, then they do so in a circle, and $s=ks'$ represents a sphere through this circle unless $k=1$, in which case the locus becomes $s-s'=0$ and is the plane containing the common circle (cf. Ex. 22 (a), no. 22 (ii)). The two second-degree equations $s=0$, $s'=0$ could be taken as equations of the circle of intersection, but this circle can be represented more simply as in (1) by the linear equation $s-s'=0$ and one second-degree equation, say $s=0$.

If s, s' touch, then $s=ks'$ represents a sphere ($k \neq 1$) or a plane ($k=1$) through their single common point, i.e. touching both s and s' at this point.

Examples

(i) *Find the equation of the sphere having the circle*
$$x^2+y^2+z^2=16, \quad 2x-3y+6z=7$$
as a great circle, and give its centre and radius.

Any sphere through the given circle has an equation of the form
$$x^2+y^2+z^2-16+k(2x-3y+6z-7)=0.$$
Since the centre of a great circle coincides with the centre of the sphere, hence the centre $(-k, \frac{3}{2}k, -3k)$ must lie in the plane $2x-3y+6z=7$. The condition for this gives $k=-\frac{2}{7}$, so that the required sphere has centre $(\frac{2}{7}, -\frac{3}{7}, \frac{6}{7})$ and equation
$$x^2+y^2+z^2-16-\tfrac{2}{7}(2x-3y+6z-7)=0,$$
i.e. $$x^2+y^2+z^2-\tfrac{4}{7}x+\tfrac{6}{7}y-\tfrac{12}{7}z-14=0.$$
Its radius is therefore $\sqrt{\{(-\frac{2}{7})^2+(\frac{3}{7})^2+(-\frac{6}{7})^2+14\}}=\sqrt{15}$.

(ii) *Find the equation of the sphere which touches the sphere*

$$x^2+y^2+z^2+x+3y-6z-3 = 0$$

at (1, 2, 3) *and passes through* (2, 0, 1).

The point (1, 2, 3) certainly lies on the given sphere. The tangent plane at (1, 2, 3) has equation†

$$x+2y+3z+\tfrac{1}{2}(x+1)+\tfrac{3}{2}(y+2)-3(z+3)-3 = 0,$$

i.e.
$$3x+7y-17 = 0.$$

Any sphere touching the given sphere at (1, 2, 3) has an equation of the form

$$x^2+y^2+z^2+x+3y-6z-3+k(3x+7y-17) = 0.$$

It passes through (2, 0, 1) if $4+1+2-6-3+k(6-17) = 0$, i.e. $k = -\frac{2}{11}$. Hence the required equation is

$$11(x^2+y^2+z^2)+5x+19y-66z+1 = 0.$$

(iii) *Find the equation of the sphere through the circle*

$$x^2+y^2-4x+2y = 1, \quad z = 0$$

which is orthogonal to the sphere $x^2+y^2+z^2+3x-5y+6z+8 = 0$.

The equation $x^2+y^2-4x+2y = 1$ does not represent a sphere but a right circular cylinder: see 22.3, ex. (i).

The given circle is that in which the plane $z = 0$ cuts the *sphere*

$$x^2+y^2+z^2-4x+2y-1 = 0.$$

Any sphere‡ through this circle can be written

$$(x^2+y^2+z^2-4x+2y-1)+kz = 0.$$

It will be orthogonal to the given sphere if (22.14, ex. (v))

$$3(-2)-5.1+3k = 8-1,$$

i.e. if $k = 6$. The required sphere therefore has equation

$$x^2+y^2+z^2-4x+2y+6z-1 = 0.$$

22.3 Surfaces in general

An equation $f(x, y, z) = 0$ explicitly involving some or all of x, y, z represents a locus in space called a *surface* (cf. 9.23). For example, when the function is linear in x, y and z, the equation represents a plane surface or *plane*; when of the type considered in 22.11, it represents a spherical surface or *sphere*. A few other special classes of equations will be noticed here because of their geometrical significance.

Examples

(i) $f(x, y) = 0$. The points in the plane xOy whose coordinates satisfy this equation lie on a curve $\mathscr{C}$ in that plane. If $P(x_1, y_1, 0)$ lies on $\mathscr{C}$, then the point $Q(x_1, y_1, z_1)$ satisfies $f(x, y) = 0$ for arbitrary z_1 (since $f(x, y)$ does not involve z). Hence Q can be any point on the line through P which is parallel to Oz. As P

† Written down by using 'alternate suffixes'.

‡ Had we used the two *given* equations, we should not be writing down the equation of a *sphere*.

varies on $\mathscr{C}$, the line PQ generates a right cylinder whose cross-section by the plane xOy is $\mathscr{C}$. Thus $f(x,y) = 0$ *represents a cylinder with generators parallel to Oz.*

Thus in ex. (iii) of 22.22 the equation $x^2+y^2-4x+2y = 1$ represents a cylinder whose section by the xy-plane is the circle with centre $(2, -1, 0)$ and radius $\sqrt{6}$.

(ii) *Homogeneous equation* $f(x,y,z) = 0$. If $f(x,y,z)$ is homogeneous of degree n in (x,y,z), then (1.52 (4))

$$f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x,y,z)$$

for all λ and x, y, z for which the functions are defined. If $P(x,y,z)$ is a point on the surface $f(x,y,z) = 0$ other than O, then $(\lambda x, \lambda y, \lambda z)$ also lies on this surface for all λ; i.e. every point of the line OP lies on the surface, which is therefore a *cone with vertex O.*

(iii) *Surface of revolution.* Let the curve $f(x,y) = 0$ in the xy-plane be rotated about Ox, thereby generating a surface of revolution with axis Ox. Let $Q(x_0, y_0, 0)$ be any point on the curve, and $P(x,y,z)$ be its position in space at some stage of the rotation; then P lies on a circle whose plane is perpendicular to Ox and passes through Q, and whose centre C lies on Ox. Thus

$$y_0 = CQ = CP = \sqrt{(y^2+z^2)}, \quad x_0 = OC = x.$$

Since Q lies on the curve, we have $f(x_0, y_0) = 0$, and hence

$$f\left(x, \sqrt{(y^2+z^2)}\right) = 0,$$

which is the equation of the surface of revolution. Cf. the Remark in 7.5.

If two surfaces intersect, they do so usually in points of a curve (although they may possess isolated points in common). In general this curve does not lie entirely in one plane (as was the case for the intersection of two spheres), and consequently is called a *twisted curve* or a *skew curve* or a *space curve.* It is determined by two equations, viz. those of the surfaces of which it is the intersection. Any other pair of surfaces through the curve would equally well serve to determine it.

Exercise 22(*b*)

1 Write down the centre and radius of the circle of intersection of the sphere $x^2+y^2+z^2 = 16$ and the plane $3x+2y-z = 0$.

2 If a sphere of radius r is cut by a plane distant c from the centre $(c < r)$, prove geometrically that the circle of intersection has radius $\sqrt{(r^2-c^2)}$. What happens when $c = r$?

3 Show that the meet of the sphere $x^2+y^2+z^2-2x-4y-6z-11 = 0$ and the plane $2x-y+2z-15 = 0$ is a circle of centre $(3, 1, 5)$, and find its radius. [Use 22.12, (i).]

4 Find the equation of the sphere which has its centre in the positive quadrant of the xy-plane, and cuts the planes $x = 0$, $y = 0$, $z = 0$ in circles of radii 3, 4, 5 respectively.

5 Obtain the equation of the plane through $A(1,0,0)$, $B(0,3,0)$, $C(0,0,-1)$, and also that of the sphere through these points and the origin. Hence find the centre of circle ABC.

6 Find the circumcentre of the triangle whose vertices are $(2, 0, 4)$, $(2, 4, 2)$, $(0, 2, 4)$.

7 Prove that the spheres

$$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$$

and $$(x-a-l)(x-a)+(y-b-m)(y-b)+(z-c-n)(z-c) = r^2$$

cut along a great circle of the former.

8 Find the equations of the two spheres through the circle

$$x^2+y^2+z^2-2x-4y = 0, \quad x+2y+3z = 8,$$

which touch the plane $4x+3y = 25$.

9 Find the equations of the spheres through the circle

$$x^2+y^2+z^2+2x+2y = 0, \quad x+y+z+4 = 0,$$

which intersect the plane $x+y = 0$ in circles of radius 3.

10 Find the equation of the sphere through the circle $x^2+y^2+2x = 0$, $z = 0$ and the point $(1, 2, 1)$.

11 Spheres are drawn through the circle $x^2+y^2 = a^2$, $z = 0$. Prove that the locus of the ends of their diameters parallel to Ox is the rectangular hyperbola $x^2-z^2 = a^2$, $y = 0$.

12 Show that every sphere through the circle $x^2+z^2 = a^2$, $y = 0$ is orthogonal to every sphere through the circle $x^2+y^2+4ax+a^2 = 0$, $z = 0$. Find the two spheres (one from each of the above systems) which cut the sphere

$$3(x^2+y^2+z^2)+2a(2x+2y-z) = a^2$$

in the same great circle.

13 Find the equation of that sphere through the intersection of the spheres

$$s \equiv x^2+y^2+z^2+4x-2y+2z+5 = 0, \quad s' \equiv x^2+y^2+z^2+2x-8y+2z+9 = 0$$

which has the smallest radius, and state this radius. [The system of spheres is best written $s+k(s-s') = 0$.]

14 Prove that the circles

$$x^2+y^2+z^2-9x+4y+5z-1 = 0, \quad 7x-2y+z = 4,$$

$$x^2+y^2+z^2+6x-10y+6z-7 = 0, \quad 4x-6y = 1$$

lie on the same sphere, and find its equation.

15 Show that the spheres $x^2+y^2+z^2 = 36$, $x^2+y^2+z^2-6x-8y-24z+120 = 0$ touch, and find the common point.

16 Find the sphere which touches the sphere $x^2+y^2+z^2-2x+6y-4z+6 = 0$ at $(3, -1, 2)$ and passes through O.

17 A system of spheres is such that any two touch at the fixed point O. Prove that any sphere which cuts each sphere of the system orthogonally will pass through O. If such spheres also pass through a fixed point A, prove that they

pass through a fixed circle. [Choose O for origin, and the common tangent plane at O for xy-plane.]

*18 Interpret the following:

(i) $f(y,z) = 0$; (ii) $f(x,z) = 0$; (iii) $f(y, \sqrt{(x^2+z^2)}) = 0$; (iv) $f(z, \sqrt{(x^2+y^2)}) = 0$.

*19 Write down the equations of the projection on the plane xOy of the circle in no. 1.

*20 Show that the right circular cone with vertex O, semi-vertical angle α, and axis Oz has equation $x^2+y^2 = z^2 \tan^2 \alpha$.

22.4 The spherical triangle

22.41 Some definitions and simple properties

Consider a sphere of centre O and radius r. In 22.12, (i) we defined a *great circle* of the sphere to be the section made by a plane through O, and a *small circle* to be any other plane section. Thus although all great circles have radius r, a small circle can have any radius between r and zero.

The ends of a diameter of the sphere are called *antipodal points*. There is a unique great circle through two non-antipodal points P, Q; for there is a unique plane through O, P, Q, and this cuts the sphere in a unique great circle. Consequently there are just two great circle arcs joining P, Q, viz. the minor and major arcs of this great circle. The minor arc is called *the arc PQ*; its length is the *spherical distance* between P and Q.

The *axis* of a circle on the sphere is that diameter of the sphere which is perpendicular to the plane of the circle. Its extremities are *poles* of the circle, and the nearer one is called *the* pole.

A circle on the sphere which passes through the poles of a given circle is called a *secondary* to that circle. Hence every secondary is a great circle; and a given circle has infinitely many secondaries. Two great circles have a unique common secondary: it lies in the plane determined by their axes.

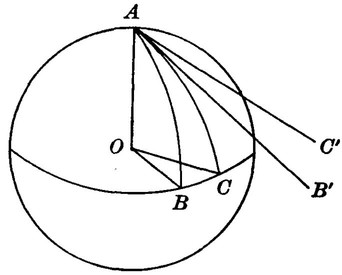

Fig. 234

Defining the *angle between two curves* at a common point to be the angle between the tangents there, let the arcs AB, AC of great circles meet their common secondary at B, C; and let AB', AC' be the tangents to these arcs at A. Since AB', AC' are both perpendicular to OA and lie in the planes AOB, AOC respectively, the angle $B'AC'$ between the arcs is the angle BOC between their planes. Hence

the angle between two great circles
= the angle between their planes
= the angle at O subtended by the arc intercepted on their common secondary.

A *lune* is the figure on a sphere formed by two great semi-circles. The angle between the semi-circles is called the *angle of the lune*. Assuming that areas of lunes on the same sphere are proportional to their angles, then for a lune of angle θ,

$$\frac{\text{area of lune}}{\text{area of sphere}} = \frac{\theta}{2\pi}$$

since the whole sphere can be regarded as a lune of angle 2π. Since the sphere has area $4\pi r^2$,

$$\text{area of lune} = 2r^2\theta. \qquad \text{(i)}$$

22.42 Sides and angles

The figure on a sphere formed by three minor arcs of three great circles is called a *spherical triangle*. Its *sides* a, b, c are the three arc-lengths, and the angles between the arcs b, c; c, a; a, b are its *angles* A, B, C.

There are two different ways of joining two non-antipodal points by an arc of a great circle. Hence if no two of the points† A, B, C are antipodal, there are eight different figures having these for vertices. Only one such figure is a spherical triangle, viz. that formed by the *minor* arcs BC, CA, AB.

Taking the sphere to have unit radius,‡ the length of an arc of a great circle is the radian measure of the angle subtended by it at the centre O. Hence the sides a, b, c are the angles at O subtended by the arcs BC, CA, AB; and since these are minor arcs, *each of a, b, c is less than π.*

The side a of the spherical triangle ABC is $B\hat{O}C$, a plane angle; and the angle A is that between the planes OBA, OCA (a dihedral angle). Therefore relations between sides, angles of a spherical triangle are equivalent to relations between plane angles, dihedral angles of a trihedral angle. It follows that

any two sides of a spherical triangle are together greater than the third, (ii)

† We use the same letter for a *vertex*, the *angle at the vertex*, and the *measure* of this angle, as in plane trigonometry; similarly for a side and its measure.

‡ Except in 22.44 and 22.54, this will be the case throughout.

since in a trihedral angle the sum of two plane angles is greater than the third;†
and

the sum of the sides of a spherical triangle is less than 2π, (iii)

(i.e. less than the circumference of a great circle) since the sum of the plane angles of a trihedral angle is less than four right-angles.† Cf. Ex. 22(*d*), no. 20.

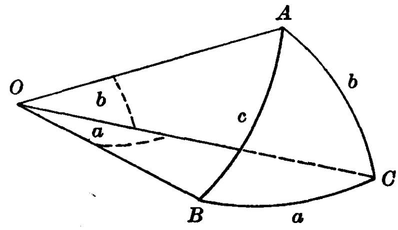

Fig. 235

Each angle of a spherical triangle is less than π. For if possible suppose $B \geqslant \pi$; let arc AB be produced to meet arc CA again at A'. Then A, A' are antipodal, and arc $CA \geqslant$ arc $AA' = \pi$, i.e. $b \geqslant \pi$, contradicting the definition of 'spherical triangle'.

