KEY

TO

PLANE TRIGONOMETRY.

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The Keys already issued to some of the Author's works have been found very useful by affording assistance to private students, and by saving the labour and time of teachers; and this has led to the issue of the present volume. Care has been taken, as in the former Keys, to present the solutions in a simple natural manner, in order to meet the difficulties which are most likely to arise, and to render the work intelligible and instructive.

September, 1874.

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PLANE TRIGONOMETRY.

CHAPTER I.

1. Let x denote the number of degrees in the larger angle, and y the number of degrees in the smaller angle. Then, since 10 grades are equal to 9 degrees, $x-y=9$; also $x+y=45$: hence we obtain $x=27$ and $y=18$.

2. In two-thirds of a right angle there are 60 degrees; let x denote the number of degrees in one part, then $60-x$ denotes the number of degrees in the other part, therefore the number of grades in this part is $\frac{10}{9}(60-x)$. Hence

$$x : \frac{10}{9}(60-x) :: 3 : 10; \text{ therefore } 10x=\frac{30}{9}(60-x);$$

therefore $9x=3(60-x)$; therefore $12x=180$; therefore $x=15$.

3. In half a right angle there are 45 degrees; let x denote the number of degrees in one part, then $45-x$ denotes the number of degrees in the other part, therefore the number of grades in this part is $\frac{10}{9}(45-x)$. Hence

$$x : \frac{10}{9}(45-x) :: 9 : 5; \text{ therefore } 5x=10(45-x);$$

therefore $\quad 15x=450$; therefore $x=30$.

4. $1`\,5`` = \cdot 0105$ of a grade; $\frac{9}{10}$ of $\cdot 0105 = \cdot 00945$.

5. Let x denote the number of degrees in one part; then $n-x$ denotes the number of degrees in the other part. In x degrees there are $60x$ English minutes. In $n-x$ degrees there are $\frac{10}{9}(n-x)$ grades, and therefore $\frac{10}{9}\times 100(n-x)$ French minutes. Therefore

$$60x=\frac{1000}{9}(n-x);$$

therefore $\quad 1540x=1000n$; therefore $77x=50n$;

therefore $\quad x=\frac{50n}{77}$, and $n-x=\frac{27n}{77}$.

6. In one-third of a right angle there are 30 degrees; if this be taken as the unit of measurement an angle of 75 degrees must be denoted by $\frac{75}{30}$, that is by $\frac{5}{2}$, that is by $2\frac{1}{2}$.

7. Let x denote the number of grades in the unit. Then an angle of $66\frac{2}{3}$ grades is denoted by $\frac{66\frac{2}{3}}{x}$; and this is equal to 20. Therefore

$$20x = 66\tfrac{2}{3} = \frac{200}{3}; \quad \text{therefore } x = \frac{10}{3}.$$

Hence the number of degrees in the unit is $\frac{9}{10} \times \frac{10}{3}$, that is 3.

8. Let $3x$ denote the number of sides in the equiangular polygon which has the greater number of sides; then $2x$ denotes the number of sides in the other equiangular polygon. All the angles of the polygon of $2x$ sides are equal to $(4x-4)$ right angles, that is to $(4x-4)\,100$ grades; therefore each angle contains $\frac{(4x-4)\,100}{2x}$ grades. All the angles of the polygon of $3x$ sides are equal to $(6x-4)$ right angles, that is to $(6x-4)\,90$ degrees; therefore each angle contains $\frac{(6x-4)\,90}{3x}$ degrees; therefore

$$\frac{(4x-4)\,100}{2x} = \frac{(6x-4)\,90}{3x};$$

therefore $(4x-4)\,5 = (6x-4)\,3$; therefore $2x = 8$; therefore $x = 4$. Thus one polygon has 8 sides and the other polygon has 12 sides.

9. It is shewn in Art. 9 that an angle expressed in centesimal seconds is transformed to English seconds by multiplying by $\frac{81}{250}$; and $\frac{81}{250} = \frac{324}{1000}$.

10. Suppose one angle to contain x English seconds, and another to contain x French minutes. The second angle then contains $100x$ French seconds, and therefore $\frac{81}{250} \times 100x$ English seconds. Hence the ratio of the former angle to the latter is that of 1 to $\frac{8100}{250}$, or of 1 to $\frac{162}{5}$, or of 5 to 162.

11.

$$\begin{array}{r|l} 60 & 3{\cdot}00 \\ 60 & 10{\cdot}05 \\ \hline & {\cdot}1675 \end{array}$$

Thus $35^\circ\,10'\,3'' = 35^\circ{\cdot}1675$.

$$\begin{array}{r} 35{\cdot}1675 \\ 3{\cdot}9075 \\ \hline 39{\cdot}0750 \end{array}$$

And $39^g{\cdot}0750 = 39^g\,7^\backprime\,50^{\backprime\backprime}$.

12. $69^g\,22^\backprime\,50^{\backprime\backprime} = 69^g{\cdot}225$.

$$\begin{array}{r} 69{\cdot}225 \\ 6{\cdot}9225 \\ \hline 62{\cdot}3025 \\ 60 \\ \hline 18{\cdot}1500 \\ 60 \\ \hline 9{\cdot}00 \end{array}$$

CHAPTER II.

1. It is shewn in Art. 8 that $\frac{D}{90}=\frac{G}{100}$; and it is shewn in Art. 22 that $\frac{D}{180}=\frac{C}{\pi}$, so that $\frac{D}{90}=\frac{2C}{\pi}$. Therefore $\frac{D}{90}=\frac{G}{100}=\frac{2C}{\pi}$.

In fact the three expressions denote the same thing, namely the ratio of the angle considered to a right angle.

2. The circular measure of the angle is $\frac{9}{10\times 12}$, that is $\frac{3}{40}$. Therefore, by Art. 22, the number of degrees in the angle is $\frac{3}{40}$ of $\frac{180}{\pi}$.

3. $5^0\,37'\,30''=337\frac{1}{2}$ minutes. Thus the circular measure

$$=\frac{337\frac{1}{2}}{180\times 60}\pi=\frac{675}{180\times 60\times 2}\pi=\frac{135}{180\times 12\times 2}\pi$$

$$=\frac{27}{36\times 12\times 2}\pi=\frac{\pi}{32}.$$

4. The angle contains 1·01 grades; therefore, by Art. 24, the circular measure is $\frac{1\cdot 01}{200}\pi$, that is $\pi\times\cdot 00505$.

5. Let x denote the number of degrees in the first angle, y the number in the second, and z the number in the third.

The circular measure of the first angle is $\frac{x\pi}{180}$, and the circular measure of the second is $\frac{y\pi}{180}$; therefore $\frac{x\pi}{180}-\frac{y\pi}{180}=\frac{\pi}{10}$; therefore $x-y=18$.

The number of grades in the second angle is $\frac{10y}{9}$, and the number of grades in the third is $\frac{10z}{9}$; therefore $\frac{10y}{9}+\frac{10z}{9}=30$; therefore $y+z=27$.

Also $x+y=36$.

From these three equations we have $x=27$, $y=9$, $z=18$.

6. The circular measure of a right angle is $\frac{\pi}{2}$; and therefore the circular measure of five-sixteenths of a right angle is $\frac{5}{16}$ of $\frac{\pi}{2}$, that is $\frac{5\pi}{32}$.

The number of degrees is $\frac{5}{16}$ of 90, that is $\frac{450}{16}$, that is 28·125.

The number of grades is $\frac{5}{16}$ of 100, that is $\frac{500}{16}$, that is 31·25.

7. Let the numbers of degrees in the three angles be denoted respectively by $x-y$, x, and $x+y$. Then $x-y+x+x+y=180$, that is $3x=180$; therefore $x=60$.

Also $x+y=2(x-y)$; therefore $3y=x=60$; therefore $y=20$.

Hence in degrees the angles are denoted by 40, 60, and 80. Therefore in grades they will be denoted by $\frac{400}{9}$, $\frac{600}{9}$, and $\frac{800}{9}$. And in circular measure they will be denoted by $\frac{40\pi}{180}$, $\frac{60\pi}{180}$, and $\frac{80\pi}{180}$; that is by $\frac{2\pi}{9}$, $\frac{\pi}{3}$, and $\frac{4\pi}{9}$.

8. Let the numbers of degrees in the three angles be denoted respectively by $x-y$, x, and $x+y$. Then $x-y+x+x+y=180$, that is $3x=180$; therefore $x=60$.

The circular measure of the greatest angle is $\frac{(x+y)\pi}{180}$; thus

$$x-y : \frac{(x+y)\pi}{180} :: 60 : \pi; \text{ therefore } (x-y)\pi = \frac{(x+y)\pi}{3};$$

therefore $3(x-y)=x+y$; therefore $y=\frac{x}{2}=30$.

Thus the angles are 30^0, 60^0, and 90^0.

9. All the angles of the polygon are equal to $(2n-4)$ right angles, that is to $(2n-4)\frac{\pi}{2}$ in circular measure, that is to $(n-2)\pi$. Hence the circular measure of each angle is $\frac{(n-2)\pi}{n}$.

10. During the quarter of an hour since twelve the long hand has described one-fourth of four right angles, that is a right angle. The short hand has described one-twelfth of this, that is $\frac{1}{12}$ of a right angle. Hence the angle between the hands at a quarter past twelve is $\frac{11}{12}$ of a right angle.

The measure in degrees $=\frac{11}{12}$ of $90=\frac{11\times 15}{2}=\frac{165}{2}=82\frac{1}{2}$.

The measure in grades $=\frac{11}{12}$ of $100=\frac{11\times 25}{3}=\frac{275}{3}=91\frac{2}{3}$.

The circular measure $=\frac{11}{12}$ of $\frac{\pi}{2}=\frac{11\pi}{24}$.

CHAPTER III.

1. Let $\sin A = \frac{3}{5}$. Then we have

$$\cos A = \sqrt{(1-\sin^2 A)} = \sqrt{\left(1-\frac{9}{25}\right)} = \sqrt{\frac{16}{25}} = \frac{4}{5};$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{3}{5} \div \frac{4}{5} = \frac{3}{5} \times \frac{5}{4} = \frac{3}{4};$$

$$\cot A = \frac{1}{\tan A} = \frac{4}{3};$$

$$\sec A = \frac{1}{\cos A} = \frac{5}{4}; \quad \operatorname{cosec} A = \frac{1}{\sin A} = \frac{5}{3};$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \frac{4}{5} = \frac{1}{5}.$$

2. Let $\tan A = \frac{4}{3}$. Then we have

$$\sin A = \frac{\tan A}{\sqrt{(1+\tan^2 A)}} = \frac{\frac{4}{3}}{\sqrt{\left(1+\frac{16}{9}\right)}} = \frac{4}{3} \div \frac{5}{3} = \frac{4}{5};$$

$$\cos A = \frac{1}{\sqrt{(1+\tan^2 A)}} = \frac{1}{\sqrt{\left(1+\frac{16}{9}\right)}} = 1 \div \frac{5}{3} = \frac{3}{5};$$

$$\sec A = \frac{1}{\cos A} = \frac{5}{3}; \quad \operatorname{cosec} A = \frac{1}{\sin A} = \frac{5}{4};$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \frac{3}{5} = \frac{2}{5}.$$

3. Let $\cos A = \sqrt{\frac{2}{3}}$. Then we have

$$\sin A = \sqrt{(1-\cos^2 A)} = \sqrt{\left(1-\frac{2}{3}\right)} = \sqrt{\frac{1}{3}};$$

$$\tan A = \frac{\sin A}{\cos A} = \sqrt{\frac{1}{3}} \div \sqrt{\frac{2}{3}} = \frac{1}{\sqrt{2}};$$

$$\cot A = \frac{1}{\tan A} = \sqrt{2};$$

$$\sec A = \frac{1}{\cos A} = \sqrt{\frac{3}{2}}; \quad \operatorname{cosec} A = \frac{1}{\sin A} = \sqrt{3};$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \sqrt{\frac{2}{3}}.$$

4. $\sec^2\theta \operatorname{cosec}^2\theta = (1+\tan^2\theta)(1+\cot^2\theta) = 1+\tan^2\theta+\cot^2\theta+(\tan\theta\cot\theta)^2$

$$=1+\tan^2\theta+\cot^2\theta+1=\tan^2\theta+\cot^2\theta+2.$$

5. $\sin^2\theta\tan\theta+\cos^2\theta\cot\theta+2\sin\theta\cos\theta=\dfrac{\sin^3\theta}{\cos\theta}+\dfrac{\cos^3\theta}{\sin\theta}+2\sin\theta\cos\theta$

$$=\frac{\sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}=\frac{(\sin^2\theta+\cos^2\theta)^2}{\sin\theta\cos\theta}=\frac{1}{\sin\theta\cos\theta}$$

$$=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\tan\theta+\cot\theta.$$

6. $2(\sin^6\theta+\cos^6\theta)=2(\sin^2\theta+\cos^2\theta)(\sin^4\theta-\sin^2\theta\cos^2\theta+\cos^4\theta)$

$$=2(\sin^4\theta-\sin^2\theta\cos^2\theta+\cos^4\theta);$$

therefore $\quad 2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1$

$$=-2\sin^2\theta\cos^2\theta-\sin^4\theta-\cos^4\theta+1$$

$$=1-(\sin^2\theta+\cos^2\theta)^2=1-1=0.$$

7. $\sin^2\theta=\frac{3}{2}\cos\theta$; therefore $1-\cos^2\theta=\frac{3}{2}\cos\theta$;

therefore $$\cos^2\theta+\frac{3}{2}\cos\theta=1.$$

By solving this quadratic in the usual way we obtain $\cos\theta=\frac{1}{2}$ or -2; but only the former value is applicable, for $\cos\theta$ cannot be numerically greater than unity. Hence $\cos\theta=\frac{1}{2}$, and therefore $\theta=\frac{\pi}{3}$.

8. $\sin\theta+\cos\theta=1$; therefore $\cos\theta=1-\sin\theta$; therefore $\cos^2\theta=(1-\sin\theta)^2$, therefore $1-\sin^2\theta=(1-\sin\theta)^2$, that is $(1-\sin\theta)(1+\sin\theta)=(1-\sin\theta)^2$. Therefore either $1-\sin\theta=0$, or $1+\sin\theta=1-\sin\theta$.

Take $1-\sin\theta=0$; thus $\sin\theta=1$, therefore $\theta=\frac{\pi}{2}$.

Next take $1+\sin\theta=1-\sin\theta$; thus $\sin\theta=0$, therefore $\theta=0$.

9. $\cot\theta=2\cos\theta$; therefore $\dfrac{\cos\theta}{\sin\theta}=2\cos\theta$.

Therefore either $\cos\theta=0$, or $\dfrac{1}{\sin\theta}=2$.

Take $\cos\theta=0$; then $\theta=\frac{\pi}{2}$. Next take $\dfrac{1}{\sin\theta}=2$; thus $\sin\theta=\frac{1}{2}$; therefore $\theta=\frac{\pi}{6}$.

10. $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$; therefore $1 - \cos^2\theta - 2\cos\theta + \frac{1}{4} = 0$; therefore $\cos^2\theta + 2\cos\theta = \frac{5}{4}$. By solving this quadratic in the ordinary way we obtain $\cos\theta = \frac{1}{2}$, or $-\frac{5}{2}$; but only the former value is applicable; therefore $\theta = \frac{\pi}{3}$.

11. $3\sec^4\theta + 8 = 10\sec^2\theta$; therefore $3\sec^4\theta - 10\sec^2\theta + 8 = 0$. By solving this quadratic in the ordinary way we obtain $\sec^2\theta = 2$ or $\frac{4}{3}$; therefore $\sec\theta = \sqrt{2}$ or $\frac{2}{\sqrt{3}}$; therefore $\theta = \frac{\pi}{4}$ or $\frac{\pi}{6}$.

12. $\tan\theta + \cot\theta = 2$; therefore $\tan\theta + \frac{1}{\tan\theta} = 2$;

therefore $\tan^2\theta - 2\tan\theta + 1 = 0$, that is $(\tan\theta - 1)^2 = 0$;

therefore $\tan\theta = 1$, therefore $\theta = \frac{\pi}{4}$.

13. $\sin(A - B) = \frac{1}{2}$; therefore $A - B = 30^0$,

$\cos(A + B) = \frac{1}{2}$; therefore $A + B = 60^0$;

from these two equations we obtain $A = 45^0$, and $B = 15^0$.

14. $\tan(A + B) = \sqrt{3}$; therefore $A + B = 60^0$,
$\tan(A - B) = 1$; therefore $A - B = 45^0$;
from these two equations we obtain $A = 52\frac{1}{2}^0$, $B = 7\frac{1}{2}^0$.

CHAPTER IV.

1. $585^0 = 360^0 + 225^0$. Thus the Trigonometrical Ratios are the same as for an angle of 225^0.

$$\sin 225^0 = \sin(180^0 + 45^0) = -\sin 45^0 = -\frac{1}{\sqrt{2}},$$

$$\cos 225^0 = \cos(180^0 + 45^0) = -\cos 45^0 = -\frac{1}{\sqrt{2}}.$$

2. $690^0 = 360^0 + 330^0$. Thus the Trigonometrical Ratios are the same as for an angle of 330^0.

$$\sin 330^0 = \sin(180^0 + 150^0) = -\sin 150^0 = -\sin 30^0 = -\frac{1}{2},$$

$$\cos 330^0 = \cos(180^0 + 150^0) = -\cos 150^0 = \quad \cos 30^0 = \frac{\sqrt{3}}{2}.$$

3. $930^0 = 720^0 + 210^0$. Thus the Trigonometrical Ratios are the same as for an angle of 210^0.

$$\sin 210^0 = \sin(180^0 + 30^0) = -\sin 30^0 = -\frac{1}{2};$$

$$\cos 210^0 = \cos(180^0 + 30^0) = -\cos 30^0 = -\frac{\sqrt{3}}{2}.$$

4. $6420^0 = 17 \times 360^0 + 300^0$. Thus the Trigonometrical Ratios are the same as for an angle of 300^0.

$$\sin 300^0 = \sin(180^0 + 120^0) = -\sin 120^0 = -\sin 60^0 = -\frac{\sqrt{3}}{2},$$

$$\cos 300^0 = \cos(180^0 + 120^0) = -\cos 120^0 = \cos 60^0 = \frac{1}{2}.$$

5. The smallest angle is 45^0; the other angles are found by increasing successively by 180^0: thus all the angles are 45^0, 225^0, 405^0, 585^0, 765^0.

6. Since $\cos^2\theta = \frac{1}{2}$, we have $\cos\theta = \pm\frac{1}{\sqrt{2}}$.

Take the upper sign; then the smallest value is 45^0, and the others are $360^0 - 45^0$, $360^0 + 45^0$, $720^0 - 45^0$, $720^0 + 45^0$.

Take the lower sign; then the smallest value is 135^0, and the others are $360^0 - 135^0$, $360^0 + 135^0$, $720^0 - 135^0$, $720^0 + 135^0$.

7. $\operatorname{vers}\frac{n\pi}{4} = 1 - \cos\frac{n\pi}{4}$.

Suppose $n=0$; then we have $1 - \cos 0$, that is $1-1$, that is 0; next suppose $n=1$, then we have $1 - \cos\frac{\pi}{4}$, that is $1 - \frac{1}{\sqrt{2}}$; next suppose $n=2$, then we have $1 - \cos\frac{\pi}{2}$, that is $1-0$, that is 1; next suppose $n=3$, then we have $1 - \cos\frac{3\pi}{4}$; that is $1 + \frac{1}{\sqrt{2}}$; next suppose $n=4$, then we have $1 - \cos\pi$, that is $1+1$, that is 2. Then the values begin to recur in the inverse order; for $\cos\frac{5\pi}{4} = \cos\frac{3\pi}{4}$, $\cos\frac{6\pi}{4} = \cos\frac{2\pi}{4}$, $\cos\frac{7\pi}{4} = \cos\frac{\pi}{4}$, $\cos\frac{8\pi}{4} = \cos 2\pi = \cos 0$.

Then the whole series recurs. For $\cos\frac{9\pi}{4} = \cos\frac{\pi}{4}$, and so on.

8. Suppose $n=0$, then we have $\sin\frac{\pi}{6}$, that is $\frac{1}{2}$; next suppose $n=1$, then we have $\sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$, that is $\sin\frac{\pi}{3}$, that is $\frac{\sqrt{3}}{2}$; next suppose $n=2$, then we have $\sin\left(\pi + \frac{\pi}{6}\right)$, that is $-\sin\frac{\pi}{6}$, that is $-\frac{1}{2}$; next suppose $n=3$,

then we have $\sin\left(\frac{3\pi}{2}-\frac{\pi}{6}\right)$, that is $-\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$, that is $-\sin\frac{\pi}{3}$, that is $-\frac{\sqrt{3}}{2}$.

Then the values recur; for suppose $n=4$; then we have $\sin\left(2\pi+\frac{\pi}{6}\right)$, that is $\sin\frac{\pi}{6}$, and so on.

9. $\sin^3\theta=-\cos^3\theta$. Extract the cube root of both sides; thus $\sin\theta=-\cos\theta$, therefore $\frac{\sin\theta}{\cos\theta}=-1$, that is $\tan\theta=-1$; therefore $\theta=\frac{3\pi}{4}$.

10. $2\sin^2\theta-5\cos\theta-4=0$; therefore $2(1-\cos^2\theta)-5\cos\theta-4=0$; therefore $2\cos^2\theta+5\cos\theta+2=0$. By solving this quadratic in the usual way we obtain $\cos\theta=-\frac{1}{2}$ or -2; but only the former value is applicable; therefore $\theta=\frac{2\pi}{3}$.

11. When $\theta=0$ we have $\cos\theta=1$ and $\sin\theta=0$, so that $\cos\theta-\sin\theta=1$. Let θ change from 0 to $\frac{\pi}{2}$, then $\cos\theta$ changes from 1 to 0, and $\sin\theta$ from 0 to 1; therefore $\cos\theta-\sin\theta$ changes from 1 to -1, vanishing when $\theta=\frac{\pi}{4}$.

Let θ change from $\frac{\pi}{2}$ to π, then $\cos\theta$ changes from 0 to -1 and $\sin\theta$ from 1 to 0; thus $\cos\theta-\sin\theta$ remains negative. It has its greatest numerical value, namely $-\sqrt{2}$, when $\theta=\frac{3\pi}{4}$. For we have

$$(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2=2(\cos^2\theta+\sin^2\theta)=2;$$

and thus $(\cos\theta-\sin\theta)^2$ has its greatest value when $\cos\theta+\sin\theta$ vanishes, that is when $\tan\theta=-1$, that is when $\theta=\frac{3\pi}{4}$.

Let θ change from π to $\frac{3\pi}{2}$; then $\cos\theta-\sin\theta$ goes through the same numerical values, with a *contrary* sign, as when θ changes from 0 to $\frac{\pi}{2}$: this follows from Art. 50.

Let θ change from $\frac{3\pi}{2}$ to 2π; then $\cos\theta-\sin\theta$ goes through the same numerical values, with a *contrary* sign, as when θ changes from $\frac{\pi}{2}$ to π: this follows from Art. 50.

12. Let θ change from 0 to $\frac{\pi}{2}$; then $\cos^2\theta$ changes from 1 to 0, and $\sin^2\theta$ from 0 to 1; therefore $\cos^2\theta-\sin^2\theta$ changes from 1 to -1.

Let θ change from $\frac{\pi}{2}$ to π; then $\cos^2\theta - \sin^2\theta$ changes from -1 to 1.

Let θ change from π to $\frac{3\pi}{2}$; then $\cos^2\theta - \sin^2\theta$ goes through the same values as when θ changes from 0 to $\frac{\pi}{2}$.

Let θ change from $\frac{3\pi}{2}$ to 2π; then $\cos^2\theta - \sin^2\theta$ goes through the same values as when θ changes from $\frac{\pi}{2}$ to π.

13. $\tan\theta + \cot\theta = \tan\theta + \frac{1}{\tan\theta}$. Let θ change from 0 to $\frac{\pi}{2}$; then $\tan\theta$ changes from 0 to infinity. Thus $\tan\theta + \frac{1}{\tan\theta}$ is always positive, and is infinite both when $\theta = 0$, and when $\theta = \frac{\pi}{2}$. The least value is when $\theta = \frac{\pi}{4}$; for we have

$$\left(\tan\theta + \frac{1}{\tan\theta}\right)^2 = \left(\tan\theta - \frac{1}{\tan\theta}\right)^2 + 4,$$

and thus the least value is when $\tan\theta - \frac{1}{\tan\theta}$ vanishes, that is when $\tan^2\theta = 1$. Thus $\tan\theta + \cot\theta$ diminishes from infinity to 2, as θ changes from 0 to $\frac{\pi}{4}$; and then increases from 2 to infinity, as θ changes from $\frac{\pi}{4}$ to $\frac{\pi}{2}$.

Let θ change from $\frac{\pi}{2}$ to π; then $\tan\theta + \cot\theta$ goes in reverse order through the same numerical values, with a *contrary* sign, as when θ changes from 0 to $\frac{\pi}{2}$: this follows from Art. 48.

Let θ change from π to 2π; then $\tan\theta + \cot\theta$ goes through the same values as when θ changes from 0 to π: this follows from Art. 50.

14. We know by Algebra that if a and b are unequal $2ab$ is less than $a^2 + b^2$, and therefore $4ab$ is less than $a^2 + b^2 + 2ab$, that is $4ab$ is less than $(a+b)^2$. Therefore $\frac{4ab}{(a+b)^2}$ is less than unity; and cannot be equal to the secant of any angle, for a secant is never less than unity.

15. $\tan(A + 90^0) = \frac{\sin(A + 90^0)}{\cos(A + 90^0)} = \frac{\cos A}{-\sin A}$, by Art. 52, $= -\cot A$,

$$\cot(A + 90^0) = \frac{1}{\tan(A + 90^0)} = -\frac{1}{\cot A} = -\tan A,$$

$$\sec(A+90^0)=\frac{1}{\cos(A+90^0)}=\frac{1}{-\sin A},\text{ by Art. 52, }=-\operatorname{cosec} A,$$

$$\operatorname{cosec}(A+90^0)=\frac{1}{\sin(A+90^0)}=\frac{1}{\cos A},\text{ by Art. 52, }=\sec A,$$

$$\operatorname{vers}(A+90^0)=1-\cos(A+90^0)=1+\sin A,\text{ by Art. 52.}$$

16. $\sin(270^0-A)=-\sin(90^0-A)$, by Art. 50, $=-\cos A$.
$\cos(270^0-A)=-\cos(90^0-A)$, by Art. 50, $=-\sin A$.

17. $\sin(270^0+A)=-\sin(90^0+A)$, by Art. 50,
$=-\cos A$, by Art. 52.
$\cos(270^0+A)=-\cos(90^0+A)$, by Art. 50,
$=-(-\sin A)$, by Art. 52, $=\sin A$.

18. $\sin(360^0-A)=-\sin(180^0-A)$, by Art. 50,
$=-\sin A$, by Art. 48.
$\cos(360^0-A)=-\cos(180^0-A)$, by Art. 50,
$=-(-\cos A)$, by Art. 48, $=\cos A$.

CHAPTER V.

1. $\tan\theta=1$; the smallest value of θ is $\frac{\pi}{4}$, and the general value is $n\pi+\frac{\pi}{4}$, by Art. 68.

2. $\sin\theta=1$; the smallest value of θ is $\frac{\pi}{2}$, and the general value is $n\pi+(-1)^n\frac{\pi}{2}$, by Art. 66. This expression may be simplified; for first suppose n even, denote it by $2m$, so that we have $2m\pi+\frac{\pi}{2}$; next suppose n odd, denote it by $2m+1$, so that we have $(2m+1)\pi-\frac{\pi}{2}$, that is $2m\pi+\frac{\pi}{2}$. Hence both cases are included in the expression $2m\pi+\frac{\pi}{2}$, that is $(4m+1)\frac{\pi}{2}$.

3. $\cos\theta=1$; the smallest value of θ is 0, and the general value is $2n\pi$, by Art. 67.

4. $\cos\theta=-\frac{1}{2}$; the smallest value of θ is $\frac{2\pi}{3}$, and the general value is $2n\pi\pm\frac{2\pi}{3}$, by Art. 67.

5. $\sin^2\theta = \sin^2\alpha$; therefore $\sin\theta = \pm\sin\alpha$. Take the upper sign, then the simplest solution is $\theta = \alpha$, and the general solution is $\theta = n\pi + (-1)^n\alpha$. Take the lower sign, then the simplest solution is $\theta = -\alpha$, and the general solution is $\theta = n\pi - (-1)^n\alpha$. The two expressions are included in the single expression $\theta = n\pi \pm \alpha$.

This might also be obtained from a diagram in the manner of Arts. 66, 67, and 68.

6. Since $\operatorname{cosec}^2\theta = \frac{4}{3}$ we have $\sin^2\theta = \frac{3}{4} = \sin^2\frac{\pi}{3}$: hence, by Example 5, the general solution is $\theta = n\pi \pm \frac{\pi}{3}$.

7. $\cos^2\theta = \cos^2\alpha$; therefore $\cos\theta = \pm\cos\alpha$. Take the upper sign, then the simplest solution is $\theta = \alpha$, and the general solution is $\theta = 2n\pi \pm \alpha$. Take the lower sign, then the simplest solution is $\theta = \pi - \alpha$, and the general solution is $\theta = 2n\pi \pm (\pi - \alpha)$. The two expressions are included in the single expression $\theta = m\pi \pm \alpha$.

It will be seen that the result is the same as for Example 5, and this should be the case; for if $\cos^2\theta = \cos^2\alpha$, then $1 - \cos^2\theta = 1 - \cos^2\alpha$, that is $\sin^2\theta = \sin^2\alpha$.

8. Since $\sec^2\theta = 2$, we have $\cos^2\theta = \frac{1}{2} = \cos^2\frac{\pi}{4}$; hence, by Example 7, the general solution is $\theta = n\pi \pm \frac{\pi}{4}$.

9. $\tan^2\theta = \tan^2\alpha$; therefore $\tan\theta = \pm\tan\alpha$. Take the upper sign, then the simplest solution is $\theta = \alpha$, and the general solution is $\theta = n\pi + \alpha$. Take the lower sign, then the simplest solution is $\theta = -\alpha$, and the general solution is $\theta = n\pi - \alpha$. The two expressions are included in the single expression $\theta = n\pi \pm \alpha$.

The result is the same as for Example 7, and this should be the case; for if $\tan^2\theta = \tan^2\alpha$ then $1 + \tan^2\theta = 1 + \tan^2\alpha$; therefore $\sec^2\theta = \sec^2\alpha$, by Art. 34; therefore $\cos^2\theta = \cos^2\alpha$.

10. $\tan^2\theta = \frac{1}{3} = \tan^2\frac{\pi}{6}$; hence, by Example 9, the general solution is $\theta = n\pi \pm \frac{\pi}{6}$.

11. All the angles included in the expression $2n\pi \pm \alpha$ have the same cosine as α, by Art. 67.

Now by Art. 45 $\sin(2n\pi + \alpha) = \sin\alpha$; and $\sin(2n\pi - \alpha) = \sin(-\alpha) = -\sin\alpha$. Thus the angles which have both the same sine and the same cosine as α are all comprised in the expression $2n\pi + \alpha$.

12. $-\frac{1}{2} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$, and $-\frac{\sqrt{3}}{2} = \cos\left(\pi + \frac{\pi}{6}\right) = \cos\frac{7\pi}{6}$; hence, by Example 11, the required general value is $\theta = 2n\pi + \frac{7\pi}{6}$.

CHAPTER VI.

1. $$\frac{\cos A+\sin A}{\cos A-\sin A}=\frac{(\cos A+\sin A)^2}{(\cos A-\sin A)(\cos A+\sin A)}$$
$$=\frac{\cos^2 A+\sin^2 A+2\sin A\cos A}{\cos^2 A-\sin^2 A}=\frac{1+\sin 2A}{\cos 2A}$$
$$=\frac{\sin 2A}{\cos 2A}+\frac{1}{\cos 2A}=\tan 2A+\sec 2A.$$

2. $2\sin^2 A\sin^2 B+2\cos^2 A\cos^2 B$
$$=\frac{(1-\cos 2A)(1-\cos 2B)}{2}+\frac{(1+\cos 2A)(1+\cos 2B)}{2}$$
$$=\frac{1-\cos 2A-\cos 2B+\cos 2A\cos 2B}{2}+\frac{1+\cos 2A+\cos 2B+\cos 2A\cos 2B}{2}$$
$$=1+\cos 2A\cos 2B.$$

3. $\tan(45^0+A)-\tan(45^0-A)$
$$=\frac{\tan 45^0+\tan A}{1-\tan 45^0\tan A}-\frac{\tan 45^0-\tan A}{1+\tan 45^0\tan A}=\frac{1+\tan A}{1-\tan A}-\frac{1-\tan A}{1+\tan A}$$
$$=\frac{(1+\tan A)^2-(1-\tan A)^2}{1-\tan^2 A}=\frac{4\tan A}{1-\tan^2 A}=2\tan 2A.$$

4. $\sin 3A\operatorname{cosec} A-\cos 3A\sec A$
$$=\frac{\sin 3A}{\sin A}-\frac{\cos 3A}{\cos A}=\frac{3\sin A-4\sin^3 A}{\sin A}-\frac{4\cos^3 A-3\cos A}{\cos A}$$
$$=3-4\sin^2 A-(4\cos^2 A-3)=6-4(\sin^2 A+\cos^2 A)=6-4=2.$$

5. $3\sin A-\sin 3A=3\sin A-(3\sin A-4\sin^3 A)$
$$=4\sin^3 A=2\sin A\times 2\sin^2 A=2\sin A(1-\cos 2A).$$

6. $$\frac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A}=\frac{\sin A+\sin 5A+2\sin 3A}{\sin 3A+\sin 7A+2\sin 5A}$$
$$=\frac{2\sin 3A\cos 2A+2\sin 3A}{2\sin 5A\cos 2A+2\sin 5A},\text{ by Art. 84,}$$
$$=\frac{2\sin 3A(1+\cos 2A)}{2\sin 5A(1+\cos 2A)}=\frac{\sin 3A}{\sin 5A}.$$

7. $$\frac{\sin(2A+B)}{\sin A}-2\cos(A+B)=\frac{\sin(A+B+A)-2\sin A\cos(A+B)}{\sin A}$$
$$=\frac{\sin(A+B)\cos A+\cos(A+B)\sin A-2\sin A\cos(A+B)}{\sin A}$$
$$=\frac{\sin(A+B)\cos A-\cos(A+B)\sin A}{\sin A}=\frac{\sin(A+B-A)}{\sin A}=\frac{\sin B}{\sin A}.$$

8. $4 \sin A \cos^3 A - 4 \cos A \sin^3 A = 4 \sin A \cos A (\cos^2 A - \sin^2 A)$

$$= 2 \sin 2A \cos 2A = \sin 4A.$$

9. $$\frac{\cos A - \cos 3A}{\sin 3A - \sin A} = \frac{2 \sin 2A \sin A}{2 \cos 2A \sin A}, \text{ by Art. 84,}$$

$$= \frac{\sin 2A}{\cos 2A} = \tan 2A.$$

10. $$\frac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \frac{2 \sin 3A \sin A}{2 \cos 3A \sin A}, \text{ by Art. 84,}$$

$$= \frac{\sin 3A}{\cos 3A} = \tan 3A.$$

11. $$\operatorname{cosec} 2A + \cot 4A = \frac{1}{\sin 2A} + \frac{\cos 4A}{\sin 4A}$$

$$= \frac{2 \cos 2A}{2 \cos 2A \sin 2A} + \frac{\cos 4A}{\sin 4A} = \frac{2 \cos 2A + \cos 4A}{\sin 4A}$$

$$= \frac{2 \cos 2A + 2 \cos^2 2A - 1}{\sin 4A} = \frac{2 \cos 2A (1 + \cos 2A) - 1}{\sin 4A}$$

$$= \frac{2 \cos 2A (1 + \cos 2A)}{2 \sin 2A \cos 2A} - \frac{1}{\sin 4A} = \frac{1 + \cos 2A}{\sin 2A} - \frac{1}{\sin 4A}$$

$$= \frac{2 \cos^2 A}{2 \sin A \cos A} - \frac{1}{\sin 4A} = \frac{\cos A}{\sin A} - \frac{1}{\sin 4A} = \cot A - \operatorname{cosec} 4A.$$

12. $\cos^2 (A - B) + \cos^2 B - 2 \cos (A - B) \cos A \cos B$

$$= \cos (A - B) \{\cos (A - B) - \cos A \cos B\}$$
$$+ \cos B \{\cos B - \cos (A - B) \cos A\}$$
$$= \cos (A - B) \sin A \sin B$$
$$+ \cos B \{\cos (A - \overline{A - B}) - \cos (A - B) \cos A\}$$
$$= \cos (A - B) \sin A \sin B + \cos B \sin A \sin (A - B)$$
$$= \sin A \{\cos (A - B) \sin B + \sin (A - B) \cos B\}$$
$$= \sin A \sin (A - B + B) = \sin A \sin A = \sin^2 A.$$

13. $\sin^2 (A - B) + \sin^2 B + 2 \sin (A - B) \sin B \cos A$

$$= \sin (A - B) \{\sin (A - B) + \sin B \cos A\}$$
$$+ \sin B \{\sin B + \sin (A - B) \cos A\}$$
$$= \sin (A - B) \sin A \cos B$$
$$+ \sin B \{\sin (A - \overline{A - B}) + \sin (A - B) \cos A\}$$
$$= \sin (A - B) \sin A \cos B + \sin B \sin A \cos (A - B)$$
$$= \sin A \{\sin (A - B) \cos B + \cos (A - B) \sin B\}$$
$$= \sin A \sin (A - B + B) = \sin A \sin A = \sin^2 A.$$