However, a spherical triangle may have one, two, or three of its angles obtuse; and one, two, or three may be right-angles. Also see 22.43 (3).

22.43 Polar triangle; supplemental relations

(1) If A' is that pole of BC which is on the same side of the plane OBC as A and B', C' are defined similarly by cyclic interchange of the letters, then $A'B'C'$ is a spherical triangle‡ called the *polar triangle* of ABC. We now prove the following reciprocal property.

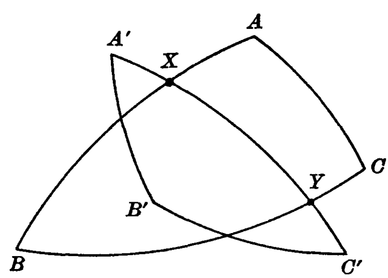

Fig. 236

If $A'B'C'$ is the polar triangle of ABC, then ABC is the polar triangle of $A'B'C'$.

Proof. Since B' is the pole of CA, $B'A = \frac{1}{2}\pi$; and since C' is the pole of AB, $C'A = \frac{1}{2}\pi$. Therefore A is a pole of $B'C'$.

† This is proved in solid geometry (Euclid XI, 20, 21).
‡ The reader should verify this statement.

As A' and A are on the same side of plane OBC, and the distance of A' from any point on BC is $\frac{1}{2}\pi$, hence $AA' < \frac{1}{2}\pi$. Thus, since the distance of A from any point on $B'C'$ is $\frac{1}{2}\pi$, A and A' must be on the same side of the plane $OB'C'$.

Therefore A is that pole of $B'C'$ which is on the same side of plane $OB'C'$ as A'; and B, C have the corresponding properties obtained by cyclic interchange. Hence ABC is the polar triangle of $A'B'C'$.

(2) *Sides, angles of the polar triangle are respectively the supplements of the corresponding angles, sides of the original triangle.*

Proof. Let $C'A'$ cut AB at X, and BC at Y. Since $C'A'$ is secondary to AB and BC, therefore $XY = \hat{B}$. Also $C'X = \frac{1}{2}\pi = YA'$. As $YA' = XY + XA'$, then $C'X + XY + XA' = \pi$, so that $C'A' = \pi - XY$, i.e. $b' = \pi - B$. We infer the *supplemental relations*

$$a' = \pi - A, \quad b' = \pi - B, \quad c' = \pi - C. \tag{iv}$$

Since ABC is the polar triangle of $A'B'C'$, we also have

$$a = \pi - A', \quad b = \pi - B', \quad c = \pi - C'. \tag{v}$$

(3) *The sum of the angles of a spherical triangle lies between* 2 *and* 6 *right-angles.*

Proof. By applying (iii) to the polar triangle $A'B'C'$,

$$0 < a' + b' + c' < 2\pi.$$

From (iv) this becomes $\quad 0 < 3\pi - (A + B + C) < 2\pi,$

i.e.
$$\pi < A + B + C < 3\pi. \tag{vi}$$

The expression $E = A + B + C - \pi$ is called the *spherical excess* of the spherical triangle ABC; it is the amount by which the sum of the angles of the spherical triangle exceeds the sum of those of any plane triangle. By (vi), $0 < E < 2\pi$.

22.44 Area

Given the spherical triangle ABC on a sphere of centre O and radius r, produce the sides AC, BC to cut the great circle AB again at A', B'. By considering areas on the hemisphere shown (fig. 237), and using (i),

$$ABC + s_1 = \text{lune } ABA'C = 2Ar^2,$$

$$ABC + s_2 = \text{lune } BAB'C = 2Br^2,$$

$$ABC + CB'A' = \text{lune with angle } C = 2Cr^2.$$

Adding, $\quad 2ABC + \text{area of hemisphere} = 2(A + B + C)\,r^2.$

$$\therefore \quad \text{area of triangle } ABC = (A + B + C - \pi)\,r^2$$

$$= Er^2. \tag{vii}$$

22.5 Triangle formulae

We continue to assume that the sphere has unit radius until 22.54.

22.51 Cosine rule

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \qquad \text{(viii)}$$

and two similar formulae which can be obtained from this by cyclic interchange of the letters.

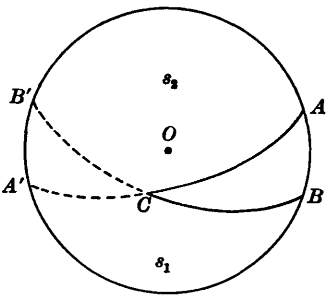

Fig. 237

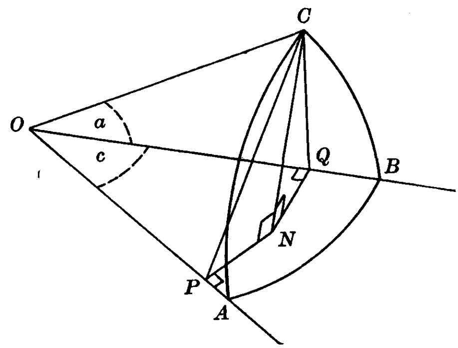

Fig. 238

The cosine rule is important because all the remaining triangle formulae can be deduced from it. In calculation it can be used to find (*a*) a side of the triangle when the other two sides and the included angle are known; (*b*) an angle when the three sides are known; this is exactly the situation in the plane case.

First proof, using projection (fig. 238).

Let N be the projection of C on the plane AOB, and P, Q be the projections of N on OA, OB respectively.

Since OP is perpendicular to CN and NP, therefore OP is perpendicular to the plane CNP, and hence $OP \perp PC$ because PC is a line in this plane. Similarly $OQ \perp QC$.

Projecting on OB,

$$\text{proj. of } ON = \text{proj. of } OP + \text{proj. of } PN,$$

i.e.
$$OQ = OP \cos c + PN \cos(\tfrac{1}{2}\pi - c).$$

Now
$$OQ = OC \cos a, \quad OP = OC \cos b,$$

and since $C\hat{P}N$ is the angle between the planes OAB, OAC, viz. A,

$$PN = PC\cos A = OC\sin b \,.\cos A.$$

Substituting, we get formula (viii).

Second proof, using coordinates (fig. 239).

Choose axes at the centre O of the sphere, with Oz along OA, and let the other vertices B, C have cartesian coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) and spherical polar coordinates $(1, \theta_1, \phi_1)$, $(1, \theta_2, \phi_2)$: see 21.12 (2) and Ex. 21 (*a*), no. 15.

Since OB, OC have direction cosines $\{x_1, y_1, z_1\}$, $\{x_2, y_2, z_2\}$,

$$\begin{aligned}\cos a &= x_1x_2 + y_1y_2 + z_1z_2\\ &= \sin\theta_1\cos\phi_1 .\sin\theta_2\cos\phi_2 + \sin\theta_1\sin\phi_1 .\sin\theta_2\sin\phi_2 + \cos\theta_1 .\cos\theta_2\\ &= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1 - \phi_2)\\ &= \cos c\cos b + \sin c\sin b\cos A\end{aligned}$$

because $\theta_1 = c$, $\theta_2 = b$, and $\phi_1 - \phi_2 = \pm A$.

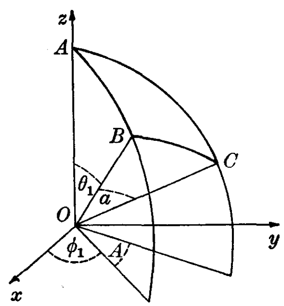

Fig. 239

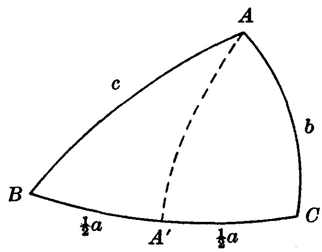

Fig. 240

Example

Median of a spherical triangle.

The arc of the great circle joining a vertex A to the mid-point A' of the opposite side BC is called a *median* of the triangle ABC (fig. 240).

Since angles $CA'A$, $AA'B$ are supplementary,

$$\cos CA'A + \cos AA'B = 0.$$

By applying the cosine rule to each of triangles $AA'C$, ABA', this becomes

$$\frac{\cos b - \cos AA' \cos \frac{1}{2}a}{\sin AA' \sin \frac{1}{2}a} + \frac{\cos c - \cos AA' \cos \frac{1}{2}a}{\sin AA' \sin \frac{1}{2}a} = 0,$$

and on solving for $\cos AA'$,

$$\cos AA' = \frac{\cos b + \cos c}{2\cos \frac{1}{2}a}.$$

22.52 Sine rule

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \qquad \text{(ix)}$$

Geometrical proof.

From fig. 238 we have

$$NC = PC \sin A = OC \sin b \sin A,$$

and
$$NC = QC \sin B = OC \sin a \sin B$$

since $C\hat{Q}N$ is the angle between the planes OBA, OBC, viz. B. Hence

$$\sin b \sin A = \sin a \sin B,$$

from which
$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}.$$

Similarly, by projecting from A onto plane OBC, we can prove

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Deduction from the cosine rule.

$$\begin{aligned}\sin^2 A &= 1 - \cos^2 A\\ &= 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}\\ &= \frac{(1-\cos^2 b)(1-\cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}\\ &= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 b \sin^2 c}.\end{aligned}$$

$$\therefore \quad \frac{\sin A}{\sin a} = \frac{+\sqrt{(1 - \Sigma \cos^2 a + 2\cos a \cos b \cos c)}}{\sin a \sin b \sin c},$$

and this expression is symmetrical in a, b, c. Hence it is also equal to $\sin B/\sin b$ and to $\sin C/\sin c$. The sine rule can therefore be written in the 'completed form'

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{+\sqrt{(1 - \Sigma \cos^2 a + 2\cos a \cos b \cos c)}}{\sin a \sin b \sin c}, \quad \text{(x)}$$

where the positive sign of the square root is taken because all the sines are positive (all sides and angles lying between 0 and π, 22.42). The reader may verify that the expression under the square root sign can be written

$$\begin{vmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{vmatrix}.$$

Examples

(i) *If A' is the mid-point of BC, $\beta = B\hat{A}A'$ and $\gamma = A'\hat{A}C$, prove*

$$\cot\beta = \cot A + \frac{\sin C}{\sin A \sin B}$$

and
$$\cot\gamma - \cot\beta = \frac{\sin(b+c)\sin(b-c)}{\sin b \sin c \sin A}.$$

By applying the sine rule to triangles ABA', $AA'C$ in fig. 240,

$$\frac{\sin\beta}{\sin B} = \frac{\sin\frac{1}{2}a}{\sin AA'} = \frac{\sin\gamma}{\sin C}.$$

$$\therefore \quad \sin\beta \sin C = \sin(A-\beta)\sin B,$$

$$\sin\beta(\cos A \sin B + \sin C) = \cos\beta \sin A \sin B,$$

and so
$$\cot\beta = \cot A + \frac{\sin C}{\sin A \sin B}.$$

Similarly,
$$\cot\gamma = \cot A + \frac{\sin B}{\sin A \sin C}.$$

$$\therefore \quad \cot\gamma - \cot\beta = \frac{\sin^2 B - \sin^2 C}{\sin A \sin B \sin C}$$

$$= \frac{\sin^2 b - \sin^2 c}{\sin A \sin b \sin c} \quad \text{by the sine rule,}$$

$$= \frac{\sin(b+c)\sin(b-c)}{\sin b \sin c \sin A} \quad \text{after reduction.}$$

(ii) *Prove that*
$$\frac{\sin(B+C)}{\sin A} = \frac{\cos b + \cos c}{1 + \cos a}.$$

$$\frac{\sin(B+C)}{\sin A} = \frac{\sin B}{\sin A}\cos C + \frac{\sin C}{\sin A}\cos B, \quad \text{and by using both rules,}$$

$$= \frac{\sin b}{\sin a}\,\frac{\cos c - \cos a \cos b}{\sin a \sin b} + \frac{\sin c}{\sin a}\,\frac{\cos b - \cos c \cos a}{\sin c \sin a}$$

$$= \frac{\sin b \sin c(\cos c - \cos a \cos b) + \sin b \sin c(\cos b - \cos c \cos a)}{\sin^2 a \sin b \sin c}$$

$$= \frac{\sin b \sin c(\cos b + \cos c)(1 - \cos a)}{(1 - \cos^2 a)\sin b \sin c}$$

$$= \frac{\cos b + \cos c}{1 + \cos a}.$$

22.53 Supplemental formulae

If the formulae (viii), (x) are applied to the polar triangle $A'B'C'$ and the relations (iv), (v) are used, new formulae are obtained for triangle ABC. Thus the cosine rule

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'$$

with
$$a' = \pi - A, \quad A' = \pi - a, \quad \text{etc.}$$

gives $$-\cos A = \cos B \cos C - \sin B \sin C \cos a,$$

i.e. $$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \tag{xi}$$

Similarly the sine rule in its completed form gives

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \frac{+\sqrt{(1-\Sigma\cos^2 A - 2\cos A \cos B \cos C)}}{\sin A \sin B \sin C}. \tag{xii}$$

22.54 Triangle on the general sphere

When the sphere has radius $r \neq 1$, the sides of the spherical triangle ABC are no longer equal to the angles which they subtend at the centre. If we continue to denote these *angles* by a, b, c, and call the sides α, β, γ, then

$$a = \frac{\alpha}{r}, \quad b = \frac{\beta}{r}, \quad c = \frac{\gamma}{r}.$$

All the preceding formulae remain valid because their proofs used the angles at the centre and not the actual sides of the triangle.

Example

By using the expansions of $\cos x$, $\sin x$ (Ex. 6(*b*), nos. 21, 20), the cosine formula (viii) can be written

$$1-\tfrac{1}{2}a^2+\ldots = (1-\tfrac{1}{2}b^2+\ldots)(1-\tfrac{1}{2}c^2+\ldots)+(b-\ldots)(c-\ldots)\cos A,$$

where the dots denote terms of order 3 or 4.

On introducing the sides α, β, γ and simplifying, this becomes

$$\alpha^2 = \beta^2+\gamma^2-2\beta\gamma\cos A + O\left(\frac{1}{r^2}\right).$$

When $r \to \infty$, the spherical triangle ABC tends to the plane triangle ABC with sides α, β, γ, and in the limit we obtain the cosine rule of plane trigonometry. The sine rule can be treated similarly.

Exercise 22(*c*)

1 Write out the formulae for $\cos b$, $\cos c$ corresponding to formula (viii).

2 If $C = \frac{1}{2}\pi$, $a = \frac{1}{6}\pi$, $b = \frac{1}{2}\pi$, find c and the area of the triangle.

3 $A = \frac{1}{4}\pi$, $b = c = \frac{1}{2}\pi$, and D is the mid-point of side AC. If $B\hat{D}C = \alpha$, show that $\cos\alpha = 1/\sqrt{3}$ and that the areas of the triangles ABD, ABC are in the ratio $(3\pi - 8\alpha):\pi$.

4 If B, C are points on the equator whose longitudes differ by 90°, and A is a point in latitude λ on the meridian through C, prove that in the triangle ABC the side AB and the angle A are both right-angles.

If M, the mid-point of AB, has latitude α, and the difference of the longitudes of M and B is β, show that $\sqrt{2}\sin\alpha = \sin\lambda$ and $\tan\beta = \cos\lambda$.

5 Prove that if C is a right-angle, then $\sin b = \tan a \cot A$.

A ship sails from a place A on the equator along a great circle which makes an angle of 60° with the equator. Find the difference in the longitudes of A and the place B on the path of the ship where it first reaches the latitude 30°.

Taking the radius of the Earth to be 3960 miles, find (correct to three significant figures) the area included by the path of the ship, the meridian through B, and the equator.

6 Two points A, B on the Earth's surface have latitude λ, and their difference in longitude is δ radians. If r is the radius of the Earth, show that (i) their distance apart along the parallel of latitude joining them is $r\delta\cos\lambda$; (ii) their distance apart along the great circle joining them is $2r\sin^{-1}(\cos\lambda\sin\frac{1}{2}\delta)$. [For (ii) use half the isosceles spherical triangle ABC, where C is the north pole.]

7 In no. 6 (ii) prove that the greatest latitude L reached on the great circle course is given by $\tan L = \tan\lambda\sec\frac{1}{2}\delta$. [If D is the point of latitude L, the spherical triangle ACD is right-angled at D.]

8 If D is any point on BC, prove that

$$\cos AD\sin a = \cos b\sin BD + \cos c\sin DC.$$

[Method of 22.51, ex.]

9 The internal and external bisectors of angle A meet BC at D, D'. Prove

$$\frac{\sin BD}{\sin DC} = \frac{\sin c}{\sin b} = \frac{\sin BD'}{\sin D'C}.$$

10 The arc of the great circle through the vertex A which meets the opposite side BC at right-angles is called the *altitude* from A. If α, β, γ are the altitudes from A, B, C, prove that

$$\sin\alpha\sin a = \sin\beta\sin b = \sin\gamma\sin c = 2n,$$

where $$n = \tfrac{1}{2}\sqrt{(1-\Sigma\cos^2 a + 2\cos a\cos b\cos c)}.$$

*11 Prove that $$\frac{\sin(b+c)}{\sin a} = \frac{\cos B + \cos C}{1-\cos A}.$$

*12 Verify as in 22.54 that the sine rule of plane trigonometry is the limit of the spherical case.

Miscellaneous Exercise 22(*d*)

1 A rectangular box has edges of lengths a, b, c, and P moves so that the sum of the squares of its distances from the six faces is equal to m^2d^2, where d is the length of a diagonal of the box and m is constant. Prove that the locus of P is a sphere if $2m^2 > 1$.

2 Through a point P three mutually perpendicular lines are drawn: one passes through a fixed point C on Oz, while the others meet Ox, Oy respectively. Prove that the locus of P is a sphere with centre C.

3 Find the coordinates of the point of the sphere $x^2+y^2+z^2-4x+2y-4=0$ which is nearest to the plane $x+4y+z=19$, and calculate its distance from this plane.

4 Find the equations of the diameter of the sphere $x^2+y^2+z^2=29$ such that a rotation about it will transfer the point $(4, -3, 2)$ to $(5, 0, -2)$ along a great circle. Find also the angle through which the sphere must be rotated.

5 A line in direction $l:m:n$ is drawn through $(0, 0, a)$ to touch the sphere $x^2+y^2+z^2-2ax=0$. Prove that $m^2+2nl=0$.

Find the coordinates of the point P in which this line meets the plane $z=0$, and prove that as the line varies, P traces the parabola $y^2=2ax$, $z=0$.

6 If the plane $ax+by+cz+d = 0$ touches the sphere

$$(x-1)^2+y^2+(z-1)^2 = 1,$$

prove that $$a^2+b^2+c^2 = (a+c+d)^2.$$

If this plane also passes through $(0,0,2)$, show that it cuts the plane $z = 0$ in a line which touches the parabola $y^2 = 4x$, $z = 0$.

7 Show that there are two spheres through $(0,0,0)$, $(2a,0,0)$, $(0,2b,0)$ which touch the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n},$$

and that if $l^2+m^2+n^2 = 1$ the distance between their centres is

$$\frac{2}{n^2}\{c^2-n^2(a^2+b^2+c^2)\}^{\frac{1}{2}}.$$

8 Prove that the centre of a sphere which touches each of the lines

$$y-mx = 0 = z-c, \quad y+mx = 0 = z+c$$

lies on the surface $mxy+(1+m^2)cz = 0$.

9 A line λ passes through O and touches each of the spheres

$$x^2+y^2+z^2+2ax+p = 0, \quad x^2+y^2+z^2+2by+q = 0.$$

Show that the angle ϕ between λ and Oz is given by $\sin^2\phi = p/a^2+q/b^2$. Also find the distance between the contacts of λ with the two spheres.

10 Find the condition for $x/l = y/m = z/n$ to touch the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0,$$

and find the point of contact. Hence show that the tangent cone from O has equation $d(x^2+y^2+z^2) = (ux+vy+wz)^2$.

11 Two skew lines AP, $A'P'$ are met by their common perpendicular at A, A', and O is the mid-point of AA'; 2α is the angle between the lines, and $AA' = 2c$. If $AP.A'P'$ has either of the values $c^2\sec^2\alpha$, $-c^2\operatorname{cosec}^2\alpha$, prove that PP' touches the sphere with centre O and radius c.

12 Prove that there are eight spheres of radius 5 which are orthogonal to the sphere $x^2+y^2+z^2 = 16$, touch Ox, and cut off a segment of length 2 from Oy.

13 Through the circle of intersection of $x^2+y^2+z^2 = 25$ and $x+2y+2z = 9$, two spheres s_1, s_2 are drawn to touch the plane $4x+3y = 30$. Find the equations of these spheres, and the coordinates of the point through which pass all the common tangent planes of s_1 and s_2.

14 Prove that in general two spheres can be drawn to pass through a given circle and to touch a given plane. If the circle lies in the plane $z = 0$ and has a given radius r, and if the plane is $x\cos\theta+z\sin\theta = 0$, show that when the distance between the centres of the two spheres is constant and equal to $2c$, then the locus of the centre of the circle is the line-pair $x = \pm\sqrt{(r^2+c^2\cos^2\theta)}$, $z = 0$.

15 Given a sphere of centre O and radius a, let π be a plane through O and N be an extremity of the diameter normal to π. If P is any point of the sphere other than N, let NP meet π at P'. Choosing axes at O with Oz along ON, let P' have coordinates $(u,v,0)$. Show that NP' has equations $x = ut$, $y = vt$, $z = a(1-t)$, and that NP' meets the sphere at points for which $t = 0$ or

$2a^2/(u^2+v^2+w^2)$. Hence write down the coordinates of P. (This gives a rational parametric representation of the sphere $x^2+y^2+z^2 = a^2$, the parameters being u, v.)

16 (i) From the cosine formulae for $\cos c$ and $\cos b$, eliminate $\cos b$, and then substitute $\sin b = \sin B \sin c/\sin C$, to prove the 'four-parts formula'

$$\cos a \cos B = \sin a \cot c - \sin B \cot C.$$

(ii) Similarly prove

$$\cos a \cos C = \sin a \cot b - \sin C \cot B.$$

17 If AD bisects angle A internally, prove $\cot AD = (\cot b + \cot c)/2\cos \frac{1}{2}A$. [Apply no. 16 to triangles ABD, ADC; angles $A\hat{D}B$, $C\hat{D}A$ are supplementary.]

18 If D is a point on BC, $B\hat{A}D = \beta$, and $D\hat{A}C = \gamma$, prove that

$$\cot AD \sin A = \cot b \sin \beta + \cot c \sin \gamma.$$

[Method of no. 17.]

19 If a spherical triangle has area one-quarter that of the whole sphere, prove that $\cos a + \cos b + \cos c = -1$, and deduce that each median is the supplement of half the side which it bisects. [Use 22.52, ex. (ii); and 22.51, ex.]

20 (i) Prove (ii) of 22.42 by using the cosine formula. [Since $A < \pi$, $\cos a > \cos b \cos c - \sin b \sin c = \cos(b+c)$, which implies $a < b+c$ if $b+c \leqslant \pi$. If $b+c > \pi$, then $b+c > a$ since $a < \pi$.]

(ii) Use the result of 22.44 to infer that $A+B+C-\pi > 0$. By writing down the supplemental inequality (cf. 22.53), deduce (iii) of 22.42.

ANSWERS TO VOLUME II

Exercise 10(*a*), p. 369

1 $(x+1)^2(2x-1)$. 2 $(x-2)(3x+1)(2x+3)$.

3 $(x-1)(x+2)(x^2+x+2)$. 4 $-1, 2, 3$.

5 $a=-1, b=-2$. 6 $a=-2, b=3$. 7 7, 1.

8 $\frac{1}{5}(9x-1)$. 9 $3x^3+x^2+7x+1$. 10 x^3+x^2+3x+4.

11 1, 3, 6, 0. 12 1, -6, 0, -2.

13 -1, 25, $\pm(x^2-3x-5)$. 14 $a=3, b=-1$.

15 $a=4, b=-2, c=1$. 17 Put $x=-(b+c)$, etc.

19 $$A\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}+B\frac{(x-a)(x-c)(x-d)}{(b-a)(b-c)(b-d)}+C\frac{(x-a)(x-b)(x-d)}{(c-a)(c-b)(c-d)}+D\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}.$$

21 (i) $q=3a^4$.

Exercise 10(*b*), p. 373

1 $(x-2y+4)(2x+y-1)$. 2 $(x+y+8z)(x-y-6z)$.

3 10, $-7\frac{1}{2}$. 5 $3(y+z)(z+x)(x+y)$.

6 $-(y-z)(z-x)(x-y)(x+y+z)$. 7 $(y-z)(z-x)(x-y)(x+y+z)$.

8 $5(y+z)(z+x)(x+y)(x^2+y^2+z^2+yz+zx+xy)$.

9 $\Sigma yz(y^2-z^2)$, $\Sigma x(y-z)^3$.

10 (i) $x^2+y^2+z^2$; (ii) $x^2y^2+y^2z^2+z^2x^2$; (iii) $bc(b-c)+ca(c-a)+ab(a-b)$; (iv) $a^2bc+b^2ca+c^2ab$; (iv) $bc^2+ca^2+ab^2+b^2c+c^2a+a^2b$.

15 (ii) $3abc(b-c)(c-a)(a-b)$;
(iii) $5(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)$.

Exercise 10(*c*), p. 378

1 (i) 15; (ii) $\frac{5}{3}$; (iii) -1; (iv) 19; (v) 13; (vi) $\frac{19}{3}$; (vii) 80; (viii) 343.

2 (i) $4x^2-2x-3=0$; (ii) $9x^2-28x+16=0$; (iii) $12x^2+2x-1=0$.

3 $b/a<0$, $c/a>0$.

4 (i) 10; (ii) $-\frac{7}{12}$; (iii) 29; (iv) $\frac{61}{6}$; (v) 29; (vi) 353; (vii) -58.

5 (i) $x^3+8x^2+17x+6=0$; (ii) $4x^3+8x^2+x-1=0$;
(iii) $x^3-2x^2-7x+12=0$; (iv) $x^3-14x^2+17x-4=0$;
(v) $4x^3-17x^2+14x-1=0$.

6 $12y^3-20y^2-y=0$; $x=-\frac{1}{2}$, $\frac{1}{3}(1\pm\sqrt{7})$.

9 (i) $2p^3+27r = 9pq$; (ii) $p^3r = q^3$.

10 $x^3+(3q-p^2)x^2+q(3q-p^2)x+q^3-p^3r = 0$;

(i) $\alpha^2-\beta\gamma = \beta^2-\gamma\alpha = \gamma^2-\alpha\beta$; (ii) $\alpha^2-\beta\gamma = -p\alpha$, etc.; p^3r-q^3.

11 $x^4+3px^3+3p^2x^2+p^3x+q = 0$.

Exercise 10(*d*), p. 382

1 $y = x^2+2$.

2 $x^2/a^2+y^2/b^2 = 1$.

3 $(a^2-b^2)^2 = 2(b^4-c^4)$.

6 $b^4c^4+c^4a^4+a^4b^4 = a^2b^2c^2d^2$.

7 (i) $l+m+n = 0$; (ii) see 16.32.

8 (i) $y^2-4ax = c^2(x+a)^2$;

(ii) the locus of points from which the tangents to $y^2 = 4ax$ include an angle $\tan^{-1}c$.

9 $y^2 = ab^2x+a^2b^2(b-2)$.

10 $x^2+y^2 = 2(a^2+b^2)$.

11 $x = abc$, $y = -\Sigma bc$, $z = \Sigma a$.

12 1, 1, 3 in any order.

13 1, 2, -5 in any order.

14 1. -2, 3 in any order.

16 $4p^3+27q^2 = 0$.

17 $27p^4 = 256q^3$.

18 $(br-cq)(aq-bp)^2 = (cp-ar)^3$.

19 If $p(x) \equiv (x^2+2bx+c)^{r-1}g(x)$, then

$$p'(x) \equiv (r-1)(x^2+2bx+c)^{r-2}\,2(x+b)\,g(x)+(x^2+2bx+c)^{r-1}g'(x).$$

Hence $x^2+2bx+c$ is a factor of $(x+b)g(x)$. If $b^2 \neq c$, then $x+b$ is not a factor of $x^2+2bx+c$, so $x^2+2bx+c$ is a factor of $g(x)$, and the result follows. However, if $b^2 = c$, then $x^2+2bx+c = (x+b)^2$, so that $x+b$ (but *not necessarily* $(x+b)^2$) is a factor of $g(x)$.

Exercise 10(*e*), p. 388

1 $x+3$.

2 $2x+7$.

3 x^2-2x+5.

4 $(x-2)^2(x+1)(x+3)$.

5 $(x+2)(x^2-x+1)^2$.

6 $(x-1)^3(x+1)^2(x-2)$.

7 $-\frac{3}{2}$ twice.

8 2; $\frac{1}{2}(-1\pm\sqrt{5})$ twice.

9 $-\frac{1}{94}(12x+35)$, $\frac{1}{94}(24x^2-14x+43)$.

Miscellaneous Exercise 10(*f*), p. 388

2 No.

6 (i) $(b-c)(c-a)(a-b)(a+b+c)$;

(ii) $(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$.

8 $y^2 = (x+2)(x-1)^2$.

9 $a^4-4ac^3+3b^4 = 0$.

11 (i) $\lambda = 3(1-2\mu)/(\mu-2)^2$; (ii) roots of $\lambda\mu^2+2\mu(3-2\lambda)+(4\lambda-3) = 0$. We find $\mu = 3$ or $\frac{7}{5}$, and $x = \frac{19}{4}$ or $\frac{13}{4}$, respectively.

12 $2ac'+2a'c-bb' = 0$.

14 (i) Coeff. of x^2 zero; (ii) constant term zero; (iii) coeff. of x^3 equal to minus the constant term; (iv) coeff. of x^2 equal to constant term.

17 $2p^2$.
18 $y^4-5y^2+6=0$; $\pm\sqrt{2}-3$, $\pm\sqrt{3}-3$.

19 $b^2x^4+2(ab-2h^2)x^2+a^2=0$.
20 $x^2/a+y^2/b+z^2/c=2$.

24 (i) $k=3$; $x=1$ twice; (ii) $k=-\frac{3}{8}$; $x=\frac{3}{2}$, $-\frac{1}{2}$ thrice.

25 (i) $m=-20$, $c=7$; $m=108$, $c=135$; (ii) $m=44$, $c=-121$.

27 p/q is constant.