14. $$\frac{1-\tan^2(45^0-A)}{1+\tan^2(45^0-A)}=\frac{1-\dfrac{\sin^2(45^0-A)}{\cos^2(45^0-A)}}{1+\dfrac{\sin^2(45^0-A)}{\cos^2(45^0-A)}}$$

$$=\frac{\cos^2(45^0-A)-\sin^2(45^0-A)}{\cos^2(45^0-A)+\sin^2(45^0-A)}=\frac{\cos 2(45^0-A)}{1}$$

$$=\cos(90^0-2A)=\sin 2A.$$

15. $$\frac{4\tan A(1-\tan^2 A)}{(1+\tan^2 A)^2}=\frac{\dfrac{4\sin A}{\cos A}\left(1-\dfrac{\sin^2 A}{\cos^2 A}\right)}{\left(1+\dfrac{\sin^2 A}{\cos^2 A}\right)^2}$$

$$=\frac{4\sin A\cos A(\cos^2 A-\sin^2 A)}{(\cos^2 A+\sin^2 A)^2}=2\sin 2A\cos 2A$$

$$=\sin 4A.$$

16. $$\sin A(1+\tan A)+\cos A(1+\cot A)$$

$$=\sin A\left(1+\frac{\sin A}{\cos A}\right)+\cos A\left(1+\frac{\cos A}{\sin A}\right)$$

$$=\sin A+\frac{\sin^2 A}{\cos A}+\cos A+\frac{\cos^2 A}{\sin A}$$

$$=\sin A+\frac{1-\cos^2 A}{\cos A}+\cos A+\frac{1-\sin^2 A}{\sin A}$$

$$=\sin A+\frac{1}{\cos A}-\cos A+\cos A+\frac{1}{\sin A}-\sin A$$

$$=\frac{1}{\cos A}+\frac{1}{\sin A}=\sec A+\operatorname{cosec} A.$$

17. $$\frac{\sin 3A+\cos 3A}{\sin 3A-\cos 3A}=\frac{3\sin A-4\sin^3 A+4\cos^3 A-3\cos A}{3\sin A-4\sin^3 A-4\cos^3 A+3\cos A}$$

$$=\frac{3(\sin A-\cos A)-4(\sin^3 A-\cos^3 A)}{3(\sin A+\cos A)-4(\sin^3 A+\cos^3 A)}$$

$$=\frac{\sin A-\cos A}{\sin A+\cos A}\times\frac{3-4(\sin^2 A+\cos^2 A+\sin A\cos A)}{3-4(\sin^2 A+\cos^2 A-\sin A\cos A)}$$

$$=\frac{\sin A-\cos A}{\sin A+\cos A}\times\frac{-1-4\sin A\cos A}{-1+4\sin A\cos A}$$

$$=\frac{\dfrac{\sin A}{\cos A}-1}{\dfrac{\sin A}{\cos A}+1}\,\frac{1+2\sin 2A}{1-2\sin 2A}$$

$$=\frac{\tan A-1}{\tan A+1}\,\frac{1+2\sin 2A}{1-2\sin 2A}$$

$$=\tan(A-45^0)\,\frac{1+2\sin 2A}{1-2\sin 2A}.$$

18. $\cos A+\cos(120^0-A)+\cos(120^0+A)$

$=\cos A+\cos 120^0\cos A+\sin 120^0\sin A+\cos 120^0\cos A-\sin 120^0\sin A$

$=\cos A+2\cos 120^0\cos A=\cos A-\cos A=0.$

19. $4\sin A\sin(60^0-A)\sin(60^0+A)$

$=4\sin A(\sin^2 60^0-\sin^2 A)$, by Art. 83,

$$=4\sin A\left(\frac{3}{4}-\sin^2 A\right)$$

$=3\sin A-4\sin^3 A=\sin 3A.$

20. $4\cos A\cos(60^0+A)\cos(60^0-A)$

$=4\cos A(\cos^2 A-\sin^2 60^0)$, by Art. 83,

$$=4\cos A\left(\cos^2 A-\frac{3}{4}\right)$$

$=4\cos^3 A-3\cos A=\cos 3A.$

21. $\tan A\tan(60^0+A)\tan(120^0+A)$

$$=\frac{\sin A\sin(60^0+A)\sin(120^0+A)}{\cos A\cos(60^0+A)\cos(120^0+A)}$$

$$=-\frac{\sin A\sin(60^0+A)\sin(60^0-A)}{\cos A\cos(60^0+A)\cos(60^0-A)},\text{ by Art. 48,}$$

$$=-\frac{\sin 3A}{\cos 3A},\text{ by Examples 19 and 20, }=-\tan 3A.$$

22. $\tan A+\tan(60^0+A)+\tan(120^0+A)$

$=\tan A+\tan(60^0+A)-\tan(60^0-A)$, by Art. 48,

$$=\tan A+\frac{\tan 60^0+\tan A}{1-\tan 60^0\tan A}-\frac{\tan 60^0-\tan A}{1+\tan 60^0\tan A}$$

$$=\tan A+\frac{(\tan 60^0+\tan A)(1+\tan 60^0\tan A)-(\tan 60^0-\tan A)(1-\tan 60^0\tan A)}{1-\tan^2 60^0\tan^2 A}$$

$$=\tan A+\frac{2\tan^2 60^0\tan A+2\tan A}{1-\tan^2 60^0\tan^2 A}$$

$$=\tan A+\frac{8\tan A}{1-3\tan^2 A}=\frac{9\tan A-3\tan^3 A}{1-3\tan^2 A}$$

$=3\tan 3A.$

23. $\cot A + \cot(60^0 + A) + \cot(120^0 + A)$

$$= \frac{1}{\tan A} + \frac{1}{\tan(60^0 + A)} - \frac{1}{\tan(60^0 - A)}$$

$$= \frac{1}{\tan A} + \frac{1 - \tan 60^0 \tan A}{\tan 60^0 + \tan A} - \frac{1 + \tan 60^0 \tan A}{\tan 60^0 - \tan A}$$

$$= \frac{1}{\tan A} + \frac{(1 - \tan 60^0 \tan A)(\tan 60^0 - \tan A) - (1 + \tan 60^0 \tan A)(\tan 60^0 + \tan A)}{\tan^2 60^0 - \tan^2 A}$$

$$= \frac{1}{\tan A} - \frac{2\tan^2 60^0 \tan A + 2\tan A}{\tan^2 60^0 - \tan^2 A}$$

$$= \frac{1}{\tan A} - \frac{8\tan A}{3 - \tan^2 A} = \frac{3 - 9\tan^2 A}{3\tan A - \tan^3 A}$$

$$= \frac{3}{\tan 3A} = 3\cot 3A.$$

24. $\cot A \cot(60^0 + A) + \cot(60^0 + A)\cot(120^0 + A) + \cot(120^0 + A)\cot A$

$$= \frac{1}{\tan A \tan(60^0 + A)} + \frac{1}{\tan(60^0 + A)\tan(120^0 + A)} + \frac{1}{\tan(120^0 + A)\tan A}$$

$$= \frac{\tan(120^0 + A) + \tan A + \tan(60^0 + A)}{\tan A \tan(60^0 + A)\tan(120^0 + A)}$$

$$= \frac{3\tan 3A}{-\tan 3A}, \text{ by Examples 21 and 22, } = -3.$$

25. $\sin^3 A = \frac{1}{4}\{3\sin A - \sin 3A\}$,

$$\sin^3(120^0 + A) = \frac{1}{4}\{3\sin(120^0 + A) - \sin 3(120^0 + A)\}$$

$$= \frac{1}{4}\{3\sin(120^0 + A) - \sin 3A\},$$

$$\sin^3(240^0 + A) = \frac{1}{4}\{3\sin(240^0 + A) - \sin 3(240^0 + A)\}$$

$$= \frac{1}{4}\{3\sin(240^0 + A) - \sin 3A\}.$$

By addition we obtain

$$\frac{3}{4}\{\sin A + \sin(120^0 + A) + \sin(240^0 + A)\} - \frac{3}{4}\sin 3A,$$

that is $-\frac{3}{4}\sin 3A$; for

$$\sin A + \sin(120^0 + A) + \sin(240^0 + A)$$

$$= \sin A + \sin(60^0 - A) - \sin(60^0 + A)$$

$$= \sin A + \sin 60^0 \cos A - \cos 60^0 \sin A - \sin 60^0 \cos A - \cos 60^0 \sin A$$

$$= \sin A - 2\cos 60^0 \sin A = \sin A - \sin A = 0.$$

26. $\sin 3A \sin^3 A + \cos 3A \cos^3 A$

$$= (3\sin A - 4\sin^3 A)\sin^3 A + (4\cos^3 A - 3\cos A)\cos^3 A$$
$$= 3(\sin^4 A - \cos^4 A) - 4\sin^6 A + 4\cos^6 A$$
$$= 3(\sin^4 A - \cos^4 A)(\sin^2 A + \cos^2 A) - 4\sin^6 A + 4\cos^6 A$$
$$= \cos^6 A - 3\cos^4 A\sin^2 A + 3\cos^2 A\sin^4 A - \sin^6 A$$
$$= (\cos^2 A - \sin^2 A)^3 = \cos^3 2A.$$

27. $\dfrac{\cos^3 A\sin 3A}{3} + \dfrac{\sin^3 A\cos 3A}{3}$

$$= \frac{1}{12}(3\cos A + \cos 3A)\sin 3A + \frac{1}{12}(3\sin A - \sin 3A)\cos 3A$$
$$= \frac{1}{4}(\sin 3A\cos A + \cos 3A\sin A)$$
$$= \frac{1}{4}\sin(3A + A) = \frac{1}{4}\sin 4A.$$

28. $\cos nA\cos(n+2)A = \cos\{(n+1)A - A\}\cos\{(n+1)A + A\}$

$$= \cos^2(n+1)A - \sin^2 A, \text{ by Art. 83;}$$

therefore $\cos nA\cos(n+2)A - \cos^2(n+1)A + \sin^2 A = 0.$

29. $\dfrac{\sin A \pm \sin nA + \sin(2n-1)A}{\cos A \pm \cos nA + \cos(2n-1)A}$

$$= \frac{\sin A + \sin(2n-1)A \pm \sin nA}{\cos A + \cos(2n-1)A \pm \cos nA}$$
$$= \frac{2\sin nA\cos(n-1)A \pm \sin nA}{2\cos nA\cos(n-1)A \pm \cos nA}, \text{ by Art. 84,}$$
$$= \frac{\sin nA\{2\cos(n-1)A \pm 1\}}{\cos nA\{2\cos(n-1)A \pm 1\}} = \frac{\sin nA}{\cos nA} = \tan nA.$$

30. $\sin nA\operatorname{cosec}^2 A\sec A - \cos nA\sec^2 A\operatorname{cosec} A$

$$= \frac{\sin nA}{\cos A\sin^2 A} - \frac{\cos nA}{\cos^2 A\sin A}$$
$$= \frac{\sin nA\cos A - \cos nA\sin A}{\sin^2 A\cos^2 A} = \frac{4\sin(nA - A)}{4\sin^2 A\cos^2 A}$$
$$= \frac{4\sin(nA - A)}{\sin^2 2A} = 4\sin(nA - A)\operatorname{cosec}^2 2A.$$

31. $\cos 10A + \cos 8A + 3\cos 4A + 3\cos 2A$

$$= 2\cos 9A\cos A + 6\cos 3A\cos A, \text{ by Art. 84,}$$
$$= 2\cos A(\cos 9A + 3\cos 3A)$$
$$= 2\cos A(4\cos^3 3A - 3\cos 3A + 3\cos 3A)$$
$$= 8\cos A\cos^3 3A.$$

32. $\cot A + \cot 2A + \cot 4A = \dfrac{\cos A}{\sin A} + \dfrac{\cos 2A}{\sin 2A} + \dfrac{\cos 4A}{\sin 4A}$

$$= \frac{2\cos^2 A}{2\sin A \cos A} + \frac{\cos 2A}{\sin 2A} + \frac{\cos 4A}{\sin 4A} = \frac{1 + 2\cos 2A}{\sin 2A} + \frac{\cos 4A}{\sin 4A}$$

$$= \frac{2\cos 2A\,(1 + 2\cos 2A)}{2\sin 2A \cos 2A} + \frac{\cos 4A}{\sin 4A}$$

$$= \frac{1}{\sin 4A}\{2\cos 2A + 4\cos^2 2A + \cos 4A\}$$

$$= \frac{1}{\sin 4A}\{2\cos 2A + 2(1 + \cos 4A) + \cos 4A\}$$

$$= \operatorname{cosec} 4A\,\{2 + 2\cos 2A + 3\cos 4A\}.$$

33. $\dfrac{2\sin 2A + 2\cos 2A}{\cos A - \sin A - \cos 3A + \sin 3A} = \dfrac{2(\sin 2A + \cos 2A)}{\cos A - \cos 3A + \sin 3A - \sin A}$

$$= \frac{2(\sin 2A + \cos 2A)}{2\sin 2A \sin A + 2\cos 2A \sin A}, \text{ by Art. 84,}$$

$$= \frac{2(\sin 2A + \cos 2A)}{2(\sin 2A + \cos 2A)\sin A} = \frac{1}{\sin A}.$$

34. $(\cos A - \sin 3A)^2 + 2\cos A \sin 3A\,(\cos A - \sin A)^2$

$$= \cos^2 A + \sin^2 3A - 2\cos A \sin 3A + 2\cos A \sin 3A\,(1 - 2\sin A \cos A)$$

$$= \cos^2 A + \sin^2 3A - 2\cos A \sin 3A \sin 2A$$

$$= \cos A\,\{\cos A - \sin 3A \sin 2A\} + \sin 3A\,\{\sin 3A - \cos A \sin 2A\}$$

$$= \cos A\,\{\cos(3A - 2A) - \sin 3A \sin 2A\}$$

$$+ \sin 3A\,\{\sin(2A + A) - \cos A \sin 2A\}$$

$$= \cos A \cos 3A \cos 2A + \sin 3A \sin A \cos 2A$$

$$= \cos 2A\,\{\cos 3A \cos A + \sin 3A \sin A\}$$

$$= \cos 2A \cos(3A - A) = \cos 2A \cos 2A = \cos^2 2A.$$

35. $\cos^6 A - \sin^6 A = (\cos^2 A - \sin^2 A)(\cos^4 A + \sin^4 A + \sin^2 A \cos^2 A)$

$$= \cos 2A\,(\cos^4 A + \sin^4 A + \sin^2 A \cos^2 A)$$

$$= \cos 2A\,\{(\cos^2 A + \sin^2 A)^2 - \sin^2 A \cos^2 A\}$$

$$= \cos 2A\,\{1 - \sin^2 A \cos^2 A\} = \cos 2A\left\{1 - \frac{\sin^2 2A}{4}\right\}.$$

36. $\sin 5A = \sin(3A + 2A) = \sin 3A \cos 2A + \cos 3A \sin 2A$

$$= (3\sin A - 4\sin^3 A)(1 - 2\sin^2 A) + (4\cos^3 A - 3\cos A)\,2\sin A \cos A$$

$$= (3\sin A - 4\sin^3 A)(1 - 2\sin^2 A) + (4\cos^2 A - 3)\,2\sin A \cos^2 A$$

$$= (3\sin A - 4\sin^3 A)(1 - 2\sin^2 A) + (1 - 4\sin^2 A)\,2\sin A\,(1 - \sin^2 A)$$

$$= 5\sin A - 20\sin^3 A + 16\sin^5 A.$$

37. $$\tan\left(\frac{\pi}{4}-\theta\right)+\cot\left(\frac{\pi}{4}-\theta\right)=4;$$

therefore $$\frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\cos\left(\frac{\pi}{4}-\theta\right)}+\frac{\cos\left(\frac{\pi}{4}-\theta\right)}{\sin\left(\frac{\pi}{4}-\theta\right)}=4;$$

therefore $$\sin^2\left(\frac{\pi}{4}-\theta\right)+\cos^2\left(\frac{\pi}{4}-\theta\right)=4\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\theta\right);$$

therefore $$1=2\sin\left(\frac{\pi}{2}-2\theta\right)=2\cos 2\theta;$$

therefore $$\cos 2\theta=\frac{1}{2};$$

therefore $$2\theta=2n\pi\pm\frac{\pi}{3},$$

therefore $$\theta=n\pi\pm\frac{\pi}{6}.$$

38. $\sin 4\theta+\sin\theta=0$, therefore $2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}=0$ by Art. 84; therefore either $\sin\frac{5\theta}{2}=0$, or $\cos\frac{3\theta}{2}=0$. The former gives $\frac{5\theta}{2}=n\pi$; and the latter gives $\frac{3\theta}{2}=2n\pi\pm\frac{\pi}{2}$, which may be expressed more simply as $\frac{3\theta}{2}=m\pi+\frac{\pi}{2}$.

Or we might proceed thus: $\sin 4\theta=-\sin\theta$, therefore $\sin 4\theta=\sin(\pi+\theta)$. Thus 4θ and $\pi+\theta$ must be angles which have the same sine; and therefore all the solutions are contained in $4\theta=n\pi+(-1)^n(\pi+\theta)$.

39. $\sin 7\theta-\sin\theta=\sin 3\theta$; therefore $2\sin 3\theta\cos 4\theta=\sin 3\theta$; therefore either $\sin 3\theta=0$, or $2\cos 4\theta=1$. The former gives $3\theta=n\pi$; and the latter gives $4\theta=2n\pi\pm\frac{\pi}{3}$.

40. $\sin\theta+\cos\theta=\frac{1}{\sqrt{2}}$; therefore $\frac{\cos\theta}{\sqrt{2}}+\frac{\sin\theta}{\sqrt{2}}=\frac{1}{2}$;

therefore $\cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2}$; therefore $\theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{3}$.

41. By Example 36 we have

$$\sin 5\theta=5\sin\theta-20\sin^3\theta+16\sin^5\theta.$$

Thus $$5 \sin \theta - 20 \sin^3 \theta + 16 \sin^5 \theta = 16 \sin^5 \theta,$$

therefore $$5 \sin \theta - 20 \sin^3 \theta = 0,$$

therefore $$\text{either } \sin \theta = 0 \text{ or } \sin^2 \theta = \frac{1}{4}.$$

The former gives $\theta = n\pi$; the latter gives $\sin^2 \theta = \sin^2 \frac{\pi}{6}$; and therefore $\theta = n\pi \pm \frac{\pi}{6}$, by Example V. 5.

42. $\cos 3\theta + \cos 2\theta + \cos \theta = 0$, therefore $\cos 2\theta + 2 \cos 2\theta \cos \theta = 0$, therefore either $\cos 2\theta = 0$, or $\cos \theta = -\frac{1}{2}$. The former gives $2\theta = n\pi + \frac{\pi}{2}$, as in Example 38; and the latter gives $\theta = 2n\pi \pm \frac{2\pi}{3}$.

43. $\sin 3\theta + \sin 2\theta + \sin \theta = 0$, therefore $\sin 2\theta + 2 \sin 2\theta \cos \theta = 0$, therefore either $\sin 2\theta = 0$, or $\cos \theta = -\frac{1}{2}$. The former gives $2\theta = n\pi$; and the latter gives $\theta = 2n\pi \pm \frac{2\pi}{3}$.

44. $\tan \theta + \tan\left(\frac{\pi}{4} + \theta\right) = 2$; therefore $\tan \theta + \frac{1 + \tan \theta}{1 - \tan \theta} = 2$,

therefore $$\tan \theta - \tan^2 \theta + 1 + \tan \theta = 2 - 2 \tan \theta,$$

therefore $$\tan^2 \theta - 4 \tan \theta + 1 = 0,$$

therefore $$\frac{\sin^2 \theta}{\cos^2 \theta} - \frac{4 \sin \theta}{\cos \theta} + 1 = 0,$$

therefore $$\sin^2 \theta + \cos^2 \theta = 4 \sin \theta \cos \theta,$$

therefore $$1 = 4 \sin \theta \cos \theta = 2 \sin 2\theta,$$

therefore $$\sin 2\theta = \frac{1}{2}, \text{ therefore } 2\theta = n\pi + (-1)^n \frac{\pi}{6}.$$

45. $\tan 2\theta + \cot \theta = 8 \cos^2 \theta$; therefore $\frac{\sin 2\theta}{\cos 2\theta} + \frac{\cos \theta}{\sin \theta} = 8 \cos^2 \theta$,

therefore $$\sin 2\theta \sin \theta + \cos 2\theta \cos \theta = 8 \cos^2 \theta \sin \theta \cos 2\theta,$$

therefore $$\cos (2\theta - \theta) = 8 \cos^2 \theta \sin \theta \cos 2\theta;$$

therefore $$\text{either } \cos \theta = 0, \text{ or } 1 = 8 \cos \theta \sin \theta \cos 2\theta.$$

The former gives $\theta = n\pi + \frac{\pi}{2}$; the latter gives

$$1 = 4 \sin 2\theta \cos 2\theta = 2 \sin 4\theta,$$

so that $$\sin 4\theta = \frac{1}{2}, \text{ and } 4\theta = n\pi + (-1)^n \frac{\pi}{6}.$$

46. $\tan\left(\frac{\pi}{4}+\theta\right)=3\tan\left(\frac{\pi}{4}-\theta\right)$,

therefore $$\tan\left(\frac{\pi}{4}+\theta\right)=3\cot\left(\frac{\pi}{4}+\theta\right)=\frac{3}{\tan\left(\frac{\pi}{4}+\theta\right)},$$

therefore $$\tan^2\left(\frac{\pi}{4}+\theta\right)=3=\tan^2\frac{\pi}{3},$$

therefore $$\frac{\pi}{4}+\theta=n\pi\pm\frac{\pi}{3}, \text{ by Example V. 9.}$$

CHAPTER VII.

1. Here $\frac{A}{2}$ lies between 225^0 and 315^0; thus $\sin\frac{A}{2}$ is negative, and is numerically greater than $\cos\frac{A}{2}$; hence

$$\sin\frac{A}{2}+\cos\frac{A}{2}=-\sqrt{(1+\sin A)},\quad \sin\frac{A}{2}-\cos\frac{A}{2}=-\sqrt{(1-\sin A)}:$$

therefore $$2\sin\frac{A}{2}=-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}.$$

2. Here $\frac{A}{2}$ lies between 405^0 and 495^0; thus $\sin\frac{A}{2}$ is positive, and is numerically greater than $\cos\frac{A}{2}$; hence

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{(1+\sin A)},\quad \sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{(1-\sin A)}:$$

therefore $$2\cos\frac{A}{2}=\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}.$$

3. Here $\frac{A}{2}$ lies between -45^0 and -135^0; thus $\sin\frac{A}{2}$ is negative, and is numerically greater than $\cos\frac{A}{2}$; hence

$$\sin\frac{A}{2}+\cos\frac{A}{2}=-\sqrt{(1+\sin A)},\quad \sin\frac{A}{2}-\cos\frac{A}{2}=-\sqrt{(1-\sin A)}:$$

therefore $$2\sin\frac{A}{2}=-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}.$$

4. The proposed formula must have arisen from

$$\sin A+\cos A=-\sqrt{(1+\sin 2A)},\quad \sin A-\cos A=\sqrt{(1-\sin 2A)};$$

the former shews that A must lie between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$, and the latter shews that A must lie between $2m\pi+\frac{\pi}{4}$ and $2m\pi+\frac{5\pi}{4}$; hence, by combining these results, it follows that A must lie between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$. See Art. 101.

5. The proposed formula must have arisen from

$$\sin A+\cos A=-\sqrt{(1+\sin 2A)},\quad \sin A-\cos A=-\sqrt{(1-\sin 2A)};$$

the former shews that A must lie between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$, and the latter shews that A must lie between $2m\pi+\frac{5\pi}{4}$ and $2m\pi+\frac{9\pi}{4}$; hence, by combining these results, it follows that A must lie between $2n\pi+\frac{5\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$.

6. The proposed formula must have arisen from

$$\sin A+\cos A=\sqrt{(1+\sin 2A)},\quad \sin A-\cos A=-\sqrt{(1-\sin 2A)};$$

the former shews that A must lie between $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$, and the latter shews that A must lie between $2m\pi+\frac{5\pi}{4}$ and $2m\pi+\frac{9\pi}{4}$, that is, between $2(m+1)\pi-\frac{3\pi}{4}$ and $2(m+1)\pi+\frac{\pi}{4}$: hence, by combining these results, it follows that A must lie between $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{\pi}{4}$.

7. Let A denote the given angle, and m the given ratio. Let x denote one of the two parts, and therefore $A-x$ the other. Then $\sin x=m\sin(A-x)$; thus $\sin x=m(\sin A\cos x-\cos A\sin x)$. Divide by $\cos x$; thus

$$\tan x=m(\sin A-\cos A\tan x),$$

therefore
$$\tan x=\frac{m\sin A}{1+m\cos A}.$$

Thus $\tan x$ is known, and therefore x is known.

8. Let A denote the given angle, and m the given ratio. Let x denote one of the two parts, and therefore $A-x$ the other. Then $\cos x=m\cos(A-x)$; thus $\cos x=m(\cos A\cos x+\sin A\sin x)$. Divide by $\cos x$; thus

$$1=m(\cos A+\sin A\tan x),$$

therefore
$$\tan x=\frac{1-m\cos A}{m\sin A}.$$

Thus $\tan x$ is known, and therefore x is known.

9. Let A denote the given angle, and m the given ratio. Let x denote one of the two parts, and therefore $A-x$ the other. Then $\tan x = m \tan (A-x)$;

thus $$\tan x = \frac{m(\tan A - \tan x)}{1+\tan A \tan x};$$

therefore $$\tan x (1+\tan A \tan x) = m(\tan A - \tan x).$$

Thus we have a quadratic equation from which the value of $\tan x$ may be found.

Or we may proceed thus,

$$\tan x = m \tan (A-x), \text{ therefore } \frac{\sin x}{\cos x} = \frac{m \sin (A-x)}{\cos (A-x)},$$

therefore $$2 \sin x \cos (A-x) = 2m \sin (A-x) \cos x,$$

therefore $$\sin A + \sin (2x-A) = m\{\sin A + \sin (A-2x)\}$$
$$= m\{\sin A - \sin (2x-A)\},$$

therefore $$(m+1) \sin (2x-A) = (m-1) \sin A.$$

Thus $\sin (2x-A)$ is known, and therefore $2x-A$ is known, and therefore x is known.

10. By Art. 87, $$\sin A = \frac{2 \tan \frac{A}{2}}{1+\tan^2 \frac{A}{2}} = \frac{2(2-\sqrt{3})}{1+(2-\sqrt{3})^2}$$

$$= \frac{2(2-\sqrt{3})}{1+4+3-4\sqrt{3}} = \frac{2(2-\sqrt{3})}{4(2-\sqrt{3})} = \frac{1}{2}.$$

11. $$\sin 105^0 + \cos 105^0 = \sqrt{(1+\sin 210^0)},$$

and $$\sin 105^0 - \cos 105^0 = \sqrt{(1-\sin 210^0)};$$

therefore $$2 \cos 105^0 = \sqrt{(1+\sin 210^0)} - \sqrt{(1-\sin 210^0)}$$

$$= \sqrt{\left(1-\frac{1}{2}\right)} - \sqrt{\left(1+\frac{1}{2}\right)} = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}};$$

thus $$2 \cos 105^0 = \frac{1-\sqrt{3}}{\sqrt{2}}, \text{ and } \cos 105^0 = \frac{1-\sqrt{3}}{2\sqrt{2}} = -\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

12. $$\tan 2A = \frac{2 \tan A}{1-\tan^2 A}; \text{ thus } \frac{2 \tan A}{1-\tan^2 A} = -\frac{24}{7};$$

therefore $14 \tan A = -24(1-\tan^2 A)$; therefore $24 \tan^2 A - 14 \tan A = 24$.

By solving this quadratic in the ordinary way we obtain

$$\tan A = \frac{4}{3}, \text{ or } -\frac{3}{4}.$$

Also $$\sin A = \frac{\tan A}{\sqrt{(1+\tan^2 A)}}, \text{ and } \cos A = \frac{1}{\sqrt{(1+\tan^2 A)}}.$$

If $\tan A = \frac{4}{3}$ we get $\sin A = \pm\frac{4}{5}$, and $\cos A = \pm\frac{3}{5}$.

If $\tan A = -\frac{3}{4}$ we get $\sin A = \pm\frac{3}{5}$, and $\cos A = \mp\frac{4}{5}$.

13. $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$. Let $2A = 330^0$, then $\tan 2A = -\frac{1}{\sqrt{3}}$;

therefore $$-\frac{1}{\sqrt{3}} = \frac{2\tan A}{1-\tan^2 A},$$

therefore $-1+\tan^2 A = 2\sqrt{3}\tan A$, therefore $\tan^2 A - 2\sqrt{3}\tan A = 1$.

By solving this quadratic in the ordinary way we obtain $\tan A = \sqrt{3}\pm 2$. But $\tan 165^0$ must be a negative quantity, and is therefore equal to $\sqrt{3}-2$.

14. $$\frac{2\sin A - \sin 2A}{2\sin A + \sin 2A} = \frac{2\sin A - 2\sin A\cos A}{2\sin A + 2\sin A\cos A} = \frac{2\sin A\,(1-\cos A)}{2\sin A\,(1+\cos A)}$$

$$= \frac{1-\cos A}{1+\cos A} = \frac{2\sin^2\frac{A}{2}}{2\cos^2\frac{A}{2}} = \tan^2\frac{A}{2}.$$

15. $2\,\text{vers}\,\frac{1}{2}(180^0+A)\,\text{vers}\,\frac{1}{2}(180^0-A)$

$$= 2\left\{1-\cos\left(90^0+\frac{A}{2}\right)\right\}\left\{1-\cos\left(90^0-\frac{A}{2}\right)\right\}$$

$$= 2\left(1+\sin\frac{A}{2}\right)\left(1-\sin\frac{A}{2}\right) = 2\left(1-\sin^2\frac{A}{2}\right) = 2\cos^2\frac{A}{2};$$

and $$\text{vers}\,(180^0-A) = 1-\cos(180^0-A) = 1+\cos A = 2\cos^2\frac{A}{2}.$$

Thus the proposed expressions are equal.

16. $(\cos A+\cos B)^2 + (\sin A+\sin B)^2$

$= \cos^2 A + \cos^2 B + 2\cos A\cos B + \sin^2 A + \sin^2 B + 2\sin A\sin B$

$= 2 + 2(\cos A\cos B + \sin A\sin B) = 2 + 2\cos(A-B)$

$= 2\{1+\cos(A-B)\} = 4\cos^2\frac{1}{2}(A-B)$.

17. $(\cos A-\cos B)^2 + (\sin A-\sin B)^2$

$= \cos^2 A + \cos^2 B - 2\cos A\cos B + \sin^2 A + \sin^2 B - 2\sin A\sin B$

$= 2 - 2(\cos A\cos B + \sin A\sin B) = 2 - 2\cos(A-B)$

$= 2\{1-\cos(A-B)\} = 4\sin^2\frac{1}{2}(A-B)$.

18. $2\sin^2 22\frac{1}{2}^0 = 1-\cos 45^0$; therefore

$$4\sin^2 22\tfrac{1}{2}^0 = 2-2\cos 45^0 = 2-\frac{2}{\sqrt{2}} = 2-\sqrt{2},$$

therefore $$2\sin 22\tfrac{1}{2}^0 = \sqrt{(2-\sqrt{2})}.$$

And $2\cos^2 22\tfrac{1}{2}^0 = 1 + \cos 45^0$; therefore

$$4\cos^2 22\tfrac{1}{2}^0 = 2 + 2\cos 45^0 = 2 + \frac{2}{\sqrt{2}} = 2 + \sqrt{2},$$

therefore $$2\cos 22\tfrac{1}{2}^0 = \sqrt{(2+\sqrt{2})}.$$

Hence $$\frac{\sin 22\tfrac{1}{2}^0}{\cos 22\tfrac{1}{2}^0} = \frac{\sqrt{(2-\sqrt{2})}}{\sqrt{(2+\sqrt{2})}} = \frac{\sqrt{(2-\sqrt{2})}}{\sqrt{(2+\sqrt{2})}} \cdot \frac{\sqrt{(2-\sqrt{2})}}{\sqrt{(2-\sqrt{2})}}$$

$$= \frac{2-\sqrt{2}}{\sqrt{(4-2)}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1,$$

that is $$\tan 22\tfrac{1}{2}^0 = \sqrt{2} - 1.$$

19. $(\tan A + \cot A)\, 2\tan\frac{A}{2}\left(1 - \tan^2\frac{A}{2}\right)$

$$= \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right) 2\tan\frac{A}{2}\left(1 - \tan^2\frac{A}{2}\right)$$

$$= \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} \cdot \frac{2\sin\frac{A}{2}}{\cos\frac{A}{2}} \cdot \frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}$$

$$= \frac{1}{\sin A \cos A} \cdot \frac{2\sin\frac{A}{2}}{\cos\frac{A}{2}} \cdot \frac{\cos A}{\cos^2\frac{A}{2}}$$

$$= \frac{2\sin\frac{A}{2}}{\sin A \cos^3\frac{A}{2}} = \frac{2\sin\frac{A}{2}}{2\sin\frac{A}{2}\cos^4\frac{A}{2}} = \frac{1}{\cos^4\frac{A}{2}}$$

$$= \left\{\frac{1}{\cos^2\frac{A}{2}}\right\}^2 = \left(1 + \tan^2\frac{A}{2}\right)^2.$$

20. $\tan^2\left(\frac{\pi}{4} + \frac{A}{2}\right) = \left\{\frac{1 + \tan\frac{A}{2}}{1 - \tan\frac{A}{2}}\right\}^2 = \left\{\frac{\cos\frac{A}{2} + \sin\frac{A}{2}}{\cos\frac{A}{2} - \sin\frac{A}{2}}\right\}^2$

$$= \frac{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{1 + \sin A}{1 - \sin A}$$

$$= \frac{\frac{1}{\cos A} + \frac{\sin A}{\cos A}}{\frac{1}{\cos A} - \frac{\sin A}{\cos A}} = \frac{\sec A + \tan A}{\sec A - \tan A}.$$

21. $\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

$$=\sin\frac{\pi}{4}\cos\frac{\theta}{2}-\cos\frac{\pi}{4}\sin\frac{\theta}{2}+\cos\frac{\pi}{4}\cos\frac{\theta}{2}+\sin\frac{\pi}{4}\sin\frac{\theta}{2}$$

$$=2\sin\frac{\pi}{4}\cos\frac{\theta}{2}=\frac{2\cos\frac{\theta}{2}}{\sqrt{2}}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sqrt{2}\sin\frac{\theta}{2}}$$

$$=\frac{\sin\theta}{\sqrt{\left(2\sin^2\frac{\theta}{2}\right)}}=\frac{\sin\theta}{\sqrt{(1-\cos\theta)}}=\frac{\sin\theta}{\sqrt{(\text{vers}\,\theta)}}.$$

22. $4\sin^2\frac{\theta}{4}\left(1-\sin\frac{\theta}{2}\right)=4\sin^2\frac{\theta}{4}\left(\sin^2\frac{\theta}{4}+\cos^2\frac{\theta}{4}-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)$

$$=4\sin^2\frac{\theta}{4}\left(\sin\frac{\theta}{4}-\cos\frac{\theta}{4}\right)^2$$

$$=\left(2\sin^2\frac{\theta}{4}-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)^2$$

$$=\left(1-\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2.$$

And $\sqrt{(1+\sin\theta)}=\sqrt{\left(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}=\sin\frac{\theta}{2}+\cos\frac{\theta}{2}$;

therefore $\{1-\sqrt{(1+\sin\theta)}\}^2=\left(1-\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2.$

23. $2\cos^2\frac{\theta}{2}=1+\cos\theta$; therefore

$$4\cos^4\frac{\theta}{2}=(1+\cos\theta)^2=1+2\cos\theta+\cos^2\theta$$

$$=1+2\cos\theta+\frac{1+\cos2\theta}{2}=\frac{1}{2}(3+4\cos\theta+\cos2\theta);$$

therefore $\cos^4\frac{\theta}{2}=\frac{1}{8}(3+4\cos\theta+\cos2\theta).$

Use this formula for each of the terms; thus

$$\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$$

$$=\frac{12}{8}+\frac{1}{2}\left(\cos\frac{\pi}{4}+\cos\frac{3\pi}{4}+\cos\frac{5\pi}{4}+\cos\frac{7\pi}{4}\right)$$

$$+\frac{1}{8}\left(\cos\frac{\pi}{2}+\cos\frac{3\pi}{2}+\cos\frac{5\pi}{2}+\cos\frac{7\pi}{2}\right)$$

$=\frac{3}{2}$: see Art. 50.

24. $$\tan 7\tfrac{1}{2}^0 = \frac{\sin 15^0}{1+\cos 15^0} = \frac{\dfrac{\sqrt{3}-1}{2\sqrt{2}}}{1+\dfrac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{\sqrt{3}-1}{2\sqrt{2}+\sqrt{3}+1}$$

$$= \frac{(\sqrt{3}-1)(2\sqrt{2}+1-\sqrt{3})}{(2\sqrt{2}+\sqrt{3}+1)(2\sqrt{2}+1-\sqrt{3})} = \frac{2\sqrt{6}-2\sqrt{2}-4+2\sqrt{3}}{6+4\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}-2+\sqrt{3}}{3+2\sqrt{2}}.$$

Multiply both numerator and denominator by $3-2\sqrt{2}$; then we obtain unity for denominator, and for numerator $\sqrt{6}-\sqrt{3}+\sqrt{2}-2$.

25. $$\tan 142\tfrac{1}{2}^0 = \frac{\sin 285^0}{1+\cos 285^0} = -\frac{\sin 105^0}{1-\cos 105^0} = -\frac{\cos 15^0}{1+\sin 15^0}$$

$$= -\frac{\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{1+\dfrac{\sqrt{3}-1}{2\sqrt{2}}} = -\frac{\sqrt{3}+1}{2\sqrt{2}-1+\sqrt{3}}$$

$$= -\frac{(\sqrt{3}+1)(2\sqrt{2}-1-\sqrt{3})}{(2\sqrt{2}-1+\sqrt{3})(2\sqrt{2}-1-\sqrt{3})}$$

$$= -\frac{\sqrt{6}+\sqrt{2}-2-\sqrt{3}}{3-2\sqrt{2}} = \frac{2+\sqrt{3}-\sqrt{2}-\sqrt{6}}{3-2\sqrt{2}}.$$

Multiply both numerator and denominator by $3+2\sqrt{2}$; then we obtain unity for denominator, and for numerator $2+\sqrt{2}-\sqrt{3}-\sqrt{6}$.

26. $$\tan x = \frac{3\tan\dfrac{x}{3} - \tan^3\dfrac{x}{3}}{1-3\tan^2\dfrac{x}{3}};$$

and since this is equal to $(2+\sqrt{3})\tan\dfrac{x}{3}$ we obtain

$$\frac{3-\tan^2\dfrac{x}{3}}{1-3\tan^2\dfrac{x}{3}} = 2+\sqrt{3};$$

therefore $$3-\tan^2\frac{x}{3} = (2+\sqrt{3})\left(1-3\tan^2\frac{x}{3}\right);$$

therefore $$(6+3\sqrt{3}-1)\tan^2\frac{x}{3} = 2+\sqrt{3}-3;$$

therefore $$\tan^2\frac{x}{3}=\frac{\sqrt{3}-1}{5+3\sqrt{3}}=\frac{(\sqrt{3}-1)(5-3\sqrt{3})}{(5+3\sqrt{3})(5-3\sqrt{3})}$$
$$=\frac{8\sqrt{3}-14}{25-27}=7-4\sqrt{3};$$

therefore $$\tan\frac{x}{3}=\sqrt{(7-4\sqrt{3})}=\pm(2-\sqrt{3}).$$

Hence $$\tan x=\pm(2+\sqrt{3})(2-\sqrt{3})=\pm 1.$$

27. $$\tan\alpha+\cot\alpha=\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}=\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha}=\frac{1}{\sin\alpha\cos\alpha}$$
$$=\frac{2}{2\sin\alpha\cos\alpha}=\frac{2}{\sin 2\alpha}.$$

Put for α its value; then the expression

$$=\frac{2}{\sin 2\left(n+\frac{1}{4}\pm\frac{1}{6}\right)\pi}=\frac{2}{\sin\left(\frac{\pi}{2}\pm\frac{\pi}{3}\right)}=\frac{2}{\cos\frac{\pi}{3}}$$
$$=2\div\frac{1}{2}=4.$$

28. $$\frac{\cos\alpha\cos 13\alpha}{\cos 3\alpha+\cos 5\alpha}=\frac{\cos\alpha\cos 13\alpha}{2\cos\alpha\cos 4\alpha}=\frac{\cos 13\alpha}{2\cos 4\alpha}=-\frac{1}{2};$$

for $13\alpha+4\alpha=\pi$, and therefore $\cos 13\alpha=-\cos 4\alpha$.