Exercise 11(*a*), p. 399

1 2.
2 40.
3 82.
4 0.

5 $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)$.

6 $a^3+b^3+c^3-3abc$.
11 0.
12 -90.

13 0.
14 0.
15 18.
16 0.
17 0.

18 $x(x-1)^2(x^2-1)$.
19 $x^4(x-1)^2(x^2-1)$.

20 $\Sigma a^3-3abc=(\Sigma a)(\Sigma a^2-\Sigma bc)$.
22 (i) and (ii) Δ.
23 9, $\pm\sqrt{3}$.

24 0.
26 $\Sigma\cos^2 A+2\cos A\cos B\cos C=1$.
28 (ii) 3^3.

Exercise 11(*b*), p. 410

1 $x=-\frac{2}{3}$, $y=\frac{5}{3}$, $z=1$.
2 4, 3, -1.
3 2, -1, -2.

4 $x=\dfrac{(b-d)(c-d)}{(b-a)(c-a)}$, etc.

8 $\lambda=1$, $x:y:z=1:-1:0$; $\lambda=-2$, $\sqrt{2}:\sqrt{2}:1$; $\lambda=3$, $1:1:-2\sqrt{2}$.

9 $\lambda=3$, $x=\frac{1}{6}$, $y=\frac{3}{2}$; $\lambda=14$, $x=-\frac{1}{5}$, $y=\frac{2}{5}$.

10 (i) $x=-3$, $y=3/(1-k)$, $z=(1-4k)/(1-k)$;
(ii) $x=\lambda$, $y=-\lambda$, $z=-2-\lambda$; (iii) inconsistent.

11 $bc+ca+ab=2abc+1$.

Exercise 11(*c*), p. 413

1 $(b-c)(c-a)(a-b)$.
2 $(b-c)(c-a)(a-b)(a+b+c)$.

3 $(b-c)(c-a)(a-b)(a+b+c)$.
4 $(b-c)(c-a)(a-b)(bc+ca+ab)$.

5 $(b-c)(c-a)(a-b)$.
6 $-(b-c)(c-a)(a-b)$.

7 $(b^2-c^2)(c^2-a^2)(a^2-b^2)$. [Use result of 11.5, ex. (i).]

8 $-(b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2)$.

9 $(b-c)(c-a)(a-b)\Sigma(a^2+bc)$.
10 $x=abc$, $y=-\Sigma bc$, $z=\Sigma a$.

11 $x=\dfrac{(b-1)\,c-1)}{a(b-a)(c-a)}$, etc.
12 $x=\dfrac{(a-b)(a-c)}{p^2+bc}$, etc.

Exercise 11(*d*), p. 415

1 $\begin{vmatrix} 1 & 2 & 3 \\ x & x^2 & x^3 \\ 0 & 2 & 6x \end{vmatrix}$. 2 $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2x \\ 1 & 3 & 3x^2 \end{vmatrix}$. [Derive by columns.]

3 $\begin{vmatrix} \log x & x & x^2 \\ 1/x & 1 & 2x \\ -2/x^3 & 0 & 0 \end{vmatrix}$.

8 $2(y-z)(z-x)(x-y)$.

9 56. [First interchange $\mathbf{c}_1$ and $\mathbf{c}_2$, then $\mathbf{r}_1$ and $\mathbf{r}_2$.]

10 -32. 11 -550. 13 $x = -8$, $y = 14$, $z = -2$, $t = 4$.

15 $-(x-a)(x-b)(x-c)(b-c)(c-a)(a-b)$. With notation of Ex. 11(*e*), no. 22, $V_3 = (b-c)(c-a)(a-b)$, $V_2 = (b-c)(c-a)(a-b)(a+b+c)$,
$V_1 = (b-c)(c-a)(a-b)(bc+ca+ab)$.

Miscellaneous Exercise 11(*e*), p. 416

1 The equation is *linear* in x, y, and is satisfied by (x_1, y_1) and (x_2, y_2).

4 $(b-c)(c-a)(a-b)$. 5 $4(b-c)(c-a)(a-b)(a+b+c)$.

6 $(b-c)(c-a)(a-b)(abc+1)$.

7 $(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)$.

8 $2abc(a+b+c)^3$. 10 $\sin^2\theta - 2\cos^2\theta + 2\sin\theta\cos\theta$.

11 $\begin{vmatrix} ap & aq \\ cp & cq \end{vmatrix} + \begin{vmatrix} ap & aq \\ dr & ds \end{vmatrix} + \begin{vmatrix} br & bs \\ cp & cq \end{vmatrix} + \begin{vmatrix} br & bs \\ dr & ds \end{vmatrix}$.

12 $x = 1$ twice, 2. [$\mathbf{c}_2 \to \mathbf{c}_2 - \mathbf{c}_1$, remove $x-1$; $\mathbf{c}_3 \to \mathbf{c}_3 - \mathbf{c}_2$.]

13 $x = b, c, a^3/bc$.

15 Three distinct collinear points cannot be concyclic.

17 (i) Unique solution $x = 3$, $y = 1$, $z = 0$; (ii) $x = 3-2\lambda$, $y = 1-\lambda$, $z = \lambda$; (iii) inconsistent.

18 (i) $\lambda \neq 3$; (ii) $\lambda = 3$, $\mu \neq 10$: (iii) $\lambda = 3$, $\mu = 10$.

19 $\lambda = -1$, $k = 18$, $x = 6-4\alpha-3\beta$, $y = \alpha$, $z = \beta$; $\lambda = -19$, $k = -18$, $x = \alpha$, $y = 2\alpha$, $z = 3\alpha+2$.

20 $\lambda = 1$, $a:b:c = 0:-2:1$; $\lambda = \frac{1}{2}$, $4:4:-3$; $\lambda = \frac{9}{2}$, $4:4:5$.

22 (ii) $\begin{vmatrix} 1 & 1 & 1 \\ (x+a)^2 & (x+b)^2 & (x+c)^2 \\ (x+a)^3 & (x+b)^3 & (x+c)^3 \end{vmatrix}$, $2\begin{vmatrix} 1 & 1 & 1 \\ x+a & x+b & x+c \\ (x+a)^3 & (x+b)^3 & (x+c)^3 \end{vmatrix}$,

$6\begin{vmatrix} 1 & 1 & 1 \\ x+a & x+b & x+c \\ (x+a)^2 & (x+b)^2 & (x+c)^2 \end{vmatrix}$.

Exercise 12(*a*), p. 422

1 $81x^4-216x^3+216x^2-96x+16$. 2 $x^5+5x^3+10x+5/x^3+1/x^5$.

3 $1+3x+2x^2-2x^3-3x^4-x^5$. 4 $1+12x+54x^2+100x^3+15x^4+\ldots$.

5 60. 6 969. 7 -105×2^{-13}. 8 7920.

9 $3^7\times 154x$. 10 -16×217. 11 (i) 1·2166; (ii) 0·9039.

12 7. 13 5th term $=\frac{11}{27}\times 5^8\times 2^{13}$.

14 Coefficient of the 6th term $=231\times 4^{11}\times 5^6$.

15 ${}^nC_r+2\,{}^nC_{r-1}+{}^nC_{r-2}={}^{n+2}C_r\ (r\geqslant 2)$.

18 ${}^{n+1}C_r={}^nC_r+{}^{n-1}C_{r-1}+{}^{n-2}C_{r-2}+\ldots+{}^{n-r+1}C_1+1$. 19 724.

Exercise 12(*b*), p. 425

3 $n.2^{n-1}$. 4 0. 8 0.

10 $\dfrac{(2n-1)!}{\{(n-1)!\}^2}$. 11 $\dfrac{(1+x)^{n+2}-(n+2)x-1}{(n+1)(n+2)}$.

13 (i) 4^6; (ii) 2080, 2016.

14 (i) $(1+x+x^2)^n=a_0x^{2n}+a_1x^{2n-1}+\ldots+a_{2n}$,

$(1-x+x^2)^n=a_0-a_1x+a_2x^2-\ldots+a_{2n}x^{2n}$;

(ii) $a_0+a_1x^2+a_2x^4+\ldots+a_{2n}x^{4n}$; (iv) $\frac{1}{2}(3^n-1)$; (v) $(n+1)3^n$.

Exercise 12(*c*), p. 432

1 $(n+1)^2(2n+1)^2$. 2 $\frac{1}{3}n(4n^2-1)$.

3 $(-1)^{n-1}\frac{1}{2}n(n+1)$. 7 $\frac{1}{3}n(n+1)(7n+2)$.

8 $a=2,\ b=-3,\ c=-3,\ d=0$; $\frac{1}{2}n(n+1)(n^2+3n-1)$.

9 $\frac{1}{6}n(n+1)(n-4)$. 10 $\frac{1}{12}n(n+1)(n+2)(3n+1)$.

11 $\frac{1}{8}(2n+1)(2n+3)(2n+5)(2n+7)-\frac{1}{8}.3.5.7$.

12 $\frac{1}{4}n(n+1)(n+6)(n+7)$. 13 $\frac{1}{48}n(n+1)(n+2)(3n+5)$.

14 $\dfrac{1}{3}-\dfrac{1}{3(3n+1)}$; $\dfrac{1}{3}$. 15 $\dfrac{1}{4}\left\{\dfrac{1}{3.5}-\dfrac{1}{(2n+3)(2n+5)}\right\}$; $\dfrac{1}{60}$.

16 $\dfrac{5}{4}-\dfrac{4n+5}{2(n+1)(n+2)}$; $\dfrac{5}{4}$. 17 $\dfrac{11}{120}-\dfrac{12n+11}{8(2n+3)(2n+5)}$; $\dfrac{11}{120}$.

18 $\dfrac{3}{4}-\dfrac{1}{2(n+1)}-\dfrac{1}{2(n+2)}$; $\dfrac{3}{4}$. 19 $\dfrac{17}{36}-\dfrac{1}{6(n+1)}-\dfrac{2}{3(n+3)}$; $\dfrac{17}{36}$.

20 $\frac{1}{2}n\sin\theta+\frac{1}{2}\sin(n+2)\theta\sin n\theta\operatorname{cosec}\theta$.

21 $\frac{1}{4}(2n-1)+\frac{1}{4}\sin(2n+1)\theta\operatorname{cosec}\theta$. 22 $\cot\theta-\cot(2^n\theta)$.

23 $\cot\theta-2^n\cot(2^n\theta)$. 24 $2^{2n}\operatorname{cosec}^2(2^n\theta)-\operatorname{cosec}^2\theta$.

25 $2\operatorname{cosec}2\theta\sin n\theta\sec(n+1)\theta$. 26 $\tan^{-1}\{n/(n+2)\}$; $\frac{1}{4}\pi$.

Exercise 12(*d*), p. 435

8 $\frac{1}{3}(a+2b)$.

Exercise 12(*e*), p. 442

4 D.	5 C.	6 D.	7 C.	8 C.	9 C.
10 D.	11 C.	12 D.	13 D.	14 C.	15 C.

Exercise 12(*f*), p. 446

1 C.
2 D.
4 C for all $x \geqslant 0$.
5 $0 \leqslant x \leqslant 1$, C; $x > 1$, D.
6 $0 \leqslant x < 1$, C; $x \geqslant 1$, D.
7 D for all $x > 0$.
8 $0 \leqslant x < 1$, C; $x \geqslant 1$, D.
9 $0 \leqslant x < \frac{1}{3}$, C; $x \geqslant \frac{1}{3}$, D.
10 $0 \leqslant x < 1$, C; $x \geqslant 1$, D.
11 $0 \leqslant x < 1$, C; $x > 1$, D.
12 $\{nx^{n+1}-(n+1)x^n+1\}(x-1)^{-2}$ if $x \neq 1$; $\frac{1}{2}n(n+1)$ if $x = 1$.

Exercise 12(*g*), p. 450

2 $p > 1$, C; $p \leqslant 1$, D.
4 $\pi/2a$, $\pi/2a+1/a^2$.
5 e^{-1}.

Exercise 12(*h*), p. 459

1 C.
2 C.
3 C.
4 D.
5 C.
6 No. 2.
7 A.C. by comparison with $\Sigma(1/r^2)$.
9 $|x| \leqslant 1$, C; $|x| > 1$, D.
10 C for all x.
11 $|x| \leqslant 1$, C; $|x| > 1$, D.
12 $|x| < 1$, C; $|x| \geqslant 1$, D.
13 $|x| < 2$, C; $|x| > 2$, D; ($x = 2$, D; $x = -2$, C).

Exercise 12(*i*), p. 465

1 $\log 2$.
2 $\frac{1}{4}\pi$.
3 See 12.72(2).
4 $\sum_{r=1}^{\infty} 2^{\frac{1}{2}r}\sin(\frac{1}{4}r\pi)\frac{x^r}{r!}$, all x.
5 $\sum_{r=0}^{\infty} 2^{\frac{1}{2}r}\cos(\frac{3}{4}r\pi)\frac{x^r}{r!}$, all x.
6 $\sum_{r=0}^{\infty} 2^{\frac{1}{2}r-1}\{1+(-1)^r\}\cos(\frac{1}{4}r\pi)\frac{x^r}{r!}$, all x.
9 $\sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \ldots$, all x and h.
10 $\{4(n-1)^2+1\}\{4(n-2)^2+1\}\ldots\{2^2+1\}/(2n)!$.
12 $1-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{12}x^4+\frac{1}{3}x^5-\frac{19}{72}x^6+\ldots$.

Exercise 12(j), p. 468

1 $\frac{1}{2}(r+1)(r+2)(-x)^r$; $1-3x+6x^2-10x^3+\ldots$; $|x|<1$.

2 $\dfrac{-2.1.4\ldots(3r-5)}{r!}\left(\dfrac{x}{3}\right)^r$; $1-\frac{2}{3}x-\frac{1}{9}x^2-\frac{4}{81}x^3-\ldots$; $|x|<1$.

3 $\dfrac{1.3.5\ldots(2r-1)}{r!}(-\frac{3}{2}x)^r$, $r\geqslant 1$; $1-\frac{3}{2}x+\frac{27}{8}x^2-\frac{135}{16}x^3+\ldots$; $|x|<\frac{1}{3}$.

4 $\dfrac{1.3.5\ldots(2r-3)}{r!}(-1)^{r-1}\dfrac{x^{2r}}{2^{3r-1}}$, $r\geqslant 2$; $2+\frac{1}{4}x^2-\frac{1}{64}x^4+\frac{1}{512}x^6-\ldots$; $|x|<2$.

5 $(r+1)x^{r+1}$; $x+2x^2+3x^3+4x^4+\ldots$; $|x|<1$.

6 $(3/2^{r+1})(-x)^r$, $r\geqslant 1$; $\frac{1}{2}-\frac{3}{4}x+\frac{3}{8}x^2-\frac{3}{16}x^3+\ldots$; $|x|<2$.

7 $(r^2+r+1)x^r$; $1+3x+7x^2+13x^3+\ldots$; $|x|<1$.

8 $672x^5$. 9 $-10(3x)^{-11}$. 10 $-\frac{7}{72}\sqrt{3}$.

12 5th term. 13 27th and 28th terms.

14 1st term. 15 33·6241. 16 9·8995.

17 $1-\frac{1}{2}x-\frac{5}{8}x^2+\frac{19}{16}x^3+\ldots$. 18 $|x|<\frac{1}{2}$; $\frac{1}{3}\{5+(-2)^r\}$.