29. $\sec(\phi+\alpha)+\sec(\phi-\alpha)=2\sec\phi$, therefore

$$\frac{1}{\cos(\phi+\alpha)}+\frac{1}{\cos(\phi-\alpha)}=\frac{2}{\cos\phi};$$

therefore $$\frac{\cos(\phi-\alpha)+\cos(\phi+\alpha)}{\cos(\phi+\alpha)\cos(\phi-\alpha)}=\frac{2}{\cos\phi};$$

therefore $$\frac{2\cos\phi\cos\alpha}{\cos^2\phi-\sin^2\alpha}=\frac{2}{\cos\phi};$$

therefore $$\cos^2\phi\cos\alpha=\cos^2\phi-\sin^2\alpha;$$

therefore $$\cos^2\phi=\frac{\sin^2\alpha}{1-\cos\alpha}=\frac{1-\cos^2\alpha}{1-\cos\alpha}=1+\cos\alpha=2\cos^2\frac{\alpha}{2};$$

therefore $$\cos\phi=\sqrt{2}\cos\frac{\alpha}{2}.$$

30. $\tan^2\frac{\theta}{2} = \frac{1+c}{1-c}\tan^2\frac{\phi}{2}$; therefore

$$\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-\frac{1+c}{1-c}\cdot\frac{\sin^2\frac{\phi}{2}}{\cos^2\frac{\phi}{2}}}{1+\frac{1+c}{1-c}\cdot\frac{\sin^2\frac{\phi}{2}}{\cos^2\frac{\phi}{2}}} = \frac{(1-c)\cos^2\frac{\phi}{2}-(1+c)\sin^2\frac{\phi}{2}}{(1-c)\cos^2\frac{\phi}{2}+(1+c)\sin^2\frac{\phi}{2}};$$

therefore, by Art. 87,

$$\cos\theta = \frac{\cos^2\frac{\phi}{2}-\sin^2\frac{\phi}{2} - c\left(\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2}\right)}{\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2} - c\left(\cos^2\frac{\phi}{2}-\sin^2\frac{\phi}{2}\right)} = \frac{\cos\phi - c}{1-c\cos\phi}.$$

CHAPTER VIII.

1. By Art. 113 we have

$$\cos(\alpha+\beta+\gamma) = \cos\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma - \cos\beta\sin\gamma\sin\alpha - \cos\gamma\sin\alpha\sin\beta;$$

divide both sides by $\cos\alpha\cos\beta\cos\gamma$; thus

$$\frac{\cos(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma} = 1 - \tan\beta\tan\gamma - \tan\gamma\tan\alpha - \tan\alpha\tan\beta.$$

2. By Art. 113 we have

$$\sin(\alpha+\beta+\gamma) = \sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\gamma\cos\alpha + \sin\gamma\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma;$$

divide both sides by $\cos\alpha\cos\beta\cos\gamma$; thus

$$\frac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma} = \tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma.$$

3. $\sin(\alpha-\beta)+\sin(\beta-\gamma) = 2\sin\frac{\alpha-\gamma}{2}\cos\frac{\alpha-2\beta+\gamma}{2}$

$$= -2\sin\frac{\gamma-\alpha}{2}\cos\frac{\alpha-2\beta+\gamma}{2};$$

$\sin(\gamma-\alpha) = 2\sin\frac{\gamma-\alpha}{2}\cos\frac{\gamma-\alpha}{2}$;

therefore $\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)$

$$=2\sin\frac{\gamma-\alpha}{2}\left\{\cos\frac{\gamma-\alpha}{2}-\cos\frac{\alpha-2\beta+\gamma}{2}\right\}$$

$$=2\sin\frac{\gamma-\alpha}{2}\,2\sin\frac{\gamma-\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$=-4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2};$$

therefore $\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)$

$$+4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}=0.$$

4. $4\sin(\theta-\alpha)\sin(m\theta-\alpha)\cos(\theta-m\theta)$

$=2\cos(\theta-m\theta)\{\cos(\theta-m\theta)-\cos(\theta+m\theta-2\alpha)\}$, by Art. 84,

$=2\cos^2(\theta-m\theta)-2\cos(\theta-m\theta)\cos(\theta+m\theta-2\alpha)$

$=1+\cos 2(\theta-m\theta)-\{\cos(2\theta-2\alpha)+\cos(2m\theta-2\alpha)\}$

$=1+\cos 2(\theta-m\theta)-\cos(2\theta-2\alpha)-\cos(2m\theta-2\alpha)$.

5. $\sin(\alpha+\beta)\cos\beta=\sin(\alpha+\beta+\gamma-\gamma)\cos\beta$

$=\{\sin(\alpha+\beta+\gamma)\cos\gamma-\cos(\alpha+\beta+\gamma)\sin\gamma\}\cos\beta$,

$\sin(\alpha+\gamma)\cos\gamma=\sin(\alpha+\beta+\gamma-\beta)\cos\gamma$

$=\{\sin(\alpha+\beta+\gamma)\cos\beta-\cos(\alpha+\beta+\gamma)\sin\beta\}\cos\gamma$;

therefore $\sin(\alpha+\beta)\cos\beta-\sin(\alpha+\gamma)\cos\gamma$

$=\cos(\alpha+\beta+\gamma)\{\sin\beta\cos\gamma-\sin\gamma\cos\beta\}$

$=\cos(\alpha+\beta+\gamma)\sin(\beta-\gamma)$.

6. $\cos(\alpha+\beta+\gamma)+\cos(\alpha+\beta-\gamma)=2\cos(\alpha+\beta)\cos\gamma$,

$\cos(\alpha-\beta+\gamma)+\cos(\beta+\gamma-\alpha)=2\cos(\alpha-\beta)\cos\gamma$;

hence the sum $=2\cos\gamma\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}$

$=4\cos\alpha\cos\beta\cos\gamma$.

7. $\cos 2\alpha+\cos 2\beta=2\cos(\alpha+\beta)\cos(\alpha-\beta)$,

$\cos 2\gamma+\cos 2(\alpha+\beta+\gamma)=2\cos(2\gamma+\alpha+\beta)\cos(\alpha+\beta)$;

hence the sum $=2\cos(\alpha+\beta)\{\cos(\alpha-\beta)+\cos(2\gamma+\alpha+\beta)\}$

$=2\cos(\alpha+\beta)\,2\cos(\alpha+\gamma)\cos(\beta+\gamma)$

$=4\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha)$.

8. Reduce the three fractions to have the common denominator

$$\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha);$$

then the whole numerator

$=-\sin\alpha\sin(\beta-\gamma)-\sin\beta\sin(\gamma-\alpha)-\sin\gamma\sin(\alpha-\beta)$

$=-\frac{1}{2}\{\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta-\gamma)\}-\frac{1}{2}\{\cos(\beta+\alpha-\gamma)-\cos(\beta+\gamma-\alpha)\}$

$-\frac{1}{2}\{\cos(\gamma-\alpha+\beta)-\cos(\gamma+\alpha-\beta)\}=0.$

9. $\cos(\alpha+\beta)\sin\beta - \cos(\alpha+\gamma)\sin\gamma$

$$=\frac{1}{2}\{\sin(\alpha+\beta+\beta)-\sin(\alpha+\beta-\beta)\}-\frac{1}{2}\{\sin(\alpha+\gamma+\gamma)-\sin(\alpha+\gamma-\gamma)\}$$

$$=\frac{1}{2}\sin(\alpha+2\beta)-\frac{1}{2}\sin(\alpha+2\gamma);$$

$\sin(\alpha+\beta)\cos\beta-\sin(\alpha+\gamma)\cos\gamma$

$$=\frac{1}{2}\{\sin(\alpha+\beta+\beta)+\sin(\alpha+\beta-\beta)\}-\frac{1}{2}\{\sin(\alpha+\gamma+\gamma)+\sin(\alpha+\gamma-\gamma)\}$$

$$=\frac{1}{2}\sin(\alpha+2\beta)-\frac{1}{2}\sin(\alpha+2\gamma).$$

Thus the two expressions are equal.

10. $\sin(\alpha+\beta-2\gamma)\cos\beta-\sin(\alpha+\gamma-2\beta)\cos\gamma$

$$=\frac{1}{2}\{\sin(\alpha+2\beta-2\gamma)+\sin(\alpha-2\gamma)-\sin(\alpha+2\gamma-2\beta)-\sin(\alpha-2\beta)\};$$

$\sin(\beta-\gamma)\{\cos(\beta+\gamma-\alpha)+\cos(\alpha+\gamma-\beta)+\cos(\alpha+\beta-\gamma)\}$

$$=\frac{1}{2}\{\sin(2\beta-\alpha)+\sin(\alpha-2\gamma)\}+\frac{1}{2}\{\sin\alpha+\sin(2\beta-2\gamma-\alpha)\}$$

$$+\frac{1}{2}\{-\sin\alpha+\sin(2\beta-2\gamma+\alpha)\}$$

$$=\frac{1}{2}\{\sin(2\beta-\alpha)+\sin(\alpha-2\gamma)+\sin(2\beta-2\gamma-\alpha)+\sin(2\beta-2\gamma+\alpha)\}.$$

Thus the two expressions are equal.

11. $\sin(\alpha+\beta+\gamma)\sin\beta=\frac{1}{2}\{\cos(\alpha+\gamma)-\cos(\alpha+2\beta+\gamma)\}$,

$$\sin(\alpha+\beta)\sin(\beta+\gamma)=\frac{1}{2}\{\cos(\alpha-\gamma)-\cos(\alpha+2\beta+\gamma)\},$$

$$\sin\alpha\sin\gamma=\frac{1}{2}\{\cos(\alpha-\gamma)-\cos(\alpha+\gamma)\}.$$

Hence $\sin(\alpha+\beta)\sin(\beta+\gamma)-\sin\alpha\sin\gamma$

$$=\frac{1}{2}\{\cos(\alpha+\gamma)-\cos(\alpha+2\beta+\gamma)\}$$

$$=\sin(\alpha+\beta+\gamma)\sin\beta.$$

12. $\sin\alpha\sin\beta\sin(\beta-\alpha)=\frac{1}{2}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}\sin(\beta-\alpha)$

$$=\frac{1}{2}\cos(\beta-\alpha)\sin(\beta-\alpha)-\frac{1}{4}\{\sin 2\beta-\sin 2\alpha\}$$

$$=\frac{1}{4}\sin 2(\beta-\alpha)-\frac{1}{4}\sin 2\beta+\frac{1}{4}\sin 2\alpha.$$

Similarly we may transform $\sin\beta\sin\gamma\sin(\gamma-\beta)$ and $\sin\gamma\sin\alpha\sin(\alpha-\gamma)$.

Also, by Example 3, we have

$$\sin(\beta-\alpha)\sin(\gamma-\beta)\sin(\alpha-\gamma)=\frac{1}{4}\{\sin 2(\alpha-\beta)+\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)\}.$$

Hence the sum of the four expressions is zero.

13. $$\cos(\alpha+\beta)\sin(\alpha-\beta)=\frac{1}{2}(\sin 2\alpha-\sin 2\beta),$$

$$\cos(\beta+\gamma)\sin(\beta-\gamma)=\frac{1}{2}(\sin 2\beta-\sin 2\gamma),$$

$$\cos(\gamma+\delta)\sin(\gamma-\delta)=\frac{1}{2}(\sin 2\gamma-\sin 2\delta),$$

$$\cos(\delta+\alpha)\sin(\delta-\alpha)=\frac{1}{2}(\sin 2\delta-\sin 2\alpha);$$

hence the sum of the four expressions is zero.

14. $$\sin(\delta-\beta)\sin(\alpha-\gamma)=\frac{1}{2}\{\cos(\alpha+\beta-\gamma-\delta)-\cos(\alpha-\beta-\gamma+\delta)\},$$

$$\sin(\beta-\gamma)\sin(\alpha-\delta)=\frac{1}{2}\{\cos(\alpha-\beta+\gamma-\delta)-\cos(\alpha+\beta-\gamma-\delta)\},$$

$$\sin(\gamma-\delta)\sin(\alpha-\beta)=\frac{1}{2}\{\cos(\alpha-\beta-\gamma+\delta)-\cos(\alpha-\beta+\gamma-\delta)\};$$

hence the sum of the three expressions is zero.

15. $$\cot\frac{A}{2}+\cot\frac{B}{2}=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}}=\frac{\sin\frac{B}{2}\cos\frac{A}{2}+\sin\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}$$

$$=\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{A}{2}\sin\frac{B}{2}}=\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}};$$

$$\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}+\cot\frac{C}{2}=\cos\frac{C}{2}\left\{\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right\}$$

$$= \frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left\{\sin\frac{C}{2} + \sin\frac{A}{2}\sin\frac{B}{2}\right\}$$

$$= \frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left\{\cos\frac{1}{2}(A+B) + \sin\frac{A}{2}\sin\frac{B}{2}\right\}$$

$$= \frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

16. $\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) = 2\cos\frac{C}{2}\cos\frac{1}{2}(A-B)$

$$\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2} = 2\cos\frac{C}{2}\cos\frac{1}{2}(A+B);$$

therefore $\sin A + \sin B + \sin C = 2\cos\frac{C}{2}\left\{\cos\frac{1}{2}(A-B) + \cos\frac{1}{2}(A+B)\right\}$

$$= 2\cos\frac{C}{2}\; 2\cos\frac{A}{2}\cos\frac{B}{2}$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

17. $\sin A + \sin C = 2\sin\frac{1}{2}(A+C)\cos\frac{1}{2}(A-C) = 2\cos\frac{B}{2}\cos\frac{1}{2}(A-C),$

$$\sin B = 2\sin\frac{B}{2}\cos\frac{B}{2} = 2\cos\frac{B}{2}\cos\frac{1}{2}(A+C);$$

therefore $\sin A - \sin B + \sin C = 2\cos\frac{B}{2}\left\{\cos\frac{1}{2}(A-C) - \cos\frac{1}{2}(A+C)\right\}$

$$= 2\cos\frac{B}{2}\; 2\sin\frac{A}{2}\sin\frac{C}{2}$$

$$= 4\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

18. $\cos 2A + \cos 2B = 2\cos(A+B)\cos(A-B) = -2\cos C\cos(A-B)$

$$\cos 2C = 2\cos^2 C - 1 = -2\cos C\cos(A+B) - 1;$$

therefore $\cos 2A + \cos 2B + \cos 2C = -2\cos C\{\cos(A-B) + \cos(A+B)\} - 1$

$$= -2\cos C \,.\, 2\cos A\cos B - 1$$

$$= -4\cos A\cos B\cos C - 1,$$

therefore $\cos 2A + \cos 2B + \cos 2C + 4\cos A\cos B\cos C + 1 = 0.$

19. $\cos 4A + \cos 4B = 2\cos 2(A+B)\cos 2(A-B) = 2\cos 2C\cos 2(A-B)$,

$$\cos 4C = 2\cos^2 2C - 1 = 2\cos 2C\cos 2(A+B) - 1;$$

therefore $\cos 4A + \cos 4B + \cos 4C = 2\cos 2C\{\cos 2(A-B) + \cos 2(A+B)\} - 1$

$$= 2\cos 2C \,.\, 2\cos 2A\cos 2B - 1$$

$$= 4\cos 2A\cos 2B\cos 2C - 1;$$

therefore $\cos 4A + \cos 4B + \cos 4C + 1 = 4\cos 2A\cos 2B\cos 2C$.

20. Let $\alpha = \frac{1}{2}(\pi - A),\ \beta = \frac{1}{2}(\pi - B),\ \gamma = \frac{1}{2}(\pi - C);$

therefore $\alpha + \beta + \gamma = \frac{1}{2}(3\pi - A - B - C) = \frac{1}{2}2\pi = \pi;$

hence, by Example 16,

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2},$$

that is $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi - A}{4}\cos\frac{\pi - B}{4}\cos\frac{\pi - C}{4}.$

21. Let $\alpha = \frac{1}{2}(\pi - A),\ \beta = \frac{1}{2}(\pi - B),\ \gamma = \frac{1}{2}(\pi - C);$

therefore $\alpha + \beta + \gamma = \frac{1}{2}(3\pi - A - B - C) = \frac{1}{2}2\pi = \pi;$

hence, by Example 17,

$$\sin\alpha - \sin\beta + \sin\gamma = 4\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2},$$

that is $\cos\frac{A}{2} - \cos\frac{B}{2} + \cos\frac{C}{2} = 4\sin\frac{\pi - A}{4}\cos\frac{\pi - B}{4}\sin\frac{\pi - C}{4}.$

22. Let $\alpha = \frac{1}{2}(\pi - A),\ \beta = \frac{1}{2}(\pi - B),\ \gamma = \frac{1}{2}(\pi - C);$

therefore $\alpha + \beta + \gamma = \frac{1}{2}(3\pi - A - B - C) = \frac{1}{2}2\pi = \pi;$

hence, by Art. 114,

$$\cos\alpha + \cos\beta + \cos\gamma - 1 = 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2},$$

that is $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - 1 = 4\sin\frac{\pi - A}{4}\sin\frac{\pi - B}{4}\sin\frac{\pi - C}{4}.$

23. $\sin^2 A + \sin^2 B + \sin^2 C = \frac{1}{2}\{1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C\}$

$$= \frac{3}{2} - \frac{1}{2}\{\cos 2A + \cos 2B + \cos 2C\}$$

$$= \frac{3}{2} + \frac{1}{2}\{1 + 4\cos A \cos B \cos C\}, \text{ by Example 18,}$$

$$= 2 + 2\cos A \cos B \cos C;$$

therefore $\sin^2 A + \sin^2 B + \sin^2 C - 2\cos A \cos B \cos C = 2.$

24. $\sin^2 2A + \sin^2 2B + \sin^2 2C = \frac{1}{2}\{3 - \cos 4A - \cos 4B - \cos 4C\}$

$$= \frac{3}{2} - \frac{1}{2}\{4\cos 2A \cos 2B \cos 2C - 1\}; \text{ by Example 19,}$$

$$= 2 - 2\cos 2A \cos 2B \cos 2C;$$

therefore $\sin^2 2A + \sin^2 2B + \sin^2 2C + 2\cos 2A \cos 2B \cos 2C = 2.$

25. $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2}$

$$= \frac{1}{\cot\frac{A}{2}\cot\frac{B}{2}} + \frac{1}{\cot\frac{B}{2}\cot\frac{C}{2}} + \frac{1}{\cot\frac{C}{2}\cot\frac{A}{2}}$$

$$= \frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}} = 1, \text{ by Example 15.}$$

26. $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$, by Example 17;

$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, by Example 16;

therefore, by division,

$$\frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \tan\frac{A}{2}\tan\frac{B}{2}.$$

27. $\cos A \sin B \sin C + \cos B \sin A \sin C + \cos C \sin A \sin B$

$$= \sin C(\cos A \sin B + \cos B \sin A) + \cos C \sin A \sin B$$

$$= \sin C \sin(A + B) + \cos C \sin A \sin B$$

$$= \sin^2 C + \cos C \sin A \sin B$$

$$= 1 - \cos^2 C + \cos C \sin A \sin B$$

$$= 1 + \cos C\{\cos(A + B) + \sin A \sin B\}$$

$$= 1 + \cos C \cos A \cos B.$$

28. Take Example 27, and divide by $\sin A \sin B \sin C$;

therefore $$\frac{1}{\sin A \sin B \sin C} + \frac{\cos A \cos B \cos C}{\sin A \sin B \sin C} = \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}:$$

thus we obtain the required result.

29. By Example 17 we have

$$\frac{(\sin B + \sin C - \sin A)(\sin C + \sin A - \sin B)}{4 \sin A \sin B}$$

$$= \frac{16 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{A}{2} \sin \frac{C}{2} \sin \frac{A}{2} \cos \frac{B}{2}}{16 \sin \frac{A}{2} \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{B}{2}} = \sin^2 \frac{C}{2}.$$

30. $$\cot A + \frac{\sin A}{\sin B \sin C} = \frac{\cos A}{\sin A} + \frac{\sin A}{\sin B \sin C}$$

$$= \frac{\cos A \sin B \sin C + \sin^2 A}{\sin A \sin B \sin C} = \frac{1 - \cos^2 A + \cos A \sin B \sin C}{\sin A \sin B \sin C}$$

$$= \frac{1 + \cos A \{\cos (B + C) + \sin B \sin C\}}{\sin A \sin B \sin C} = \frac{1 + \cos A \cos B \cos C}{\sin A \sin B \sin C}.$$

We have thus an expression which involves A, B, and C symmetrically; and we shall in the same manner obtain the same result if in the original expression any two of the quantities A, B, C be interchanged.

31. By Art. 114, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$;

by Example 16, $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$;

therefore, by division,

$$\frac{\tan A + \tan B + \tan C}{(\sin A + \sin B + \sin C)^2} = \frac{\tan A \tan B \tan C}{16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}}$$

$$= \frac{8 \sin \frac{A}{2} \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{B}{2} \sin \frac{C}{2} \cos \frac{C}{2}}{16 \cos A \cos B \cos C \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} = \frac{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}{2 \cos A \cos B \cos C}.$$

32. $$\sin nA + \sin nB = 2 \sin \frac{n}{2}(A + B) \cos \frac{n}{2}(A - B)$$

$$= 2 \sin \frac{n}{2}(\pi - C) \cos \frac{n}{2}(A - B)$$

$$= 2 \left\{\sin \frac{n\pi}{2} \cos \frac{nC}{2} - \cos \frac{n\pi}{2} \sin \frac{nC}{2}\right\} \cos \frac{n}{2}(A - B)$$

$$= 2 \sin \frac{n\pi}{2} \cos \frac{nC}{2} \cos \frac{n}{2}(A - B);\ \text{since } \cos \frac{n\pi}{2} = 0.$$

Also $\sin nC = 2\sin\frac{nC}{2}\cos\frac{nC}{2} = 2\sin\frac{n}{2}(\pi - A - B)\cos\frac{nC}{2}$

$$= 2\left\{\sin\frac{n\pi}{2}\cos\frac{n}{2}(A+B) - \cos\frac{n\pi}{2}\sin\frac{n}{2}(A+B)\right\}\cos\frac{nC}{2}$$

$$= 2\sin\frac{n\pi}{2}\cos\frac{n}{2}(A+B)\cos\frac{nC}{2}.$$

Therefore $\sin nA + \sin nB + \sin nC$

$$= 2\sin\frac{n\pi}{2}\cos\frac{nC}{2}\left\{\cos\frac{n}{2}(A-B) + \cos\frac{n}{2}(A+B)\right\}$$

$$= 4\sin\frac{n\pi}{2}\cos\frac{nA}{2}\cos\frac{nB}{2}\cos\frac{nC}{2}.$$

33. Proceed as in Example 32. Thus

$$\sin nA + \sin nB = 2\left\{\sin\frac{n\pi}{2}\cos\frac{nC}{2} - \cos\frac{n\pi}{2}\sin\frac{nC}{2}\right\}\cos\frac{n}{2}(A-B)$$

$$= -2\cos\frac{n\pi}{2}\sin\frac{nC}{2}\cos\frac{n}{2}(A-B);\ \text{since } \sin\frac{n\pi}{2} = 0.$$

Also $\sin nC = 2\sin\frac{nC}{2}\cos\frac{nC}{2} = 2\cos\frac{n}{2}(\pi - A - B)\sin\frac{nC}{2}$

$$= 2\left\{\cos\frac{n\pi}{2}\cos\frac{n}{2}(A+B) + \sin\frac{n\pi}{2}\sin\frac{n}{2}(A+B)\right\}\sin\frac{nC}{2}$$

$$= 2\cos\frac{n\pi}{2}\sin\frac{nC}{2}\cos\frac{n}{2}(A+B).$$

Therefore $\sin nA + \sin nB + \sin nC$

$$= -2\cos\frac{n\pi}{2}\sin\frac{nC}{2}\left\{\cos\frac{n}{2}(A-B) - \cos\frac{n}{2}(A+B)\right\}$$

$$= -4\cos\frac{n\pi}{2}\sin\frac{nC}{2}\sin\frac{nA}{2}\sin\frac{nB}{2}.$$

34. By Example 20,

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi - A}{4}\cos\frac{\pi - B}{4}\cos\frac{\pi - C}{4}$$

$$= 4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}.$$

35. $\frac{\tan B}{\tan A} + \frac{\tan C}{\tan A} = \frac{1}{\tan A}\left(\frac{\sin B}{\cos B} + \frac{\sin C}{\cos C}\right) = \frac{\sin(B+C)}{\tan A\cos B\cos C}$

$$= \frac{\sin A}{\tan A\cos B\cos C} = \frac{\cos A}{\cos B\cos C}.$$

In this way we see that the given expression

$$=\frac{\cos A}{\cos B\cos C}+\frac{\cos B}{\cos C\cos A}+\frac{\cos C}{\cos A\cos B}$$

$$=\frac{\cos^2 A+\cos^2 B+\cos^2 C}{\cos A\ \cos B\ \cos C}=\frac{3-\sin^2 A-\sin^2 B-\sin^2 C}{\cos A\ \cos B\cos C}$$

$$=\frac{1-2\cos A\cos B\cos C}{\cos A\cos B\cos C},\text{ by Example 23,}$$

$$=\sec A\ \sec B\sec C-2.$$

36. Suppose $A+B+C+D=180^0$; then $A+B=180^0-C-D$;

therefore $\tan(A+B)=-\tan(C+D)$, by Art. 48;

therefore $$\frac{\tan A+\tan B}{1-\tan A\ \tan B}=-\frac{\tan C+\tan D}{1-\tan C\tan D};$$

therefore $(\tan A+\tan B)(1-\tan C\tan D)=-(\tan C+\tan D)(1-\tan A\ \tan B)$,

therefore $\tan A+\tan B+\tan C+\tan D$

$$=(\tan A+\tan B)\tan C\tan D+(\tan C+\tan D)\tan A\ \tan B$$

$$=\tan B\tan C\tan D+\tan A\tan C\tan D+\tan A\ \tan B\tan D$$
$$+\tan A\tan B\tan C.$$

37. $$\frac{\sin^2 C}{\sin^2 A}=1-\frac{\tan(A-B)}{\tan A}=1-\frac{\sin(A-B)\cos A}{\cos(A-B)\sin A}$$

$$=\frac{\sin A\cos(A-B)-\cos A\sin(A-B)}{\cos(A-B)\sin A}=\frac{\sin\{A-(A-B)\}}{\cos(A-B)\sin A}$$

$$=\frac{\sin B}{\cos(A-B)\sin A};\quad\text{therefore }\sin^2 C=\frac{\sin A\ \sin B}{\cos(A-B)}.$$

Hence $$\cos^2 C=1-\sin^2 C=1-\frac{\sin A\ \sin B}{\cos(A-B)}$$

$$=\frac{\cos(A-B)-\sin A\ \sin B}{\cos(A-B)}=\frac{\cos A\ \cos B}{\cos(A-B)}.$$

Therefore $$\frac{\sin^2 C}{\cos^2 C}=\frac{\sin A\sin B}{\cos(A-B)}\div\frac{\cos A\ \cos B}{\cos(A-B)}=\frac{\sin A\ \sin B}{\cos A\cos B},$$

that is $$\tan^2 C=\tan A\ \tan B.$$

38. $$\frac{\tan^2\alpha}{\tan^2\beta}=\frac{\cos\beta(\cos x-\cos\alpha)}{\cos\alpha(\cos x-\cos\beta)},$$

therefore $$\frac{\cos x-\cos\alpha}{\cos x-\cos\beta}=\frac{\tan^2\alpha\cos\alpha}{\tan^2\beta\cos\beta}=\frac{\sin^2\alpha\cos\beta}{\sin^2\beta\cos\alpha};$$

therefore $\cos x = \frac{\sin^2\beta\cos^2\alpha - \sin^2\alpha\cos^2\beta}{\sin^2\beta\cos\alpha - \sin^2\alpha\cos\beta}$

$$= \frac{(1-\cos^2\beta)\cos^2\alpha - (1-\cos^2\alpha)\cos^2\beta}{(1-\cos^2\beta)\cos\alpha - (1-\cos^2\alpha)\cos\beta}$$

$$= \frac{\cos^2\alpha - \cos^2\beta}{(\cos\alpha - \cos\beta)(1+\cos\alpha\cos\beta)} = \frac{\cos\alpha + \cos\beta}{1+\cos\alpha\cos\beta}.$$

Hence $\frac{1-\cos x}{1+\cos x} = \frac{1+\cos\alpha\cos\beta - \cos\alpha - \cos\beta}{1+\cos\alpha\cos\beta + \cos\alpha + \cos\beta} = \frac{(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)}$;

therefore $\tan^2\frac{x}{2} = \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}$, by Art. 82.

39. $\frac{\tan^2\theta}{\tan^2\theta'} = \frac{\tan^2\alpha}{\tan^2\alpha'}$; but $\tan^2\theta = \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{\cos\beta - \cos\alpha}{\cos\alpha}$,

and $\tan^2\theta' = \frac{1-\cos^2\theta'}{\cos^2\theta'} = \frac{\cos\beta - \cos\alpha'}{\cos\alpha'}$;

therefore $\frac{\cos\beta - \cos\alpha}{\cos\beta - \cos\alpha'} \cdot \frac{\cos\alpha'}{\cos\alpha} = \frac{\tan^2\alpha}{\tan^2\alpha'}$;

therefore $\frac{\cos\beta - \cos\alpha}{\cos\beta - \cos\alpha'} = \frac{\sin^2\alpha\cos\alpha'}{\sin^2\alpha'\cos\alpha}$,

therefore $\cos\beta = \frac{\sin^2\alpha'\cos^2\alpha - \sin^2\alpha\cos^2\alpha'}{\sin^2\alpha'\cos\alpha - \sin^2\alpha\cos\alpha'}$

$$= \frac{(1-\cos^2\alpha')\cos^2\alpha - (1-\cos^2\alpha)\cos^2\alpha'}{(1-\cos^2\alpha')\cos\alpha - (1-\cos^2\alpha)\cos\alpha'}$$

$$= \frac{\cos^2\alpha - \cos^2\alpha'}{(\cos\alpha - \cos\alpha')(1+\cos\alpha\cos\alpha')} = \frac{\cos\alpha + \cos\alpha'}{1+\cos\alpha\cos\alpha'}.$$

Hence $\frac{1-\cos\beta}{1+\cos\beta} = \frac{1+\cos\alpha\cos\alpha' - \cos\alpha - \cos\alpha'}{1+\cos\alpha\cos\alpha' + \cos\alpha + \cos\alpha'} = \frac{(1-\cos\alpha)(1-\cos\alpha')}{(1+\cos\alpha)(1+\cos\alpha')}$;

therefore $\tan^2\frac{\beta}{2} = \tan^2\frac{\alpha}{2}\tan^2\frac{\alpha'}{2}$.

40. $\cos\phi = \frac{\cos\alpha}{\cos\beta}$, $\cos\phi' = \frac{\cos\alpha}{\cos\beta'}$;

therefore $1-\cos\phi = \frac{\cos\beta - \cos\alpha}{\cos\beta}$, $1-\cos\phi' = \frac{\cos\beta' - \cos\alpha}{\cos\beta'}$;

therefore $2\sin^2\frac{\phi}{2} = \frac{\cos\beta - \cos\alpha}{\cos\beta}$, $2\sin^2\frac{\phi'}{2} = \frac{\cos\beta' - \cos\alpha}{\cos\beta'}$;

therefore $4\sin^2\frac{\phi}{2}\sin^2\frac{\phi'}{2} = \frac{(\cos\beta - \cos\alpha)(\cos\beta' - \cos\alpha)}{\cos\beta\cos\beta'}$.

Thus $$\sin^2 \alpha = \frac{(\cos\beta - \cos\alpha)(\cos\beta' - \cos\alpha)}{\cos\beta\cos\beta'};$$

therefore $$\cos\beta\cos\beta'\sin^2\alpha = \cos\beta\cos\beta' - \cos\alpha(\cos\beta + \cos\beta') + \cos^2\alpha,$$

therefore $$\cos\beta\cos\beta'\cos^2\alpha = \cos\alpha(\cos\beta + \cos\beta') - \cos^2\alpha;$$

therefore $$\cos\alpha(1 + \cos\beta\cos\beta') = \cos\beta + \cos\beta';$$

therefore $$\cos\alpha = \frac{\cos\beta + \cos\beta'}{1 + \cos\beta\cos\beta'}.$$

Hence $$\frac{1 - \cos\alpha}{1 + \cos\alpha} = \frac{(1 - \cos\beta)(1 - \cos\beta')}{(1 + \cos\beta)(1 + \cos\beta')};$$

therefore $$\tan^2\frac{\alpha}{2} = \tan^2\frac{\beta}{2}\tan^2\frac{\beta'}{2}.$$

41. The proposed result is true if

$$\cot\beta - \cot(\alpha + \theta) = \cot\theta + \cot(\alpha - \beta),$$

that is if

$$\frac{\cos\beta}{\sin\beta} - \frac{\cos(\alpha + \theta)}{\sin(\alpha + \theta)} = \frac{\cos\theta}{\sin\theta} + \frac{\cos(\alpha - \beta)}{\sin(\alpha - \beta)},$$

that is if

$$\frac{\sin(\alpha + \theta)\cos\beta - \cos(\alpha + \theta)\sin\beta}{\sin\beta\sin(\alpha + \theta)} = \frac{\sin(\alpha - \beta)\cos\theta + \cos(\alpha - \beta)\sin\theta}{\sin\theta\sin(\alpha - \beta)},$$

that is if

$$\frac{\sin(\alpha + \theta - \beta)}{\sin\beta\sin(\alpha + \theta)} = \frac{\sin(\alpha - \beta + \theta)}{\sin\theta\sin(\alpha - \beta)},$$

that is if

$$\sin\theta\sin(\alpha - \beta) = \sin\beta\sin(\alpha + \theta);$$

and this is true by supposition.

42. $\left(\frac{\tan\alpha - \cos\theta\tan\beta}{\sin\theta}\right)^2 = \tan^2\alpha - \tan^2\beta$; therefore

$$(\tan\alpha - \cos\theta\tan\beta)^2 = (1 - \cos^2\theta)(\tan^2\alpha - \tan^2\beta);$$

therefore

$$\tan^2\alpha - 2\cos\theta\tan\alpha\tan\beta + \cos^2\theta\tan^2\beta = (1 - \cos^2\theta)(\tan^2\alpha - \tan^2\beta);$$

therefore $$\tan^2\beta - 2\cos\theta\tan\alpha\tan\beta + \cos^2\theta\tan^2\alpha = 0,$$

that is $$(\tan\beta - \cos\theta\tan\alpha)^2 = 0;$$

therefore $\tan\beta - \cos\theta\tan\alpha = 0$; therefore $\cos\theta = \frac{\tan\beta}{\tan\alpha}$.

43. $\cos\theta = \frac{\tan\phi}{\tan\alpha}$; therefore $\tan^2\theta = \frac{\tan^2\alpha - \tan^2\phi}{\tan^2\phi}$;

therefore $$\frac{\tan^2\alpha - \tan^2\phi}{\tan^2\phi} = \frac{\tan^2\alpha'}{\sin^2\phi};$$

therefore $$\frac{\cos^2\phi\tan^2\alpha-\sin^2\phi}{\sin^2\phi}=\frac{\tan^2\alpha'}{\sin^2\phi};$$

therefore $$\cos^2\phi\tan^2\alpha-(1-\cos^2\phi)=\tan^2\alpha';$$

therefore $$\cos^2\phi=\frac{1+\tan^2\alpha'}{1+\tan^2\alpha}=\frac{\cos^2\alpha}{\cos^2\alpha'};$$

therefore $$\cos\phi=\pm\frac{\cos\alpha}{\cos\alpha'}.$$

Take the upper sign; thus $\cos\phi=\frac{\cos\alpha}{\cos\alpha'}$; therefore

$$\frac{1-\cos\phi}{1+\cos\phi}=\frac{\cos\alpha'-\cos\alpha}{\cos\alpha'+\cos\alpha}=\frac{2\sin\frac{1}{2}(\alpha-\alpha')\sin\frac{1}{2}(\alpha+\alpha')}{2\cos\frac{1}{2}(\alpha-\alpha')\cos\frac{1}{2}(\alpha+\alpha')}$$

$$=\tan\frac{1}{2}(\alpha-\alpha')\tan\frac{1}{2}(\alpha+\alpha').$$

44. $1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$

$$=1-(\cos\alpha-\cos\beta\cos\gamma)^2+\cos^2\beta\cos^2\gamma-\cos^2\beta-\cos^2\gamma$$

$$=(1-\cos^2\beta)(1-\cos^2\gamma)-(\cos\alpha-\cos\beta\cos\gamma)^2$$

$$=\sin^2\beta\sin^2\gamma-(\cos\alpha-\cos\beta\cos\gamma)^2$$

$$=(\sin\beta\sin\gamma-\cos\alpha+\cos\beta\cos\gamma)(\sin\beta\sin\gamma+\cos\alpha-\cos\beta\cos\gamma)$$

$$=\{-\cos\alpha+\cos(\beta-\gamma)\}\{\cos\alpha-\cos(\beta+\gamma)\}$$

$$=4\sin\frac{\alpha+\beta-\gamma}{2}\sin\frac{\alpha-\beta+\gamma}{2}\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\beta+\gamma-\alpha}{2}.$$

Hence in order that the proposed expression may be zero one of the four sines last written must be zero, and thus one of the four angles must be zero or a multiple of two right angles.

45. Let $\frac{1}{k}$ denote the common value of the three fractions; so that

$$x=k\tan(\theta+\alpha),\quad y=k\tan(\theta+\beta),\quad z=k\tan(\theta+\gamma).$$

Then $$\frac{x+y}{x-y}\sin^2(\alpha-\beta)=\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}\sin^2(\alpha-\beta)$$

$$=\frac{\sin(\theta+\alpha)\cos(\theta+\beta)+\sin(\theta+\beta)\cos(\theta+\alpha)}{\sin(\theta+\alpha)\cos(\theta+\beta)-\sin(\theta+\beta)\cos(\theta+\alpha)}\sin^2(\alpha-\beta)$$

$$=\frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}\sin^2(\alpha-\beta)=\sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)$$

$$=\frac{1}{2}\{\cos(2\theta+2\beta)-\cos(2\theta+2\alpha)\}.$$

Similarly $\frac{y+z}{y-z}\sin^2(\beta-\gamma)=\frac{1}{2}\{\cos(2\theta+2\gamma)-\cos(2\theta+2\beta)\}$,

and $\frac{z+x}{z-x}\sin^2(\gamma-\alpha)=\frac{1}{2}\{\cos(2\theta+2\alpha)-\cos(2\theta+2\gamma)\}$.

Thus the sum of the three terms is zero.