19 $|x|<2$; $\frac{1}{5}\{2^{-r}-(-3)^{-r}\}$. 20 $|x|<1$; $2^{1-r}-r-4$.

21 $|x|<1$; $+1, -1, 0$ according as $r=3p, 3p+1, 3p+2$ $(p=0,1,2,\ldots)$.

22 $|x|<1$; $\dfrac{(n+r-1)!}{(n-1)!\,r!}$. 23 Each is $\dfrac{1.3.5\ldots(2r-1)}{2.4.6\ldots(2r)}$, $r\geqslant 1$.

24 $\frac{2}{3}\sqrt{3}$. 25 $\sqrt[3]{\frac{4}{7}}$. 26 $10\sqrt[3]{\frac{10}{7}}-11$.

27 1. 28 $\dfrac{1+4x-3x^2}{(1-x)^3}$.

Exercise 12(k), p. 472

1 $\dfrac{3^r}{r!}e^2$. 2 $(-1)^{r-1}\dfrac{r-1}{r!}$. 3 $(-1)^r\dfrac{r^2-4r+1}{r!}$.

4 $\dfrac{1+(-1)^r}{r!}$.

5 (i) $1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+\ldots$; (ii) $1+\dfrac{2}{1!}+\dfrac{2^2}{2!}+\dfrac{2^3}{3!}+\ldots$;

(iii) $4+\dfrac{2^3}{2!}+\dfrac{2^5}{4!}+\dfrac{2^7}{6!}+\ldots$.

6 $1-e^{-3}$. 7 $\frac{1}{8}(e^4-e^{-4})$. 8 $2e$. 9 $\frac{3}{2}e$.

10 $2\sqrt{e}$. 11 $(e^x-1-x)/x$ if $x\neq 0$; 0 if $x=0$.

12 $\{(x-1)e^x+1-\frac{1}{2}x^2\}/x^2$ if $x\neq 0$; 0 if $x=0$.

Exercise 12(l), p. 475

1 $-\sum_{r=1}^{\infty}\frac{4^r}{r}x^r,\ -\frac{1}{4}\leqslant x<\frac{1}{4}$.

2 $\log 2-\sum_{r=1}^{\infty}\frac{x^r}{r}(-\frac{1}{2})^r,\ -2<x\leqslant 2$.

3 $-2\sum_{r=1}^{\infty}\frac{(-1)^r}{r}x^r,\ -1<x\leqslant 1$.

4 $\sum_{r=1}^{\infty}(-1)^{r-1}\frac{1}{r}(2^r+3^r)x^r,\ -\frac{1}{3}<x\leqslant\frac{1}{3}$.

5 $x+\sum_{r=2}^{\infty}\frac{2r-1}{r(1-r)}(-1)^r x^r,\ -1\leqslant x<1$.

6 $\frac{1}{r}$ if $r=3p$, $\frac{2}{r}$ if $r=3p\pm 1$; $-1\leqslant x<1$.

7 $-\frac{1}{2r-1}$ for x^{2r-1}, $-\frac{1+2(-1)^r}{2r}$ for x^{2r}; $|x|<1$.

8 $2x+x^2-\frac{10}{3}x^3+\frac{7}{2}x^4$.

9 $\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{rx^r}$.

14 $\log\frac{3}{2}$.

15 $\frac{1}{2}\log 2$.

16 $-\frac{1}{2}\log(1-x^2)$.

17 $\log 2$.

18 $\log 4-1$.

19 $\log 2-\frac{1}{2}$.

20 $\frac{1}{4}\log 2$.

21 $\frac{1}{4}\left(x-\frac{1}{x}\right)\log\frac{1+x}{1-x}+\frac{1}{2}$ if $x\neq 0$; 0 if $x=0$.

22 $-\left(1+\frac{1}{x^2}\right)\log(1-x)-\frac{1}{2}-\frac{1}{x}$ if $x\neq 0$; 0 if $x=0$.

23 $\log\frac{225}{64}$.

24 $x^2>1$.

25 Each is $-2\log\cos\alpha$.

Exercise 12(m), p. 481

5 $1{\cdot}4142136$.

8 $1+x^2+\frac{4}{3}x^3$.

12 $\frac{4}{9}+\frac{7}{3}x$.

13 $1+\frac{1}{2}x+\frac{7}{8}x^2$.

14 $0{\cdot}0037$.

15 $0{\cdot}393$.

17 $1+\frac{1}{2}p-\frac{1}{16}p^2$.

18 $\frac{1}{2}$.

19 $\log a$.

Miscellaneous Exercise 12(n), p. 482

3 $(2n)!/(n!)^2$.

5 (i) $\frac{1}{2}(r-1)r$; (ii) $(r-1)r+1,\ r(r+1)-1$; (iii) r^3; (iv) m^3.

6 $2-\frac{2}{n+1}$; 2.

7 $\frac{5}{12}+\frac{3}{2(n+2)}-\frac{7}{2(n+3)}$; $\frac{5}{12}$.

8 $\frac{1}{4}\pi$.

9 $\frac{1}{2}(n^2+n+2)$.

10 $-\pi$.

12 (i) $\log\sin\theta-\log\sin\frac{\theta}{2^n}-n\log 2$; (ii) $\frac{1}{2^n}\cot\frac{\theta}{2^n}-\cot\theta$; (iii) $\theta^{-1}-\cot\theta$.

19 (i) and (ii) 15, 18.

20 27.

21 $|x|<$ the smaller of $1/|\alpha|$, $1/|\beta|$.

22 $x=\frac{1}{10}$, $y=-\frac{9}{10}$, $z=\frac{9}{2}$.

23 2.

24 $1+\log\frac{4}{3}$.

25 e^{-1}.

26 0.

27 $e+2$.

28 $\frac{5}{3}$.

29 $3-\log 16$.

30 $\frac{1}{36}$.

31 $(1+3x+x^2)e^x$.

32 $\dfrac{3}{4}-\dfrac{1}{2x}-\dfrac{(1-x)^2}{2x^2}\log(1-x)$ if $-1 \leqslant x < 1$; $\frac{1}{4}$ if $x = 1$.

33 (i) $\frac{1}{2}(\text{ch}\,x+\cos x)$; (ii) $\frac{1}{2}(\text{sh}\,x-\sin x)$. 34 1·609.

Exercise 13(*a*), p. 493

1 $5+11i$. 2 $-1+3i$. 3 $7+3i$.

4 $7+3i$. 5 $-9+19i$. 6 $41+0i$.

7 $\frac{21}{13}+\frac{1}{13}i$. 8 $\frac{28}{13}+\frac{3}{13}i$. 9 $2i, -4, -8i$.

10 -10. 11 $\cos(\theta+\phi)+i\sin(\theta+\phi)$.

12 $\cos(\theta-\phi)+i\sin(\theta-\phi)$.

14 $X = \dfrac{x^2+y^2-1}{(x-1)^2+y^2}$, $Y = -\dfrac{2y}{(x-1)^2+y^2}$.

15 $0, \pm\left(\dfrac{1}{LC}\dfrac{L-CR_1^2}{L-CR_2^2}\right)^{\frac{1}{2}}$. 16 $2+i$.

17 $(x^3-3xy^2)+i(3x^2y-y^3)$; $(x^3-3xy^2)-i(3x^2y-y^3)$.

26 $\{ad-bc, bd\}$, $\{ad, bc\}$.

Exercise 13(*b*), p. 498

1 $2(\cos\frac{1}{3}\pi+i\sin\frac{1}{3}\pi)$. 2 $2\left(\cos(-\frac{1}{3}\pi)+i\sin(-\frac{1}{3}\pi)\right)$.

3 $\sqrt{2}(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi)$. 4 $\sqrt{2}\left(\cos(-\frac{1}{4}\pi)+i\sin(-\frac{1}{4}\pi)\right)$.

5 $2(\cos\frac{1}{6}\pi+i\sin\frac{1}{6}\pi)$. 6 $\cos\frac{1}{3}\pi+i\sin\frac{1}{3}\pi$.

7 $\cos\frac{1}{2}\pi+i\sin\frac{1}{2}\pi$.

8 $2\cos\frac{1}{2}\theta(\cos\frac{1}{2}\theta+i\sin\frac{1}{2}\theta)$ or $-2\cos\frac{1}{2}\theta\{\cos(\frac{1}{2}\theta+\pi)+i\sin(\frac{1}{2}\theta+\pi)\}$.

9 $2\cos\frac{1}{2}(\alpha-\beta)\{\cos\frac{1}{2}(\alpha+\beta)+i\sin\frac{1}{2}(\alpha+\beta)\}$ or
$-2\cos\frac{1}{2}(\alpha-\beta)\{\cos\frac{1}{2}(\alpha+\beta+2\pi)+i\sin\frac{1}{2}(\alpha+\beta+2\pi)\}$.

10 $(x^2+y^2)^n$. 13 $(u-1)^2+v^2 = 9$.

16 $(x+y)(x+\omega y)(x+\omega^2 y)$; $(x+\omega y)(x+\omega^2 y)$.

Exercise 13(*c*), p. 506

2 $2z_3+z_1 = 3z_2$. 3 $(lz_1+kz_2)/(l+k)$.

4 The region between the circles of centre $(-1, 2)$ and radii 2, 3.

5 (i) 5, 3; (ii) 9, 0. 10 Apply no. 9 after the product construction.

13 The point corresponding to the least value of $|z|$ is the foot of the perpendicular from O to the line.

15 $\sqrt{3}$. 17 $z(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi)$, $z(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi)$.

18 $r\{\cos(\alpha+\frac{2}{5}k\pi)+i\sin(\alpha+\frac{2}{5}k\pi)\}$, $k = 0, 1, 2, 3, 4$.

20 Straight line through P_1 and perpendicular to P_2P_3.

Exercise 13(*d*), p. 514

2 $1 \pm i$, $1 \pm \sqrt{3}$.   3 -2, $2 \pm \sqrt{3}$.   4 ω, ω, ω^2, ω^2, $\frac{4}{3}$.

5 $\pm\sqrt{2} \pm \sqrt{3}$, ω, ω^2.   6 The root $z = -b/a$ is real, so $\beta = 0$.

7 (ii) The division process introduces coefficients of the same type as in divisor and dividend.

9 $pr - 4s$.   11 -3.   12 ± 4, 5.

13 -4, 2, ω, ω^2.   14 -4, -1, 2, 5.

15 $y^3 + 3(a-k)y^2 + 3(k^2 - 2ak + b)y - k^3 + 3ak^2 - 3bk + c$; a.

16 $k = 1$ or -3; $x = -2, -4, 1 \pm \sqrt{15}$.

17 $k = 2$; $y^4 + 2y^3 - 9y^2 - 16y - 6 = 0$.

18 (i) $y^3 - 2qy^2 + (q^2 + pr)y + (r^2 - pqr) = 0$;
(ii) $ry^3 - (q + 3r)y^2 + (p + 2q + 3r)y - (p + q + 1) = 0$.

20 $q^2(y+1)^3 + p^3(y+2) = 0$.

Exercise 13(*e*), p. 520

1 $\frac{2}{3}$.   2 $\frac{3}{2}$, -2.   3 None.

4 $(-2, -1)$, $(0, 1)$.   5 $(-1, 0)$, $(1, 2)$.   6 $(1, 2)$.

7 Roots in (a_1, a_2), (a_3, a_4), (a_5, a_6).   11 $(1, 2)$, $(-6, -5)$.

12 $(2, 3)$.   14 One root.   15 $8 < k < 11$.

16 $-47 < k < 98$.

Exercise 13(*f*), p. 524

1 2·0946.   2 0·2261.   3 -2·0945.   4 4·25.

5 1·52.   6 0·3399, 2·2618, -2·6018.

7 1·3569, 1·6920, -3·0489.   8 0·567.   9 0·057, 1·468.

10 1·3503.   11 4·4934.   12 1·87.

14 $K + 1/2K^2 - 1/4K^4$.

15 Newton's method fails; but put $x = 2n\pi + h$ and expand.

Miscellaneous Exercise 13(*g*), p 525

2 $s + ic$.   4 $r^2 + \rho^2 - 2r\rho\cos(\theta - \alpha)$.

7 (ii) $\frac{1}{2}(11 - i)$, $-\frac{1}{2}(3 + 5i)$.

8 (i) Point dividing P_1P_2 in the ratio $k : 1-k$;
(ii) $(\sqrt{3} - 1)(1 - i)$, $\frac{1}{3}(2\sqrt{3} - 1) - \frac{1}{3}(\sqrt{3} - 2)i$.

10 (i) Straight line $y = 1$; (ii) circle $x^2 + y^2 = 4k^2$; (iii) arc of a circle through P_1, P_2 and containing angle $\frac{1}{3}\pi$; (iv) branch of a hyperbola which encloses the focus P_2 (P_1 is the other focus).

11 (ii) $x^2 - y^2 = 2$.

12 $x^2+y^2-(a/u)\,x = 0,\ x^2+y^2+(a/v)\,y = 0$ (if $u \neq 0,\ v \neq 0$).

14 If P describes $|z| = 1$ counterclockwise, Q moves from $-\infty$ to $+\infty$ along Oy.

16 $y = 0$ or $(c-a)(x^2+y^2)+2(d-b)\,x+(ad-bc) = 0$.

18 (i) $3\sum_{p=1}^{n} a_{3p}x^{3p}$; (ii) $3\sum_{p=1}^{n} a_{3p-2}x^{3p-2}$, $3\sum_{p=1}^{n} a_{3p-2}x^{3p-2}$.

20 $b^2c-abd+d^2 = 0$; *or* $b = 0 = d$ and $a^2 \geqslant 4c$.

22 $x^5-3x^4-6x^3+14x^2-7x+1 = 0$.

24 $s > p^4$.

25 $b = 1-\frac{1}{2}+\frac{1}{3}-\ldots-1/n$.

26 $-\frac{3}{2},\ -\frac{2}{3},\ \frac{1}{2},\ 2$.

27 $1,\ \frac{1}{2}(3\pm\sqrt{5}),\ \frac{1}{2}(1\pm i\sqrt{3})$.

28 $-1,\ \frac{1}{2}(1\pm i\sqrt{3}),\ \frac{1}{4}(-1\pm i\sqrt{15})$.

29 $(n+\frac{1}{2})\,\pi+4(-1)^{n-1}/(2n+1)\,\pi$.

Exercise 14(*a*), p. 532

2 $(2\cos\frac{1}{2}\theta)^5 \operatorname{cis} 7\theta$.

5 $32, -\frac{2}{3}\pi$.

6 $\sqrt{2}\operatorname{cis}\frac{1}{4}\pi,\ \sqrt{2}\operatorname{cis}\frac{11}{12}\pi,\ \sqrt{2}\operatorname{cis}\frac{19}{12}\pi$.

7 $\operatorname{cis}\left(\dfrac{8r+1}{16}\pi\right),\ r = 0, 1, 2, 3$.

8 $\operatorname{cis}(\frac{1}{2}\theta+\frac{1}{3}k\pi),\ k = 0, 1, \ldots, 5;\ \operatorname{cis}(\frac{1}{2}\theta+k\pi),\ k = 0, 1$.