46. From the second given equation

$$\sin^2\phi=\frac{\sin^2\beta\sin^2\theta}{\sin^2\alpha},$$

therefore $$\tan^2\phi=\frac{\sin^2\beta\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta}.$$

Substitute in the first given equation; thus

$$\frac{\tan^2\theta}{\tan^2\alpha}+\frac{\cos^2\beta\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta}=1;$$

therefore $$\frac{\tan^2\theta}{\tan^2\alpha}=\frac{\sin^2\alpha-\sin^2\beta\sin^2\theta-\cos^2\beta\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta}$$

$$=\frac{\sin^2\alpha-\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta};$$

therefore $$\frac{\sin^2\theta\cos^2\alpha}{(1-\sin^2\theta)\sin^2\alpha}=\frac{\sin^2\alpha-\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta};$$

therefore

$$\sin^2\theta\cos^2\alpha(\sin^2\alpha-\sin^2\beta\sin^2\theta)=(\sin^2\alpha-\sin^2\theta)(1-\sin^2\theta)\sin^2\alpha;$$

therefore

$$\sin^4\theta(\sin^2\alpha+\cos^2\alpha\sin^2\beta)-\sin^2\theta(\cos^2\alpha\sin^2\alpha+\sin^2\alpha+\sin^4\alpha)+\sin^4\alpha=0;$$

therefore $$\sin^4\theta(1-\cos^2\alpha\cos^2\beta)-2\sin^2\theta\sin^2\alpha+\sin^4\alpha=0.$$

By solving this quadratic in the ordinary way we obtain

$$\sin^2\theta=\frac{1\pm\cos\alpha\cos\beta}{1-\cos^2\alpha\cos^2\beta}\sin^2\alpha=\frac{\sin^2\alpha}{1\mp\cos\alpha\cos\beta}.$$

47. $$\frac{\sin\{\theta-\beta-(\alpha-\beta)\}}{\sin(\theta-\beta)}=\frac{a}{b},$$

therefore $$\frac{\sin(\theta-\beta)\cos(\alpha-\beta)-\cos(\theta-\beta)\sin(\alpha-\beta)}{\sin(\theta-\beta)}=\frac{a}{b};$$

therefore $$\cos(\alpha-\beta)-\sin(\alpha-\beta)\cot(\theta-\beta)=\frac{a}{b}.$$

Again, $$\frac{\cos\{\theta-\beta-(\alpha-\beta)\}}{\cos(\theta-\beta)}=\frac{\alpha'}{b'},$$

therefore $$\frac{\cos(\theta-\beta)\cos(\alpha-\beta)+\sin(\theta-\beta)\sin(\alpha-\beta)}{\cos(\theta-\beta)}=\frac{a'}{b'};$$

therefore $$\cos(\alpha-\beta)+\tan(\theta-\beta)\sin(\alpha-\beta)=\frac{a'}{b'}.$$

Hence $\sin(\alpha-\beta)\cot(\theta-\beta)\sin(\alpha-\beta)\tan(\theta-\beta)$

$$=\left\{\cos(\alpha-\beta)-\frac{a}{b}\right\}\left\{\frac{a'}{b'}-\cos(\alpha-\beta)\right\};$$

therefore $$\sin^2(\alpha-\beta)=-\frac{aa'}{bb'}+\left(\frac{a}{b}+\frac{a'}{b'}\right)\cos(\alpha-\beta)-\cos^2(\alpha-\beta);$$

therefore $$1+\frac{aa'}{bb'}=\left(\frac{a}{b}+\frac{a'}{b'}\right)\cos(\alpha-\beta);$$

therefore $$\cos(\alpha-\beta)=\frac{aa'+bb'}{ab'+a'b}.$$

48. $\tan\phi=\dfrac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}$; thus

$$\frac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}=\frac{\sin\theta\cos\theta'}{\sin\theta'+\cos\theta};$$

therefore $$2\tan\frac{\phi}{2}(\sin\theta'+\cos\theta)=\left(1-\tan^2\frac{\phi}{2}\right)\sin\theta\cos\theta';$$

therefore $$\sin\theta\cos\theta'\tan^2\frac{\phi}{2}+2\tan\frac{\phi}{2}(\sin\theta'+\cos\theta)=\sin\theta\cos\theta'.$$

By solving this quadratic in the ordinary way we obtain

$$\tan\frac{\phi}{2}=\frac{-(\sin\theta'+\cos\theta)\pm(1+\sin\theta'\cos\theta)}{\sin\theta\cos\theta'}.$$

Take the upper sign; thus $\tan\dfrac{\phi}{2}=\dfrac{(1-\sin\theta')(1-\cos\theta)}{\sin\theta\cos\theta'}$.

Now $$\frac{1-\cos\theta}{\sin\theta}=\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\tan\frac{\theta}{2},$$

and similarly $$\frac{1-\sin\theta'}{\cos\theta'}=\frac{1-\cos\left(\frac{\pi}{2}-\theta'\right)}{\sin\left(\frac{\pi}{2}-\theta'\right)}=\tan\left(\frac{\pi}{4}-\frac{\theta'}{2}\right);$$

thus $$\tan\frac{\phi}{2}=\tan\frac{\theta}{2}\tan\left(\frac{\pi}{4}-\frac{\theta'}{2}\right).$$

In like manner with the lower sign we shall find that

$$\tan\frac{\phi}{2}=-\cot\frac{\theta}{2}\cot\left(\frac{\pi}{4}-\frac{\theta'}{2}\right).$$

The product of the two values of $\tan\frac{\phi}{2}$ is -1, as it should be by the nature of quadratic equations.

49. $\cos\theta=\cos\alpha\cos\beta$;

therefore $$\frac{1-\cos\theta}{1+\cos\theta}=\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta};$$

therefore $$\tan^2\frac{\theta}{2}=\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}.$$

Similarly $$\tan^2\frac{\theta'}{2}=\frac{1-\cos\alpha'\cos\beta}{1+\cos\alpha'\cos\beta}.$$

Hence $$\frac{(1-\cos\alpha\cos\beta)(1-\cos\alpha'\cos\beta)}{(1+\cos\alpha\cos\beta)(1+\cos\alpha'\cos\beta)}=\tan^2\frac{\beta}{2}=\frac{1-\cos\beta}{1+\cos\beta};$$

therefore $$\frac{1-(\cos\alpha+\cos\alpha')\cos\beta+\cos\alpha\cos\alpha'\cos^2\beta}{1+(\cos\alpha+\cos\alpha')\cos\beta+\cos\alpha\cos\alpha'\cos^2\beta}=\frac{1-\cos\beta}{1+\cos\beta};$$

therefore $$\frac{(\cos\alpha+\cos\alpha')\cos\beta}{1+\cos\alpha\cos\alpha'\cos^2\beta}=\cos\beta;$$

therefore $$\cos\alpha+\cos\alpha'=1+\cos\alpha\cos\alpha'(1-\sin^2\beta);$$

therefore $$\sin^2\beta\cos\alpha\cos\alpha'=1-\cos\alpha-\cos\alpha'+\cos\alpha\cos\alpha'$$
$$=(1-\cos\alpha)(1-\cos\alpha');$$

therefore $$\sin^2\beta=\left(\frac{1}{\cos\alpha}-1\right)\left(\frac{1}{\cos\alpha'}-1\right)$$
$$=(\sec\alpha-1)(\sec\alpha'-1).$$

50. Here

$$\sin(C+A-B)-\sin(B+C-A)=\sin(A+B-C)-\sin(C+A-B);$$

therefore $$2\sin(A-B)\cos C=2\sin(B-C)\cos A;$$

therefore $(\sin A\cos B-\cos A\sin B)\cos C=(\sin B\cos C-\cos B\sin C)\cos A.$

Divide by $\cos A \cos B \cos C$; thus

$$\tan A - \tan B = \tan B - \tan C;$$

therefore $\tan A$, $\tan B$, and $\tan C$ are in Arithmetical Progression.

51. Suppose $\sin A$, $\sin B$, and $\sin C$ to be in Arithmetical Progression, so that $\sin B - \sin A = \sin C - \sin B$.

Thus $$2 \sin \frac{B-A}{2} \cos \frac{B+A}{2} = 2 \sin \frac{C-B}{2} \cos \frac{C+B}{2};$$

therefore $$\sin \frac{B-A}{2} \sin \frac{C}{2} = \sin \frac{C-B}{2} \sin \frac{A}{2};$$

therefore $$\left(\sin \frac{B}{2} \cos \frac{A}{2} - \cos \frac{B}{2} \sin \frac{A}{2}\right) \sin \frac{C}{2}$$
$$= \left(\sin \frac{C}{2} \cos \frac{B}{2} - \cos \frac{C}{2} \sin \frac{B}{2}\right) \sin \frac{A}{2}.$$

Divide by $\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$; thus

$$\cot \frac{A}{2} - \cot \frac{B}{2} = \cot \frac{B}{2} - \cot \frac{C}{2};$$

thus $\cot \frac{C}{2}$, $\cot \frac{B}{2}$ and $\cot \frac{A}{2}$ are in Arithmetical Progression.

52. Suppose $$\cos^2 A + \cos^2 B + \cos^2 C = 1;$$

therefore $$3 - \sin^2 A - \sin^2 B - \sin^2 C = 1;$$

therefore $$\sin^2 A + \sin^2 B + \sin^2 C = 2;$$

therefore by Example 23 we have $\cos A \cos B \cos C = 0$; therefore one of the three angles is a right angle, and this will be the largest angle. Suppose it to be A, so that $A = 90^0$; therefore $B + C = 90^0 = A$; therefore $A - C = B$.

53. $$\sin\left(A + \frac{C}{2}\right) = \sin\left(A + \frac{180^0 - A - B}{2}\right)$$
$$= \sin\left(90^0 - \frac{B-A}{2}\right) = \cos \frac{B-A}{2};$$

and $$\sin \frac{C}{2} = \cos \frac{A+B}{2};$$

thus $$\cos \frac{B-A}{2} = n \cos \frac{A+B}{2};$$

therefore $$\cos \frac{A}{2} \cos \frac{B}{2} + \sin \frac{A}{2} \sin \frac{B}{2} = n\left(\cos \frac{A}{2} \cos \frac{B}{2} - \sin \frac{A}{2} \sin \frac{B}{2}\right);$$

therefore $$(n+1) \sin \frac{A}{2} \sin \frac{B}{2} = (n-1) \cos \frac{A}{2} \cos \frac{B}{2};$$

therefore $$\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=\frac{n-1}{n+1},$$

therefore $$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{n-1}{n+1}.$$

54. Suppose $\frac{1}{k}$ to denote the value of $\frac{\sin A}{x}$, $\frac{\sin B}{y}$ and $\frac{\sin C}{z}$; then

$$x=k\sin A,\quad y=k\sin B,\quad z=k\sin C.$$

Therefore $(x-y)\cot\frac{C}{2}=k(\sin A-\sin B)\cot\frac{C}{2}$

$$=2k\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)\cot\frac{C}{2}$$

$$=2k\sin\frac{1}{2}(A-B)\sin\frac{C}{2}\cot\frac{C}{2}$$

$$=2k\sin\frac{1}{2}(A-B)\cos\frac{C}{2}$$

$$=2k\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B)$$

$$=2k\{\sin^2\frac{1}{2}A-\sin^2\frac{1}{2}B\},\text{ by Art. 83.}$$

Similarly $$(y-z)\cot\frac{A}{2}=2k\{\sin^2\frac{1}{2}B-\sin^2\frac{1}{2}C\},$$

and $$(z-x)\cot\frac{B}{2}=2k\{\sin^2\frac{1}{2}C-\sin^2\frac{1}{2}A\}.$$

Thus the sum of the three terms is zero.

55. $\tan(A+B+C)=\tan m\pi=0$; and therefore, by Art. 113,

$$\tan A+\tan B+\tan C-\tan A\tan B\tan C=0.$$

56. $\sin(2\alpha+x)+\sin(2\beta+x)=2\sin(\alpha+\beta+x)\cos(\alpha-\beta)$,

$\sin(2\gamma+x)-\sin(2\alpha+2\beta+2\gamma+3x)=-2\sin(\alpha+\beta+x)\cos(\alpha+\beta+2\gamma+2x)$;

$$2\sin(\alpha+\beta+x)\{\cos(\alpha-\beta)-\cos(\alpha+\beta+2\gamma+2x)\}$$
$$=2\sin(\alpha+\beta+x)\,2\sin(\beta+\gamma+x)\sin(\alpha+\gamma+x)$$
$$=4\sin(\alpha+\beta+x)\sin(\beta+\gamma+x)\sin(\gamma+\alpha+x).$$

57. If $x=0$ we have

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma-\sin(2\alpha+2\beta+2\gamma)=4\sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\gamma+\alpha).$$

If then $\alpha+\beta+\gamma=\pi$ we have $\sin(2\alpha+2\beta+2\gamma)=0$;

also $$\sin(\alpha+\beta)=\sin\gamma,\quad \sin(\beta+\gamma)=\sin\alpha,\quad \sin(\gamma+\alpha)=\sin\beta,$$

so that $$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=4\sin\gamma\sin\alpha\sin\beta.$$

If $x=\frac{\pi}{2}$ we have

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos(2\alpha+2\beta+2\gamma)$$
$$=4\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha).$$

If then $\alpha+\beta+\gamma=\frac{\pi}{2}$ we have $\cos(2\alpha+2\beta+2\gamma)=-1$,

also $\qquad \cos(\alpha+\beta)=\sin\gamma,\quad \cos(\beta+\gamma)=\sin\alpha,\quad \cos(\gamma+\alpha)=\sin\beta,$

so that $\qquad \cos 2\alpha+\cos 2\beta+\cos 2\gamma-1=4\sin\alpha\sin\beta\sin\gamma.$

58. $$4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}=2\cos\frac{\gamma}{2}\left\{\cos\frac{1}{2}(\alpha-\beta)+\cos\frac{1}{2}(\alpha+\beta)\right\}$$
$$=\cos\frac{1}{2}(\gamma+\alpha-\beta)+\cos\frac{1}{2}(\gamma+\beta-\alpha)+\cos\frac{1}{2}(\alpha+\beta+\gamma)+\cos\frac{1}{2}(\alpha+\beta-\gamma).$$

Thus the left-hand member of the proposed expression

$$=\sin\alpha+\sin\beta+\sin\gamma-\cos\frac{1}{2}(\alpha+\beta+\gamma)-\cos\frac{1}{2}(\beta+\gamma-\alpha)$$
$$-\cos\frac{1}{2}(\alpha+\gamma-\beta)-\cos\frac{1}{2}(\alpha+\beta-\gamma).$$

Again

$$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{3\alpha-\beta-\gamma+\pi}{4}=\sin\alpha+\sin\frac{\beta+\gamma-\alpha-\pi}{2}$$
$$=\sin\alpha-\cos\frac{\beta+\gamma-\alpha}{2};$$

so also $$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{3\beta-\alpha-\gamma+\pi}{4}=\sin\beta-\cos\frac{\alpha+\gamma-\beta}{2},$$

$$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{3\gamma-\alpha-\beta+\pi}{4}=\sin\gamma-\cos\frac{\alpha+\beta-\gamma}{2},$$

and $$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{\alpha+\beta+\gamma-\pi}{4}=\sin\frac{\alpha+\beta+\gamma-\pi}{2}$$
$$=-\cos\frac{\alpha+\beta+\gamma}{2}.$$

Thus the result is established.

59. $$\cos 5\theta=\cos(3\theta+2\theta)=\cos 3\theta\cos 2\theta-\sin 3\theta\sin 2\theta$$
$$=(4\cos^3\theta-3\cos\theta)(2\cos^2\theta-1)-(3\sin\theta-4\sin^3\theta)\,2\sin\theta\cos\theta$$
$$=(4\cos^3\theta-3\cos\theta)(2\cos^2\theta-1)-2\sin^2\theta(3-4\sin^2\theta)\cos\theta$$
$$=(4\cos^3\theta-3\cos\theta)(2\cos^2\theta-1)-2(1-\cos^2\theta)(4\cos^2\theta-1)\cos\theta$$
$$=8\cos^5\theta-10\cos^3\theta+3\cos\theta-2(-4\cos^4\theta+5\cos^2\theta-1)\cos\theta$$
$$=16\cos^5\theta-20\cos^3\theta+5\cos\theta.$$

60. $\sin 6\theta = 2\sin 3\theta \cos 3\theta = 2(3\sin\theta - 4\sin^3\theta)(4\cos^3\theta - 3\cos\theta)$

$$= 2\sin\theta(3 - 4\sin^2\theta)(4\cos^3\theta - 3\cos\theta)$$
$$= 2\sin\theta(4\cos^2\theta - 1)(4\cos^3\theta - 3\cos\theta)$$
$$= 2\sin\theta(16\cos^5\theta - 16\cos^3\theta + 3\cos\theta).$$

CHAPTER IX.

1. Let $PCB = A$, so that $BPM = \frac{1}{2}A$ and $PAM = \frac{1}{2}A$. Then

$$\frac{MB}{PM} = \tan\frac{1}{2}A, \text{ and } \frac{PM}{AM} = \tan\frac{1}{2}A,$$

so that
$$\tan^2\frac{1}{2}A = \frac{MB}{PM}\cdot\frac{PM}{AM} = \frac{MB}{AM} = \frac{CB - CM}{CA + CM}$$
$$= \frac{CP - CM}{CP + CM} = \frac{1 - \frac{CM}{CP}}{1 + \frac{CM}{CP}} = \frac{1 - \cos A}{1 + \cos A}.$$

2. $\cos\theta = \frac{a\cos\phi - b}{a - b\cos\phi}$; therefore

$$\frac{1 - \cos\theta}{1 + \cos\theta} = \frac{a - b\cos\phi - a\cos\phi + b}{a - b\cos\phi + a\cos\phi - b} = \frac{(a+b)(1-\cos\phi)}{(a-b)(1+\cos\phi)};$$

therefore
$$\tan^2\frac{\theta}{2} = \frac{a+b}{a-b}\tan^2\frac{\phi}{2};$$

therefore
$$\frac{\tan^2\frac{\theta}{2}}{a+b} = \frac{\tan^2\frac{\phi}{2}}{a-b}.$$

3. $\cos^2\theta = \frac{1}{1 + \tan^2\theta} = \frac{1}{1 + 2\tan^2\phi + 1} = \frac{1}{2(1 + \tan^2\phi)} = \frac{1}{2}\cos^2\phi$;

and
$$\cos 2\theta = 2\cos^2\theta - 1 = \cos^2\phi - 1 = -\sin^2\phi,$$

therefore
$$\cos 2\theta + \sin^2\phi = 0.$$

4. $\sec 2\theta = 2\sec\theta \operatorname{cosec}\theta$; therefore $\frac{1}{\cos 2\theta} = \frac{2}{\cos\theta\sin\theta}$;

therefore
$$1 = \frac{2\cos 2\theta}{\cos\theta\sin\theta};$$

therefore
$$\frac{1}{\sin 2\theta}=\frac{2\cos 2\theta}{\sin 2\theta\cos\theta\sin\theta}=\frac{\cos 2\theta}{\sin^2\theta\cos^2\theta}$$
$$=\frac{\cos^2\theta-\sin^2\theta}{\sin^2\theta\cos^2\theta}=\frac{1}{\sin^2\theta}-\frac{1}{\cos^2\theta}.$$

Thus
$$\operatorname{cosec} 2\theta=\operatorname{cosec}^2\theta-\sec^2\theta.$$

5. $\tan(\theta-\phi)=\dfrac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\dfrac{(n-1)\tan\phi}{1+n\tan^2\phi}=\dfrac{n-1}{\cot\phi+n\tan\phi}$;

therefore
$$\tan^2(\theta-\phi)=\frac{(n-1)^2}{\cot^2\phi+2n+n^2\tan^2\phi}=\frac{(n-1)^2}{(n\tan\phi-\cot\phi)^2+4n}.$$

The greatest value of this fraction is when the denominator is least, that is when the term $n\tan\phi-\cot\phi$ vanishes.

6. $\sin\theta+\sin\phi-\cos\theta\sin(\theta+\phi)$

$$=2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)-2\cos\theta\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta+\phi)$$
$$=2\sin\frac{1}{2}(\theta+\phi)\left\{\cos\frac{1}{2}(\theta-\phi)-\cos\theta\cos\frac{1}{2}(\theta+\phi)\right\}$$
$$=2\sin\frac{1}{2}(\theta+\phi)\left\{\cos\left(\theta-\frac{\theta+\phi}{2}\right)-\cos\theta\cos\frac{1}{2}(\theta+\phi)\right\}$$
$$=2\sin\frac{1}{2}(\theta+\phi)\sin\theta\sin\frac{1}{2}(\theta+\phi)=2\sin\theta\sin^2\frac{1}{2}(\theta+\phi).$$

7. $$\frac{\sin\beta\cos\alpha(\tan\alpha+\tan\beta)}{1-\cos(\alpha+\beta)}=\frac{\sin\beta\cos\alpha}{2\sin^2\frac{1}{2}(\alpha+\beta)}\left\{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}\right\}$$
$$=\frac{\sin\beta\cos\alpha}{2\sin^2\frac{1}{2}(\alpha+\beta)}\cdot\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$
$$=\frac{\sin\beta\,.\,2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)}{2\sin^2\frac{1}{2}(\alpha+\beta)\cos\beta}$$
$$=\frac{\sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta};$$

and
$$\frac{\sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}+\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}$$

$$=\frac{\sin\left(\frac{\alpha+\beta}{2}-\beta\right)+\sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}=\frac{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}=1.$$

8. Let x denote the height in yards; then $\frac{x}{1760}=\tan 1'$, therefore $x=1760\tan 1'$. The value of $\tan 1'$ is approximately equal to the circular measure of $1'$, that is to $\frac{\pi}{180\times 60}$; therefore $x=\frac{1760\pi}{180\times 60}$ approximately.

9. Let x denote the distance in inches; then $\frac{3}{x}=\tan\frac{1^\circ}{4}$; and taking the tangent as approximately equal to the circular measure we have

$$\frac{3}{x}=\frac{\pi}{180\times 4}; \text{ therefore } x=\frac{12\times 180}{\pi}.$$

10. We have $3\sin A-4\sin^3 A=n\sin A$; as we suppose that A is not zero nor a multiple of two right angles we may divide by $\sin A$; thus $3-4\sin^2 A=n$; therefore $\sin^2 A=\frac{3-n}{4}$, and as this must lie between zero and unity, n must lie between 3 and -1.

If $n=2$ we have $\sin^2 A=\frac{1}{4}=\sin^2\frac{\pi}{6}$; therefore $A=m\pi\pm\frac{\pi}{6}$, where m is zero or any integer.

11. $$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$=\frac{\tan\alpha-\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}{1+\tan\alpha\,.\,\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}$$

$$=\frac{\sin\alpha(1-n\sin^2\alpha)-n\sin\alpha\cos^2\alpha}{\cos\alpha(1-n\sin^2\alpha)+n\sin^2\alpha\cos\alpha}$$

$$=\frac{\sin\alpha-n\sin\alpha}{\cos\alpha}=\frac{(1-n)\sin\alpha}{\cos\alpha}=(1-n)\tan\alpha.$$

12. All the angles which have the same sine as 3θ are included in the formula $n\pi+(-1)^n 3\theta$. Therefore any expression which gives the value of $\tan\theta$ in terms of $\sin 3\theta$ may be expected to give the value of the tangent of every angle included in the formula $\tan\frac{1}{3}\{n\pi+(-1)^n 3\theta\}$.

Now n must be of one of the following forms:

$$6m,\quad 6m+1,\quad 6m+2,\quad 6m+3,\quad 6m+4,\quad 6m+5.$$

The corresponding values of $\tan\frac{1}{3}\left\{n\pi+(-1)^n 3\theta\right\}$ are, by Art. 45,

$$\tan\theta,\quad \tan\left(\frac{\pi}{3}-\theta\right),\quad \tan\left(\frac{2\pi}{3}+\theta\right),\quad \tan(\pi-\theta),$$

$$\tan\left(\pi+\frac{\pi}{3}+\theta\right),\quad \tan\left(\pi+\frac{2\pi}{3}-\theta\right).$$

Thus we have six distinct values. They may also by Arts. 48 and 50 be expressed thus:

$$\pm\tan\theta,\quad \pm\tan\left(\frac{\pi}{3}+\theta\right),\quad \pm\tan\left(\frac{2\pi}{3}+\theta\right).$$

13. $\cos^2 A=\frac{1}{2}(1+\cos 2A)$; therefore

$$\cos^4 A=\frac{1}{4}(1+2\cos 2A+\cos^2 2A)$$

$$=\frac{1}{4}+\frac{1}{2}\cos 2A+\frac{1+\cos 4A}{8}$$

$$=\frac{3}{8}+\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A.$$

Similarly $\sin^2 A=\frac{1}{2}(1-\cos 2A)$;

therefore $\sin^4 A=\frac{1}{4}(1-2\cos 2A+\cos^2 2A)$

$$=\frac{3}{8}-\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A.$$

Therefore $\cos^8 A+\sin^8 A$

$$=\left(\frac{3}{8}+\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A\right)^2+\left(\frac{3}{8}-\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A\right)^2$$

$$=2\left\{\left(\frac{3}{8}\right)^2+\left(\frac{1}{2}\cos 2A\right)^2+\left(\frac{1}{8}\cos 4A\right)^2+2\cdot\frac{3}{8}\cdot\frac{1}{8}\cos 4A\right\}$$

$$=2\left\{\frac{9}{64}+\frac{1}{4}\cos^2 2A+\frac{1}{64}\cos^2 4A+\frac{3}{32}\cos 4A\right\}$$

$$=\frac{9}{32}+\frac{1}{4}(1+\cos 4A)+\frac{1}{64}(1+\cos 8A)+\frac{3}{16}\cos 4A$$

$$=\frac{1}{64}\{\cos 8A+28\cos 4A+35\}.$$

14. $\cos\theta \cos\phi = -1.$

As the cosine of an angle is never numerically greater than unity, we must have $\cos\theta$ and $\cos\phi$ both numerically equal to unity, one being positive and the other negative. Hence one of the angles must be zero or an even multiple of π, and the other must be an odd multiple of π.

15. $\sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos(\alpha-\beta)$

$$= \sin\alpha\{\sin\alpha - \sin\beta\cos(\alpha-\beta)\} + \sin\beta\{\sin\beta - \sin\alpha\cos(\alpha-\beta)\}$$

$$= \sin\alpha\{\sin(\alpha-\beta+\beta) - \sin\beta\cos(\alpha-\beta)\}$$
$$+ \sin\beta\{\sin(\alpha - \overline{\alpha-\beta}) - \sin\alpha\cos(\alpha-\beta)\}$$

$$= \sin\alpha\sin(\alpha-\beta)\cos\beta - \sin\beta\cos\alpha\sin(\alpha-\beta)$$

$$= \sin(\alpha-\beta)\{\sin\alpha\cos\beta - \sin\beta\cos\alpha\} = \sin^2(\alpha-\beta).$$

Thus $$\sin^2(\alpha-\beta) = n^2\sin^2(\alpha+\beta);$$

therefore $$\sin(\alpha-\beta) = \pm n\sin(\alpha+\beta);$$

therefore $$\sin\alpha\cos\beta - \cos\alpha\sin\beta = \pm n(\sin\alpha\cos\beta + \cos\alpha\sin\beta);$$

divide by $\cos\alpha\cos\beta$; thus $\tan\alpha - \tan\beta = \pm n(\tan\alpha + \tan\beta)$;

therefore $$(1\mp n)\tan\alpha = (1\pm n)\tan\beta;$$

therefore $$\tan\alpha = \frac{1\pm n}{1\mp n}\tan\beta.$$

16. $$\frac{\sin 4\theta\cot\theta}{\text{vers } 2\theta\cot^2 2\theta} = \frac{\sin 4\theta\sin^2 2\theta\cos\theta}{(1-\cos 2\theta)\cos^2 2\theta\sin\theta} = \frac{2\sin^3 2\theta\cos 2\theta\cos\theta}{2\sin^3\theta\cos^2 2\theta}$$

$$= \frac{2(2\sin\theta\cos\theta)^3\cos\theta}{2\sin^3\theta\cos 2\theta} = \frac{8\cos^4\theta}{\cos 2\theta}.$$

When $\theta = 0$ the value is therefore 8.

17. $\sin\theta + \cos\theta = \sqrt{2}$; therefore $\dfrac{\sin\theta}{\sqrt{2}} + \dfrac{\cos\theta}{\sqrt{2}} = 1$;

therefore $\cos\left(\theta - \dfrac{\pi}{4}\right) = 1$; therefore $\theta - \dfrac{\pi}{4} = 2n\pi.$

18. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$; therefore $\dfrac{\sqrt{3}}{2}\sin\theta - \dfrac{1}{2}\cos\theta = \dfrac{1}{\sqrt{2}}$;

therefore $$\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = -\frac{1}{\sqrt{2}};$$

therefore $$\cos\left(\theta + \frac{\pi}{3}\right) = -\frac{1}{\sqrt{2}};$$

therefore $$\theta + \frac{\pi}{3} = 2n\pi \pm \frac{3\pi}{4}.$$

19. $\sin 2\theta = \cos\theta$; therefore $\cos\left(\frac{\pi}{2} - 2\theta\right) = \cos\theta$;

therefore $\frac{\pi}{2} - 2\theta$ and θ are angles having the same cosine; therefore all the solutions are contained in $\frac{\pi}{2} - 2\theta = 2n\pi \pm \theta$.

20. $\cos\theta - \cos 2\theta = \sin 3\theta$; therefore

$$2\sin\frac{3\theta}{2}\sin\frac{\theta}{2} = 2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2};$$

therefore either $\sin\frac{3\theta}{2} = 0$, or $\sin\frac{\theta}{2} = \cos\frac{3\theta}{2}$.

If $\sin\frac{3\theta}{2} = 0$, then $\frac{3\theta}{2} = n\pi$.

If $\sin\frac{\theta}{2} = \cos\frac{3\theta}{2}$, then $\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos\frac{3\theta}{2}$;

and therefore $$\frac{\pi}{2} - \frac{\theta}{2} = 2n\pi \pm \frac{3\theta}{2}.$$

21. $(4 - \sqrt{3})(\sec\theta + \operatorname{cosec}\theta) = 4(\sin\theta\tan\theta + \cos\theta\cot\theta)$;

therefore $$(4 - \sqrt{3})\left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right) = 4\left(\frac{\sin^2\theta}{\cos\theta} + \frac{\cos^2\theta}{\sin\theta}\right);$$

therefore $$(4 - \sqrt{3})(\sin\theta + \cos\theta) = 4(\sin^3\theta + \cos^3\theta)$$
$$= 4(\sin\theta + \cos\theta)(\sin^2\theta + \cos^2\theta - \sin\theta\cos\theta);$$

therefore either $$\sin\theta + \cos\theta = 0,$$

or $$4 - \sqrt{3} = 4(1 - \sin\theta\cos\theta).$$

If $\sin\theta + \cos\theta = 0$, then $\sin\theta = -\cos\theta$; therefore $\tan\theta = -1$;

therefore $$\theta = n\pi + \frac{3\pi}{4}.$$

If $4 - \sqrt{3} = 4(1 - \sin\theta\cos\theta)$, then $\sqrt{3} = 4\sin\theta\cos\theta = 2\sin 2\theta$;

therefore $\sin 2\theta = \frac{\sqrt{3}}{2}$; therefore $2\theta = n\pi + (-1)^n\frac{\pi}{3}$.

22. $\cot\theta - \tan\theta = \cos\theta + \sin\theta$; therefore $\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \cos\theta + \sin\theta$;

therefore $$\cos^2\theta - \sin^2\theta = \sin\theta\cos\theta(\cos\theta + \sin\theta);$$

therefore either $\cos\theta + \sin\theta = 0$, or $\cos\theta - \sin\theta = \sin\theta\cos\theta$.

If $\sin\theta+\cos\theta=0$, then $\sin\theta=-\cos\theta$; therefore $\tan\theta=-1$;

therefore $$\theta=n\pi+\frac{3\pi}{4}.$$

If $\cos\theta-\sin\theta=\sin\theta\cos\theta$, then by squaring

$$1-2\sin\theta\cos\theta=\sin^2\theta\cos^2\theta;$$

therefore $$1-\sin 2\theta=\frac{\sin^2 2\theta}{4}.$$

By solving this quadratic in the usual way we obtain $\sin 2\theta=-2\pm 2\sqrt{2}$; the upper sign must be taken, for the lower sign would make $\sin 2\theta$ numerically greater than unity.

23. $2\sin^2\theta+\sin^2 2\theta=2$; therefore $\sin^2 2\theta=2-2\sin^2\theta=2(1-\sin^2\theta)$;

therefore $$4\sin^2\theta\cos^2\theta=2\cos^2\theta;$$

therefore either $\cos^2\theta=0$, or $\sin^2\theta=\frac{1}{2}$.

If $\cos^2\theta=0$, then $$\theta=n\pi+\frac{\pi}{2}.$$

If $\sin^2\theta=\frac{1}{2}$, then $$\sin^2\theta=\sin^2\frac{\pi}{4};$$

therefore $$\theta=n\pi\pm\frac{\pi}{4}.$$

24. $\tan\theta+2\cot 2\theta=\sin\theta\left(1+\tan\theta\tan\frac{\theta}{2}\right)$; therefore

$$\frac{\sin\theta}{\cos\theta}+\frac{2\cos 2\theta}{\sin 2\theta}=\sin\theta\left(1+\frac{\sin\theta\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2}}\right);$$

therefore $$\frac{\sin^2\theta+\cos 2\theta}{\sin\theta\cos\theta}=\sin\theta\,.\,\frac{\cos\left(\theta-\frac{\theta}{2}\right)}{\cos\theta\cos\frac{\theta}{2}}=\frac{\sin\theta}{\cos\theta};$$

therefore $\sin^2\theta+\cos 2\theta=\sin^2\theta$; therefore $\cos 2\theta=0$;

therefore $$2\theta=n\pi+\frac{\pi}{2}.$$

25. $\sin^2 2\theta-\sin^2\theta=\sin^2\frac{\pi}{6}=\frac{1}{4}$; therefore

$$4\sin^2\theta\cos^2\theta-\sin^2\theta=\frac{1}{4};$$

therefore $$4\sin^2\theta\,(1-\sin^2\theta)-\sin^2\theta=\frac{1}{4};$$

therefore $$4\sin^4\theta-3\sin^2\theta+\frac{1}{4}=0.$$

By solving this quadratic in the usual way we obtain

$$\sin^2\theta=\frac{3\pm\sqrt{5}}{8}.$$

Taking the upper sign we have $\sin^2\theta=\sin^2\frac{3\pi}{10}$, and therefore

$$\theta=n\pi\pm\frac{3\pi}{10}.$$

Taking the lower sign we have $\sin^2\theta=\sin^2\frac{\pi}{10}$, and therefore

$$\theta=n\pi\pm\frac{\pi}{10}.$$

26. $\operatorname{cosec}\theta=\operatorname{cosec}\frac{\theta}{2}$; therefore $\dfrac{1}{\sin\theta}=\dfrac{1}{\sin\frac{\theta}{2}}$;

therefore $\sin\frac{\theta}{2}=\sin\theta$; therefore $\sin\frac{\theta}{2}=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$;

therefore either $\sin\frac{\theta}{2}=0$, or $\cos\frac{\theta}{2}=\frac{1}{2}$.

If $\sin\frac{\theta}{2}=0$, then $\frac{\theta}{2}=n\pi$.

If $\cos\frac{\theta}{2}=\frac{1}{2}$, then $\frac{\theta}{2}=2m\pi\pm\frac{\pi}{3}$.

27. $\cos\theta\cos3\theta=\cos5\theta\cos7\theta$; therefore

$$\cos4\theta+\cos2\theta=\cos12\theta+\cos2\theta;$$

therefore $\cos4\theta=\cos12\theta$; therefore $12\theta=2n\pi\pm4\theta$;

taking the upper sign we obtain $\theta=\frac{2n\pi}{8}=\frac{n\pi}{4}$,

and taking the lower sign we obtain $\theta=\frac{2n\pi}{16}=\frac{n\pi}{8}$.

It is obvious however that the second expression includes the first.

28. $\sin\theta\sin3\theta=\frac{1}{2}$; therefore $\sin\theta\,(3\sin\theta-4\sin^3\theta)=\frac{1}{2}$;

therefore $$4\sin^4\theta-3\sin^2\theta+\frac{1}{2}=0.$$

By solving this quadratic in the usual way we obtain

$$\sin^2\theta = \frac{3 \pm 1}{8} = \frac{1}{2} \text{ or } \frac{1}{4}.$$

If $\sin^2\theta = \frac{1}{2}$, then $\sin^2\theta = \sin^2\frac{\pi}{4}$, and $\theta = n\pi \pm \frac{\pi}{4}$.

If $\sin^2\theta = \frac{1}{4}$, then $\sin^2\theta = \sin^2\frac{\pi}{6}$, and $\theta = n\pi \pm \frac{\pi}{6}$.

See Example 5 of Chapter V.

29. $4\sin^2\theta + \sin^2 2\theta = 3$; therefore $4\sin^2\theta + 4\sin^2\theta(1 - \sin^2\theta) = 3$;

therefore $$4\sin^4\theta - 8\sin^2\theta + 3 = 0.$$

By solving this quadratic in the usual way we obtain $\sin^2\theta = \frac{1}{2}$ or $\frac{3}{2}$; and only the former value is admissible. Thus $\sin^2\theta = \sin^2\frac{\pi}{4}$; therefore $\theta = n\pi \pm \frac{\pi}{4}$.

30. $(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$;

therefore $$\left(1 - \frac{\sin\theta}{\cos\theta}\right)(\sin\theta + \cos\theta)^2 = 1 + \frac{\sin\theta}{\cos\theta};$$

therefore $$(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)^2 = \cos\theta + \sin\theta;$$

therefore either $\cos\theta + \sin\theta = 0$, or $(\cos\theta - \sin\theta)(\cos\theta + \sin\theta) = 1$.

If $\cos\theta + \sin\theta = 0$, then $\sin\theta = -\cos\theta$;

therefore $$\tan\theta = -1;$$

therefore $$\theta = n\pi + \frac{3\pi}{4}.$$

If $(\cos\theta - \sin\theta)(\cos\theta + \sin\theta) = 1$, then $\cos^2\theta - \sin^2\theta = 1$;

therefore $$\cos 2\theta = 1;$$

therefore $$2\theta = 2n\pi.$$

31. $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$;

therefore $$\sin\theta + \sin 4\theta + \sin 2\theta + \sin 3\theta = 0;$$

therefore $$2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2} + 2\sin\frac{5\theta}{2}\cos\frac{\theta}{2} = 0;$$

therefore $$2\sin\frac{5\theta}{2}\left(\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right) = 0;$$

therefore $$4\sin\frac{5\theta}{2}\cos\frac{\theta}{2}\cos\theta = 0.$$

Thus there are three cases:

If $\sin\frac{5\theta}{2}=0$, then $\frac{5\theta}{2}=n\pi$,

If $\cos\frac{\theta}{2}=0$, then $\frac{\theta}{2}=n\pi+\frac{\pi}{2}$,

If $\cos\theta=0$, then $\theta=n\pi+\frac{\pi}{2}$.

32. $\sin\theta-\cos\theta=4\sin\theta\cos^2\theta$;

therefore $$\sin\theta-4\sin\theta(1-\sin^2\theta)=\cos\theta;$$

therefore $$4\sin^3\theta-3\sin\theta=\cos\theta;$$

therefore $$\cos\theta=-\sin 3\theta=\cos\left(3\theta+\frac{\pi}{2}\right);$$

therefore $$3\theta+\frac{\pi}{2}=2n\pi\pm\theta.$$

33. $(\cot\theta-\tan\theta)^2(2-\sqrt{3})=4(2+\sqrt{3})$;

therefore $$\left(\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}\right)^2=\frac{4(2+\sqrt{3})}{2-\sqrt{3}};$$

therefore $$\left(\frac{\cos^2\theta-\sin^2\theta}{2\sin\theta\cos\theta}\right)^2=\frac{2+\sqrt{3}}{2-\sqrt{3}}=\frac{(2+\sqrt{3})^2}{(2-\sqrt{3})(2+\sqrt{3})};$$

therefore $$\left(\frac{\cos 2\theta}{\sin 2\theta}\right)^2=(2+\sqrt{3})^2;$$

therefore $$\cot^2 2\theta=\cot^2\frac{\pi}{12};$$

therefore $$2\theta=n\pi\pm\frac{\pi}{12}.$$

34. $2\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)(1+\sin\theta)=1+\cos 2\theta$;

therefore $$2\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)(1+\sin\theta)=2\cos^2\theta=2(1-\sin^2\theta);$$

therefore either $1+\sin\theta=0$, or $\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)=1-\sin\theta$.

If $1+\sin\theta=0$, then $\sin\theta=-1$; therefore $\theta=n\pi+(-1)^n\frac{3\pi}{2}$, which may be expressed more simply as $(4m+3)\frac{\pi}{2}$.

If $\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)=1-\sin\theta$, then $\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)=1-\sin\theta$;

therefore $2 \sin \theta = 1 - \cos \theta$;

therefore $4 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \sin^2 \frac{\theta}{2}$;

therefore either $\sin \frac{\theta}{2} = 0$, or $\tan \frac{\theta}{2} = 2$.

If $\sin \frac{\theta}{2} = 0$, then $\frac{\theta}{2} = n\pi$.

If $\tan \frac{\theta}{2} = 2$, then $\frac{\theta}{2} = n\pi + \alpha$, where α is such that $\tan \alpha = 2$.

35. $\sin 9\theta + \sin 5\theta + 2 \sin^2 \theta = 1$; therefore

$$2 \sin 7\theta \cos 2\theta = 1 - 2 \sin^2 \theta = \cos 2\theta;$$

therefore either $\cos 2\theta = 0$, or $\sin 7\theta = \frac{1}{2}$.

If $\cos 2\theta = 0$, then $2\theta = n\pi + \frac{\pi}{2}$.

If $\sin 7\theta = \frac{1}{2}$, then $7\theta = n\pi + (-1)^n \frac{\pi}{6}$.

CHAPTER X.

1. Let x denote the required logarithm; then

$$128 = (\sqrt[3]{4})^x, \text{ that is } 2^7 = 4^{\frac{x}{3}} = 2^{\frac{2x}{3}};$$

therefore $\frac{2x}{3} = 7$; therefore $x = \frac{21}{2}$.

2. Let x denote the required logarithm; then

$$243 \sqrt[3]{9} = (\sqrt{3})^x, \text{ that is } 3^5 \sqrt[3]{9} = 3^{\frac{x}{2}}, \text{ that is } 3^{5+\frac{2}{3}} = 3^{\frac{x}{2}};$$

therefore $\frac{x}{2} = \frac{17}{3}$; therefore $x = \frac{34}{3}$.

3. Let x denote the logarithm of 2187 to the base 3; then $2187 = 3^x$, that is $3^7 = 3^x$; therefore $x = 7$.

Let x denote the logarithm of $\cdot 0001$ to the base 10; then $\cdot 0001 = 10^x$, that is $\frac{1}{10^4} = 10^x$, that is $10^{-4} = 10^x$; therefore $x = -4$.

Let x denote the logarithm of $\cos 45^0$ to the base 2; then $\cos 45^0 = 2^x$, that is $\frac{1}{\sqrt{2}} = 2^x$, that is $2^{-\frac{1}{2}} = 2^x$; therefore $x = -\frac{1}{2}$.

4. $5^{6-4x}=2^{x+3}$; therefore $(6-4x)\log 5=(x+3)\log 2$;

therefore $$(6-4x)\log\frac{10}{2}=(x+3)\log 2;$$

therefore $$(6-4x)(1-\log 2)=(x+3)\log 2;$$

therefore $$x(4-3\log 2)=6-9\log 2;$$

therefore $$x=\frac{6-9\log 2}{4-3\log 2}=\frac{3\cdot 29073}{3\cdot 09691}=1\cdot 06....$$

5. Here $a=\log\cdot 224=\log\frac{224}{1000}=\log\frac{7\times 32}{1000}=\log 7+5\log 2-3$;

$$b=\log 125=\log\frac{1000}{8}=3-3\log 2.$$

From the second equation we have $\log 2=\frac{1}{3}(3-b)$; and then substituting in the first equation we have $\log 7=a+3-\frac{5}{3}(3-b)$.

6. 725 lies between 6^3 and 6^4; and therefore the characteristic of the logarithm of 725 to the base 6 is 3.

Then $$\log\sqrt[5]{(\cdot 0725)}=\frac{1}{5}\log\cdot 0725=\frac{1}{5}\log\frac{725}{10000};$$

and $\frac{725}{10000}$ lies between $\frac{1}{6}$ and $\frac{1}{36}$, that is between 6^{-1} and 6^{-2}. Hence $\frac{1}{5}\log\frac{725}{10000}$ to the base 6 lies between $-\frac{1}{5}$ and $-\frac{2}{5}$; and thus the characteristic will be -1, since by supposition the decimal part of a logarithm is positive.

7. $\text{Log } 405=\log(81\times 5)=\log\left(81\times\frac{10}{2}\right)=\log\frac{3^4\times 10}{2}=4\log 3+1-\log 2$;

therefore $$4\log 3=\log 405+\log 2-1=8\cdot 908485;$$

therefore $$\log 3=\cdot 477121.$$

8. $\text{Log } 98=\log(2\times 7^2)=\log 2+2\log 7=\cdot 301030+1\cdot 690196=1\cdot 991226$;

$$\log\left(\frac{4}{343}\right)^{\frac{1}{2}}=\frac{1}{2}\log\frac{4}{343}=\frac{1}{2}\log\frac{2^2}{7^3}=\frac{1}{2}(2\log 2-3\log 7)$$

$$=-\cdot 966617=\bar{1}\cdot 033383.$$

9. $$\text{Log }(\cdot 0020736)^{\frac{1}{3}}=\frac{1}{3}\log\cdot 0020736=\frac{1}{3}\log\frac{20736}{10^7}$$

$$=\frac{1}{3}\log\frac{3^4\times 2^8}{10^7}=\frac{1}{3}\{4\log 3+8\log 2-7\}$$

$$=-\cdot 89443=\bar{1}\cdot 10557.$$

10. $\frac{2}{\lfloor 3} = \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3}$, $\frac{4}{\lfloor 5} = \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5}$, $\frac{6}{\lfloor 7} = \frac{1}{\lfloor 6} - \frac{1}{\lfloor 7}$, ...

thus we see that the series $= \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5} + \ldots = e^{-1}$.

11. $$\frac{1}{\lfloor 2} = \frac{1}{2} \cdot \frac{1.2}{\lfloor 2} = \frac{1}{2},$$

$$\frac{1+2}{\lfloor 3} = \frac{1}{2} \cdot \frac{2.3}{\lfloor 3} = \frac{1}{2} \cdot \frac{1}{1},$$

$$\frac{1+2+3}{\lfloor 4} = \frac{1}{2} \cdot \frac{3.4}{\lfloor 4} = \frac{1}{2} \cdot \frac{1}{\lfloor 2},$$

$$\frac{1+2+3+4}{\lfloor 5} = \frac{1}{2} \cdot \frac{4.5}{\lfloor 5} = \frac{1}{2} \cdot \frac{1}{\lfloor 3},$$

and generally $\frac{1+2+3+\ldots+n}{\lfloor n+1} = \frac{1}{2} \cdot \frac{n(n+1)}{\lfloor n+1} = \frac{1}{2} \cdot \frac{1}{\lfloor n-1}$.

Thus we see that the series $= \frac{1}{2}\left\{1 + \frac{1}{1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\right\} = \frac{e}{2}$.

12. $4 \sin x \sin (x - a) = 2 \cos a - 1$;

therefore $2\{\cos a - \cos (2x - a)\} = 2 \cos a - 1$;

therefore $\cos (2x - a) = \frac{1}{2}$;

therefore $2x - a = 2n\pi \pm \frac{\pi}{3}$.

13. $\cos \beta \sqrt{(a^2 - x^2)} = x \sin \beta - a \sin a$;

therefore $\cos^2 \beta (a^2 - x^2) = x^2 \sin^2 \beta - 2xa \sin \beta \sin a + a^2 \sin^2 a$;

therefore $x^2 - 2xa \sin \beta \sin a = a^2 \cos^2 \beta - a^2 \sin^2 a$;

therefore $(x - a \sin \beta \sin a)^2 = a^2 \cos^2 \beta - a^2 \sin^2 a + a^2 \sin^2 \beta \sin^2 a$

$$= a^2 \cos^2 \beta - a^2 \sin^2 a \cos^2 \beta = a^2 \cos^2 \beta \cos^2 a;$$

therefore $x - a \sin \beta \sin a = \pm a \cos \beta \cos a$;

therefore $x = a (\sin \beta \sin a \pm \cos \beta \cos a) = a \cos (\beta - a)$ or $-a \cos (\beta + a)$.

14. $\sin a + \sin (x - a) + \sin (2x + a) = \sin (x + a) + \sin (2x - a)$;

therefore $\sin a = \sin (x + a) - \sin (x - a) + \sin (2x - a) - \sin (2x + a)$

$$= 2 \sin a \cos x - 2 \sin a \cos 2x;$$

therefore $$1 = 2\cos x - 2\cos 2x = 2\cos x - 2(2\cos^2 x - 1);$$

therefore $$4\cos^2 x - 2\cos x - 1 = 0.$$

By solving this quadratic in the usual way we obtain $\cos x = \frac{1 \pm \sqrt{5}}{4}$.

Taking the upper sign we have $\cos x = \cos\frac{\pi}{5}$, and therefore $x = 2n\pi \pm \frac{\pi}{5}$.

Taking the lower sign we have $\cos x = \cos\frac{3\pi}{5}$, and therefore $x = 2n\pi \pm \frac{3\pi}{5}$.

15. $\cos\left(x + \frac{3}{2}\right)\alpha + \cos\left(x + \frac{1}{2}\right)\alpha = \sin\alpha;$

therefore $$2\cos(x+1)\alpha\cos\frac{\alpha}{2} = \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2};$$

therefore $$\cos(x+1)\alpha = \sin\frac{\alpha}{2} = \cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right).$$

Hence all the solutions are contained in

$$(x+1)\alpha = 2n\pi \pm \left(\frac{\pi}{2} - \frac{\alpha}{2}\right).$$

16. $x^2\cos\alpha\cos\left(\alpha - \frac{\beta}{2}\right) + x\cos(\alpha-\beta) = 2\cos\frac{\beta}{2};$

therefore $$x^2 + \frac{x\cos(\alpha-\beta)}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)} = \frac{2\cos\frac{\beta}{2}}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)};$$

therefore $$\left\{x + \frac{\cos(\alpha-\beta)}{2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}\right\}^2 = \frac{2\cos\frac{\beta}{2}}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)} + \frac{\cos^2(\alpha-\beta)}{4\cos^2\alpha\cos^2\left(\alpha-\frac{\beta}{2}\right)}$$

$$= \frac{\cos^2(\alpha-\beta) + 8\cos\alpha\cos\frac{\beta}{2}\cos\left(\alpha-\frac{\beta}{2}\right)}{4\cos^2\alpha\cos^2\left(\alpha-\frac{\beta}{2}\right)}$$

$$= \frac{\cos^2(\alpha-\beta) + 4\cos\alpha\{\cos\alpha + \cos(\alpha-\beta)\}}{4\cos^2\alpha\cos^2\left(\alpha-\frac{\beta}{2}\right)}$$

$$= \frac{\{\cos(\alpha-\beta) + 2\cos\alpha\}^2}{4\cos^2\alpha\cos^2\left(\alpha-\frac{\beta}{2}\right)};$$

therefore $$x + \frac{\cos(\alpha-\beta)}{2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)} = \pm\frac{\cos(\alpha-\beta) + 2\cos\alpha}{2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}.$$

Taking the upper sign we have

$$x = \frac{2\cos\alpha}{2\cos\alpha\cos\left(\alpha - \frac{\beta}{2}\right)} = \sec\left(\alpha - \frac{\beta}{2}\right).$$

Taking the lower sign we have

$$x = -\frac{\cos\alpha + \cos(\alpha-\beta)}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}$$

$$= -\frac{2\cos\left(\alpha-\frac{\beta}{2}\right)\cos\frac{\beta}{2}}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)} = -2\cos\frac{\beta}{2}\sec\alpha.$$

Or we may write the proposed equation in this form

$$x\cos\alpha\left\{x\cos\left(\alpha-\frac{\beta}{2}\right)-1\right\} + 2\left\{x\cos\left(\alpha-\frac{\beta}{2}\right)-1\right\}\cos\frac{\beta}{2}=0;$$

and then the two values of x which satisfy it are obvious.

17. $\cot 2^{x-1}\alpha - \cot 2^x\alpha = \operatorname{cosec} 3\alpha$;

put y for $2^{x-1}\alpha$; thus $\cot y - \cot 2y = \operatorname{cosec} 3\alpha$;

therefore $$\frac{\cos y}{\sin y} - \frac{\cos 2y}{\sin 2y} = \operatorname{cosec} 3\alpha;$$

therefore $$\frac{\sin(2y-y)}{\sin y\sin 2y} = \operatorname{cosec} 3\alpha = \frac{1}{\sin 3\alpha};$$

therefore $\sin 2y = \sin 3\alpha$, that is $\sin 2^x\alpha = \sin 3\alpha$.

Thus the general solution is $2^x\alpha = n\pi + (-1)^n 3\alpha$.

18. $m \operatorname{vers}\theta = n\operatorname{vers}(\alpha-\theta)$;

therefore $$m(1-\cos\theta) = n\{1-\cos(\alpha-\theta)\};$$

therefore $$2m\sin^2\frac{\theta}{2} = 2n\sin^2\frac{\alpha-\theta}{2};$$

therefore $$\sin\frac{\alpha-\theta}{2} = \left(\frac{m}{n}\right)^{\frac{1}{2}}\sin\frac{\theta}{2};$$

therefore $$\sin\frac{\alpha}{2}\cos\frac{\theta}{2} - \cos\frac{\alpha}{2}\sin\frac{\theta}{2} = \left(\frac{m}{n}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}.$$

Divide by $\cos\frac{\theta}{2}$; thus we obtain a simple equation for finding $\tan\frac{\theta}{2}$.

19. $\cos n\theta + \cos(n-2)\theta = \cos\theta$;

therefore $$2\cos(n-1)\theta\cos\theta = \cos\theta;$$

therefore either $\cos\theta = 0$, or $\cos(n-1)\theta = \frac{1}{2}$.

If $\cos\theta = 0$, then $\theta = m\pi + \frac{\pi}{2}$.

If $\cos(n-1)\theta = \frac{1}{2}$, then $(n-1)\theta = 2m\pi \pm \frac{\pi}{3}$.

20. $\sin\theta + \sin 3\theta = \sin 2\theta + \sin 4\theta$;

therefore $$2\sin 2\theta\cos\theta = 2\sin 3\theta\cos\theta;$$

therefore either $\cos\theta = 0$, or $\sin 2\theta = \sin 3\theta$.

If $\cos\theta = 0$, then $\theta = n\pi + \frac{\pi}{2}$.

If $\sin 2\theta = \sin 3\theta$, then $\sin 2\theta - \sin 3\theta = 0$; therefore $2\sin\frac{\theta}{2}\cos\frac{5\theta}{2} = 0$; therefore either $\sin\frac{\theta}{2} = 0$, or $\cos\frac{5\theta}{2} = 0$: taking $\sin\frac{\theta}{2} = 0$ we have $\frac{\theta}{2} = n\pi$, and taking $\cos\frac{5\theta}{2} = 0$ we have $\frac{5\theta}{2} = n\pi + \frac{\pi}{2}$.

The seven values greater than 0 and less than 2π are

$$\frac{\pi}{5},\ \frac{3\pi}{5},\ \frac{5\pi}{5},\ \frac{7\pi}{5},\ \frac{9\pi}{5},\ \frac{\pi}{2} \text{ and } \frac{3\pi}{2}.$$

21. $\tan x = \tan\beta\tan(x+\alpha) = \dfrac{\tan\beta(\tan x + \tan\alpha)}{1 - \tan x\tan\alpha}$;

therefore $$\tan x(1 - \tan x\tan\alpha) = \tan\beta(\tan x + \tan\alpha);$$

therefore $$\tan^2 x\tan\alpha + (\tan\beta - 1)\tan x + \tan\alpha\tan\beta = 0.$$

By solving this quadratic in the usual way we obtain the values of $\tan x$. It is known by the theory of quadratic equations that for the values to be real we must have $(\tan\beta - 1)^2 - 4\tan^2\alpha\tan\beta$ positive or zero.

And $(\tan\beta - 1)^2 - 4\tan^2\alpha\tan\beta$.

$$= \tan^2\beta - 2\tan\beta - 4\tan^2\alpha\tan\beta + 1$$

$$= \{\tan\beta - (1 + 2\tan^2\alpha)\}^2 + 1 - (1 + 2\tan^2\alpha)^2$$

$$= \left\{\tan\beta - \frac{1+\sin^2\alpha}{\cos^2\alpha}\right\}^2 - \frac{4\sin^2\alpha}{\cos^4\alpha}$$

$$= \left\{\tan\beta - \frac{1+\sin^2\alpha}{\cos^2\alpha} - \frac{2\sin\alpha}{\cos^2\alpha}\right\}\left\{\tan\beta - \frac{1+\sin^2\alpha}{\cos^2\alpha} + \frac{2\sin\alpha}{\cos^2\alpha}\right\}$$

$$= \left\{\tan\beta - \left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2\right\}\left\{\tan\beta - \left(\frac{1-\sin\alpha}{\cos\alpha}\right)^2\right\}.$$

This expression then must be positive or zero, and therefore $\tan\beta$ must not lie between $\left(\frac{1-\sin\alpha}{\cos\alpha}\right)^2$ and $\left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2$.

22. $$\tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{\pi}{4}+\theta\right)$$

$$=\frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\cos\left(\frac{\pi}{4}-\theta\right)}+\frac{\sin\left(\frac{\pi}{4}+\theta\right)}{\cos\left(\frac{\pi}{4}+\theta\right)}$$

$$=\frac{\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)+\sin\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}{\cos\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)}$$

$$=\frac{\sin\frac{\pi}{2}}{\cos\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)}=\frac{1}{\sin\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)}$$

$$=\frac{2}{\sin\left(\frac{\pi}{2}+2\theta\right)}=\frac{2}{\cos 2\theta}.$$

Thus $$\frac{2}{\cos 2\theta}=\left(\frac{8\sqrt{2}}{1+\sqrt{2}}\right)^{\frac{1}{2}};$$

therefore $$\frac{\cos 2\theta}{2}=\left(\frac{1+\sqrt{2}}{8\sqrt{2}}\right)^{\frac{1}{2}};$$

therefore $$\cos^2 2\theta=\frac{1+\sqrt{2}}{2\sqrt{2}};$$

therefore $$2\cos^2 2\theta-1=\frac{1+\sqrt{2}}{\sqrt{2}}-1=\frac{1}{\sqrt{2}};$$

therefore $$\cos 4\theta=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4};$$

therefore the least value of θ is given by $4\theta=\frac{\pi}{4}$.

23. $$\sin^2(n+1)\theta=\sin^2 n\theta+\sin^2(n-1)\theta;$$

therefore $$\sin^2(n+1)\theta-\sin^2(n-1)\theta=\sin^2 n\theta;$$

therefore $$\sin 2n\theta\sin 2\theta=\sin^2 n\theta. \quad \text{(Art. 83.)}$$

But $$(n+1)\theta+(n-1)\theta+n\theta=\pi;$$

therefore $3n\theta=\pi$; therefore $n\theta=\frac{\pi}{3}$;

therefore $$\sin 2\theta \sin\frac{2\pi}{3}=\sin^2\frac{\pi}{3};$$

therefore $$\sin 2\theta=\sin\frac{\pi}{3};$$

thus $2\theta=\frac{\pi}{3}$; therefore $\theta=\frac{\pi}{6}$. But $n\theta=\frac{\pi}{3}$; and therefore $n=2$.

24. $\cos^2\theta-\cos^2\alpha=2\cos^3\theta(\cos\theta-\cos\alpha)-2\sin^3\theta(\sin\theta-\sin c)$;

therefore $$\cos^2\theta-\cos^2\alpha=\frac{\cos 3\theta+3\cos\theta}{2}(\cos\theta-\cos\alpha)$$
$$-\frac{3\sin\theta-\sin 3\theta}{2}(\sin\theta-\sin\alpha);$$

therefore $$2(\cos^2\theta-\cos^2\alpha)=\cos 3\theta\cos\theta+\sin 3\theta\sin\theta-\cos 3\theta\cos\alpha-\sin 3\theta\sin\alpha$$
$$+3\cos^2\theta-3\sin^2\theta-3\cos\theta\cos\alpha+3\sin\theta\sin\alpha;$$

therefore $\cos(3\theta-\theta)-\cos(3\theta-\alpha)-3\cos(\theta+\alpha)=3\sin^2\theta-\cos^2\theta-2\cos^2\alpha$;

therefore $$\cos 2\theta-\cos(3\theta-\alpha)-3\cos(\theta+\alpha)=3-4\cos^2\theta-2\cos^2\alpha$$
$$=3-2(1+\cos 2\theta)-(1+\cos 2\alpha)$$
$$=-2\cos 2\theta-\cos 2\alpha;$$

therefore $$3\cos 2\theta-3\cos(\theta+\alpha)-\cos(3\theta-\alpha)+\cos 2\alpha=0;$$

therefore $$3\sin\frac{3\theta+\alpha}{2}\sin\frac{\alpha-\theta}{2}+\sin\frac{(3\theta+\alpha)}{2}\sin\frac{3\theta-3\alpha}{2}=0;$$

therefore $$\sin\frac{3\theta+\alpha}{2}\left\{\sin\frac{3(\theta-\alpha)}{2}-3\sin\frac{\theta-\alpha}{2}\right\}=0;$$

therefore $$4\sin\frac{3\theta+\alpha}{2}\sin^3\frac{\theta-\alpha}{2}=0.$$

Hence either $\sin\frac{3\theta+\alpha}{2}=0$, or $\sin\frac{\theta-\alpha}{2}=0$; the former gives $\frac{3\theta+\alpha}{2}=n\pi$, and the latter gives $\frac{\theta-\alpha}{2}=n\pi$.

25. Let θ denote an angle having the same sine as α, so that $\sin\theta=\sin\alpha$; thus $\cos\left(\theta-\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}-\alpha\right)$; therefore all the solutions are comprised in $\theta-\frac{\pi}{2}=2n\pi\pm\left(\frac{\pi}{2}-\alpha\right)$.

26. Let θ denote an angle having the same cosine as α, so that $\cos\theta = \cos\alpha$; thus $\sin\left(\theta - \frac{\pi}{2}\right) = \sin\left(\alpha - \frac{\pi}{2}\right)$; therefore all the solutions are comprised in $\theta - \frac{\pi}{2} = n\pi + (-1)^n\left(\alpha - \frac{\pi}{2}\right)$.

27. By Art. 101 it follows that the upper sign ought to be taken if $\frac{A}{2}$ lies between $n\,360^0 + 225^0$ and $n\,360^0 + 405^0$; in this case A lies between $2n\,360^0+450^0$ and $2n\,360^0+810^0$, and $A+270^0$ lies between $2n\,360^0+720^0$ and $2n360^0+1080^0$, and therefore $\frac{A+270^0}{360^0}$ lies between $2n+2$ and $2n+3$: thus the integral part of this fraction is an *even* number, so that denoting it by m we have $(-1)^m$ positive.

In precisely the same manner we find that the present example agrees with Art. 101 for the case in which m is *odd*.

28. First suppose the number of degrees in A to lie between $n\,360$ and $n\,360+90$; then $\tan A$ and $\tan\frac{A}{2}$ are both positive, and therefore the upper sign must be taken in the ambiguity. Also in this case $\frac{A+90}{180}$ lies between $\frac{n\,360+90}{180}$ and $\frac{n\,360+180}{180}$, that is between $2n+\frac{1}{2}$ and $2n+1$; so that m is *even*.

Next suppose the number of degrees in A to lie between $n\,360+90$ and $n\,360+180$; then $\tan A$ is negative, and $\tan\frac{A}{2}$ is positive; and therefore the lower sign must be taken in the ambiguity. Also in this case $\frac{A+90}{180}$ lies between $2n+1$ and $2n+2$, so that m is *odd*.

Similarly we may proceed if the number of degrees in A lies between $n\,360+180$ and $n\,360+270$, or between $n\,360+270$ and $n\,360+360$.

It will be observed that in this and the preceding example the *greatest integer* in a certain expression means that integer which with a *positive* proper fraction constitutes the whole expression.

Or we might treat the example thus:

$$\pm\sqrt{(1+\tan^2 A)} = \pm\sqrt{\frac{1}{\cos^2 A}} = \pm\frac{1}{\cos A};$$

but

$$\tan\frac{A}{2} = \frac{1-\cos A}{\sin A} = \frac{\frac{1}{\cos A} - 1}{\tan A}:$$

hence the ambiguity in $\pm\sqrt{(1+\tan^2 A)}$ must be so taken as to ensure that the *sign is the same as the sign* of $\cos A$, and it is easy to shew that $(-1)^m$ is of the same sign as $\cos A$ when m has the prescribed value.

29. $$\tan(\cot x) = \cot(\tan x);$$

therefore $$\tan(\cot x) = \tan\left\{\frac{\pi}{2} - \tan x\right\};$$

therefore, by Art. 68, all the possible solutions are comprised in

$$\cot x = n\pi + \frac{\pi}{2} - \tan x;$$

therefore $$\cot x + \tan x = n\pi + \frac{\pi}{2};$$

therefore $$\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = n\pi + \frac{\pi}{2};$$

therefore $$\frac{1}{\sin x \cos x} = \frac{(2n+1)\pi}{2};$$

therefore $$\sin x \cos x = \frac{2}{(2n+1)\pi};$$

therefore $$\sin 2x = \frac{4}{(2n+1)\pi}.$$

The value $n = -1$ would make $\sin 2x$ greater than unity.

30. $$2\cos^2\frac{A}{2} = 1 + \cos A;$$

therefore $$4\cos^2\frac{A}{2} = 2 + 2\cos A;$$

therefore $$2\cos\frac{A}{2} = \sqrt{(2 + 2\cos A)}.$$

Again $$2\cos^2\frac{A}{4} = 1 + \cos\frac{A}{2};$$

therefore $$4\cos^2\frac{A}{4} = 2 + 2\cos\frac{A}{2};$$

therefore $$2\cos\frac{A}{4} = \sqrt{\left(2 + 2\cos\frac{A}{2}\right)} = \sqrt{\{2 + \sqrt{(2 + 2\cos A)}\}}.$$

Similarly $$2\cos\frac{A}{8} = \sqrt{[2 + \sqrt{\{2 + \sqrt{(2 + 2\cos A)}\}}]};$$

and this process may be continued to any extent.

31. Change x successively to $\frac{\pi}{4} - x$ and $\frac{\pi}{4} + x$; thus

$$\cos\left(x - \frac{\pi}{4}\right) = \pm\sqrt{\frac{1}{2}\left\{1 + \cos\left(2x - \frac{\pi}{2}\right)\right\}} = \pm\sqrt{\frac{1 + \sin 2x}{2}},$$

and $\cos\left(\frac{\pi}{4}+x\right) = \pm\sqrt{\frac{1}{2}\left\{1+\cos\left(\frac{\pi}{2}+2x\right)\right\}} = \pm\sqrt{\frac{1-\sin 2x}{2}}$.

Then putting for $\cos\left(\frac{\pi}{4}-x\right)$ and $\cos\left(\frac{\pi}{4}+x\right)$ their values we have

$$\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \pm\sqrt{\frac{1+\sin 2x}{2}} \quad \ldots\ldots\ldots\ldots (1),$$

and

$$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \pm\sqrt{\frac{1-\sin 2x}{2}} \quad \ldots\ldots\ldots\ldots (2).$$

Hence by subtraction we find the required expression for $\sin x$. In (1) the upper or lower sign must be taken according as $\cos\left(x-\frac{\pi}{4}\right)$ is positive or negative, that is according as $x-\frac{\pi}{4}$ lies between $2n\pi-\frac{1}{2}\pi$ and $2n\pi+\frac{1}{2}\pi$, or between $2n\pi+\frac{1}{2}\pi$ and $2n\pi+\frac{3\pi}{2}$. Similarly we can determine the sign to be taken in (2).

32. Let k denote the value which the expression retains for all values of θ, so that

$$\frac{A\cos(\theta+\alpha)+B\sin(\theta+\beta)}{A'\sin(\theta+\alpha)+B'\cos(\theta+\beta)} = k;$$

then $A\cos(\theta+\alpha)+B\sin(\theta+\beta) = k\{A'\sin(\theta+\alpha)+B'\cos(\theta+\beta)\}$;

therefore $\cos\theta(A\cos\alpha+B\sin\beta)+\sin\theta(B\cos\beta-A\sin\alpha)$

$$= k\cos\theta(A'\sin\alpha+B'\cos\beta)+k\sin\theta(A'\cos\alpha-B'\sin\beta);$$

therefore $\cos\theta\{A\cos\alpha+B\sin\beta-k(A'\sin\alpha+B'\cos\beta)\}$

$$+\sin\theta\{B\cos\beta-A\sin\alpha-k(A'\cos\alpha-B'\sin\beta)\}=0.$$

Now this is to be true for *all values* of θ. Put for θ in succession 0 and $\frac{\pi}{2}$; thus we obtain the following two results:

$$A\cos\alpha+B\sin\beta = k(A'\sin\alpha+B'\cos\beta),$$
$$B\cos\beta-A\sin\alpha = k(A'\cos\alpha-B'\sin\beta);$$

and it is obvious that if these hold the original expression does always retain the same value.

By cross multiplication we obtain

$(A\cos\alpha+B\sin\beta)(A'\cos\alpha-B'\sin\beta) = (A'\sin\alpha+B'\cos\beta)(B\cos\beta-A\sin\alpha)$;

therefore $AA'\cos^2\alpha-BB'\sin^2\beta+(A'B-AB')\cos\alpha\sin\beta$

$$= BB'\cos^2\beta-AA'\sin^2\alpha+(A'B-AB')\sin\alpha\cos\beta;$$

therefore $AA'-BB' = (A'B-AB')\sin(\alpha-\beta)$.

33. Let A denote the sum of the two angles x and y. Then

$$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=2\sin\frac{A}{2}\cos\frac{x-y}{2};$$

and the numerically greatest value of this expression is when $\cos\frac{x-y}{2}$ is greatest, that is when $x-y=0$, that is when $x=y$.

Again $$\tan x+\tan y=\frac{\sin x}{\cos x}+\frac{\sin y}{\cos y}=\frac{\sin(x+y)}{\cos x\cos y}$$

$$=\frac{\sin A}{\cos x\cos y}=\frac{2\sin A}{2\cos x\cos y}$$

$$=\frac{2\sin A}{\cos(x-y)+\cos(x+y)}=\frac{2\sin A}{\cos(x-y)+\cos A};$$

and if $\cos A$ is positive the numerically least value of this is when

$$\cos(x-y)=1,\text{ that is when } x=y.$$

34. By Art. 114 we have

$$\tan A\tan B+\tan B\tan C+\tan C\tan A=1;$$

therefore $$\tan^2 A+\tan^2 B+\tan^2 C=1+\frac{1}{2}(\tan A-\tan B)^2$$

$$+\frac{1}{2}(\tan B-\tan C)^2+\frac{1}{2}(\tan C-\tan A)^2.$$

Hence the least value of the expression is when $\tan A-\tan B$, $\tan B-\tan C$, and $\tan C-\tan A$ all vanish; and the value is then unity.

35. By Art. 114 we have

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C;$$

therefore $$\frac{1}{\cot A}+\frac{1}{\cot B}+\frac{1}{\cot C}=\frac{1}{\cot A\cot B\cot C};$$

therefore $$\cot B\cot C+\cot A\cot C+\cot A\cot B=1;$$

therefore $$\cot^2 A+\cot^2 B+\cot^2 C$$

$$=1+\frac{1}{2}(\cot A-\cot B)^2+\frac{1}{2}(\cot B-\cot C)^2+\frac{1}{2}(\cot C-\cot A)^2.$$

Hence the least value of the expression is when $\cot A-\cot B$, $\cot B-\cot C$, and $\cot C-\cot A$ all vanish; and the value is then unity.

36. $\cot B + \cot C - \operatorname{cosec} A = \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C} - \frac{1}{\sin A}$

$$= \frac{\sin(B+C)}{\sin B \sin C} - \frac{1}{\sin A} = \frac{\sin A}{\sin B \sin C} - \frac{1}{\sin A} = \frac{\sin^2 A - \sin B \sin C}{\sin A \sin B \sin C}.$$

Proceeding in this way we find that the difference of the two given expressions is equivalent to a fraction with the denominator $\sin A \sin B \sin C$, while the numerator is

$$\sin^2 A + \sin^2 B + \sin^2 C - \sin B \sin C - \sin C \sin A - \sin A \sin B,$$

that is $\frac{1}{2}(\sin A - \sin B)^2 + \frac{1}{2}(\sin B - \sin C)^2 + \frac{1}{2}(\sin C - \sin A)^2.$

This expression is never negative.

37. Suppose A, B, C to be three acute angles such that

$$\cos^2 A + \cos^2 B + \cos^2 C = 1,$$

then
$$\cos^2 A = 1 - \cos^2 C - \cos^2 B = \sin^2 C - \cos^2 B$$
$$= -\cos(C-B)\cos(C+B).$$

This shews that $C+B$ must be greater than a right angle. Now if we take $A' = 180^0 - C - B$ we shall have $\cos^2 A'$ numerically equal to $\cos^2(B+C)$, and therefore numerically *less* than $\cos(C-B)\cos(C+B)$; for we may suppose C not less than B, and then $C-B$ is less than $180^0 - C - B$. Hence $\cos^2 A$ is greater than $\cos^2 A'$, and A is less than A', and therefore $A+B+C$ is less than 180^0.

38. By Art. 113 we have $\sin A + \sin B + \sin C - \sin(A+B+C)$

$$= \sin A(1 - \cos B \cos C) + \sin B(1 - \cos C \cos A) + \sin C(1 - \cos A \cos B)$$
$$+ \sin A \sin B \sin C;$$

and as A, B, and C are acute this expression is necessarily positive.

39. Let $u = \left(\cos\frac{a}{n}\right)^{n^2}$;

therefore
$$\log u = n^2 \log\cos\frac{a}{n} = \frac{n^2}{2}\log\left(1 - \sin^2\frac{a}{n}\right)$$
$$= -\frac{n^2}{2}\left\{\sin^2\frac{a}{n} + \frac{1}{2}\sin^4\frac{a}{n} + \frac{1}{3}\sin^6\frac{a}{n} + \ldots\right\}.$$

Now $n\sin\frac{a}{n} = a\frac{\sin\frac{a}{n}}{\frac{a}{n}}$, and this is equal to a when n is indefinitely increased; and therefore $n^2\sin^2\frac{a}{n}$ is equal to a^2.

Then $n^2 \sin^4 \frac{\alpha}{n} = n^2 \sin^2 \frac{\alpha}{n} \times \sin^2 \frac{\alpha}{n}$; and this vanishes when n is indefinitely increased. Similarly the other terms in $\log u$ vanish, and as in Art. 150 their sum vanishes also; and thus $\log u = -\frac{\alpha^2}{2}$ ultimately. Therefore $u = e^{-\frac{\alpha^2}{2}}$.

40. Let $u = \left(\cos \frac{\alpha}{n}\right)^{n^3}$; therefore

$$\log u = n^3 \log \cos \frac{\alpha}{n} = \frac{n^3}{2} \log \left(1 - \sin^2 \frac{\alpha}{n}\right)$$

$$= -\frac{n^3}{2} \left\{\sin^2 \frac{\alpha}{n} + \frac{1}{2} \sin^4 \frac{\alpha}{n} + \frac{1}{3} \sin^6 \frac{\alpha}{n} + \dots\right\}.$$

Now we have shewn in solving the preceding Example that $n^2 \sin^2 \frac{\alpha}{n} = \alpha^2$ ultimately; hence $n^3 \sin^2 \frac{\alpha}{n} = n\alpha^2$, and so becomes infinite. Thus the logarithm of u is negative infinity, and therefore u vanishes ultimately.

41. $\sin\theta - (\tan\theta - \frac{1}{2}\tan^3\theta) = \sin\theta - \tan\theta + \frac{1}{2}\tan^3\theta$

$$= \sin\theta - \frac{\sin\theta}{\cos\theta} + \frac{1}{2}\frac{\sin^3\theta}{\cos^3\theta} = \frac{\sin\theta}{\cos^3\theta}\{\cos^3\theta - \cos^2\theta + \frac{1}{2}\sin^2\theta\}$$

$$= \frac{\sin\theta}{2\cos^3\theta}\{2\cos^3\theta - 2\cos^2\theta + 1 - \cos^2\theta\}$$

$$= \frac{\sin\theta\,(1-\cos\theta)}{2\cos^3\theta}\{1 + \cos\theta - 2\cos^2\theta\}$$

$$= \frac{\sin\theta\,(1-\cos\theta)\,(1-\cos\theta)\,(1+2\cos\theta)}{2\cos^3\theta}$$

$$= \frac{\sin\theta\,(1-\cos\theta)^2\,(1+2\cos\theta)}{2\cos^3\theta}, \text{ which is positive.}$$

42. Let $u = \left(\frac{x-1}{x}\right)^x$; then

$$\log u = x \log \frac{x-1}{x} = x \log \left(1 - \frac{1}{x}\right)$$

$$= -x\left\{\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \dots\right\}$$

$$= -\left\{1 + \frac{1}{2x} + \frac{1}{3x^2} + \dots\right\}.$$

Thus the logarithm is always negative, and as x increases the logarithm diminishes numerically, and so u increases; when x is infinite $\log u = -1$; and therefore $u = e^{-1}$.

CHAPTER XI.