9 $\cos 3\theta+i\sin 3\theta = \cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$;

$$\tan 3\theta = (3\tan\theta-\tan^3\theta)/(1-3\tan^2\theta).$$

10 (i) $\cos\theta-i\sin\theta$; (ii) $2\cos\theta$; (iii) $2i\sin\theta$; (iv) $2\cos n\theta$; (v) $2i\sin n\theta$.

11 $\operatorname{cis}\left(\dfrac{2k+1}{n}\pi\right),\ k = 0, 1, \ldots, n-1$.

12 $-i\cot\left(\dfrac{2k+1}{2n}\pi\right),\ k = 0, 1, \ldots, n-1$.

13 $\dfrac{1}{2}\left(1+i\cot\dfrac{k\pi}{n}\right),\ k = 1, 2, \ldots, n-1$.

14 $i\tan\dfrac{k\pi}{n}$, where $k = 0, 1, \ldots, n-1$ if n is odd, and the value $k = \frac{1}{2}n$ is excluded if n is even.

15 $\operatorname{cis}\left(\dfrac{2k\pi}{n}\pm\alpha\right),\ k = 0, 1, \ldots, n-1$.

18 $z^2+z+2 = 0$. The $+$ sign is correct since

$$\sin\tfrac{2}{7}\pi+\sin\tfrac{8}{7}\pi = \sin\tfrac{2}{7}\pi-\sin\tfrac{1}{7}\pi > 0 \quad\text{and}\quad \sin\tfrac{4}{7}\pi > 0.$$

20 (i) $|z| =$ constant; (ii) $\arg z$ is a rational multiple of π.

Exercise 14(*b*), p. 538

1 $\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$.

2 $\frac{1}{8}(\cos 4\theta-4\cos 2\theta+3)$.

3 $\frac{1}{16}(\sin 5\theta-5\sin 3\theta+10\sin\theta)$.

4 $\frac{1}{256}(\sin 9\theta-\sin 7\theta-4\sin 5\theta+4\sin 3\theta+6\sin\theta)$.

5 $-\frac{1}{128}(\sin 8\theta+2\sin 6\theta-2\sin 4\theta+6\sin 2\theta)$.

6 $\frac{1}{64}(\frac{1}{7}\cos 7\theta-\frac{7}{5}\cos 5\theta+7\cos 3\theta-35\cos\theta)+c$.

7 $-\frac{1}{2^9}(\frac{1}{10}\sin 10\theta-\frac{1}{4}\sin 8\theta-\frac{1}{2}\sin 6\theta+2\sin 4\theta+\sin 2\theta-6\theta)+c.$

8 $\frac{1}{256}(\pi+\frac{63}{80}\sqrt{3}).$

9 (i) $32c^6-48c^4+18c^2-1$; (ii) $1-18s^2+48s^4-32s^6$.

10 $7s-56s^3+112s^5-64s^7$.

11 $1-24s^2+80s^4-64s^6$.

12 $\theta=2n\pi\pm\frac{1}{3}\pi$.

13 $\theta=n\pi$ or $(-1)^n\frac{1}{6}\pi+n\pi$.

16 (i) (a) $(-1)^{\frac{1}{2}n}s^n,\ (-1)^{\frac{1}{2}n-1}ncs^{n-1}$; (b) $(-1)^{\frac{1}{2}(n-1)}ncs^{n-1},\ (-1)^{\frac{1}{2}(n-1)}s^n$.
(ii) (a) $(-1)^{\frac{1}{2}n-1}nt^{n-1},\ (-1)^{\frac{1}{2}n}t^n$; (b) $(-1)^{\frac{1}{2}(n-1)}t^n,\ (-1)^{\frac{1}{2}(n-1)}nt^{n-1}$.

17 See answer to Ex. 14(*a*), no. 9; $\tan\frac{1}{12}\pi$, $\tan\frac{5}{12}\pi$.

18 (i) $\Sigma t_1=\Sigma t_2t_3t_4$;
(ii) $(c-a)/(1-b+d)$; $\Sigma\theta_1$ is zero or an integral multiple of π.

20 (v) (a) $$2^{n-1}c^n-n.2^{n-3}c^{n-2}+\frac{n(n-3)}{2!}2^{n-5}c^{n-4}-\frac{n(n-4)(n-5)}{3!}2^{n-7}c^{n-6}+\dots;$$

(b) $$(-1)^{\frac{1}{2}n}\left\{1-\frac{n^2}{2!}c^2+\frac{n^2(n^2-2^2)}{4!}c^4-\frac{n^2(n^2-2^2)(n^2-4^2)}{6!}c^6+\dots\right\},$$
$$(-1)^{\frac{1}{2}(n-1)}\left\{nc-\frac{n(n^2-1^2)}{3!}c^3+\frac{n(n^2-1^2)(n^2-3^2)}{5!}c^5-\dots\right\}.$$

Exercise 14(*c*), p. 544

1 $(x^2-2x\cos\frac{1}{9}\pi+1)(x^2-2x\cos\frac{5}{9}\pi+1)(x^2-2x\cos\frac{7}{9}\pi+1)$.

2 $(x^2-2\sqrt{2}x\cos\frac{1}{12}\pi+2)(x^2+2\sqrt{2}x\cos\frac{1}{12}\pi+2)(x^2-2\sqrt{2}x\cos\frac{5}{12}\pi+2)$
$\times(x^2+2\sqrt{2}x\cos\frac{5}{12}\pi+2)$.

3 $(x^2-2x\cos\frac{2}{7}\pi+1)(x^2-2x\cos\frac{4}{7}\pi+1)(x^2-2x\cos\frac{6}{7}\pi+1)$.

4 $x=1,\ \alpha=2\beta+\pi/n$.

5 $x=-1,\ \alpha=2\beta$.

6 $x=-1,\ \alpha=2\beta+\pi/n$.

7 $\mp\tan n\beta=\sum_{r=0}^{n-1}\cot\left\{\beta+\frac{(2r+1)\pi}{2n}\right\}$.

8 Derive wo β the result of 14.33, ex. (iv).

10 $\sum_{r=0}^{n-1}\left\{\frac{1}{n}\operatorname{cosec} n\alpha\sin\left(\alpha+\frac{2r\pi}{n}\right)\right\}\Big/\left\{x^2-2x\cos\left(\alpha+\frac{2r\pi}{n}\right)+1\right\}$.

13 (i) $1=2^{n-\frac{1}{2}}\prod_{r=0}^{n-1}\sin\frac{2r+1}{4n}\pi$.

14 (ii) $\frac{\cot\theta-n\cot n\theta}{\sin\theta}=\sum_{r=1}^{n}\frac{1}{\cos\theta-\cos(r\pi/n)}$.

16 Equate coefficients of x^{2n-2}.

Exercise 14(*d*), p. 547

1 $8x^3-6x-\sqrt{3}=0$. 2 $16x^4-20x^2+5=0$.

3 $4x^2+2x-1=0$; $\frac{1}{4}(\sqrt{5}\pm 1)$. 5 (i) $y^6-7y^4+14y^2-1=0$.

9 $\frac{7}{4}$.

Exercise 14(*e*), p. 550

1 $-\sin n\beta \sec\frac{1}{2}\beta\cos\{\alpha+(n-\frac{1}{2})\beta\}$. 3 $2^n\cos^n\theta\sin n\theta$.

4 $\cos n\theta$. 5 $\{2nx^n(x-1)+(x+1)(1-x^n)\}/(x-1)^2$.

6 $\left\{\sin A-\left(\frac{b}{c}\right)^n\sin(n+1)A+\left(\frac{b}{c}\right)^{n+1}\sin nA\right\}\Big/\left(1-2\frac{b}{c}\cos A+\frac{b^2}{c^2}\right)$.

7 $\operatorname{cosec}\theta\cos(\alpha-\theta)$.

Exercise 14(*f*), p. 556

1 -1. 2 $\frac{1}{2}e(1+i\sqrt{3})$. 3 $r\operatorname{cis}\theta$. 4 $2\cos 1$.

5 $2e^{\cos\theta}\cos(\sin\theta)$. 6 $2ie^{ax}\sin bx$. 7 $\sum_{r=1}^{\infty}\frac{\sin r\alpha}{r!}x^r$. all x.

8 $e^{x\cos\theta}\sin(x\sin\theta)$, all x, θ. 9 $e^{-\cos^2\beta}\cos(\alpha-\sin\beta\cos\beta)$, all α, β.

10 $e^x\sin(x\tan\beta)$, all x. 11 $\cos(\sin\theta)\operatorname{sh}(\cos\theta)$, all θ.

12 $\frac{1}{2}e-\frac{1}{2}e^{\cos 2\theta}\cos(\sin 2\theta)$. 13 Line $v=0$; line $v=u$.

14 (i) $u^2+v^2=e^{2x}$; (ii) $u\sin y=v\cos y$; circle and a diameter.

Exercise 14(*g*), p. 560

4 $\operatorname{ch}x\cos y+i\operatorname{sh}x\sin y$.

5 $2(\cos x\operatorname{ch}y+i\sin x\operatorname{sh}y)/(\cos 2x+\operatorname{ch}2y)$.

6 $(\operatorname{sh}2x+i\sin 2y)/(\operatorname{ch}2x+\cos 2y)$.

8 (i) $x+n\pi i$; (ii) $x+0i$ or $n\pi+yi$. 9 $(n+\frac{1}{2})\pi+\frac{1}{2}i\log 2$.

10 $(2n\pi\pm\frac{1}{4}\pi)\pm i$, both + or both − signs being taken.

12 (i) Hyperbola, (ii) ellipse, in the (u,v)-plane.

13 $\frac{z^3}{3!}+\frac{z^7}{7!}+\frac{z^{11}}{11!}+\ldots$, all z. 14 $\operatorname{sh}(\cos\theta)\sin(\sin\theta)$, all θ.

15 $\frac{1}{2}\{\operatorname{sh}(\cos\theta)\cos(\sin\theta)+\sin(\cos\theta)\operatorname{ch}(\sin\theta)\}$, all θ.

Miscellaneous Exercise 14(*h*), p. 561

4 (i) $\operatorname{cis}\frac{1}{3}k\pi$, $k=1,2,\ldots,5$; (ii) ± 1, $\operatorname{cis}\frac{1}{3}\pi$, $\operatorname{cis}(-\frac{1}{3}\pi)$.

5 $-i\cot\frac{2r+1}{4n}\pi$, $r=0,1,\ldots,2n-1$.

8 (i) $\frac{a^2-1}{a(a-b)}\frac{1}{1-at}-\frac{b^2-1}{b(a-b)}\frac{1}{1-bt}-\frac{1}{ab}$; (ii) $\frac{2}{a^2}\frac{1}{1-at}+\frac{a^2-1}{a^2}\frac{1}{(1-at)^2}-\frac{1}{a^2}$.
Take $t=\tan\frac{1}{2}\phi$.

11 $8c^6-9c^3+1=0$; $\theta=0, \frac{1}{3}\pi, \frac{5}{3}\pi, 2\pi$.

16 $-\frac{1}{2}, -\frac{1}{2}(1+i\cot\frac{1}{8}r\pi)$, $r=\pm1, \pm2, \pm3$. Put $z=-\sin^2\theta$.

17 $z-a\operatorname{cis}\frac{1}{3}r\pi$, $r=0,1,\ldots,5$.

21 $u=\{(x-1)\cos\alpha+y\sin\alpha\}\left\{\dfrac{1}{(x-1)^2+y^2}+1\right\}$,

$$v=\{(x-1)\sin\alpha-y\cos\alpha\}\left\{\dfrac{1}{(x-1)^2+y^2}-1\right\}.$$

23 $\cos u=\cos\frac{1}{2}\theta-\sin\frac{1}{2}\theta$, $\operatorname{ch} v=\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta$.

25 Denoting the vertices $+a$, $+a+ia$, $-a+ia$, $-a$ by A, B, C, D respectively, then as z moves from O to A, w moves along the u-axis from O to $(a^2,0)$; as z moves from A to B, w moves along the parabolic arc $v^2=4a^2(a^2-u)$ from $(a^2,0)$ to $(0,2a^2)$; from B to C, w moves along the parabolic arc $v^2=4a^2(a^2+u)$ from $(0,2a^2)$ through $(-a^2,0)$ to $(0,-2a^2)$; from C to D, w moves along arc $v^2=4a^2(a^2-u)$ from $(0,-2a^2)$ to $(a^2,0)$; from D to O, w moves along the u-axis from $(a^2,0)$ to O.

27 $\frac{1}{2}e-1+\frac{1}{2}e^{\cos 2\alpha}\cos(\sin 2\alpha)$, all α. 28 $\operatorname{ch}(x\cos\theta)\sin(x\sin\theta)$, all x, θ.

29 $\sum\limits_{n=0}^{\infty}\dfrac{x^n r^n}{n!}\cos n\theta$, where $r=+\sqrt{(a^2+b^2)}$ and $\cos\theta:\sin\theta:1=a:b:r$.

30 (ii) (*a*) $\dfrac{x^2}{2!}+\dfrac{x^5}{5!}+\dfrac{x^8}{8!}+\ldots=\frac{1}{3}e^x+\frac{2}{3}e^{-\frac{1}{2}x}\cos(\frac{1}{2}x\sqrt{3}+\frac{2}{3}\pi)$,

(*b*) $x+\dfrac{x^4}{4!}+\dfrac{x^7}{7!}+\ldots=\frac{1}{3}e^x+\frac{2}{3}e^{-\frac{1}{2}x}\cos(\frac{1}{2}x\sqrt{3}-\frac{2}{3}\pi)$, for all real x.

31 $z=e^{-ikt}(Ce^{i\mu t}+De^{-i\mu t})$, where $\mu^2=n^2+k^2$ and C, D are arbitrary complex constants.

33 $x'=x\cos\theta+y\sin\theta$, $y'=-x\sin\theta+y\cos\theta$.

Exercise 15(*a*), p. 575

6 $x+y=2c\operatorname{cosec}\omega$. 7 $x+y=c\sec^2\frac{1}{2}\omega$.

8 $x^2+2xy\cos\omega+y^2=4c^2\operatorname{cosec}^2\omega$. 9 Straight line.

11 $h/x+k/y=2$. 12 Circle with centre O.

Exercise 15(*b*), p. 578

1 $x-2y=0$. 2 $3x-5y+9=0$. 3 $10x-9y=5$.

4 $7(x-y)=6$, $7(y-x)=16$, $11x+5y=0$. 5 (ii) $(-1,1)$.

6 $2x-y-1=0$. 7 Yes. 8 No.

9 Yes. 10 Yes. 11 $-\frac{7}{9}$.

13 $a+b+c=0$. 14 $\Sigma\sin\alpha=0$.

Exercise 15(*c*), p. 585

1 $\sqrt{2}$.

4 $bx^2 - 2hxy + ay^2 = 0$.

5 (i) $\tan^{-1}(2\sqrt{21})$; (ii) $11x^2 - 26xy - 11y^2 = 0$.

7 $pq(x^2+y^2) = (p^2+q^2)xy$.

8 (i) $\frac{1}{4}\pi$, $(-1, 4)$; (ii) $\tan^{-1}\frac{11}{13}$, $(-\frac{1}{11}, -\frac{3}{11})$; (iii) $\frac{1}{2}\pi$, $(-2, 3)$; (iv) $\tan^{-1}(2\sqrt{6})$, $(1, -2)$.