1.
$$\begin{array}{r} 4{\cdot}0948553 \\ 4{\cdot}0948204 \\ \hline {\cdot}0000349 \end{array} \qquad 1 : {\cdot}35 :: {\cdot}0000349 : x;$$

this gives $x = {\cdot}0000122;$

therefore $\log 12440{\cdot}35 = 4{\cdot}0948326.$

2.
$$\begin{array}{r} {\cdot}0288558 \\ {\cdot}0288152 \\ \hline {\cdot}0000406 \end{array} \quad \begin{array}{r} {\cdot}0288355 \\ {\cdot}0288152 \\ \hline {\cdot}0000203 \end{array} \qquad {\cdot}0000406 : {\cdot}0000203 :: {\cdot}0001 : x;$$

this gives $x = {\cdot}00005;$

therefore $\log 1{\cdot}06865 = {\cdot}0288355.$

3.
$$\begin{array}{r} 4{\cdot}3702725 \\ 4{\cdot}3702540 \\ \hline {\cdot}0000185 \end{array}$$

$$\begin{array}{lll} \log 23456 & = 4{\cdot}3702540 & \\ \text{add for } 3 & \phantom{=4{\cdot}370}555 & \\ \phantom{\text{add for }} 8 & \phantom{=4{\cdot}370}1480 & \\ \hline & 4{\cdot}370261030 & \end{array}$$

1	185
2	370
3	555
4	740
5	925
6	1110
7	1295
8	1480
9	1665

therefore retaining 7 places of decimals

$$\log 23456{\cdot}38 = 4{\cdot}3702610, \text{ and } \log {\cdot}2345638 = \bar{1}{\cdot}3702610.$$

4. $-(1{\cdot}8753145) = \bar{2}{\cdot}1246855.$

$$\begin{array}{r} {\cdot}1246998 \\ {\cdot}1246672 \\ \hline {\cdot}0000326 \end{array} \quad \begin{array}{r} {\cdot}1246855 \\ {\cdot}1246672 \\ \hline {\cdot}0000183 \end{array} \qquad {\cdot}0000326 : {\cdot}0000183 :: {\cdot}0001 : x;$$

this gives $x = {\cdot}000056;$

therefore $\log 1{\cdot}332556 = {\cdot}1246855,$

therefore $\log {\cdot}01332556 = \bar{2}{\cdot}1246855.$

5.
$$\begin{array}{r} {\cdot}5860356 \\ {\cdot}5860244 \\ \hline {\cdot}0000112 \end{array} \qquad {\cdot}0001 : {\cdot}00004 :: {\cdot}0000112 : x;$$

this gives $x = {\cdot}0000045;$

therefore $\log 3{\cdot}85504 = {\cdot}5860289;$

therefore $\log {\cdot}00385504 = \bar{3}{\cdot}5860289;$

therefore $\log ({\cdot}00385504)^{\frac{1}{4}} = \frac{1}{4}(\bar{3}{\cdot}5860289) = \frac{1}{4}(-4 + 1{\cdot}5860289) = \bar{1}{\cdot}3965072.$

6. $\log (24)^{\frac{1}{2}} = \frac{1}{2} \log 24 = \cdot 6901056.$

$$\begin{array}{r} \cdot 6901074 \\ \cdot 6900986 \\ \hline \cdot 0000088 \end{array} \qquad \begin{array}{r} \cdot 6901056 \\ \cdot 6900986 \\ \hline \cdot 0000070 \end{array} \qquad \cdot 0000088 : \cdot 0000070 :: \cdot 0001 : x;$$

this gives $x = \cdot 000079;$

therefore $\log 4 \cdot 898979 = \cdot 6901056;$

therefore $(24)^{\frac{1}{2}} = 4 \cdot 898979.$

7. $\log (142 \cdot 71)^{\frac{1}{7}} = \frac{1}{7} \times 2 \cdot 1544544 = \cdot 3077792.$

$$\begin{array}{r} \cdot 3077954 \\ \cdot 3077741 \\ \hline \cdot 0000213 \end{array} \qquad \begin{array}{r} \cdot 3077792 \\ \cdot 3077741 \\ \hline \cdot 0000051 \end{array} \qquad \cdot 0000213 : \cdot 0000051 :: 1 : x;$$

this gives $x = \cdot 24;$

therefore $\log 20313 \cdot 24 = 4 \cdot 3077792;$

therefore $\log 2 \cdot 031324 = \cdot 3077792;$

therefore $(142 \cdot 71)^{\frac{1}{7}} = 2 \cdot 031324.$

8. $\log (\cdot 07)^{\frac{1}{5}} = \frac{1}{5} \log \cdot 07 = \frac{1}{5} (\bar{2} \cdot 8450980) = \frac{1}{5} (-5 + 3 \cdot 8450980) = \bar{1} \cdot 7690196.$

$$\begin{array}{r} \cdot 7690227 \\ \cdot 7690153 \\ \hline \cdot 0000074 \end{array} \qquad \begin{array}{r} \cdot 7690196 \\ \cdot 7690153 \\ \hline \cdot 0000043 \end{array} \qquad \cdot 0000074 : \cdot 0000043 :: 1 : x;$$

this gives $x = \cdot 58;$

therefore $\log 58751 \cdot 58 = 4 \cdot 7690196;$

therefore $\log \cdot 5875158 = \bar{1} \cdot 7690196;$

therefore $(\cdot 07)^{\frac{1}{5}} = \cdot 5875158.$

9. $\log (\cdot 0625)^{\frac{1}{5}} = \log \left(\frac{625}{10000}\right)^{\frac{1}{5}} = \log \left(\frac{125}{2000}\right)^{\frac{1}{5}} = \log \left(\frac{25}{400}\right)^{\frac{1}{5}}$

$$= \log \left(\frac{1}{16}\right)^{\frac{1}{5}} = -\frac{1}{5} \log 16 = -\frac{4}{5} \log 2 = -\cdot 2408240$$

$$= \bar{1} \cdot 7591760 = \log \cdot 5743491;$$

therefore $(\cdot 0625)^{\frac{1}{5}} = \cdot 5743491.$

10. $\log (27)^{-\frac{1}{5}} = -\frac{1}{5} \log 27 = -\frac{1}{5} (1 \cdot 4313638) = -\cdot 2862728$

$$= \bar{1} \cdot 7137272 = \log \cdot 5172818;$$

therefore $(27)^{-\frac{1}{5}} = \cdot 5172818.$

11. $\log 71968 \cdot 6 = 4 \cdot 8571394 + \frac{6}{10}$ of $\cdot 0000060 = 4 \cdot 8571430$;

$$\log (\cdot 0719686)^{\frac{1}{8}} = \frac{1}{8} (\bar{2} \cdot 8571430) = \frac{1}{8} (-8 + 6 \cdot 8571430) = \bar{1} \cdot 8571429.$$

But $\log \cdot 719686 = \bar{1} \cdot 8571430$; therefore $(\cdot 0719686)^{\frac{1}{8}} = \cdot 719686$.

12. $\log (1 \cdot 03)^{-10} = -10 \times \cdot 0128372 = -\cdot 128372 = \bar{1} \cdot 871628 = \log \cdot 7440942$;

therefore $(1 \cdot 03)^{-10} = \cdot 7440942$.

13. $\log (1 \cdot 05)^{-20} = -20 \times \cdot 0211893 = -\cdot 423786 = \bar{1} \cdot 576214 = \log \cdot 37689$;

therefore $(1 \cdot 05)^{-20} = \cdot 37689$;

therefore $64 \{1 - (1 \cdot 05)^{-20}\} = 64 \{1 - \cdot 37689\}$

$= 64 \times \cdot 62311 = 39 \cdot 87904$.

14. Denote it by u; then $\log u = \sqrt{5} \log 5 = 2 \sqrt{5} \log \sqrt{5}$;

therefore $\log (\log u) = \log 2 + \log \sqrt{5} + \log (\log \sqrt{5})$.

Now $\log \sqrt{5} = \frac{1}{2} \log 5 = \frac{1}{2} \log \frac{10}{2} = \frac{1}{2} (1 - \log 2)$

$= \frac{1}{2} (1 - \cdot 301030) = \frac{1}{2} (\cdot 698970) = \cdot 349485$,

$\log (\log \sqrt{5}) = \log \cdot 349485 = \bar{1} \cdot 543428$.

Therefore $\log (\log u) = \cdot 301030 + \cdot 349485 + \bar{1} \cdot 543428 = \cdot 193943$.

Therefore $\log u = 1 \cdot 562944$.

·563006	·562944	
·562887	·562887	·000119 : ·000057 :: ·001 : x;
·000119	·000057	

this gives $x = \cdot 00048$; therefore $u = 36 \cdot 5548$.

15. $\log 144 = \log 12^2 = 2 \log 12 = 2 \cdot 1583624$;

$\log (1 \cdot 44)^{-6} = -6 \log 1 \cdot 44 = -6 (\cdot 1583624) = -\cdot 9501744$

$= \bar{1} \cdot 0498256 = \log \cdot 1121568$;

therefore $(1 \cdot 44)^{-6} = \cdot 1121568$.

$\log (1 \cdot 44)^{-12} = -12 \log 1 \cdot 44 = -12 (\cdot 1583624) = -1 \cdot 9003488$

$= \bar{2} \cdot 0996512 = \log \cdot 01257915$;

therefore $(1 \cdot 44)^{-12} = \cdot 01257915$;

therefore $(1 \cdot 44)^{-6} - (1 \cdot 44)^{-12} = \cdot 1121568 - \cdot 01257915 = \cdot 09957765$.

16. $$\log\frac{1}{(1\cdot05)^{13}} = -13\log 1\cdot05 = -13\,(\cdot0211893) = -\cdot2754609$$

$$= \bar{1}\cdot7245391 = \log\cdot5303214;$$

therefore $$\frac{1}{(1\cdot05)^{13}} = \cdot5303214;$$

$$\log\frac{1}{(1\cdot05)^{20}} = -20\log 1\cdot05 = -20\,(\cdot0211893) = -\cdot423786$$

$$= \bar{1}\cdot576214 = \log\cdot3768894;$$

therefore $$\frac{1}{(1\cdot05)^{20}} = \cdot3768894;$$

therefore $$\frac{1}{\cdot05}\left\{\frac{1}{(1\cdot05)^{13}} - \frac{1}{(1\cdot05)^{20}}\right\} = 20\,\{\cdot5303214 - \cdot3768894\}$$

$$= 20\times\cdot153432 = 3\cdot06864.$$

17. $$\begin{array}{r}\cdot7431448\\ \cdot7313537\\ \hline \cdot0117911\end{array} \qquad 60' : 1' :: \cdot0117911 : x;$$

this gives $$x = \cdot0001965;$$

therefore $$\sin 47^\circ\,1' = \cdot7313537 + \cdot0001965 = \cdot7315502.$$

18. $$\begin{array}{r}\cdot1270646\\ \cdot1267761\\ \hline \cdot0002885\end{array} \qquad 60'' : 25'' :: \cdot0002885 : x;$$

this gives $$x = \cdot0001202;$$

therefore $$\sin 7^\circ\,17'\,25'' = \cdot1267761 + \cdot0001202 = \cdot1268963.$$

19. $$\begin{array}{r}9\cdot4663483\\ 9\cdot4659353\\ \hline \cdot0004130\end{array} \qquad 60'' : 12'' :: \cdot0004130 : x;$$

this gives $$x = \cdot0000826;$$

therefore $$L\sin 17^\circ\,0'\,12'' = 9\cdot4659353 + \cdot0000826 = 9\cdot4660179.$$

20. $$\begin{array}{r}9\cdot6482582\\ 9\cdot6480038\\ \hline \cdot0002544\end{array} \qquad 60'' : 12'' :: \cdot0002544 : x;$$

this gives $$x = \cdot0000509;$$

therefore $$L\sin 26^\circ\,24'\,12'' = 9\cdot6480038 + \cdot0000509 = 9\cdot6480547.$$

21. $$\begin{array}{r}9\cdot5052891\\ 9\cdot5048538\\ \hline \cdot0004353\end{array} \qquad 60'' : 35'' :: \cdot0004353 : x;$$

this gives $$x = \cdot0002539;$$

therefore $$L\cot 72^\circ\,15'\,35'' = 9\cdot5052891 - \cdot0002539 = 9\cdot5050352.$$

22. $\begin{array}{r} 9{\cdot}1604569 \\ 9{\cdot}1603493 \\ \hline {\cdot}0001076 \end{array}$ $\quad {\cdot}0001486 : {\cdot}0001076 :: 10 : x;$

this gives $x=7$; therefore the required angle is $81^0\,46'\,7''$.

23. $\begin{array}{r} 9{\cdot}9713383 \\ 9{\cdot}9713351 \\ \hline {\cdot}0000032 \end{array}$ $\quad {\cdot}0000079 : {\cdot}0000032 :: 10 : x;$

this gives $x=4$; therefore the required angle is $20^0\,35'\,20''-4''$, that is $20^0\,35'\,16''$. For as the L cosine increases the angle diminishes.

24. $$60'' : 26'' :: {\cdot}0000865 : x;$$

this gives $$x={\cdot}0000375;$$

therefore $$L\cos 34^0\,24'\,26''=9{\cdot}9165137-{\cdot}0000375=9{\cdot}9164762.$$

Again $\begin{array}{r} 9{\cdot}9165646 \\ 9{\cdot}9165137 \\ \hline {\cdot}0000509 \end{array}$ $\quad {\cdot}0000865 : {\cdot}0000509 :: 60 : x;$

this gives $x=35$; therefore the required angle is $34^0\,24'-35''$, that is $34^0\,23'\,25''$.

25. Since $\sec\theta\times\cos\theta=1$, we have $\log\sec\theta+\log\cos\theta=0$;

therefore $L\sec\theta+L\cos\theta-20=0$; therefore $L\sec\theta=20-L\cos\theta$.

We shall first find $L\cos 37^0\,19'\,47''$.

$$60'' : 47'' :: {\cdot}0000963 : x;$$

this gives $$x={\cdot}0000754;$$

therefore $$L\cos 37^0\,19'\,47''=9{\cdot}9005294-{\cdot}0000754=9{\cdot}9004540.$$

Then $$L\sec 37^0\,19'\,47''=20-9{\cdot}9004540=10{\cdot}0995460.$$

Next find $L\sin 37^0\,19'\,47''$.

$$60'' : 47'' :: {\cdot}0001657 : x;$$

this gives $$x={\cdot}0001298;$$

therefore $$L\sin 37^0\,19'\,47''=9{\cdot}7826301+{\cdot}0001298=9{\cdot}7827599.$$

Then $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$; therefore $\log\tan\theta=\log\sin\theta-\log\cos\theta$;

therefore $L\tan\theta-10=L\sin\theta-10-(L\cos\theta-10)=L\sin\theta-L\cos\theta$;

therefore $$L\tan\theta=10+L\sin\theta-L\cos\theta.$$

Thus $L\tan 37^0\,19'\,47''=10+9{\cdot}7827599-9{\cdot}9004540=9{\cdot}8823059.$

26. $$60'' : 24''\cdot6 :: \cdot0001998 : x;$$

this gives $$x = \cdot0000819;$$

therefore $$L \sin 32^\circ\, 18'\, 24''\cdot6 = 9\cdot7278277 + \cdot0000819 = 9\cdot7279096.$$

$$60'' : 24''\cdot6 :: \cdot0000799 : x;$$

this gives $$x = \cdot0000328;$$

therefore $$L \cos 32^\circ\, 18'\, 24''\cdot6 = 9\cdot9269913 - \cdot0000328 = 9\cdot9269585.$$

And $$L \tan 32^\circ\, 18'\, 24''\cdot6 = 10 + L \sin 32^\circ\, 18'\, 24''\cdot6 - L \cos 32^\circ\, 18'\, 24''\cdot6$$
$$= 9\cdot8009511.$$

CHAPTER XII.

1. Let $ABCD$ denote the rectangle. From A draw AP perpendicular to the diagonal BD; and from P draw PM perpendicular to BC, and PN perpendicular to CD.

Let the angle DBA be denoted by α; then

$$AB = c \cos \alpha, \quad BP = AB \cos \alpha = c \cos^2 \alpha,$$
$$PM = BP \cos BPM = BP \cos \alpha = c \cos^3 \alpha.$$

Thus denoting PM by p we have $p = c \cos^3 \alpha$.

Similarly $AD = c \sin \alpha$, $PD = AD \sin PAD = AD \sin \alpha = c \sin^2 \alpha$,

$$PN = PD \sin PDN = PD \sin \alpha = c \sin^3 \alpha.$$

Thus $q = c \sin^3 \alpha$.

Therefore $p^{\frac{2}{3}} + q^{\frac{2}{3}} = (c \cos^3 \alpha)^{\frac{2}{3}} + (c \sin^3 \alpha)^{\frac{2}{3}} = c^{\frac{2}{3}} (\cos^2 \alpha + \sin^2 \alpha) = c^{\frac{2}{3}}$.

2. Let a denote the radius of the larger circle, and b the radius of the smaller circle. Let x denote the distance of the point of intersection of the two common tangents from the centre of the larger circle; therefore $x - a - b$ denotes the distance of this point from the centre of the smaller circle.

Then $$\sin \frac{\theta}{2} = \frac{a}{x}, \text{ and also } \sin \frac{\theta}{2} = \frac{b}{x - a - b};$$

therefore $$x = \frac{a}{\sin \frac{\theta}{2}}, \text{ and } x - a - b = \frac{b}{\sin \frac{\theta}{2}};$$

therefore, by subtraction, $$a + b = \frac{a - b}{\sin \frac{\theta}{2}};$$

therefore $$\sin \frac{\theta}{2} = \frac{a - b}{a + b}; \quad \text{therefore } \cos \frac{\theta}{2} = \frac{2 \sqrt{(ab)}}{a + b};$$

and $$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{4 (a - b) \sqrt{(ab)}}{(a + b)^2}.$$

3. $$\sec\alpha\sec\theta+\tan\alpha\tan\theta=\sec\beta;$$

therefore $$\sec\alpha\sec\theta=\sec\beta-\tan\alpha\tan\theta;$$

therefore $$\sec^2\alpha\sec^2\theta=(\sec\beta-\tan\alpha\tan\theta)^2;$$

therefore $$\sec^2\alpha(1+\tan^2\theta)=\sec^2\beta-2\sec\beta\tan\alpha\tan\theta+\tan^2\alpha\tan^2\theta;$$

therefore $$(\sec^2\alpha-\tan^2\alpha)\tan^2\theta+2\sec\beta\tan\alpha\tan\theta=\sec^2\beta-\sec^2\alpha;$$

therefore $$\tan^2\theta+2\sec\beta\tan\alpha\tan\theta=\sec^2\beta-\sec^2\alpha;$$

therefore $$(\tan\theta+\tan\alpha\sec\beta)^2=\sec^2\beta-\sec^2\alpha+\tan^2\alpha\sec^2\beta$$
$$=\sec^2\beta\sec^2\alpha-\sec^2\alpha=\tan^2\beta\sec^2\alpha;$$

therefore $$\tan\theta+\tan\alpha\sec\beta=\pm\tan\beta\sec\alpha;$$

therefore $$\tan\theta=-\tan\alpha\sec\beta\pm\tan\beta\sec\alpha$$
$$=-\frac{\sin\alpha}{\cos\alpha\cos\beta}\pm\frac{\sin\beta}{\cos\beta\cos\alpha}=\frac{-\sin\alpha\pm\sin\beta}{\cos\alpha\cos\beta}.$$

4. $$\frac{\sin\frac{\theta}{2}\cos2\theta}{\text{vers}\,\theta\cot\theta}=\frac{\sin\frac{\theta}{2}\cos2\theta\sin\theta}{\text{vers}\,\theta\cos\theta}=\frac{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\cos2\theta}{(1-\cos\theta)\cos\theta}$$
$$=\frac{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\cos2\theta}{2\sin^2\frac{\theta}{2}\cos\theta}=\frac{\cos\frac{\theta}{2}\cos2\theta}{\cos\theta};$$

and the value of this is unity when $\theta=0$.

$$\frac{\tan^2\theta}{\sec2\theta-1}=\frac{\sin^2\theta}{\cos^2\theta(\sec2\theta-1)}=\frac{\sin^2\theta\cos2\theta}{\cos^2\theta(1-\cos2\theta)}$$
$$=\frac{\sin^2\theta\cos2\theta}{2\cos^2\theta\sin^2\theta}=\frac{\cos2\theta}{2\cos^2\theta};$$

and the value of this is $\frac{1}{2}$ when $\theta=0$.

5. $$\cot\frac{\theta}{2}-(1+\cot\theta)=\cot\frac{\theta}{2}-\cot\theta-1=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}-\frac{\cos\theta}{\sin\theta}-1$$
$$=\frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}\sin\theta}-1=\frac{\sin\left(\theta-\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\sin\theta}-1=\frac{1}{\sin\theta}-1;$$

now this is always positive as θ changes from 0 to π, except when $\theta=\frac{\pi}{2}$, and then it is zero.

6. $$\tan\frac{\theta}{2}=\frac{\tan\theta+c-1}{\tan\theta+c+1};$$

therefore $$\tan\frac{\theta}{2}(\tan\theta+c+1)=\tan\theta+c-1;$$

therefore $$\tan\frac{\theta}{2}\left(\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}+c+1\right)=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}+c-1;$$

therefore $$2\tan^2\frac{\theta}{2}+(c+1)\left(1-\tan^2\frac{\theta}{2}\right)\tan\frac{\theta}{2}$$

$$=2\tan\frac{\theta}{2}+(c-1)\left(1-\tan^2\frac{\theta}{2}\right);$$

therefore $$(c+1)\tan^3\frac{\theta}{2}-(1+c)\tan^2\frac{\theta}{2}+(1-c)\tan\frac{\theta}{2}+(c-1)=0;$$

therefore $$(c+1)\tan^2\frac{\theta}{2}(\tan\frac{\theta}{2}-1)=(c-1)(\tan\frac{\theta}{2}-1);$$

therefore either $\tan\frac{\theta}{2}-1=0$, or $(c+1)\tan^2\frac{\theta}{2}=c-1$.

Thus $$\tan\frac{\theta}{2}=1, \text{ or } \pm\sqrt{\frac{c-1}{c+1}}.$$

7. $a\sec^2\theta-b\cos\theta=2a$, therefore $a-b\cos^3\theta=2a\cos^2\theta$;

therefore $$b\cos^3\theta=a-2a\cos^2\theta;$$

again $b\cos^2\theta-a\sec\theta=2b$, therefore $b\cos^3\theta=2b\cos\theta+a$;

thus $$a-2a\cos^2\theta=2b\cos\theta+a;$$

therefore $-a\cos\theta=b$; therefore $\cos\theta=-\frac{b}{a}$.

Substitute this value of $\cos\theta$ in either of the given equations, for instance the first; thus $\frac{a^3}{b^2}+\frac{b^2}{a}=2a$; therefore $a^4+b^4-2a^2b^2=0$; therefore $a^2=b^2$.

8. $$a^2+b^2=(\sin\alpha\cos\beta\sin\theta+\cos\alpha\cos\theta)^2+(\sin\alpha\cos\beta\cos\theta-\cos\alpha\sin\theta)^2$$
$$=\sin^2\alpha\cos^2\beta+\cos^2\alpha;$$

$$\frac{c^2}{\sin^2\theta}=\sin^2\alpha\sin^2\beta;$$

therefore $$a^2+b^2+\frac{c^2}{\sin^2\theta}=\sin^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta+\cos^2\alpha$$
$$=\sin^2\alpha+\cos^2\alpha=1.$$

9. $$b+c\cos\alpha=u\cos(\alpha-\theta),\quad b+c\cos\beta=u\cos(\beta-\theta),$$
therefore $$2b+c(\cos\alpha+\cos\beta)=u\cos(\alpha-\theta)+u\cos(\beta-\theta);$$
therefore $$b+c\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=u\cos\left(\frac{\alpha+\beta}{2}-\theta\right)\cos\frac{\alpha-\beta}{2};$$
therefore $$b\sec\frac{\alpha-\beta}{2}=u\cos\left(\frac{\alpha+\beta}{2}-\theta\right)-c\cos\frac{\alpha+\beta}{2}\ \ldots\ldots\ldots\ldots\ (1).$$

Again from the first two equations, by subtraction,
$$c(\cos\alpha-\cos\beta)=u\cos(\alpha-\theta)-u\cos(\beta-\theta);$$
therefore $$c\sin\frac{\beta-\alpha}{2}\sin\frac{\alpha+\beta}{2}=u\sin\frac{\beta-\alpha}{2}\sin\left(\frac{\alpha+\beta}{2}-\theta\right);$$
therefore $$0=u\sin\left(\frac{\alpha+\beta}{2}-\theta\right)-c\sin\frac{\alpha+\beta}{2}\ldots\ldots\ldots\ldots\ldots\ (2).$$

Square and add (1) and (2); thus
$$b^2\sec^2\delta=u^2+c^2-2uc\left\{\cos\left(\frac{\alpha+\beta}{2}-\theta\right)\cos\frac{\alpha+\beta}{2}+\sin\left(\frac{\alpha+\beta}{2}-\theta\right)\sin\frac{\alpha+\beta}{2}\right\}$$
$$=u^2+c^2-2uc\cos\theta.$$

10. $$a\tan^2\theta-x=\frac{2a\tan\theta\tan2\alpha\tan2\alpha'}{\tan2\alpha+\tan2\alpha'},$$
$$a-x=\frac{2a\tan\theta}{\tan2\alpha+\tan2\alpha'};$$
therefore, by subtraction,
$$a(1-\tan^2\theta)=\frac{2a\tan\theta(1-\tan2\alpha\tan2\alpha')}{\tan2\alpha+\tan2\alpha'};$$
therefore $$\frac{\tan2\alpha+\tan2\alpha'}{1-\tan2\alpha\tan2\alpha'}=\frac{2\tan\theta}{1-\tan^2\theta};$$
therefore $$\tan(2\alpha+2\alpha')=\tan2\theta.$$

11. $$\sin\theta+\sin\phi=a,\quad\cos\theta+\cos\phi=b;$$
square and add; thus
$$2+2(\cos\theta\cos\phi+\sin\theta\sin\phi)=a^2+b^2;$$
therefore $$2+2\cos(\theta-\phi)=a^2+b^2;$$
therefore $$2+2c=a^2+b^2.$$

12. $x\cos\theta + y\sin\theta = a, \quad x\cos(\theta+2\phi) - y\sin(\theta+2\phi) = a;$

therefore, by subtraction,

$$x\{\cos\theta - \cos(\theta+2\phi)\} + y\{\sin\theta + \sin(\theta+2\phi)\} = 0;$$

therefore $$x\sin(\theta+\phi)\sin\phi + y\sin(\theta+\phi)\cos\phi = 0;$$

therefore $$x\sin\phi + y\cos\phi = 0 \quad\text{.............................}\quad (1).$$

Again, by addition,

$$x\{\cos\theta + \cos(\theta+2\phi)\} + y\{\sin\theta - \sin(\theta+2\phi)\} = 2a;$$

therefore $$x\cos(\theta+\phi)\cos\phi - y\cos(\theta+\phi)\sin\phi = a;$$

therefore $$x\cos\phi - y\sin\phi = \frac{a}{\cos(\theta+\phi)} \quad\text{.........................}\quad (2).$$

Square and add (1) and (2): thus

$$x^2 + y^2 = \frac{a^2}{\cos^2(\theta+\phi)} = \frac{a^2}{1-\sin^2(\theta+\phi)} = \frac{a^2}{1-\frac{a^2}{b^2}\sin^2\phi};$$

therefore $$(x^2+y^2)\left(1-\frac{a^2}{b^2}\sin^2\phi\right) = a^2.$$

But from (1) $$x^2\sin^2\phi = y^2\cos^2\phi = y^2(1-\sin^2\phi);$$

therefore $$\sin^2\phi = \frac{y^2}{x^2+y^2}.$$

Therefore $$(x^2+y^2)\left\{1-\frac{a^2y^2}{b^2(x^2+y^2)}\right\} = a^2;$$

therefore $$x^2+y^2 = a^2 + \frac{a^2y^2}{b^2}.$$

13. $$\tan c = \tan(x+y) = \frac{\tan x + \tan y}{1-\tan x\tan y}.$$

Now $$\tan x + \tan y = a, \quad\text{and}\quad \cot x + \cot y = b;$$

therefore $$\frac{1}{\tan x} + \frac{1}{\tan y} = b;$$

therefore $$\tan x + \tan y = b\tan x\tan y;$$

therefore $a = b\tan x\tan y$; therefore $\tan x\tan y = \dfrac{a}{b}$;

therefore $$\tan c = \frac{a}{1-\frac{a}{b}} = \frac{ab}{b-a};$$

therefore $$\cot c = \frac{b-a}{ab} = \frac{1}{a} - \frac{1}{b}.$$

14. $$\frac{x}{a} = \frac{\sec^2\theta - \cos^2\theta}{\sec^2\theta + \cos^2\theta} = \frac{1-\cos^4\theta}{1+\cos^4\theta},$$

$$\frac{2b}{y} = \sec^2\theta + \cos^2\theta = \frac{1+\cos^4\theta}{\cos^2\theta};$$

therefore $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{1-\cos^4\theta}{1+\cos^4\theta}\right)^2 + \frac{4\cos^4\theta}{(1+\cos^4\theta)^2} = \left(\frac{1+\cos^4\theta}{1+\cos^4\theta}\right)^2 = 1.$$

15. $$(a+b)\tan(\theta-\phi) = (a-b)\tan(\theta+\phi),$$

therefore $$(a+b)\sin(\theta-\phi)\cos(\theta+\phi) = (a-b)\sin(\theta+\phi)\cos(\theta-\phi);$$

therefore $$b\{\sin(\theta+\phi)\cos(\theta-\phi) + \sin(\theta-\phi)\cos(\theta+\phi)\}$$
$$= a\{\sin(\theta+\phi)\cos(\theta-\phi) - \sin(\theta-\phi)\cos(\theta+\phi)\};$$

therefore $$b\sin 2\theta = a\sin 2\phi,$$

and $$b\cos 2\theta = c - a\cos 2\phi;$$

square and add; thus $$b^2 = c^2 + a^2 - 2ac\cos 2\phi.$$

16. $$x = \frac{z\sin(\theta+\theta')}{\sin 2\theta}, \quad y = \frac{z\sin(\theta-\theta')}{\sin 2\theta}.$$

Square and substitute in the first given equation; thus

$$\frac{z^2\sin^2(\theta+\theta')}{a^2\sin^2 2\theta}\cos\theta = \frac{z^2\sin^2(\theta-\theta')}{a^2\sin^2 2\theta}\cos\theta + \frac{z^2}{b^2}\cos\theta';$$

therefore $$\frac{\sin^2(\theta+\theta')\cos\theta - \sin^2(\theta-\theta')\cos\theta}{a^2\sin^2 2\theta} = \frac{\cos\theta'}{b^2};$$

therefore $$\frac{(\sin\theta\cos\theta' + \cos\theta\sin\theta')^2 - (\sin\theta\cos\theta' - \cos\theta\sin\theta')^2}{4a^2\sin^2\theta\cos^2\theta}\cos\theta = \frac{\cos\theta'}{b^2};$$

therefore $$\frac{4\sin\theta\cos^2\theta\sin\theta'\cos\theta'}{4a^2\sin^2\theta\cos^2\theta} = \frac{\cos\theta'}{b^2};$$

therefore $$\frac{\sin\theta'}{\sin\theta} = \frac{a^2}{b^2}.$$

17. $$y\cos\phi - x\sin\phi = a\cos 2\phi \text{ .... } (1),$$
$$y\sin\phi + x\cos\phi = 2a\sin 2\phi \text{ .... } (2).$$

Multiply (1) by $\cos\phi$, and (2) by $\sin\phi$, and add; thus

$$y = a\cos 2\phi\cos\phi + 2a\sin 2\phi\sin\phi$$
$$= a\cos\phi(\cos 2\phi + 4\sin^2\phi) = a\cos\phi(\cos^2\phi + 3\sin^2\phi).$$

Again multiply (2) by $\cos\phi$, and (1) by $\sin\phi$, and subtract: thus

$$x = 2a\sin 2\phi\cos\phi - a\cos 2\phi\sin\phi$$
$$= a\sin\phi(4\cos^2\phi - \cos 2\phi) = a\sin\phi(3\cos^2\phi + \sin^2\phi).$$

Thus $x+y=a(\sin^3\phi+\cos^3\phi+3\sin^2\phi\cos\phi+3\cos^2\phi\sin\phi)$

$=a(\sin\phi+\cos\phi)^3$;

and $x-y=a(\sin^3\phi-3\sin^2\phi\cos\phi+3\cos^2\phi\sin\phi-\cos^3\phi)$

$=a(\sin\phi-\cos\phi)^3$.

Therefore $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=a^{\frac{2}{3}}\{(\sin\phi+\cos\phi)^2+(\sin\phi-\cos\phi)^2\}=2a^{\frac{2}{3}}$.

18. $\cos\theta\cos\phi+\sin\theta\sin\phi=\sin\beta\sin\gamma$;

therefore $\sin^2\theta\sin^2\phi=(\sin\beta\sin\gamma-\cos\theta\cos\phi)^2$;

therefore $(1-\cos^2\theta)(1-\cos^2\phi)=(\sin\beta\sin\gamma-\cos\theta\cos\phi)^2$;

therefore $\left(1-\frac{\sin^2\beta}{\sin^2\alpha}\right)\left(1-\frac{\sin^2\gamma}{\sin^2\alpha}\right)=\left(\sin\beta\sin\gamma-\frac{\sin\beta\sin\gamma}{\sin^2\alpha}\right)^2$;

therefore $(\sin^2\alpha-\sin^2\beta)(\sin^2\alpha-\sin^2\gamma)=\sin^2\beta\sin^2\gamma(\sin^2\alpha-1)^2$;

therefore $\sin^4\alpha-\sin^2\alpha(\sin^2\beta+\sin^2\gamma)=\sin^2\beta\sin^2\gamma(\sin^4\alpha-2\sin^2\alpha)$;

therefore $\sin^2\alpha-\sin^2\beta-\sin^2\gamma=\sin^2\beta\sin^2\gamma(\sin^2\alpha-2)$;

therefore $\sin^2\alpha(1-\sin^2\beta\sin^2\gamma)=\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma$;

therefore $$\sin^2\alpha=\frac{\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma};$$

therefore $$\cos^2\alpha=\frac{1-\sin^2\beta\sin^2\gamma-\sin^2\beta\cos^2\gamma-\cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$$

$$=\frac{(\sin^2\beta+\cos^2\beta)(\sin^2\gamma+\cos^2\gamma)-\sin^2\beta\sin^2\gamma-\sin^2\beta\cos^2\gamma-\cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$$

$$=\frac{\cos^2\beta\cos^2\gamma}{1-\sin^2\beta\sin^2\gamma}.$$

Therefore $$\tan^2\alpha=\frac{\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma}{\cos^2\beta\cos^2\gamma}$$

$$=\frac{\sin^2\beta}{\cos^2\beta}+\frac{\sin^2\gamma}{\cos^2\gamma}=\tan^2\beta+\tan^2\gamma.$$

19. $$m=\operatorname{cosec}\theta-\sin\theta=\frac{1}{\sin\theta}-\sin\theta=\frac{1-\sin^2\theta}{\sin\theta}=\frac{\cos^2\theta}{\sin\theta},$$

$$n=\sec\theta-\cos\theta=\frac{1}{\cos\theta}-\cos\theta=\frac{1-\cos^2\theta}{\cos\theta}=\frac{\sin^2\theta}{\cos\theta};$$

therefore $$mn=\frac{\cos^2\theta\sin^2\theta}{\sin\theta\cos\theta}=\cos\theta\sin\theta;$$

therefore $$\sin\theta=\frac{mn}{\cos\theta},\ \text{and}\ \cos\theta=\frac{mn}{\sin\theta};$$

therefore $$m = \frac{\cos^3\theta}{mn}, \text{ and } n = \frac{\sin^3\theta}{mn};$$

therefore $$\cos\theta = (m^2n)^{\frac{1}{3}}, \text{ and } \sin\theta = (mn^2)^{\frac{1}{3}};$$

therefore $$\cos^2\theta + \sin^2\theta = (m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}};$$

therefore $$1 = (mn)^{\frac{2}{3}}\{m^{\frac{2}{3}} + n^{\frac{2}{3}}\}.$$

20. $$(x\sin\theta - y\cos\theta)^2 = x^2 + y^2;$$

therefore $$x^2 + y^2 - (x\sin\theta - y\cos\theta)^2 = 0;$$

therefore $$x^2\cos^2\theta + 2xy\sin\theta\cos\theta + y^2\sin^2\theta = 0;$$

therefore $$(x\cos\theta + y\sin\theta)^2 = 0;$$

therefore $$x\cos\theta + y\sin\theta = 0;$$

therefore $$\tan\theta = -\frac{x}{y}.$$

Hence we obtain $\cos^2\theta = \dfrac{y^2}{x^2+y^2}$ and $\sin^2\theta = \dfrac{x^2}{x^2+y^2}$.

Substitute in the second given equation: thus

$$\frac{1}{x^2+y^2}\left(\frac{y^2}{a^2} + \frac{x^2}{b^2}\right) = \frac{1}{x^2+y^2};$$

therefore $$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

21. $$a\sin^2\theta + a'\cos^2\theta = b;$$

therefore $$a\sin^2\theta + a'(1 - \sin^2\theta) = b;$$

therefore $$\sin^2\theta = \frac{b-a'}{a-a'};$$

therefore $$\cos^2\theta = \frac{a-b}{a-a'};$$

therefore $$\tan^2\theta = \frac{b-a'}{a-b}.$$

Similarly we find $$\tan^2\theta' = \frac{b'-a}{a'-b'}.$$

But $$a^2\tan^2\theta = a'^2\tan^2\theta';$$

therefore $$a^2\frac{b-a'}{a-b} = a'^2\frac{b'-a}{a'-b'};$$

therefore $$a^2(b-a')(b'-a') = a'^2(b'-a)(b-a);$$

therefore $$a^2\{bb'-a'(b+b')\} = a'^2\{bb'-a(b+b')\};$$

therefore $$bb'(a^2-a'^2) = aa'(a-a')(b+b');$$

therefore $$bb'(a+a') = aa'(b+b').$$

Divide by $aa'bb'$; thus $\frac{1}{a'}+\frac{1}{a}=\frac{1}{b'}+\frac{1}{b}$.

22. $$y^2\cos^2\theta + x^2\sin^2\theta = \frac{x^2y^2}{a^2} = \frac{a^2b^2\sin^2\alpha}{a^2} = b^2\sin^2\alpha;$$

therefore $$\frac{y^2}{2}(1+\cos 2\theta)+\frac{x^2}{2}(1-\cos 2\theta) = b^2\sin^2\alpha;$$

therefore $$x^2+y^2+(y^2-x^2)\cos 2\theta = 2b^2\sin^2\alpha;$$

therefore $$a^2+b^2+\{(y^2+x^2)^2-4y^2x^2\}^{\frac{1}{2}}\cos 2\theta = 2b^2\sin^2\alpha;$$

therefore $$a^2+b^2+\{(a^2+b^2)^2-4a^2b^2\sin^2\alpha\}^{\frac{1}{2}}\cos 2\theta = 2b^2\sin^2\alpha;$$

therefore $$\cos 2\theta = \frac{2b^2\sin^2\alpha - b^2 - a^2}{\{(a^2+b^2)^2-4a^2b^2\sin^2\alpha\}^{\frac{1}{2}}};$$

therefore $$\sin^2 2\theta = \frac{-4a^2b^2\sin^2\alpha + 4b^2\sin^2\alpha(b^2+a^2) - 4b^4\sin^4\alpha}{(a^2+b^2)^2-4a^2b^2\sin^2\alpha}$$

$$= \frac{4b^4\sin^2\alpha(1-\sin^2\alpha)}{(a^2+b^2)^2-4a^2b^2\sin^2\alpha};$$

therefore $$\pm\sin 2\theta = \frac{2b^2\sin\alpha\cos\alpha}{\{(a^2+b^2)^2-4a^2b^2\sin^2\alpha\}^{\frac{1}{2}}}.$$

Hence by division

$$\pm\cot 2\theta = \frac{2b^2\sin^2\alpha - b^2 - a^2}{2b^2\sin\alpha\cos\alpha} = -\frac{a^2+b^2\cos 2\alpha}{b^2\sin 2\alpha}$$

$$= -\cot 2\alpha - \frac{a^2}{b^2}\operatorname{cosec} 2\alpha,$$

which we may also express thus

$$\pm\cot 2\theta = \cot 2\alpha + \frac{a^2}{b^2}\operatorname{cosec} 2\alpha.$$

23. Let $\frac{\cos x}{a_1}$, $\frac{\cos 2x}{a_2}$, and $\frac{\cos 3x}{a_3}$ each be equal to $\frac{1}{k}$; then

$$a_1 = k\cos x,\quad a_2 = k\cos 2x,\quad \text{and}\quad a_3 = k\cos 3x.$$

Therefore $$\frac{2a_2-a_1-a_3}{4a_2} = \frac{2\cos 2x - \cos x - \cos 3x}{4\cos 2x}$$

$$= \frac{2\cos 2x - 2\cos 2x\cos x}{4\cos 2x} = \frac{1-\cos x}{2} = \sin^2\frac{x}{2}.$$

24. Let $\frac{\sin x}{a_1}$, $\frac{\sin 3x}{a_3}$, and $\frac{\sin 5x}{a_5}$ each be equal to $\frac{1}{k}$; then

$$a_1 = k\sin x,\quad a_3 = k\sin 3x,\ \text{and}\ a_5 = k\sin 5x.$$

Therefore $\frac{a_1 - 2a_3 + a_5}{a_3} = \frac{\sin x - 2\sin 3x + \sin 5x}{\sin 3x} = \frac{2\sin 3x\cos 2x - 2\sin 3x}{\sin 3x}$

$$= 2(\cos 2x - 1) = -4\sin^2 x;$$

and $\frac{a_3 - 3a_1}{a_1} = \frac{\sin 3x - 3\sin x}{\sin x} = \frac{3\sin x - 4\sin^3 x - 3\sin x}{\sin x} = -4\sin^2 x.$

Thus $$\frac{a_1 - 2a_3 + a_5}{a_3} = \frac{a_3 - 3a_1}{a_1}.$$

25. Let $\frac{1}{k}$ denote the value of the fractions which are given equal; thus

$$a_1 = k\cos x,\quad a_2 = k\cos(x+\theta),\quad a_3 = k\cos(x+2\theta),\quad a_4 = k\cos(x+3\theta);$$

therefore $\frac{a_1 + a_3}{a_2} = \frac{\cos x + \cos(x+2\theta)}{\cos(x+\theta)} = \frac{2\cos(x+\theta)\cos\theta}{\cos(x+\theta)} = 2\cos\theta,$

and $\frac{a_2 + a_4}{a_3} = \frac{\cos(x+\theta) + \cos(x+3\theta)}{\cos(x+2\theta)} = \frac{2\cos(x+2\theta)\cos\theta}{\cos(x+2\theta)} = 2\cos\theta;$

thus the required result is established.

26. $$\sin^2\phi = \frac{\cos 2\alpha\cos 2\alpha'}{\cos^2(\alpha+\alpha')};$$

therefore $$\cos^2\phi = \frac{\cos^2(\alpha+\alpha') - \cos 2\alpha\cos 2\alpha'}{\cos^2(\alpha+\alpha')}$$

$$= \frac{1 + \cos 2(\alpha+\alpha') - \cos 2(\alpha+\alpha') - \cos 2(\alpha-\alpha')}{2\cos^2(\alpha+\alpha')} = \frac{\sin^2(\alpha-\alpha')}{\cos^2(\alpha+\alpha')};$$

therefore $$\cos\phi = \pm\frac{\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')}.$$

Take the upper sign; then $\cos\phi = \frac{\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')}$; therefore

$$\frac{1-\cos\phi}{1+\cos\phi} = \frac{\cos(\alpha+\alpha') - \sin(\alpha-\alpha')}{\cos(\alpha+\alpha') + \sin(\alpha-\alpha')} = \frac{\sin\left(\frac{\pi}{2} - \alpha - \alpha'\right) - \sin(\alpha-\alpha')}{\sin\left(\frac{\pi}{2} - \alpha - \alpha'\right) + \sin(\alpha-\alpha')}$$

$$= \frac{2\sin\left(\frac{\pi}{4} - \alpha\right)\cos\left(\frac{\pi}{4} - \alpha'\right)}{2\sin\left(\frac{\pi}{4} - \alpha'\right)\cos\left(\frac{\pi}{4} - \alpha\right)} = \frac{\tan\left(\frac{\pi}{4} - \alpha\right)}{\tan\left(\frac{\pi}{4} - \alpha'\right)};$$

therefore $$\tan^2\frac{\phi}{2} = \frac{\tan\left(\frac{\pi}{4} - \alpha\right)}{\tan\left(\frac{\pi}{4} - \alpha'\right)}.$$

Take the lower sign; then $\cos\phi = -\frac{\sin(\alpha - \alpha')}{\cos(\alpha + \alpha')}$; therefore

$$\frac{1 - \cos\phi}{1 + \cos\phi} = \frac{\cos(\alpha + \alpha') + \sin(\alpha - \alpha')}{\cos(\alpha + \alpha') - \sin(\alpha - \alpha')} = \frac{\sin\left(\frac{\pi}{2} - \alpha - \alpha'\right) + \sin(\alpha - \alpha')}{\sin\left(\frac{\pi}{2} - \alpha - \alpha'\right) - \sin(\alpha - \alpha')}$$

$$= \frac{2\sin\left(\frac{\pi}{4} - \alpha'\right)\cos\left(\frac{\pi}{4} - \alpha\right)}{2\sin\left(\frac{\pi}{4} - \alpha\right)\cos\left(\frac{\pi}{4} - \alpha'\right)} = \frac{\cot\left(\frac{\pi}{4} - \alpha\right)}{\cot\left(\frac{\pi}{4} - \alpha'\right)} = \frac{\tan\left(\frac{\pi}{4} + \alpha\right)}{\tan\left(\frac{\pi}{4} + \alpha'\right)};$$

therefore $$\tan^2\frac{\phi}{2} = \frac{\tan\left(\frac{\pi}{4} + \alpha\right)}{\tan\left(\frac{\pi}{4} + \alpha'\right)}.$$

27. $$\frac{\sin(\theta - \beta)\cos\alpha}{\sin(\phi - \alpha)\cos\beta} + \frac{\cos(\alpha + \theta)\sin\beta}{\cos(\phi - \beta)\sin\alpha} = 0;$$

therefore $$\frac{\sin(\theta - \beta)\cos\alpha}{\cos(\alpha + \theta)\cos\beta} + \frac{\sin(\phi - \alpha)\sin\beta}{\cos(\phi - \beta)\sin\alpha} = 0;$$

therefore $$\frac{(\sin\theta\cos\beta - \cos\theta\sin\beta)\cos\alpha}{(\cos\alpha\cos\theta - \sin\alpha\sin\theta)\cos\beta} + \frac{(\sin\phi\cos\alpha - \cos\phi\sin\alpha)\sin\beta}{(\cos\phi\cos\beta + \sin\phi\sin\beta)\sin\alpha} = 0;$$

therefore $$\frac{(\tan\theta\cos\beta - \sin\beta)\cos\alpha}{(\cos\alpha - \sin\alpha\tan\theta)\cos\beta} + \frac{(\tan\phi\cos\alpha - \sin\alpha)\sin\beta}{(\cos\beta + \tan\phi\sin\beta)\sin\alpha} = 0;$$

therefore $$\frac{\tan\theta - \tan\beta}{1 - \tan\alpha\tan\theta} + \frac{\tan\phi\cot\alpha - 1}{\cot\beta + \tan\phi} = 0;$$

therefore $(\tan\theta - \tan\beta)(\cot\beta + \tan\phi) + (\tan\phi\cot\alpha - 1)(1 - \tan\alpha\tan\theta) = 0$;

therefore $\tan\theta(\cot\beta + \tan\alpha) + \tan\phi(\cot\alpha - \tan\beta) = 2$.

But $\tan\theta = -\tan\phi \cdot \frac{\tan\beta}{\tan\alpha}\frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)}$; therefore

$$-\tan\phi \cdot (\cot\beta + \tan\alpha)\frac{\tan\beta}{\tan\alpha}\frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} + \tan\phi(\cot\alpha - \tan\beta) = 2;$$

therefore $$-\tan\phi(\cot\alpha + \tan\beta)\cos(\alpha - \beta) + \tan\phi(\cot\alpha - \tan\beta)\cos(\alpha + \beta)$$
$$= 2\cos(\alpha + \beta);$$

therefore $\tan\phi\,\{\cot\alpha\,[\cos(\alpha+\beta)-\cos(\alpha-\beta)]-\tan\beta\,[\cos(\alpha+\beta)+\cos(\alpha-\beta)]\}$
$$=2\cos(\alpha+\beta);$$

therefore $\tan\phi\,\{\cot\alpha\sin\alpha\sin\beta+\tan\beta\cos\alpha\cos\beta\}=-\cos(\alpha+\beta);$

therefore $$\tan\phi=-\frac{\cos(\alpha+\beta)}{2\cos\alpha\sin\beta}=\frac{1}{2}(\tan\alpha-\cot\beta);$$

and $$\tan\theta=-\tan\phi\,\frac{\tan\beta}{\tan\alpha}\,\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$$
$$=\frac{\cos(\alpha-\beta)}{2\sin\alpha\cos\beta}=\frac{1}{2}(\cot\alpha+\tan\beta).$$

28. $$\frac{2}{1+x}=\frac{\sin\beta\sin\theta}{\cos(\beta-\theta)}=\frac{\sin\beta\sin\theta}{\cos\beta\cos\theta+\sin\beta\sin\theta}=\frac{1}{\cot\beta\cot\theta+1};$$

therefore $$\cot\beta\cot\theta+1=\frac{1+x}{2};$$

therefore $$\cot\beta\cot\theta=\frac{1+x}{2}-1=\frac{x-1}{2}\;\ldots\ldots\ldots\ldots\ldots\;(1).$$

Again $$\frac{2}{1+x}=\frac{\tan(\theta-\alpha)}{\cot\beta}=\frac{(\tan\theta-\tan\alpha)\tan\beta}{1+\tan\theta\tan\alpha};$$

therefore $2(1+\tan\theta\tan\alpha)=(1+x)(\tan\theta-\tan\alpha)\tan\beta;$

therefore $$\tan\theta=\frac{2+(1+x)\tan\alpha\tan\beta}{(1+x)\tan\beta-2\tan\alpha}\;\ldots\ldots\ldots\ldots\ldots\;(2).$$

From (1) and (2) by multiplication
$$\cot\beta=\frac{2+(1+x)\tan\alpha\tan\beta}{(1+x)\tan\beta-2\tan\alpha}\cdot\frac{x-1}{2};$$

therefore $2\cot\beta\,\{(1+x)\tan\beta-2\tan\alpha\}=2(x-1)+(x^2-1)\tan\alpha\tan\beta;$

therefore $2(1+x)-4\cot\beta\tan\alpha=2(x-1)+(x^2-1)\tan\alpha\tan\beta;$

therefore $x^2\tan\alpha\tan\beta=4-4\cot\beta\tan\alpha+\tan\alpha\tan\beta;$

therefore $$x^2=4\cot\alpha\cot\beta-4\cot^2\beta+1$$
$$=2\left(\cot\frac{\alpha}{2}-\tan\frac{\alpha}{2}\right)\cot\beta-4\cot^2\beta+1$$
$$=\left(\cot\frac{\alpha}{2}-2\cot\beta\right)\left(\tan\frac{\alpha}{2}+2\cot\beta\right).$$

29. $\sin\theta\sin\phi=\sin\alpha\sin\beta$; therefore $2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\frac{\sin\alpha\sin\beta}{\sin\phi}$;

therefore $$4\sin^2\frac{\theta}{2}-4\sin^4\frac{\theta}{2}=\frac{\sin^2\alpha\sin^2\beta}{\sin^2\phi};$$

therefore $$4\sin^4\frac{\theta}{2} - 4\sin^2\frac{\theta}{2} + 1 = 1 - \frac{\sin^2\alpha\sin^2\beta}{\sin^2\phi},$$

but $$\sin^2\phi = \frac{\cot^2\frac{\alpha}{2}}{\cot^2\frac{\alpha}{2} + \cos^2\beta};$$

therefore $$4\sin^4\frac{\theta}{2} - 4\sin^2\frac{\theta}{2} + 1 = 1 - \frac{\sin^2\alpha\left(\cot^2\frac{\alpha}{2} + \cos^2\beta\right)\sin^2\beta}{\cot^2\frac{\alpha}{2}}$$

$$= 1 - 4\sin^4\frac{\alpha}{2}\left(\cot^2\frac{\alpha}{2} + \cos^2\beta\right)\sin^2\beta$$

$$= 1 - 4\sin^4\frac{\alpha}{2}\left(\cot^2\frac{\alpha}{2} + 1 - \sin^2\beta\right)\sin^2\beta$$

$$= 1 - 4\sin^2\frac{\alpha}{2}\sin^2\beta + 4\sin^4\frac{\alpha}{2}\sin^4\beta;$$

therefore $$2\sin^2\frac{\theta}{2} - 1 = \pm\left(1 - 2\sin^2\frac{\alpha}{2}\sin^2\beta\right).$$

Taking the lower sign we have $\sin^2\frac{\theta}{2} = \sin^2\frac{\alpha}{2}\sin^2\beta$.

30. $\sin\phi = n\sin\theta$; therefore $\cos\phi = \sqrt{(1 - n^2\sin^2\theta)}$;

therefore $$\tan\phi = \frac{n\sin\theta}{\sqrt{(1 - n^2\sin^2\theta)}}.$$

Hence we have $$\frac{n\sin\theta}{\sqrt{(1 - n^2\sin^2\theta)}} = 2\tan\theta = \frac{2\sin\theta}{\cos\theta};$$

therefore $$n\cos\theta = 2\sqrt{(1 - n^2\sin^2\theta)};$$

therefore $$n^2(1 - \sin^2\theta) = 4(1 - n^2\sin^2\theta);$$

therefore $3n^2\sin^2\theta = 4 - n^2$; therefore $\sin^2\theta = \frac{4 - n^2}{3n^2}$.

This must lie between 0 and 1, so that $4 - n^2$ must lie between 0 and $3n^2$, therefore 4 must lie between n^2 and $4n^2$; therefore n^2 must lie between 1 and 4.

31. Assume $x = \tan A$ and $y = \tan B$; then by Art. 114 we have $z = \tan C$, where $A + B + C = 180^0$.

Therefore $2A + 2B + 2C = 360^0$; therefore $\tan(2A + 2B + 2C) = 0$;

and therefore, as in Art. 114,

$$\tan 2A + \tan 2B + \tan 2C = \tan 2A\tan 2B\tan 2C;$$

therefore $$\frac{2\tan A}{1-\tan^2 A}+\frac{2\tan B}{1-\tan^2 B}+\frac{2\tan C}{1-\tan^2 C}$$
$$=\frac{2\tan A}{1-\tan^2 A}\cdot\frac{2\tan B}{1-\tan^2 B}\cdot\frac{2\tan C}{1-\tan^2 C}.$$

32. $$v\sin c=\sin z=\sin(2\pi-x-y)=-\sin(x+y)$$
$$=-\sin x\cos y-\cos x\sin y=-v\sin a\cos y-v\sin b\cos x.$$

Therefore either $v=0$; or $\sin c=-\sin a\cos y-\sin b\cos x$.

Take the latter, thus $\sin a\cos y=-\sin c-\sin b\cos x$;

but $\sin a\sin y=\sin b\sin x$;

square and add, thus $\sin^2 a=\sin^2 b+\sin^2 c+2\sin b\sin c\cos x$;

therefore $$\cos x=\frac{\sin^2 a-\sin^2 b-\sin^2 c}{2\sin b\sin c}.$$

Similarly $\cos y$ and $\cos z$ may be found, and then v.

If $v=0$, we have $\sin x=0$, $\sin y=0$, and $\sin z=0$. This will give us three solutions; $x=0$, $y=\pi$, $z=\pi$; $x=\pi$, $y=0$, $z=\pi$; $x=\pi$, $y=\pi$, $z=0$: and also three solutions, $x=0$, $y=0$, $z=2\pi$; $x=0$, $y=2\pi$, $z=0$; $x=2\pi$, $y=0$, $z=0$.

33. Let $u=(\cos\alpha x)^{\operatorname{cosec}^2\beta x}$; therefore

$$\log u=\operatorname{cosec}^2\beta x\log\cos\alpha x=\frac{1}{2}\operatorname{cosec}^2\beta x\log(1-\sin^2\alpha x)$$
$$=-\frac{1}{2\sin^2\beta x}\left\{\sin^2\alpha x+\frac{1}{2}\sin^4\alpha x+\frac{1}{3}\sin^6\alpha x+\dots\right\}.$$

Now $$\frac{\sin\alpha x}{\sin\beta x}=\frac{\alpha}{\beta}\cdot\frac{\sin\alpha x}{\alpha x}\cdot\frac{\beta x}{\sin\beta x};$$

when x is zero the value of $\frac{\sin\alpha x}{\alpha x}$ is unity, and so also is the value of $\frac{\beta x}{\sin\beta x}$; thus $\frac{\sin\alpha x}{\sin\beta x}=\frac{\alpha}{\beta}$; therefore $\frac{\sin^2\alpha x}{\sin^2\beta x}=\frac{\alpha^2}{\beta^2}$.

The limit of $\frac{\sin^4\alpha x}{\sin^2\beta x}$ is zero, and so also the other terms in $\log u$ vanish, and as in Art. 150 their sum vanishes also. Hence $\log u=-\frac{\alpha^2}{2\beta^2}$, and therefore $u=e^{-\frac{\alpha^2}{2\beta^2}}$.

34. By Art. 188 if h is very small we have $\tan(\theta+h)-\tan\theta=h\sec^2\theta$; thus if θ be nearly equivalent to 60^0 we have approximately

$$\tan(\theta+h)-\tan\theta=4h.$$

Since the tables extend to 7 places of decimals it follows that we can discriminate angles which are near 60^0, by means of their tangents, when the circular measure h of the difference is such that $4h = \cdot0000001$. Thus $h = \frac{1}{4}$ of $\frac{1}{10^7}$; the corresponding value in seconds is $\frac{1}{4} \times \frac{1}{10^7} \times \frac{180}{\pi} \times 60 \times 60$, that is $\frac{18 \times 9}{10000\pi}$, that is about $\frac{1}{200}$.

35. By Art. 196 if h is very small we have

$$L \sin(\theta + h) - L \sin\theta = \mu h \cot\theta = \frac{h}{(\log_e 10)\tan\theta};$$

thus if θ be nearly equivalent to $64^0\,36'$ we have approximately

$$L \sin(\theta + h) - L \sin\theta = \frac{h}{4\cdot8492}.$$

Since the tables extend to 7 places of decimals it follows that we can discriminate angles which are near $64^0\,36'$ by means of their L sines, when the circular measure of the difference is such that $\frac{h}{4\cdot8492} = \cdot0000001$. Thus $h = \frac{4\cdot8492}{10^7}$; the corresponding value in seconds is $\frac{4\cdot8492}{10^7} \times \frac{180}{\pi} \times 60 \times 60$: this will be found to be about $\frac{1}{10}$.

36.
$$1 - \tan^2\frac{\alpha}{2} = \frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}} = \frac{\cos\alpha}{\cos^2\frac{\alpha}{2}},$$

$$1 - \tan^2\frac{\alpha}{4} = \frac{\cos^2\frac{\alpha}{4} - \sin^2\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}} = \frac{\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{4}},$$

and so on.

In this way we find that the proposed expression

$$= \frac{\cos\alpha\cos\frac{\alpha}{2}\cos\frac{\alpha}{2^2}\cos\frac{\alpha}{2^3}\cdots}{\cos^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2^2}\cos^2\frac{\alpha}{2^3}\cdots}$$

$$= \frac{\cos\alpha}{\cos\frac{\alpha}{2}\cos\frac{\alpha}{2^2}\cos\frac{\alpha}{2^3}\cdots}$$

$$= \cos\alpha \div \frac{\sin\alpha}{\alpha} = \frac{\alpha}{\tan\alpha}.$$ See Art. 129.

37. We have universally

$$\sin^2(A+B) = (\sin A\cos B + \cos A\sin B)^2$$

$$= \sin^2 A\cos^2 B + \cos^2 A\sin^2 B + 2\sin A\cos A\sin B\cos B$$

$$= \sin^2 A(1-\sin^2 B) + \sin^2 B(1-\sin^2 A) + 2\sin A\cos A\sin B\cos B$$

$$= \sin^2 A + \sin^2 B + 2\sin A\sin B\{\cos A\cos B - \sin A\sin B\}$$

$$= \sin^2 A + \sin^2 B + 2\sin A\sin B\cos(A+B) \;\ldots\ldots\ldots\ldots (1).$$

Also in the present case

$$\sin^2 A + \sin^2 B = 1 - \sin^2 C = \cos^2 C \;\ldots\ldots\ldots\ldots (2).$$

If $A+B$ is greater than 90^0, then *a fortiori* $A+B+C$ is greater than 90^0.

If $A+B$ is less than 90^0, then $\sin^2(A+B)$ is greater than $\sin^2 A + \sin^2 B$ by (1), and therefore greater than $\cos^2 C$ by (2); and therefore $A+B$ is greater than $90^0 - C$, so that $A+B+C$ is greater than 90^0.

38. Take the diagram of Art. 71. Let α be the angle PAB. Suppose a circle having its centre O within the space bounded by PB, BT, and TP; let it touch the arc PB, the tangent BT, and the secant APT. Let ρ denote the radius of this circle, and r the radius of the original circle.

OT will bisect the angle ATB, and OA will pass through the point of contact of the circles. Let N be the point of contact of the secant APT and the circle with centre O. Then

$$NT = \rho\cot\frac{1}{2}\left(\frac{\pi}{2} - \alpha\right); \quad OA = r + \rho;$$

therefore $$AN = \sqrt{(r+\rho)^2 - \rho^2} = \sqrt{(r^2 + 2r\rho)}.$$

Hence $$\sqrt{(r^2+2r\rho)} + \rho\cot\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = AT = r\sec\alpha;$$

therefore $$\sqrt{(r^2+2r\rho)} = r\sec\alpha - \rho\cot\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

By squaring we obtain a quadratic equation for determining ρ. The reason why we have a quadratic equation is that another circle can also be drawn, which may be said to fulfil the conditions. For produce PA through A to meet the original circle again at p; then we may have a circle outside the arc Bp, touching this arc, touching TB produced through B, and touching Tp produced through p. The corresponding equation would be

$$\rho\cot\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) - \sqrt{(r^2+2r\rho)} = r\sec\alpha;$$

this differs from the former only in the sign of the radical, and therefore leads to the same quadratic equation.

Suppose $\rho = r$; then $\pm\sqrt{3} = \dfrac{1}{\cos\alpha} - \dfrac{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}$

$$= \frac{1}{\cos\alpha} - \frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$$

$$= \frac{1}{\cos\alpha} - \frac{\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)^2}{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}$$

$$= \frac{1}{\cos\alpha} - \frac{1 + \sin\alpha}{\cos\alpha} = -\tan\alpha.$$

Hence taking $\sqrt{3} = \tan\alpha$ we have $\alpha = \dfrac{\pi}{3}$.

39. Let x denote the value of $l\sin(\theta - \beta) - m\sin(\theta - \alpha)$; so that $l\cos(\theta - \beta) - m\cos(\theta - \alpha) = n$, $l\sin(\theta - \beta) - m\sin(\theta - \alpha) = x$.

Square and add; thus

$$l^2 + m^2 - 2lm\{\cos(\theta - \beta)\cos(\theta - \alpha) + \sin(\theta - \beta)\sin(\theta - \alpha)\} = n^2 + x^2,$$

that is
$$l^2 + m^2 - 2lm\cos(\alpha - \beta) = n^2 + x^2;$$

therefore
$$x = \sqrt{l^2 + m^2 - n^2 - 2lm\cos(\alpha - \beta)}.$$

40. $\theta - \sin\theta$ is less than $\tan\theta - \theta$ if 2θ is less than $\sin\theta + \tan\theta$, that is if 2θ is less than $\tan\theta(1 + \cos\theta)$, that is if 2θ is less than $\dfrac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}} \times \dfrac{2}{1 + \tan^2\frac{\theta}{2}}$, that is if $\dfrac{\theta}{2}$ is less than $\dfrac{\tan\frac{\theta}{2}}{1 - \tan^4\frac{\theta}{2}}$: and this is obviously the case, because $\dfrac{\theta}{2}$ is less than $\tan\dfrac{\theta}{2}$.

XIII.

1. The greatest angle is opposite to the greatest side; thus the cosine

$$= \frac{(x^2 - 1)^2 + (2x + 1)^2 - (x^2 + x + 1)^2}{2(x^2 - 1)(2x + 1)}$$

$$= \frac{x^4 - 2x^2 + 1 + 4x^2 + 4x + 1 - (x^4 + x^2 + 1 + 2x^3 + 2x^2 + 2x)}{2(x^2 - 1)(2x + 1)}$$

$$= \frac{-2x^3 - x^2 + 2x + 1}{2(2x^3 + x^2 - 2x - 1)} = -\frac{1}{2}.$$

Therefore the angle is 120^0.

2. $2\sin C\cos B = \sin A = \sin(B+C) = \sin B\cos C + \cos B\sin C$;

therefore $\sin C\cos B = \sin B\cos C$;

therefore $\sin(C-B) = 0$; therefore $B = C$.

3. We have $\cos A = \frac{b}{c}$, $\sin A = \frac{a}{c}$;

therefore $\frac{1+\cos A}{\sin A} = \frac{c+b}{a}$; therefore $\cot\frac{A}{2} = \frac{c+b}{a}$.

4. $a\tan A + b\tan B = (a+b)\tan\frac{A+B}{2}$;

therefore $a\left(\tan A - \tan\frac{A+B}{2}\right) = b\left(\tan\frac{A+B}{2} - \tan B\right)$;

therefore
$$\frac{a\left(\sin A\cos\frac{A+B}{2} - \cos A\sin\frac{A+B}{2}\right)}{\cos A\cos\frac{A+B}{2}}$$
$$= \frac{b\left(\sin\frac{A+B}{2}\cos B - \cos\frac{A+B}{2}\sin B\right)}{\cos B\cos\frac{A+B}{2}};$$

therefore $\frac{a\sin\frac{A-B}{2}}{\cos A} = \frac{b\sin\frac{A-B}{2}}{\cos B}$; therefore $\frac{a}{b} = \frac{\cos A}{\cos B}$.

But $\frac{a}{b} = \frac{\sin A}{\sin B}$; therefore $\frac{\sin A}{\sin B} = \frac{\cos A}{\cos B}$;

therefore $\tan A = \tan B$; therefore $A = B$.

5. Let 2α denote the least angle; then the other angles are 4α and 8α respectively: therefore $2\alpha + 4\alpha + 8\alpha = \pi$; therefore $\alpha = \frac{\pi}{14}$.

Then by Art. 214 the ratio of the greatest side to the perimeter

$$= \frac{\sin 8\alpha}{\sin 2\alpha + \sin 4\alpha + \sin 8\alpha}$$

$$= \frac{\sin 8\alpha}{\sin 2\alpha + \sin 4\alpha + \sin 6\alpha} = \frac{2\sin 4\alpha\cos 4\alpha}{2\sin 3\alpha\cos\alpha + 2\sin 3\alpha\cos 3\alpha};$$

but $4\alpha + 3\alpha = \frac{\pi}{2}$, therefore $\cos 4\alpha = \sin 3\alpha$; hence this expression

$$= \frac{\sin 4\alpha}{\cos\alpha + \cos 3\alpha} = \frac{2\sin 2\alpha\cos 2\alpha}{2\cos\alpha\cos 2\alpha} = \frac{\sin 2\alpha}{\cos\alpha} = 2\sin\alpha.$$

6. $2bc \text{ vers } A' + 2ca \text{ vers } B' + 2ab \text{ vers } C'$

$$= 2bc(1 - \cos A') + 2ca(1 - \cos B') + 2ab(1 - \cos C')$$

$$= 2bc(1 + \cos A) + 2ca(1 + \cos B) + 2ab(1 + \cos C)$$

$$= 4bc \cos^2 \frac{A}{2} + 4ca \cos^2 \frac{B}{2} + 4ab \cos^2 \frac{C}{2}$$

$$= 4s(s - a) + 4s(s - b) + 4s(s - c)$$

$$= 4s(3s - a - b - c) = 4s^2 = (2s)^2 = (a + b + c)^2.$$

7. Let $AD = p$. Suppose the angles B and C to be acute, as in the left-hand diagram of Art. 214. Then

$$AE = p \cos(90^0 - B) = p \sin B,$$

$$DE = p \sin(90^0 - B) = p \cos B,$$

$$EB = DE \cot B = p \cos B \cot B;$$

therefore $$AE \,.\, EB = p^2 \cos^2 B.$$

Similarly $$AF \,.\, FC = p^2 \cos^2 C.$$

Therefore $$AE \,.\, EB \cos^2 C = AF \,.\, FC \cos^2 B.$$

Next suppose one of the angles B and C to be obtuse, say the angie C, as in the right-hand diagram of Art. 214.

Then $$AE \,.\, EB = p^2 \cos^2 B \text{ as before,}$$

$$AF = p \cos(C - 90^0) = p \sin C,$$

$$DF = p \sin(C - 90^0) = -p \cos C,$$

$$FC = DF \cot(180^0 - C) = -DF \cot C = p \cos C \cot C;$$

therefore $$AF \,.\, FC = p^2 \cos^2 C, \text{ as before.}$$

8. $\dfrac{\sin 2\theta + \sin 4\theta}{\sin 3\theta} = \dfrac{a+c}{b}$; therefore $2\cos\theta = \dfrac{a+c}{b}$; therefore $\cos\theta = \dfrac{a+c}{2b}$;

therefore $\tan^2\theta = \dfrac{1}{\cos^2\theta} - 1 = \left(\dfrac{2b}{a+c}\right)^2 - 1.$

9. Since C is obtuse, $A + B$ is less than 90^0; therefore $\cos(A+B)$ is positive, therefore $\cos A \cos B - \sin A \sin B$ is positive; therefore $\sin A \sin B$ is less than $\cos A \cos B$; therefore $\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}$ is less than unity, that is $\tan A \tan B$ is less than unity.

10. Since a, b, c are in Arithmetical Progression, so are $\sin A$, $\sin B$, $\sin C$; hence $\sin A + \sin C = 2 \sin B$;

therefore $\sin\frac{A+C}{2}\cos\frac{A-C}{2}=2\sin\frac{B}{2}\cos\frac{B}{2}=2\sin\frac{B}{2}\sin\frac{A+C}{2}$;

therefore $\cos\frac{A-C}{2}=2\sin\frac{B}{2}$.

Again $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{a}{2}(1+\cos C)+\frac{c}{2}(1+\cos A)$

$=\frac{1}{2}(a+c)+\frac{1}{2}(a\cos C+c\cos A)=\frac{1}{2}(a+c)+\frac{b}{2}$, by Art. 216,

$=b+\frac{b}{2}$, by hypothesis, $=\frac{3b}{2}$.

11. From the triangle ABD we have

$$\frac{\sin ADB}{\sin BAD}=\frac{AB}{BD}=\frac{2c}{a}.$$

Put ϕ for BAD; thus

$$\frac{\sin(\phi+B)}{\sin\phi}=\frac{2c}{a}=\frac{2\sin C}{\sin A};$$

therefore $\frac{\sin\phi\cos B+\cos\phi\sin B}{\sin\phi}=\frac{2\sin C}{\sin A}$;

therefore $\cot B+\cot\phi=\frac{2\sin C}{\sin A\sin B}=\frac{2\sin(A+B)}{\sin A\sin B}$

$=2\cot A+2\cot B$;

therefore $\cot\phi-\cot B=2\cot A$.

12. Let the angle A of a triangle be divided into two parts by a straight line AD; denote BAD by ϕ and CAD by ψ, and suppose that $\frac{\sin\phi}{\sin\psi}=\frac{c}{b}$.

Thus $\frac{\sin(A-\psi)}{\sin\psi}=\frac{c}{b}=\frac{\sin C}{\sin B}$;

therefore $\sin A\cot\psi-\cos A=\frac{\sin C}{\sin B}$;

therefore $\cot\psi=\cot A+\frac{\sin(A+B)}{\sin A\sin B}=2\cot A+\cot B$.

Similarly $\cot\phi=2\cot A+\cot C$.

Therefore $\cot\psi-\cot\phi=\cot B-\cot C$.

13. Suppose $\cot A+\cot C=2\cot B$;

thus $\frac{\cos A}{\sin A}+\frac{\cos C}{\sin C}=\frac{2\cos B}{\sin B}$;

therefore $$\frac{\sin(A+C)}{\sin A \sin C} = \frac{2\cos B}{\sin B};$$

therefore $$\frac{\sin^2 B}{\sin A \sin C} = 2\cos B;$$

therefore $$\frac{b^2}{ac} = \frac{a^2+c^2-b^2}{ac};$$

therefore $$2b^2 = a^2 + c^2.$$

Thus a^2, b^2, c^2 are in Arithmetical Progression.

14. Let a perpendicular AD be drawn from the angle A of a triangle on the base BC. Let $BAD=\phi$, and $CAD=\psi$. Let m denote the ratio of the base BC to the perpendicular AD.

Then in the case of the left-hand diagram of Art. 214 we have

$$\tan\phi = \frac{BD}{AD}, \quad \tan\psi = \frac{CD}{AD};$$

therefore $$\tan\phi + \tan\psi = \frac{BD+CD}{AD} = \frac{BC}{AD} = m \text{ .... (1).}$$

Also $\phi + \psi = A$; thus

$$\tan A = \tan(\phi+\psi) = \frac{\tan\phi + \tan\psi}{1 - \tan\phi\tan\psi} \text{ .... (2).}$$

Hence from (1) and (2) we can find $\tan\phi$ and $\tan\psi$.

Similarly in the case of the right-hand diagram of Art. 214 we have

$$\tan\phi - \tan\psi = m,$$

and $$\tan A = \tan(\phi - \psi) = \frac{\tan\phi - \tan\psi}{1 + \tan\phi\tan\psi}.$$

15. Let the base BC of a triangle be divided at D and E, so that $BD=DE=EC$. Let the angle BAD be denoted by ϕ_1, the angle DAE by ϕ_2, and the angle EAC by ϕ_3.

Then from the triangle AEB we have $\frac{\sin(\phi_1+\phi_2)}{\sin AEB} = \frac{BE}{AB} = \frac{2}{3} \cdot \frac{a}{c}$, and from the triangle AEC we have $\frac{\sin\phi_3}{\sin AEC} = \frac{EC}{AC} = \frac{1}{3} \cdot \frac{a}{b}$;

therefore by division $$\frac{\sin(\phi_1+\phi_2)}{\sin\phi_3} = \frac{2b}{c}.$$

In the same manner we see that

$$\frac{\sin(\phi_3+\phi_2)}{\sin\phi_1} = \frac{2c}{b}.$$

Therefore $$\frac{\sin(\phi_1+\phi_2)\sin(\phi_3+\phi_2)}{\sin\phi_1 \sin\phi_3} = 4 = 4(\sin^2\phi_2 + \cos^2\phi_2);$$

therefore $(\cos\phi_2 + \sin\phi_2\cot\phi_1)(\cos\phi_2 + \sin\phi_2\cot\phi_3) = 4(\sin^2\phi_2 + \cos^2\phi_2)$;

therefore $$(\cot\phi_2 + \cot\phi_1)(\cot\phi_2 + \cot\phi_3) = 4(1 + \cot^2\phi_2).$$

16. Suppose that $\sin A + \sin C = 2 \sin B$,

then $$2 \sin \frac{A+C}{2} \cos \frac{A-C}{2} = 4 \sin \frac{B}{2} \cos \frac{B}{2} = 4 \cos \frac{A+C}{2} \sin \frac{A+C}{2};$$

therefore $$\cos \frac{A-C}{2} = 2 \cos \frac{A+C}{2};$$

therefore $$\cos \frac{A}{2} \cos \frac{C}{2} + \sin \frac{A}{2} \sin \frac{C}{2} = 2 \cos \frac{A}{2} \cos \frac{C}{2} - 2 \sin \frac{A}{2} \sin \frac{C}{2};$$

therefore $$3 \sin \frac{A}{2} \sin \frac{C}{2} = \cos \frac{A}{2} \cos \frac{C}{2};$$

therefore $$\tan \frac{A}{2} \tan \frac{C}{2} = \frac{1}{3}.$$

17. Denote ADB by ϕ. From the triangle ABD we have

$$\frac{\sin BAD}{\sin ADB} = \frac{BD}{AB} = \frac{a}{2c};$$

therefore $$\frac{\sin(\phi + B)}{\sin \phi} = \frac{a}{2c};$$

therefore $$\cos B + \sin B \cot \phi = \frac{a}{2c};$$

therefore $$\cot \phi = \frac{\frac{a}{2c} - \cos B}{\sin B};$$

therefore $$\tan \phi = \frac{2c \sin B}{a - 2c \cos B} = \frac{2ac \sin B}{a^2 - (a^2 + c^2 - b^2)}$$

$$= \frac{2ac \sin B}{b^2 - c^2} = \frac{2bc \sin A}{b^2 - c^2}.$$

18. Here $\cot \frac{A}{2} + \cot \frac{C}{2} = 2 \cot \frac{B}{2}$;

therefore $$\frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} + \frac{\cos \frac{C}{2}}{\sin \frac{C}{2}} = \frac{2 \cos \frac{B}{2}}{\sin \frac{B}{2}} = \frac{2 \sin \frac{A+C}{2}}{\cos \frac{A+C}{2}};$$

therefore $$\frac{\sin \frac{A+C}{2}}{\sin \frac{A}{2} \sin \frac{C}{2}} = \frac{2 \sin \frac{A+C}{2}}{\cos \frac{A+C}{2}};$$

therefore $$\cos \frac{A+C}{2} = 2 \sin \frac{A}{2} \sin \frac{C}{2};$$

therefore $$\cos\frac{A}{2}\cos\frac{C}{2}-\sin\frac{A}{2}\sin\frac{C}{2}=2\sin\frac{A}{2}\sin\frac{C}{2};$$

therefore $$\cos\frac{A}{2}\cos\frac{C}{2}=3\sin\frac{A}{2}\sin\frac{C}{2};$$

therefore $$\cot\frac{A}{2}\cot\frac{C}{2}=3.$$

19. First suppose that $$\frac{\sin DAC}{\sin DAB}=\frac{1}{n},$$

and that $$\frac{\sin DBC}{\sin DBA}=\frac{1}{n}.$$

We have $$\frac{\sin DCB}{\sin DBC}=\frac{BD}{DC},$$

and $$\frac{\sin DBC}{\sin DBA}=\frac{1}{n};$$

therefore $$\frac{\sin DCB}{\sin DBA}=\frac{BD}{DC}\cdot\frac{1}{n}.$$

Similarly $$\frac{\sin DCA}{\sin DAB}=\frac{AD}{DC}\cdot\frac{1}{n}.$$

Therefore $$\frac{\sin DCB}{\sin DCA}\cdot\frac{\sin DAB}{\sin DBA}=\frac{BD}{DA};$$

therefore $$\frac{\sin DCB}{\sin DCA}\cdot\frac{DB}{DA}=\frac{BD}{DA};$$

therefore $$\frac{\sin DCB}{\sin DCA}=1.$$

In this case the angle C is bisected by DC.