9 (i) $3x^2 - 7xy + 2y^2 = 0$; (ii) $2x^2 + xy - 15y^2 = 0$; (iii) $xy = 0$; (iv) $2x^2 + 4xy - y^2 = 0$.

10 $3x^2 + 40xy - 16y^2 = 0$.

12 $2(g-g')x + 2(f-f')y + (c-c') = 0$.

Exercise 15(*d*), p. 592

1 $x^2 + y^2 - 2px - 2qy = 0$.

4 (i) $(x_1+x_2)x + (y_1+y_2)y = x_1x_2 + y_1y_2 - a^2$, $xx_1 + yy_1 = a^2$.

5 $(a^2l/n, a^2m/n)$.

6 $c^2 = a^2(1+m^2)$.

7 (i) $x_1^2 + y_1^2 = 2a^2$; (ii) $x^2 + y^2 = 2a^2$.

8 (i) Transverse common tangents divide the line of centres *internally* in the ratio of the radii.

(ii) $5x \pm 12y = 29$, $11x \pm 4\sqrt{3}y = 61$.

9 $\{(r_1+r_2)x - (a_1r_2 + a_2r_1)\}^2 = \{(a_1-a_2)^2 - (r_1+r_2)^2\}y^2$.

13 $x^2 + y^2 - 3x - 5y + 1 = 0$.

14 $x^2 + y^2 + 2x - 8y + 5 = 0$.

16 Straight line.

17 $s - s' = 0$ is a diameter of σ.

20 $x^2 + y^2 + x - 4y - 2 = 0$.

21 $x^2 + y^2 + x + y - 2 = 0$.

Exercise 15(*e*), p. 600

1 $-2, 0$.

2 $x^2 - xy - 6y^2 = 1$.

3 $xy = -\frac{1}{2}a^2$.

4 New equation is $x = p$.

5 $\frac{1}{2}\tan^{-1}\frac{1}{2}$.

6 $x^2 + 3y^2 = 1$.

Miscellaneous Exercise 15(*f*), p. 601

2 $t_1t_2t_3t_4 = 1$.

7 $3(a+b)^2 = 4(h^2 - ab)$.

8 $(gh - af)x = (hf - bg)y$, $2gx + 2fy + c = 0$.

11 $(a, 3a)$; $\frac{3}{5}a$.

12 $x^2 + y^2 - 5x + 5y = 0$.

13 $(a + r\cos\theta, a\sin\theta)$.

15 (i) $(x-x_1)(x-x_2) = 0$; (ii) $(y-y_1)(y-y_2) = 0$.

17 $3x^2 - 8xy - 3y^2 = 0$; $x^2 + y^2 - 2x - 4y = 0$.

19 $(1+2gl+cl^2)x^2+2(fl+gm+clm)xy+(1+2fm+cm^2)y^2 = 0$;
$$\left(\frac{b(b-a)-2mh}{2(am^2-2hlm+bl^2)}, \frac{m(a-b)-2lh}{2(am^2-2hlm+bl^2)}\right); \; mx+ly = 0.$$

20 $(1-h)(x^2+y^2) = h(x+y)$; $(a+k)(x^2+y^2)+k(x-y) = 0$.

21 $b(x^3+y^3) = 3axy(x-y)$.

Exercise 16(*a*), p. 605

1 $2/(t_1+t_2)$. 2 $a(t+1/t)^2$. 10 Focus A, directrix l.

Exercise 16(*b*), p. 610

5 $y = x+a$, $x+2y+4a = 0$. 10 The directrix.

11 Each is equivalent to $t_1 t_2 = -1$. 12 $y^2 = 4a(x+a)$.

15 2:1. 17 (i) Equally inclined to the axis.

19 $xa^{\frac{1}{3}}+yb^{\frac{1}{3}}+(a^2b^2)^{\frac{1}{3}} = 0$.

20 $\frac{7}{6}y = 2(x+\frac{1}{3})$; $2y = 4x+1$, $6y = 9x+4$.

25 The directrix.

Exercise 16(*c*), p. 614

3 $y^2 = a(x-3a)$. 4 $s = -t-2/t$. 6 $(-2a, 0)$.

7 $x-y = 3a$, $3x-y = 33a$, $4x+y = 72a$.

13 $x+4a = 0$, $y^2 = 16a(x-6a)$, $y^2 = 2a(x-4a)$.

Exercise 16(*d*), p. 616

1 $y = 3x$. 2 $-\frac{2}{3}$.

Miscellaneous Exercise 16(*e*), p. 616

1 Mid-point of OC, where O is the point of contact and C is the centre; line perpendicular to OC at distance $\frac{1}{2}OC$ from O on the side remote from C.

2 $x^2+y^2-(p+q)x-\dfrac{r^2+pq}{r}y+pq = 0$.

4 $\left(-\dfrac{b}{2a}, \dfrac{4ac-b^2}{4a}\right)$; $\dfrac{1}{|a|}$. 8 $\frac{1}{3}$.

10 $x = 2a$ if the normals meet on the curve at a point distinct from P and Q; the locus stated if the normal at one passes through the other.

12 $y^2 = a(x-3a)$. 20 $(2a-b)y^2 = 4a^2x$. 22 $yy_1-2a(x+x_1) = 0$.

23 $\{y_1(y-y_1)-2a(x-x_1)\}^2 = (y-y_1)^2(y_1^2-4ax_1)$, which can be reduced to the form in 16.26, ex. (v).

24 (ii) $k = 1$. 25 $x+2y-8 = 0$; $(4, 2)$.

Exercise 17(*a*), p. 629

1 $2b^2/a$. 2 $PN'^2 : B'N'.N'B = OA^2 : OB^2$.

5 Ellipse with semi-axes a, b. 8 $\frac{1}{2}a$, $\frac{1}{2}b$.

10 (i) $ay = bx \tan\alpha$; (ii) $x^2/a^2 + y^2/b^2 = \cos^2\alpha$. 15 (i) -3ϕ.

17 $(x/a)\cos\phi + (y/b)\sin\phi = 1$. 20 πab.

Exercise 17(*b*), p. 634

2 $\theta - \phi = \pm\pi$.

4 Tangents at corresponding points meet on the major axis.

11 $ON'.OT' = b^2$. 19 $\big(a(a^2-b^2)/(a^2+b^2), 0\big)$.

22 $\frac{1}{6}x + \frac{2}{3}y = 1$; $x + y - 3 = 0$, $x - 5y + 9 = 0$.

24 Tangents at the extremities of a focal chord meet on the corresponding directrix.

27 $\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$; $x^2 + y^2 = a^2 + b^2$.

Exercise 17(*c*), p. 637

1 $b^2.OG' = -a^2e^2.PN$.

4 $\dfrac{a^2x^2}{(2a^2-b^2)^2} + \dfrac{y^2}{b^2} = \dfrac{1}{4}$. 5 $(1, 2)$.

7 $a^2x^2 + b^2y^2 = a^4b^4/(a^2-b^2)^2$. NN' is the normal at the point

$\big(\lambda b\cos(\pi-\phi), \lambda a \sin(\pi - \phi)\big)$, where $\lambda = ab/(a^2-b^2)$.

Exercise 17(*d*), p. 641

1 $5x + 3y = 0$. 2 $-\frac{1}{4}$. 4 $\sqrt{\{\frac{1}{2}(a^2+b^2)\}}$.

10 (ii) $a^2l^2 + b^2m^2 = 2n^2$; (ii) $x^2/a^2 + y^2/b^2 = \frac{1}{2}$.

Miscellaneous Exercise 17(*e*), p. 642

1 $x^2/a^2 + y^2/b^2 = 1$.

8 A hyperbola having focus S and the given circle for auxiliary circle; p passes through the point on the circle diametrically opposite to S.

9 $4y = \pm x \pm 5\sqrt{17}$. 12 (i) $x^2 + y^2 = a^2 + b^2$; (ii) $ky^2 = b^2(k - x)$.

13 If T is (f, g),

$(g^2 - b^2)x^2 - 2fgxy + (f^2 - a^2)y^2 = 0$, $\{gx - y(f - ae)\}\{gx - y(f + ae)\} = 0$.

14 $x^2/a^2 + y^2/b^2 = k^2$.

20 $b^4(h^2-a^2)x^2+2a^2b^2hkxy+a^4(k^2-b^2)y^2=0$.

(i) $\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}=\dfrac{1}{a^2}+\dfrac{1}{b^2}$; (ii) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2$. $(\pm a, \pm b)$.

23 $(-2,1)$; $2, \sqrt{3}$; $\frac{1}{2}$; $(-1,1)$, $(-3,1)$; $x=2$, $x=-6$.

24 $(1,\frac{1}{3})$; $3, 2\sqrt{2}$; $\frac{1}{3}$; $(2,\frac{1}{3})$, $(0,\frac{1}{3})$; $x=10$, $x=-8$.

25 $(0,1)$; $2, \sqrt{3}$; $\frac{1}{2}$; $(\frac{1}{2},\frac{1}{2})$, $(-\frac{1}{2},\frac{3}{2})$; $x-y-3=0$, $x-y+5=0$.

Exercise 18(*a*), p. 651

2 A branch of a hyperbola with foci A, B: if $AB > nv$, the branch enclosing B; if $AB < nv$, the branch enclosing A. If $AB = nv$, the perpendicular bisector of AB.

3 That branch of a hyperbola with foci A, B which contains the centre of the *smaller* circle.

4 (i) Referring to fig. 182: $-\infty < t < -1$, $A'Q'$; $-1 < t < 0$, PA; $0 < t < 1$, AQ; $1 < t < +\infty$, $P'A'$. (ii) (*a*) $(1-t_1^2)(1-t_2^2) > 0$; (*b*) $t_1t_2 = 1$.

13 $(1+t_1t_2)x/a+(1-t_1t_2)y/b = t_1+t_2$.

14 $(1+t^2)x/a+(1-t^2)y/b = 2t$.

15 $(1-t^2)ax-(1+t^2)by = (a^2+b^2)(1-t^4)/2t$.

Exercise 18(*b*), p. 656

12 $3x^2-7xy+2y^2+5x-5y+3=0$.

13 $2x+y-3=0$, $x-3y+2=0$; $2x^2-5xy-3y^2+x+11y-4=0$.

Exercise 18(*c*), p. 659

4 Write the equation as $xy+yx = 2c^2$.

31 $(-d/c, a/c)$.

33 $x=\frac{1}{2}p$, $y=\frac{1}{2}q$.

Miscellaneous Exercise 18(*d*), p. 661

3 $x^2/a^2-y^2/b^2 = 1$ $(a > b)$.

5 Branch of a hyperbola which contains the centre of the smaller circle, if both contacts are external or both internal. Ellipse if one contact is external and the other internal.

6 Foci A, B in either case.

10 $(u+w)x/a+(u-w)y/b+v=0$.

12 $(x_1, -y_1)$.

14 See 18.53, example.

18 (i) (*a*) Repeated line $y=0$; (*b*) repeated line $x=0$; (ii) $\left(\pm\sqrt{(a^2-b^2)}, 0\right)$.

20 (i) $pa^2 = qb^2$; (ii) $r=0$.

21 (i) $(a^2\sin^2\theta-b^2\cos^2\theta)r^2+2(a^2y_1\sin\theta-b^2x_1\cos\theta)r$
$+(a^2y_1^2-b^2x_1^2+a^2b^2k)=0$;

(iii) OK, OK' are the semi-diameters drawn in the given directions to the hyperbola or to its conjugate.

22 The case $k = 0$ in no. 21 (ii): the same argument applies.

23 $(0, 0)$; $2, 3$; $\frac{1}{2}\sqrt{13}$; $(0, \pm\sqrt{13})$; $y = \pm 4/\sqrt{13}$; $y = \pm\frac{2}{3}x$.

24 $(-1, -2)$; $\frac{3}{2}, 2$; $\frac{5}{3}$; $(\frac{3}{2}, -2), (-\frac{7}{2}, -2)$; $x = -\frac{1}{10}, x = -\frac{19}{10}$;
$4x - 3y = 2,\ 4x + 3y + 10 = 0$.

25 $(0, 1)$; $1, \frac{12}{5}$; $\frac{13}{5}$; $(\pm\frac{13}{5}, 1)$; $x = \pm\frac{5}{13}$; $y = 1 \pm \frac{12}{5}x$.

Exercise 19(*a*), p. 672

1 $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1,\ \dfrac{x_1x_2}{a^2}+\dfrac{y_1y_2}{b^2}-1.$ 2 $y_1^2 - 4ax_1,\ y_1y_2 - 2a(x_1+x_2)$.

3 $x_1y_1 - c^2,\ \frac{1}{2}(x_1y_2 + x_2y_1) - c^2$.

4 $(lx_1+my_1+n)(l'x_1+m'y_1+n')$,
$\frac{1}{2}(lx_1+my_1+n)(l'x_2+m'y_2+n') + \frac{1}{2}(lx_2+my_2+n)(l'x_1+m'y_1+n')$.

5 $s = 0$ passes through the mid-point of P_1P_2.

8 $pb - 2rh + qa = 0$. 9 $(ax^2+2hxy+by^2)(a+b) = 2(ab-h^2)(x^2+y^2)$.

11 $(ax_1+hy_1+g)(x-x_1) + (hx_1+by_1+f)(y-y_1) = 0$.

13 (i) $ax^2+2hxy+by^2+gx+fy+c = 0$; (ii) $ax^2+2hxy+by^2+c = 0$;
(iii) $gx+fy+c = 0$.

14 $(x/a)\sin\phi + (y/b)\cos\phi = 0$; at the point at the end of the question.

15 Either $h \neq 0$, or $h = 0$ and $l = 0$.

Exercise 19(*b*), p. 681

1 $3x^2+3xy+2y^2-9x-2y = 0$.

2 $165x^2-294xy+120y^2-273x+691y+1572 = 0$.

3 $(a-b)h' = (a'-b')h$.

4 $11x^2-25xy+11y^2 = 0,\ (x+y)^2 = 47,\ (x-y)^2 = 3$.

8 $x^2+y^2 \pm 2ae^3x = a^2(1-e^2-e^4)$. 9 $\left(x-\dfrac{a^2-b^2}{a}\right)^2 + y^2 = \left(\dfrac{b^2}{a}\right)^2$.

11 $\dfrac{x}{a}\cos\phi - \dfrac{y}{b}\sin\phi = \cos 2\phi$; $\left(\dfrac{a^2-b^2}{a}\cos^3\phi,\ -\dfrac{a^2-b^2}{b}\sin^3\phi\right)$;
$\dfrac{(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{3}{2}}}{ab}$.

12 $t^3(x^2+y^2) - c(3t^4+1)x - ct^2(t^4+3)y + 3c^2t(t^4+1) = 0$;
$\left(\frac{1}{2}c\left(\dfrac{1}{t^3}+3t\right), \frac{1}{2}c\left(t^3+\dfrac{3}{t}\right)\right)$; $\frac{1}{2}c\left(t^2+\dfrac{1}{t^2}\right)^{\frac{3}{2}}$.

15 (i) $\alpha\beta = k\gamma\delta$.

17 $k_1 : k_2 : k_3 = (t_2-t_3)(1+t_1^2) : (t_3-t_1)(1+t_2^2) : (t_1-t_2)(1+t_3^2)$. The circumcircle of a triangle circumscribed to a parabola passes through the focus.

Miscellaneous Exercise 19(*c*), p. 682

8 $(a^2m^2+b^2)(x^2-y^2) = (1-m^2)a^2b^2$.

10 $25x^2-(k+1)y^2-(k+26)y+(6k+169) = 0$; $y = x^2+\frac{169}{25}$.

12 $s_{12}s_1-s_{11}s_2 = 0$.

Exercise 20(*a*), p. 688

2 $r_2r_3\sin(\theta_2-\theta_3)+r_3r_1\sin(\theta_3-\theta_1)+r_1r_2\sin(\theta_1-\theta_2) = 0$.

5 (i) (*a*) $r^2-2\rho r\cos(\theta-\alpha)+\rho^2\cos^2\alpha = 0$; (*b*) $r = \pm 2a\sin\theta$;
(ii) tangents at O are Oy, Ox.

6 $\left(\frac{1}{2}\sqrt{(a^2+b^2)}, \tan^{-1}(b/a)\right)$.

10 $r\cos(\theta-\theta_1-\theta_2+\alpha) = 2a\cos(\theta_1-\alpha)\cos(\theta_2-\alpha)$;
$r\cos(\theta-2\theta_1+\alpha) = 2a\cos^2(\theta_1-\alpha)$.

11 Circle through O.

12 $b^2c^2+2ac = 1$.

Exercise 20(*b*), p. 696

1 $a = l/(e^2-1)$, $b = l/\sqrt{(e^2-1)}$.

2 Parabola $2a/r = 1-\cos\theta$.

3 Conic with $l = k/a$, $e = (b^2+c^2)^{\frac{1}{2}}/a$.

5 $(2-e^2)/2l$.

8 $l/r = e\cos(\theta-\gamma)+\sec\frac{1}{2}(\alpha-\beta)\cos\{\theta-\frac{1}{2}(\alpha+\beta)\}$, which is therefore obtainable by writing $\alpha-\gamma$, $\beta-\gamma$, $\theta-\gamma$ for α, β, θ in the standard equation.

9 $(l, \frac{1}{2}\pi)$.

15 Concavities of the arcs LL' may be the same $(-)$ or opposite $(+)$ in sense.

18 (ii) $2l\sec\phi$, $\sec\phi$.

21 The directrix $l/r = \cos\theta$ of the parabola $l/r = 1+\cos\theta$.

25 $(1/\rho)\cos(\theta-\alpha)+(d\theta/dr)_\alpha\sin(\theta-\alpha) = 1/r$.

Miscellaneous Exercise 20(*c*), p. 698

3 $(a\sin\alpha-b\sin\beta)\sin\theta = (b\cos\beta-a\cos\alpha)\cos\theta$;
$(ab/r)\sin(\alpha-\beta) = a\sin(\alpha-\theta)-b\sin(\beta-\theta)$.

4 $r^2-\{a\cos(\theta-\alpha)+b\cos(\theta-\beta)\}r+ab\cos(\alpha-\beta) = 0$.

6 $(2-e^2)/l$.

8 (ii) $r = a\sec^3\frac{1}{2}\alpha\cos(\theta-\frac{1}{2}\alpha)$.

9 If $e < 1$, $\alpha = \pm\cos^{-1}(-e)$; if $e \geqslant 1$, there is no such α; $r = l\operatorname{cosec}^2\theta$.

12 $(1-e^2)r^2+2elr\cos\theta = l^2$; the auxiliary circle if $e \neq 1$, the directrix if $e = 1$.

14 $ey^2+lx = 0$.

Exercise 21(*a*), p. 704

1 $(+,+,+)$, $(-,+,+)$, $(-,-,+)$, $(+,-,+)$, $(+,+,-)$, $(-,+,-)$, $(-,-,-)$, $(+,-,-)$.

2 (i) Plane yOz; (ii) zOx; (iii) xOy; (iv) x-axis; (v) y-axis; (vi) z-axis.

3 $PB = \sqrt{(z^2+x^2)}$, $PC = \sqrt{(x^2+y^2)}$.

4 $15\sqrt{2}$.

7 $2\sqrt{46}$, $2\sqrt{38}$.

8 $3:4$, $5:7$, $-2:1$.

13 $x = \rho\cos\phi$, $y = \rho\sin\phi$; $\rho = \sqrt{(x^2+y^2)}$, $\cos\phi : \sin\phi : 1 = x : y : \rho$.

14 (i) On a cylinder with axis Oz, radius k; (ii) on a half-plane through Oz; (iii) on a line parallel to Oz.

15 $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$;
$r = \sqrt{(x^2+y^2+z^2)}$, $\cos\phi : \sin\phi : 1 = x : y : \sqrt{(x^2+y^2)}$,
$\cos\theta : \sin\theta : 1 = z : \sqrt{(x^2+y^2)} : r$.

16 (i) On a sphere with centre O, radius k; (ii) on a cone with vertex O and semi-vertical angle α; (iii) on a half-plane through Oz; (iv) generator of cone; (v) great semi-circle of sphere; (vi) small circle of sphere.

Exercise 21(*b*), p. 708

5 $1:1:1$, $1:-1:1$, $1:1:-1$, $-1:1:1$.

6 $\cos^{-1}\frac{1}{3}$.

7 $\cos^{-1}\{(b^2+c^2-a^2)/(a^2+b^2+c^2)\}$, etc.

8 $A = \cos^{-1}\frac{12}{13}$, $B = \cos^{-1}\dfrac{109}{13\sqrt{74}}$, $C = \cos^{-1}\dfrac{-7}{\sqrt{74}}$.

10 $\left(\dfrac{p^2x}{OP^2}, \dfrac{p^2y}{OP^2}, \dfrac{p^2z}{OP^2}\right)$.

15 Each is equal to OP^2.

Exercise 21(*c*), p. 713

1 $x-4y+5z = 0$.

2 $6x+2y-10z = 31$.

3 $3x+3y-z = 6$.

4 $5x+2y-3z = 17$.

5 $6x+3y-2z = 18$, $2x-3y-6z = 6$.

6 (i) $\cos^{-1}(1/\sqrt{14})$; (ii) $\sin^{-1}(1/\sqrt{6})$.

7 $\cos^{-1}[c/\sqrt{\{(p-q)^2+(p-r)^2+c^2\}}]$, $cp/(p-q)$, $cp/(p-r)$.

8 $\cos^{-1}\{(a^{-2}+b^{-2}-c^{-2})/(a^{-2}+b^{-2}+c^{-2})\}$.

9 $(x_1-x_2)x+(y_1-y_2)y+(z_1-z_2)z = \frac{1}{2}(x_1^2-x_2^2+y_1^2-y_2^2+z_1^2-z_2^2)$.

13 $-(Ax_1+By_1+Cz_1+D)/(Ax_2+By_2+Cz_2+D)$.

Exercise 21(*d*), p. 721

1 $(11, 22, 19)$; $5\sqrt{38}$.

2 $(0, b-am/l, c-an/l)$, $(a-bl/m, 0, c-bn/m)$, $(a-cl/n, b-cm/n, 0)$.

4 21.

5 $\dfrac{x-1}{0} = \dfrac{y-2}{1} = \dfrac{z-3}{-5}$; $(1, 3, -2)$.

6 $x-4 = -9y-9 = -3z+21$.

7 $\dfrac{x-1}{4} = \dfrac{y-3}{3} = \dfrac{z+3}{-4}$.

8 $5x-22y-19z+27 = 0$.

9 $(4, 2, -1)$, $x-y-z = 3$.

12 $3x+y-4z = 0$.

13 12.

14 60°.

15 $x = \frac{1}{2}y = -z$, $x = -y = \frac{1}{2}z$.

16 $\dfrac{x-3}{\sqrt{2}\pm 1}=\dfrac{y-2}{\sqrt{2}\mp 1}=\dfrac{z+3}{-\sqrt{2}}$ (both upper signs, or both lower).

17 2.

18 $2x+60y-162z+27=0$, $54x-18y-6z+1=0$.

19 $\pm\dfrac{p^2}{2lmn}\sqrt{(l^2+m^2+n^2)}$.

20 $6\times$ volume of $OP_1P_2P_3$.

Exercise 21(*e*), p. 726

1 $12x-11y+4z=1$.

2 $3x+4y-2z=7$.

3 $9x-3y-z+14=0$.

4 $2:-1:1$.

5 $3x-8y+7z+4=0=3x+2y+z$, $\dfrac{x+1}{-11}=\dfrac{y-1}{9}=\dfrac{z-1}{15}$.

6 $7:-11:10$.

7 $3x+2(3\pm\sqrt{3})\,y-3(4\pm\sqrt{3})\,z=0$ (both upper or both lower signs).

8 $2x-y-z+3=0=x+y+z-6$; $1:0:-1$ or $1:-1:0$.

9 $x-y+z+1=0$, $4x-6y-z-3=0$, $7x-9y+2z=0$.

Exercise 21(*f*), p. 732

1 $10:3:-4$.

2 $\dfrac{x+6}{5}=\dfrac{y+4}{3}=\dfrac{z+6}{6}$; $(4, 2, 6)$, $(-1, -1, 0)$.

3 $(3,0,0)$, $(\frac{81}{17},0,0)$.

5 $2\sqrt{3}$.

6 5.

7 13; $\dfrac{x-1}{3}=\dfrac{y-2}{4}=\dfrac{z-3}{12}$.

8 $\dfrac{x-2}{3}=\dfrac{y-\frac{54}{17}}{3}=\dfrac{z-\frac{22}{17}}{5}$; $\sqrt{\frac{32}{17}}$.

9 See 21.52, ex. (iv).

Miscellaneous Exercise 21(*g*), p. 733

3 $-18:5$.

4 $\left(\dfrac{3t}{13}, \dfrac{4t}{13}, \dfrac{156-12t}{13}\right)$; $\dfrac{144}{13}$; $\dfrac{\sqrt{709}}{325}$.

6 $\dfrac{x}{-2}=\dfrac{y}{1}=\dfrac{z}{4}$; $(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3})$.

7 60°; $x+y-z=0$.

8 $(l\pm l')\,x+(m\pm m')\,y+(n\pm n')\,z=0=\Sigma(mn'-m'n)\,x$ (all $+$ or all $-$).

9 $(0,0,0)$, $(1, 1, -1)$.

10 $x-4y+6z=106$; $\sqrt{293}$.

11 $7x-8y+3z=0$; $13x=122$, $20y+23z=0$.

12 $\left(x-\dfrac{a+d}{2}\right)(a-d)+\left(y-\dfrac{b+e}{2}\right)(b-e)+\left(z-\dfrac{c+f}{2}\right)(c-f)=0$.

15 $\cos^{-1}\frac{1}{4}$, $\cos^{-1}\frac{2}{3}$.

Exercise 22(*a*), p. 740

1 $7(x^2+y^2+z^2)+16x-31y-8z+34=0$, $(-\frac{8}{7}, \frac{31}{14}, \frac{4}{7})$.

2 A sphere if $\lambda \neq 1$; the plane bisecting AB at right-angles if $\lambda=1$.

3 $2(x^2+y^2+z^2)-7x-6y-9z+13=0$.

4 $x^2+y^2+z^2+2x-2y+2z-6=0$. 5 $3(x^2+y^2+z^2)-2x-2y-2z-1=0$.

6 $6(x^2+y^2+z^2)-18x-36y-12z=0$; 7.

7 $3(x^2+y^2+z^2)-12x-6y-4z=0$. 9 $(x-5)^2+(y\pm 4)^2+(z\pm 4)^2=41$.

10 $(2,4,4)$, $(\frac{9}{7},4,\frac{64}{21})$. 11 $(-\frac{7}{2},-5,\frac{5}{2})$, $\frac{3}{2}\sqrt{14}$.

12 $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $\frac{1}{2}$; $(2,2,2)$, 2.

13 $81(x^2+y^2+z^2)-126(x+y+z)+98=0$. 15 $2x+2y\pm z=3$.

16 $x+2y-2z+15=0$, $4y+3z-11=0$. 17 $(lr^2/p, mr^2/p, nr^2/p)$.

18 $(lu+mv+nw-p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$.

19 $(x-5)^2+(y+2)^2+(z-3)^2=49$; $2x+3y+6z=71$. 21 $(\frac{4}{19},\frac{1}{38},-\frac{7}{38})$.

22 (ii) Tangents from a point on the common circle have equal lengths zero.

Exercise 22(*b*), p. 745

1 $(0,0,0)$. 2 Plane touches circle.

3 4. 4 $x^2+y^2+z^2-8x-6y=0$.

5 $3x+y-3z=3$, $x^2+y^2+z^2-x-3y+z=0$; $(\frac{5}{19},\frac{27}{19},-\frac{5}{19})$.

6 $(2,2,3)$.

8 $x^2+y^2+z^2+6z-16=0$, $5(x^2+y^2+z^2)-14x-28y-12z+32=0$.

9 $x^2+y^2+z^2-2z-8=0$, $x^2+y^2+z^2+20x+20y+18z+72=0$.

10 $x^2+y^2+z^2+2x-8z=0$.

12 $x^2+y^2+z^2+2ay-a^2=0$, $x^2+y^2+z^2+4ax-2az+a^2=0$.

13 $5(x^2+y^2+z^2)+19x-13y+10z+27=0$; $3/\sqrt{10}$.

14 $x^2+y^2+z^2-2x+2y+6z-5=0$. 15 $(\frac{18}{13},\frac{24}{13},\frac{72}{13})$.

16 $x^2+y^2+z^2+x+9y-4z=0$.

18 Cylinder with generators parallel to (i) Ox, (ii) Oy; surface of revolution with axis (iii) Oy, (iv) Oz.

19 $10x^2+12xy+5y^2=16$.

Exercise 22(*c*), p. 755

1 $\cos b=\cos c\cos a+\sin c\sin a\cos B$, $\cos c=\cos a\cos b+\sin a\sin b\cos C$.

2 $\frac{1}{2}\pi$, $\frac{1}{6}\pi r^2$. 5 $19°\,28'$; 1,440,000 sq. miles.

Miscellaneous Exercise 22(*d*), p. 756

2 $x^2+y^2+z^2-2cz = 0$.

3 $(2+\frac{1}{2}\sqrt{2},\ -1+2\sqrt{2},\ \frac{1}{2}\sqrt{2})$, $\frac{1}{2}(7\sqrt{2}-6)$.

4 $\dfrac{x}{2}=\dfrac{y}{6}=\dfrac{z}{5}$, $\cos^{-1}\frac{16}{29}$.   5 $(-al/n,\ -am/n)$.   9 $|\sqrt{p}-\sqrt{q}|$.

10 $(\Sigma ul)^2 = d\Sigma l^2$; $(\lambda l, \lambda m, \lambda n)$ where $\lambda = -d/(\Sigma ul)$.

12 $(x\pm 4)^2+(y\pm 4)^2+(z\pm 3)^2 = 25$.

13 $x^2+y^2+z^2-2x-4y-4z-7 = 0$,
$5(x^2+y^2+z^2)+22x+44y+44z-323 = 0$; $(3, 6, 6)$.

15 $\left(\dfrac{2a^2u}{u^2+v^2+a^2}, \dfrac{2a^2v}{u^2+v^2+a^2}, \dfrac{a(u^2+v^2-a^2)}{u^2+v^2+a^2}\right)$.

INDEX TO VOLUME II

Numbers refer to pages.

* *means 'Also see this entry in the Index of Vol. I'.*

F. GERRISH

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