Next suppose that $$\frac{\sin DAC}{\sin DAB}=\frac{1}{n},$$

and that $$\frac{\sin DBA}{\sin DBC}=\frac{1}{n};$$

thus the angle B is divided into two parts equal to the two former, but differently situated.

Then proceeding as before we have

$$\frac{\sin DCB}{\sin DBC}=\frac{BD}{DC},$$

and $$\frac{\sin DBC}{\sin DBA}=n;$$

therefore $$\frac{\sin DCB}{\sin DBA} = \frac{n \, . \, BD}{DC}.$$

Also $$\frac{\sin DCA}{\sin DAB} = \frac{AD}{DC}\frac{1}{n}.$$

Hence we find that

$$\frac{\sin DCB}{\sin DCA} = n^2.$$

20. Let the straight line which bisects the angle A of a triangle meet the base at D. Then

the angle ADC = the angle B + the angle BAD;

thus $$\sin\theta = \sin\left(B + \frac{A}{2}\right).$$

Hence $$s\left(\sin\theta - \sin\frac{A}{2}\right) = s\left\{\sin\left(B + \frac{A}{2}\right) - \sin\frac{A}{2}\right\}$$

$$= 2s\cos\frac{B+A}{2}\sin\frac{B}{2} = 2s\sin\frac{C}{2}\sin\frac{B}{2};$$

put for $\sin\frac{C}{2}$ and $\sin\frac{B}{2}$ their values by Art. 217; thus we have

$$2s\sin\frac{C}{2}\sin\frac{B}{2} = \frac{2s}{a}(s-a)\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$= \frac{2s(s-a)}{a}\sin\frac{A}{2} = \frac{2bc}{a}\cos^2\frac{A}{2}\sin\frac{A}{2}$$

$$= \frac{bc}{a}\cos\frac{A}{2}\sin A.$$

Again $l\sin\theta$ = the perpendicular from A on BC

$= b\sin C$.

Therefore $$l\sin\theta\cos\frac{A}{2} = b\sin C\cos\frac{A}{2}$$

$$= \frac{bc}{a}\sin A\cos\frac{A}{2}, \text{ by Art. 214.}$$

Therefore $$s\left(\sin\theta - \sin\frac{A}{2}\right) = l\sin\theta\cos\frac{A}{2}.$$

21. The third angle of the triangle will be $\pi - \theta - \phi$; and as the sines of the angles must be in Arithmetical Progression, we have

$$\sin\theta + \sin\phi = 2\sin(\pi - \theta - \phi) = 2\sin(\theta+\phi);$$

therefore $$2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = 4\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2};$$

therefore $$2\cos\frac{\theta+\phi}{2} = \cos\frac{\theta-\phi}{2};$$

therefore $$2\left(\cos\frac{\theta}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2}\right) = \cos\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\theta}{2}\sin\frac{\phi}{2};$$

therefore $$\cos\frac{\theta}{2}\cos\frac{\phi}{2} = 3\sin\frac{\theta}{2}\sin\frac{\phi}{2};$$

therefore $$\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2} = 9\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}$$

$$= 9\left(1 - \cos^2\frac{\theta}{2}\right)\left(1 - \cos^2\frac{\phi}{2}\right);$$

therefore $$8\left(1-\cos^2\frac{\theta}{2}\right)\left(1-\cos^2\frac{\phi}{2}\right)$$

$$= \cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2} - \left(1-\cos^2\frac{\theta}{2}\right)\left(1-\cos^2\frac{\phi}{2}\right)$$

$$= \cos^2\frac{\theta}{2} + \cos^2\frac{\phi}{2} - 1 = \cos^2\frac{\theta}{2} - \sin^2\frac{\phi}{2};$$

therefore $$8\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2} = \cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2};$$

therefore $$4(1-\cos\theta)(1-\cos\phi) = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$

$$= \cos\theta + \cos\phi.$$

Or thus, $\cos\theta = \dfrac{a^2+b^2-c^2}{2ab}$, and $b = \dfrac{a+c}{2}$; therefore

$$\cos\theta = \frac{a-c}{a} + \frac{b}{2a} = \frac{a-c}{a} + \frac{a+c}{4a} = \frac{5a-3c}{4a}.$$

Similarly $$\cos\phi = \frac{5c-3a}{4c}.$$

Hence $$4(1-\cos\theta)(1-\cos\phi) = \frac{(3c-a)(3a-c)}{4ac} = \frac{10ac-3a^2-3c^2}{4ac};$$

and $$\cos\theta + \cos\phi = \frac{5a-3c}{4a} + \frac{5c-3a}{4c} = \frac{10ac-3a^2-3c^2}{4ac}.$$

22. Draw from A, B, C respectively straight lines to meet the opposite sides at D, E, F, so that the angle BAD = the angle CBE = the angle $ACF = \alpha$. Let LMN be the triangle formed by the straight lines thus drawn: so that A, L, M, D are in one straight line; B, M, N, E on another; and C, N, L, F on a third. Then will the triangle LMN be similar to the triangle ABC.

For the angle MLN = the angle MAC + the angle $LCA = A - \alpha + \alpha = A$; similarly the angle $NML = B$, and the angle $LNM = C$. Thus the triangle LMN is equiangular to the original triangle, and therefore similar to it.

Again $$\frac{BN}{BC} = \frac{\sin BCN}{\sin BNC} = \frac{\sin(C-\alpha)}{\sin(\pi - C)} = \frac{\sin(C-\alpha)}{\sin C};$$

therefore $$BN = \frac{a\sin(C-\alpha)}{\sin C},$$

and $$\frac{BM}{BA} = \frac{\sin BAM}{\sin BMA} = \frac{\sin\alpha}{\sin(\pi - B)} = \frac{\sin\alpha}{\sin B};$$

therefore $$BM = \frac{c\sin\alpha}{\sin B}.$$

Hence $$MN = \frac{a\sin(C-\alpha)}{\sin C} - \frac{c\sin\alpha}{\sin B}$$

$$= a\cos\alpha - a\cot C\sin\alpha - \frac{a\sin C}{\sin A\sin B}\sin\alpha$$

$$= a\cos\alpha - a\cot C\sin\alpha - \frac{a\sin(A+B)}{\sin A\sin B}\sin\alpha$$

$$= a\cos\alpha - a\sin\alpha\,(\cot C + \cot B + \cot A).$$

The ratio of this to a is the same as the ratio of

$$\cos\alpha - \sin\alpha\,(\cot A + \cot B + \cot C) \text{ to unity.}$$

23. $$ab\cos C - ac\cos B = \frac{a^2+b^2-c^2}{2} - \frac{a^2+c^2-b^2}{2} = b^2 - c^2.$$

24. $$a(\cos B\cos C + \cos A) = a\{\cos B\cos C - \cos(B+C)\}$$

$$= a\sin B\sin C = \frac{a}{\sin A}\sin A\sin B\sin C.$$

Similarly $$b(\cos A\cos C + \cos B) = \frac{b}{\sin B}\sin A\sin B\sin C;$$

and $$c(\cos A\cos B + \cos C) = \frac{c}{\sin C}\sin A\sin B\sin C.$$

Thus the three expressions are equal by Art. 214.

25. $$(b+c-a)\tan\frac{A}{2} = 2(s-a)\tan\frac{A}{2} = 2(s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \frac{2\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}}.$$

Similarly the other two proposed expressions reduce to the same symmetrical form.

26. $$b \cos B + c \cos C = \frac{a \sin B}{\sin A} \cos B + \frac{a \sin C}{\sin A} \cos C$$
$$= \frac{a}{2 \sin A} (\sin 2B + \sin 2C)$$
$$= \frac{2a \sin (B+C) \cos (B-C)}{2 \sin (B+C)} = a \cos (B-C).$$

27. By Art. 216
$$c \cos B + b \cos C = a, \quad a \cos C + c \cos A = b, \quad b \cos A + a \cos B = c;$$
therefore by addition
$$c (\cos B + \cos A) + b (\cos A + \cos C) + a (\cos C + \cos B) = a + b + c.$$

28. Let k stand for $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, and $\frac{c}{\sin C}$ which we know are all equal. Then
$$(a^2 - b^2) \cot C + (b^2 - c^2) \cot A + (c^2 - a^2) \cot B$$
$$= k^2 \{(\sin^2 A - \sin^2 B) \cot C + (\sin^2 B - \sin^2 C) \cot A + (\sin^2 C - \sin^2 A) \cot B\}$$
$$= k^2 \{\sin (A+B) \sin (A-B) \cot C + \sin (B+C) \sin (B-C) \cot A + \sin (C+A) \sin (C-A) \cot B\}$$
$$= k^2 \{\sin (A-B) \cos C + \sin (B-C) \cos A + \sin (C-A) \cos B\}$$
$$= -k^2 \{\sin (A-B) \cos (A+B) + \sin (B-C) \cos (B+C) + \sin (C-A) \cos (C+A)\}$$
$$= -\frac{k^2}{2} \{\sin 2A - \sin 2B + \sin 2B - \sin 2C + \sin 2C - \sin 2A\}$$
$$= 0.$$

29. Let k have the same meaning as in the preceding solution; then
$$(a-b) \cot \frac{C}{2} + (c-a) \cot \frac{B}{2} + (b-c) \cot \frac{A}{2}$$
$$= k \left\{(\sin A - \sin B) \cot \frac{C}{2} + (\sin C - \sin A) \cot \frac{B}{2} + (\sin B - \sin C) \cot \frac{A}{2}\right\}$$
$$= 2k \left\{\sin \frac{A-B}{2} \sin \frac{A+B}{2} + \sin \frac{C-A}{2} \sin \frac{C+A}{2} + \sin \frac{B-C}{2} \sin \frac{B+C}{2}\right\}$$
$$= 2k \left\{\sin^2 \frac{A}{2} - \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} - \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2}\right\}$$
$$= 0.$$

30. $$1 - \tan \frac{A}{2} \tan \frac{B}{2} = 1 - \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$
$$= 1 - \frac{s-c}{s} = 1 - \frac{a+b-c}{a+b+c} = \frac{2c}{a+b+c}.$$

31. $(a+b+c)(\cos A+\cos B+\cos C)$

$$=a\cos A+b\cos B+c\cos C$$
$$+a\cos B+b\cos A+a\cos C+c\cos A+b\cos C+c\cos B$$
$$=a\cos A+b\cos B+c\cos C+c+b+a, \text{ by Art. 216,}$$
$$=a(1+\cos A)+b(1+\cos B)+c(1+\cos C)$$
$$=2a\cos^2\frac{A}{2}+2b\cos^2\frac{B}{2}+2c\cos^2\frac{C}{2}.$$

32. Let k have the same meaning as in the solution of Example 28; then

$$\frac{\cos A\cos B}{ab}+\frac{\cos A\cos C}{ac}+\frac{\cos B\cos C}{bc}$$
$$=\frac{1}{k^2}\left\{\frac{\cos A\cos B}{\sin A\sin B}+\frac{\cos A\cos C}{\sin A\sin C}+\frac{\cos B\cos C}{\sin B\sin C}\right\}$$
$$=\frac{1}{k^2}\{\cot A\cot B+\cot A\cot C+\cot B\cot C\}$$
$$=\frac{1}{k^2}\frac{\tan A+\tan B+\tan C}{\tan A\tan B\tan C}=\frac{1}{k^2}, \text{ by Art. 114,}$$
$$=\frac{\sin^2 A}{a^2}.$$

33. $a\cos A+b\cos B+c\cos C=a\cos A+\frac{a\sin B}{\sin A}\cos B+\frac{a\sin C}{\sin A}\cos C$

$$=a\cos A+\frac{a(\sin 2B+\sin 2C)}{2\sin A}=a\cos A+\frac{2a\sin(B+C)\cos(B-C)}{2\sin A}$$
$$=a\cos A+a\cos(B-C)=-a\cos(B+C)+a\cos(B-C)$$
$$=2a\sin B\sin C.$$

34. $\dfrac{2a\sin B\sin C}{a+b+c}=\dfrac{2\sin B\sin C}{1+\dfrac{b}{a}+\dfrac{c}{a}}=\dfrac{2\sin B\sin C}{1+\dfrac{\sin B}{\sin A}+\dfrac{\sin C}{\sin A}}$

$$=\frac{2\sin A\sin B\sin C}{\sin A+\sin B+\sin C}=\frac{2\sin A\sin B\sin C}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}, \text{ by Example VIII. 16,}$$
$$=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\cos A+\cos B+\cos C-1, \text{ by Art. 114;}$$

therefore $\qquad \cos A+\cos B+\cos C=1+\dfrac{2a\sin B\sin C}{a+b+c}.$

35. $$a^2 - 2ab\cos(60^\circ + C) = a^2 - 2ab(\cos 60^\circ \cos C - \sin 60^\circ \sin C)$$

$$= a^2 - ab\cos C + 2ab\sin 60^\circ \sin C$$

$$= a^2 - \frac{a^2 + b^2 - c^2}{2} + 2cb\sin 60^\circ \sin A$$

$$= c^2 - \frac{c^2 + b^2 - a^2}{2} + 2bc\sin 60^\circ \sin A$$

$$= c^2 - bc\cos A + 2bc\sin 60^\circ \sin A$$

$$= c^2 - 2bc\cos(60^\circ + A).$$

36. $$\frac{b+c-a}{2a} = \frac{\sin B + \sin C - \sin A}{2\sin A}$$

$$= \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}}{4\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \frac{\cos\frac{B-C}{2} - \sin\frac{A}{2}}{2\sin\frac{A}{2}} = \frac{\cos\frac{B-C}{2} - \cos\frac{B+C}{2}}{2\sin\frac{A}{2}} = \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}.$$

Again $$\cot\frac{A}{4} - \operatorname{cosec}\frac{A}{2} = \frac{\cos\frac{A}{4}}{\sin\frac{A}{4}} - \frac{1}{\sin\frac{A}{2}} = \frac{2\cos^2\frac{A}{4} - 1}{\sin\frac{A}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}};$$

and $$\cot\frac{B}{2} + \cot\frac{C}{2} = \frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}} = \frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}};$$

therefore $$\frac{\cot\frac{A}{4} - \operatorname{cosec}\frac{A}{2}}{\cot\frac{B}{2} + \cot\frac{C}{2}} = \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}} = \frac{b+c-a}{2a}.$$

37. $$4\Sigma\left(\Sigma - \cos\frac{A}{2}\right)\left(\Sigma - \cos\frac{B}{2}\right)\left(\Sigma - \cos\frac{A}{2}\right) = \text{the product of}$$

$$\frac{1}{4}\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)\left(\cos\frac{B}{2} + \cos\frac{C}{2} - \cos\frac{A}{2}\right)$$

into $$\left(\cos\frac{A}{2} + \cos\frac{C}{2} - \cos\frac{B}{2}\right)\left(\cos\frac{A}{2} + \cos\frac{B}{2} - \cos\frac{C}{2}\right).$$

Now substitute for the trinomial expressions results given by Examples VIII. 20 and 21; thus we obtain

$$\left\{8\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+C}{4}\right\}^2,$$

that is $$\left\{8\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}\right\}^2,$$

that is $$\left\{\sin\frac{\pi-A}{2}\sin\frac{\pi-B}{2}\sin\frac{\pi-C}{2}\right\}^2,$$

that is $$\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}.$$

38. The perimeter $=a+b+c=\dfrac{c\sin A}{\sin C}+\dfrac{c\sin B}{\sin C}+c$

$$=\frac{c(\sin A+\sin B+\sin C)}{\sin C}=\frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin C}, \text{ by Example VIII. 16,}$$

$$=\frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}=\frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A+B}{2}}$$

$$=2\cos\frac{A}{2}\cos\frac{B}{2}\sec\frac{A+B}{2}.$$

39. Let $h=y\sin^2 A+x\sin^2 B=z\sin^2 B+y\sin^2 C=x\sin^2 C+z\sin^2 A$.

Thus $h(\sin^2 C-\sin^2 A)=x\sin^2 B\sin^2 C-z\sin^2 A\sin^2 B$,

and $h=x\sin^2 C+z\sin^2 A$;

therefore $h(\sin^2 C-\sin^2 A)+h\sin^2 B=2x\sin^2 B\sin^2 C$,

therefore $h\sin(C-A)\sin(C+A)+h\sin^2 B=2x\sin^2 B\sin^2 C$;

therefore $h\sin(C-A)+h\sin(C+A)=2x\sin B\sin^2 C$;

therefore $x\sin B\sin^2 C=h\sin C\cos A$,

therefore $$x=\frac{h\cos A}{\sin B\sin C}=\frac{h\sin 2A}{2\sin A\sin B\sin C}.$$

Similarly $$y=\frac{h\sin 2B}{2\sin A\sin B\sin C}, \text{ and } z=\frac{h\sin 2C}{2\sin A\sin B\sin C}.$$

40. Since $A+B+C=\pi$, we may shew that $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ has its greatest value when A, B, and C are all equal.

For $$\sin\frac{A}{2}\sin\frac{B}{2}=\sin\left(\frac{A+B}{4}+\frac{A-B}{4}\right)\sin\left(\frac{A+B}{4}-\frac{A-B}{4}\right)$$

$$=\sin^2\frac{A+B}{4}-\sin^2\frac{A-B}{4};$$

thus, whatever may be the value of C, it follows that $\sin\frac{A}{2}\sin\frac{B}{2}$ has its greatest value when $A=B$; for $\sin\frac{A+B}{4}$ does not change while A and B change in such a manner as to leave C unchanged. In this way we see that the greatest value of the expression is when all the angles are equal, and the value then is $8\sin^3\frac{\pi}{6}$, that is 1.

41. Let k have the same meaning as in the solution of Example 28; then

$$a\sin(B-C)\cos(B+C-A)=k\sin A\sin(B-C)\cos(180^\circ-2A)$$
$$=-k\sin(B+C)\sin(B-C)\cos 2A=k(\sin^2 B-\sin^2 C)(2\sin^2 A-1)$$
$$=2k\sin^2 A(\sin^2 B-\sin^2 C)-k(\sin^2 B-\sin^2 C).$$

Similarly the other two terms of the proposed expression may be transformed; and then the whole vanishes because

$$\sin^2 A(\sin^2 B-\sin^2 C)+\sin^2 B(\sin^2 C-\sin^2 A)+\sin^2 C(\sin^2 A-\sin^2 B)=0,$$

and $$\sin^2 B-\sin^2 C+\sin^2 C-\sin^2 A+\sin^2 A-\sin^2 B=0.$$

42. $$\frac{\sin A}{\cos B}+\frac{\sin B}{\cos A}=\frac{\sin A\cos A+\sin B\cos B}{\cos A\cos B}=\frac{\sin 2A+\sin 2B}{2\cos A\cos B}$$

$$=\frac{2\sin(A+B)\cos(A-B)}{2\cos A\cos B}=\frac{\sin C}{\cos A\cos B}(\cos A\cos B+\sin A\sin B)$$

$$=\sin C+\cos C\tan A\tan B\tan C.$$

Similarly $$\frac{\sin B}{\cos C}+\frac{\sin C}{\cos B}=\sin A+\cos A\tan A\tan B\tan C,$$

and $$\frac{\sin C}{\cos A}+\frac{\sin A}{\cos C}=\sin B+\cos B\tan A\tan B\tan C.$$

Hence by addition we obtain the required result.

CHAPTER XIV.

1. $\sin A=\frac{a}{b}\sin B=\frac{5}{2\cdot 5}\cdot 25=\frac{1}{2}$; therefore $A=30^\circ$ or 150°.

2. Suppose $c=\frac{1}{2}b$, and $A=60^0$; then, by Art. 229,

$$\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\cot 30^0=\frac{1}{3}\cdot\sqrt{3}=\frac{1}{\sqrt{3}};$$

therefore $\frac{1}{2}(B-C)=30^0$; and $\frac{1}{2}(B+C)=60^0$.

Hence $B=90^0$ and $C=30^0$.

3. Let a, b, c denote these sides in order. Then

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{6+(1+\sqrt{3})^2-4}{2(1+\sqrt{3})\sqrt{6}}=\frac{6+2\sqrt{3}}{2(1+\sqrt{3})\sqrt{6}}$$

$$=\frac{\sqrt{3}(1+\sqrt{3})}{(1+\sqrt{3})\sqrt{6}}=\frac{1}{\sqrt{2}};\quad \text{therefore } A=45^0.$$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{4+(1+\sqrt{3})^2-6}{4(1+\sqrt{3})}=\frac{2+2\sqrt{3}}{4(1+\sqrt{3})}=\frac{1}{2};$$

therefore $B=60^0$.

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{4+6-(1+\sqrt{3})^2}{4\sqrt{6}}=\frac{6-2\sqrt{3}}{4\sqrt{6}}=\frac{3-\sqrt{3}}{2\sqrt{6}}$$

$$=\frac{\sqrt{3}-1}{2\sqrt{2}};\quad \text{therefore } C=75^0.$$

4. $\sin B=\frac{b}{a}\sin A=\frac{100}{40}\cdot\frac{1}{2}=\frac{5}{4}$; but this is impossible, for a sine cannot be greater than unity.

5. $$\sin B=\frac{b}{a}\sin A=\frac{4+\sqrt{(80)}}{4}\sin 18^0=(1+\sqrt{5})\sin 18^0$$

$$=\frac{(1+\sqrt{5})(\sqrt{5}-1)}{4}=1;\quad \text{therefore } B=90^0.$$

Thus $C=72^0$; and $c^2=b^2-a^2=\{4+\sqrt{(80)}\}^2-16$

$$=80+8\sqrt{(80)}=16(5+2\sqrt{5});$$

therefore $c=4\sqrt{(5+2\sqrt{5})}$.

6. $$\sin B = \frac{b}{a} \sin A = \frac{4+\sqrt{(48)}}{4} \sin 15^\circ = (1+\sqrt{3}) \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{1}{\sqrt{2}};$$

therefore $$B = 45^\circ \text{ or } 135^\circ.$$

If $B=45^\circ$, then $C=120^\circ$; and $c=\dfrac{a \sin C}{\sin A} = 4 \cdot \dfrac{2\sqrt{2}}{\sqrt{3}-1} \cdot \dfrac{\sqrt{3}}{2}$

$$= \frac{4\sqrt{6}}{\sqrt{3}-1} = \frac{4\sqrt{6}(\sqrt{3}+1)}{2} = 2\sqrt{6}(\sqrt{3}+1).$$

If $B=135^\circ$, then $C=30^\circ$; and $c=\dfrac{a \sin C}{\sin A} = 4 \cdot \dfrac{2\sqrt{2}}{\sqrt{3}-1} \cdot \dfrac{1}{2}$

$$= \frac{4\sqrt{2}}{\sqrt{3}-1} = \frac{4\sqrt{2}(\sqrt{3}+1)}{2} = 2\sqrt{2}(\sqrt{3}+1).$$

7. With the first diagram of Art. 234 we may put $c = AB$ and $c' = AB'$; thus

$$c = b \cos A - a \cos CBB', \text{ and } c' = b \cos A + a \cos CBB';$$

therefore $$c + c' = 2b \cos A,$$

and $$cc' = b^2 \cos^2 A - a^2 \cos^2 CBB' = b^2 \cos^2 A - a^2 \cos^2 B$$
$$= b^2 (1 - \sin^2 A) - a^2 (1 - \sin^2 B) = b^2 - a^2.$$

Hence $$(c+c')^2 = 4b^2 \cos^2 A,$$
$$4cc' \cos^2 A = 4(b^2 - a^2) \cos^2 A;$$

therefore $$c^2 + 2cc' + c'^2 - 4cc' \cos^2 A = 4a^2 \cos^2 A,$$

that is $$c^2 - 2cc' \cos 2A + c'^2 = 4a^2 \cos^2 A.$$

8. With the notation of the preceding solution the area of the smaller triangle is $\dfrac{c}{2} b \sin A$, and the area of the larger triangle is $\dfrac{c'}{2} b \sin A$; hence the sum of the areas $= \dfrac{1}{2}(c+c') b \sin A = b^2 \sin A \cos A$.

9. With the notation of the two preceding solutions we have

$$\frac{\sin C_1}{\sin B_1} = \frac{c}{b} \text{ and } \frac{\sin C_2}{\sin B_2} = \frac{c'}{b};$$

therefore $$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = \frac{c+c'}{b} = \frac{2b \cos A}{b} = 2 \cos A.$$

10. As in the solution of Example 8, we have

$$\frac{1}{2} c' b \sin A = \frac{n}{2} cb \sin A;$$

therefore $c' = nc$.

And as in the solution of Example 7,

$$\frac{c'+c}{c'-c}=\frac{2b\cos A}{2a\cos CBB'};$$

therefore
$$\frac{b}{a}=\frac{n+1}{n-1}\cdot\frac{\cos CBB'}{\cos A};$$

but the angle CBB' is greater than A, and therefore $\frac{\cos CBB'}{\cos A}$ is less than unity. Hence $\frac{b}{a}$ is less than $\frac{n+1}{n-1}$.

11. $\sin B=\frac{b}{a}\sin A$; therefore $L\sin B-10=\log b+L\sin A-10-\log a$.

Thus if $\log a+10=\log b+L\sin A$ we have $L\sin B-10=0$; therefore $L\sin B=10$, therefore $\log\sin B=0$, therefore $\sin B=1$, therefore $B=90^\circ$, and the triangle is not ambiguous.

12. $$\frac{a+b}{c}=\frac{\sin A+\sin B}{\sin C}=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C};$$

therefore
$$\cos\frac{1}{2}(A-B)=\frac{(a+b)\sin\frac{C}{2}}{c}.$$

Now assume $\cos\theta=\frac{a-b}{c}$; therefore

$$\sin^2\theta=\frac{c^2-(a-b)^2}{c^2}=\frac{a^2+b^2-2ab\cos C-(a-b)^2}{c^2}$$

$$=\frac{2ab(1-\cos C)}{c^2}=\frac{4ab}{c^2}\sin^2\frac{C}{2};$$

therefore
$$\sin\theta=\frac{2\sqrt{(ab)}}{c}\sin\frac{C}{2};$$

therefore
$$\cos\frac{1}{2}(A-B)=\frac{(a+b)\sin\theta}{2\sqrt{(ab)}}.$$

And
$$\sin\theta=\frac{2\sqrt{(ab)}}{c}\sin\frac{C}{2}=\frac{2\sqrt{(ab)}}{c}\cos\frac{1}{2}(A+B);$$

therefore
$$\cos\frac{1}{2}(A+B)=\frac{c\sin\theta}{2\sqrt{(ab)}}.$$

13. $$c^2 = a^2 + b^2 - 2ab\cos C = a^2 + b^2 - 2ab\left(1 - 2\sin^2\frac{C}{2}\right)$$

$$= (a-b)^2 + 4ab\sin^2\frac{C}{2} = (a-b)^2 + (a-b)^2\tan^2\phi$$

$$= (a-b)^2\{1 + \tan^2\phi\} = (a-b)^2\sec^2\phi.$$

14. Here $s = 30$, $s-a = 12$, $s-b = 10$, $s-c = 8$.

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{10\times 8}{30\times 12}} = \sqrt{\frac{8}{36}} = \sqrt{\frac{2}{9}};$$

therefore $$L\tan\frac{A}{2} = 10 + \log\sqrt{\frac{2}{9}} = 10 + \frac{1}{2}\log 2 - \log 3 = 9\cdot6733937.$$

15. The greatest angle is opposite to the side 66; denote this angle by C. Then

$$\cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}.$$

Here $s = 69$, $s-a = 37$, $s-b = 29$, $s-c = 3$;

therefore $$\cot\frac{C}{2} = \sqrt{\frac{69\times 3}{37\times 29}} = \sqrt{\frac{207}{1073}};$$

therefore $$L\cot\frac{C}{2} = 10 + \log\sqrt{\frac{207}{1073}}$$

$$= 10 + \frac{1}{2}(\log 207 - \log 1073) = 9\cdot6426853.$$

$$\begin{array}{r} 9\cdot6426853 \\ 9\cdot6424342 \\ \hline \cdot0002511 \end{array} \qquad \cdot0003431 : \cdot0002511 :: 60'' : x'';$$

this gives $x = 44$; therefore $\frac{C}{2} = 66^\circ\,18' - 44'' = 66^\circ\,17'\,16''$;

therefore $$C = 132^\circ\,34'\,32''.$$

16. Here $s = \frac{15}{2}$, $s-a = \frac{7}{2}$, $s-b = \frac{5}{2}$, $s-c = \frac{3}{2}$.

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} = \sqrt{\frac{15\times 5}{8\times 12}} = \sqrt{\frac{25}{32}} = \sqrt{\frac{100}{2^7}};$$

therefore $$L\cos\frac{B}{2} = 10 + \log\sqrt{\frac{100}{2^7}} = 10 + \frac{1}{2}(\log 100 - \log 2^7)$$

$$= 10 + 1 - \frac{7}{2}\log 2 = 9\cdot9463950.$$

$$\begin{array}{r} 9\cdot 9464040 \\ 9\cdot 9463950 \\ \hline \cdot 0000090 \end{array} \qquad \cdot 0000669 : \cdot 0000090 :: 60'' : x'';$$

this gives $x=8$; therefore $\frac{B}{2}=27^0\, 53'\, 8''$; therefore $B=55^0\, 46'\, 16''$.

17. Here $a=7$, $s=9$, $s-a=2$; therefore

$$\cos\frac{A}{2}=\sqrt{\frac{9\times 2}{5\times 6}}=\sqrt{\frac{3}{5}}=\sqrt{\frac{6}{10}};$$

therefore $$L\cos\frac{A}{2}=10+\log\sqrt{\frac{6}{10}}=10+\frac{1}{2}(\log 6-\log 10)$$

$$=10+\frac{1}{2}\log 6-\frac{1}{2}=9\cdot 8890756.$$

$$\begin{array}{r} 9\cdot 8890756 \\ 9\cdot 8890644 \\ \hline \cdot 0000112 \end{array} \qquad \cdot 0001032 : \cdot 0000112 :: 60'' : x'';$$

this gives $x=6\cdot 5$; therefore $\frac{A}{2}=39^0\, 14'-6''\cdot 5=39^0\, 13'\, 53''\cdot 5$; therefore $A=78^0\, 27'\, 47''$.

18. As in Art. 229 we have

$$\tan\frac{1}{2}(B-C)=\frac{18-2}{18+2}\cot\frac{A}{2}=\frac{8}{10}\cot 27^0\, 30';$$

therefore $$L\tan\frac{1}{2}(B-C)=L\cot 27^0\, 30'+\log 8-\log 10$$

$$=L\cot 27^0\, 30'+3\log 2-1=10\cdot 1866133.$$

$$\begin{array}{r} 10\cdot 1866133 \\ 10\cdot 1863769 \\ \hline \cdot 0002364 \end{array} \qquad \cdot 0002763 : \cdot 0002364 :: 60'' : x'';$$

this gives $x=51$; therefore $\frac{1}{2}(B-C)=56^0\, 56'\, 51''$.

And $\frac{1}{2}(B+C)=62^0\, 30'$; therefore $B=119^0\, 26'\, 51''$, $C=5^0\, 33'\, 9''$.

19. $$\tan\frac{1}{2}(B-C)=\frac{9-7}{9+7}\cot\frac{A}{2}=\frac{1}{8}\cot 32^0\, 6';$$

therefore $$L\tan\frac{1}{2}(B-C)=L\cot 32^0\, 6'-\log 8$$

$$=L\tan 57^0\, 54'-3\log 2=9\cdot 2994355.$$

$$\begin{array}{r} 9\cdot2999804 \\ 9\cdot2993216 \\ \hline \cdot0006588 \end{array} \qquad \begin{array}{r} 9\cdot2994355 \\ 9\cdot2993216 \\ \hline \cdot0001139 \end{array} \qquad \cdot0006588 : \cdot0001139 :: 60'' : x'';$$

this gives $x=10$; therefore $\frac{1}{2}(B-C)=11^0\,16'\,10''$.

And $\frac{1}{2}(B+C)=57^0\,54'$; therefore $B=69^0\,10'\,10''$, $C=46^0\,37'\,50''$.

20. $\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{70-35}{70+35}\cot\frac{C}{2}=\frac{1}{3}\cot 18^0\,26'\,6''$;

therefore $L\tan\frac{1}{2}(A-B)=L\cot 18^0\,26'\,6''-\log 3=10$;

therefore $\log\tan\frac{1}{2}(A-B)=0$;

therefore $\tan\frac{1}{2}(A-B)=1$; therefore $\frac{1}{2}(A-B)=45^0$.

And $\frac{1}{2}(A+B)=71^0\,33'\,54''$; therefore $A=116^0\,33'\,54''$, $B=26^0\,33'\,54''$.

21. $\tan\frac{1}{2}(B-C)=\frac{9-7}{9+7}\cot\frac{A}{2}=\frac{1}{8}\cot 23^0\,42'\,30''$;

therefore $L\tan\frac{1}{2}(B-C)=L\cot 23^0\,42'\,30''-\log 8$

$=L\tan 66^0\,17'\,30''-3\log 2=9\cdot4543042.$

$$\begin{array}{r} 9\cdot4543042 \\ 9\cdot4541479 \\ \hline \cdot0001563 \end{array} \qquad \cdot0004797 : \cdot0001563 :: 60'' : x'';$$

this gives $x=20''$; therefore $\frac{1}{2}(B-C)=15^0\,53'\,20''$.

And $\frac{1}{2}(B+C)=66^0\,17'\,30''$; therefore $B=82^0\,10'\,50''$, $C=50^0\,24'\,10''$.

22. $\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{30-20}{30+20}\cot\frac{C}{2}=\frac{2}{10}\cot 11^0$;

therefore $L\tan\frac{1}{2}(A-B)=L\cot 11^0+\log 2-\log 10$

$=L\cot 11^0+\log 2-1=10\cdot0123777.$

$$\begin{array}{r} 10\cdot0123821 \\ 10\cdot0121294 \\ \hline \cdot0002527 \end{array} \qquad \begin{array}{r} 10\cdot0123777 \\ 10\cdot0121294 \\ \hline \cdot0002483 \end{array} \qquad \cdot0002527 : \cdot0002483 :: 60'' : x'';$$

this gives $x=59$; therefore $\frac{1}{2}(A-B)=45^0\,48'\,59''$.

And $\frac{1}{2}(A+B)=79^0$; therefore $A=124^0\,48'\,59''$, $B=33^0\,11'\,1''$.

23. $$\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{3}{25}\cot 30^0=\frac{3\sqrt{3}}{25};$$

therefore $$L\tan\frac{1}{2}(B-C)=10+\log\frac{3\sqrt{3}}{25}=10+\frac{3}{2}\log 3-\log 25$$

$$=10+\frac{3}{2}\log 3-\log\frac{100}{4}=10+\frac{3}{2}\log 3-2+2\log 2=9\cdot 31774;$$

therefore $$\frac{1}{2}(B-C)=11^0\,44'\,29''.$$

And $\frac{1}{2}(B+C)=60^0$; therefore $B=71^0\,44'\,29''$.

24. Let $a=7$, $b=8$, $c=9$; then $s=12$, $s-a=5$, $s-b=4$, $s-c=3$.

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{4\times 3}{12\times 5}}=\sqrt{\frac{1}{5}}=\sqrt{\frac{2}{10}}.$$

$$L\tan\frac{A}{2}=10+\log\sqrt{\frac{2}{10}}=10+\frac{1}{2}(\log 2-\log 10)$$

$$=10+\frac{1}{2}(\log 2-1)=9\cdot 6505150.$$

$$\begin{array}{r} 9\cdot 6505634 \\ 9\cdot 6505069 \\ \hline \cdot 0000565 \end{array}\qquad \begin{array}{r} 9\cdot 6505150 \\ 9\cdot 6505069 \\ \hline \cdot 0000081 \end{array}\qquad \cdot 0000565 : \cdot 0000081 :: 10'' : x'';$$

this gives $x=1\cdot 5$; therefore $\frac{A}{2}=24^0\,5'\,41''\cdot 5$; therefore $A=48^0\,11'\,23''$.

$$\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\sqrt{\frac{5\times 3}{12\times 4}}=\sqrt{\frac{5}{16}}=\sqrt{\frac{10}{32}}.$$

$$L\tan\frac{B}{2}=10+\log\sqrt{\frac{10}{32}}=10+\frac{1}{2}(\log 10-\log 32)$$

$$=10+\frac{1}{2}-\frac{5}{2}\log 2=9\cdot 7474250.$$

$$\begin{array}{r} 9\cdot 7474677 \\ 9\cdot 7474183 \\ \hline \cdot 0000494 \end{array}\qquad \begin{array}{r} 9\cdot 7474250 \\ 9\cdot 7474183 \\ \hline \cdot 0000067 \end{array}\qquad \cdot 0000494 : \cdot 0000067 :: 10'' : x'';$$

this gives $x=1\cdot 5$; therefore $\frac{B}{2}=29^0\,12'\,21''\cdot 5$; therefore $B=58^0\,24'\,43''$.

Hence $$C=180^0-48^0\,11'\,23''-58^0\,24'\,43''=73^0\,23'\,54''.$$

25. As in Art. 238 we have

$$\sin\left(45^0 - \frac{B}{2}\right) = \sqrt{\frac{1-\sin B}{2}} = \sqrt{\frac{1}{2}\left(1 - \frac{3}{6953}\right)}$$

$$= \sqrt{\frac{1}{2} \times \frac{6950}{6953}} = \sqrt{\frac{3475}{6953}};$$

therefore $$L\sin\left(45^0 - \frac{B}{2}\right) = 10 + \log\sqrt{\frac{3475}{6953}}$$

$$= 10 + \frac{1}{2}\log(3475 - \log 6953)$$

$$= 10 - \frac{1}{2}(\cdot 3012174) = 9{\cdot}8493913.$$

$$\begin{array}{r} 9{\cdot}8493913 \\ 9{\cdot}8493902 \\ \hline \cdot 0000011 \end{array} \qquad \cdot 0000021 : \cdot 0000011 : 1'' : x'';$$

this gives $x = \cdot 5$; therefore $45^0 - \frac{B}{2} = 44^0\,59'\,15''{\cdot}5$; therefore $\frac{B}{2} = 44''{\cdot}5$; therefore $B = 1'\,29''$.

26. Let $$b = 100, \quad c = 80;$$

$$\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{1}{9}\cot 30^0 = \frac{\sqrt{3}}{9} = 3^{-\frac{3}{2}};$$

therefore $$L\tan\frac{1}{2}(B-C) = 10 + \log 3^{-\frac{3}{2}} = 10 - \frac{3}{2}\log 3 = 9{\cdot}28432;$$

therefore $$\frac{1}{2}(B-C) = 10^0\,53'\,36''.$$

And $\frac{1}{2}(B+C) = 60^0$; therefore $B = 70^0\,53'\,36''$, $C = 49^0\,6'\,24''$.

27. Let $$b = 5, \quad c = 3;$$

$$\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{1}{4}\cot 60^0 = \frac{1}{4\sqrt{3}} = \frac{1}{\sqrt{(48)}};$$

therefore $$L\tan\frac{1}{2}(B-C) = 10 + \log\frac{1}{\sqrt{(48)}} = 10 - \frac{1}{2}\log 48$$

$$= 10 - \frac{1}{2}(1{\cdot}6812412) = 9{\cdot}1593794.$$

$$\begin{array}{r} 9{\cdot}1593794 \\ 9{\cdot}1586706 \\ \hline \cdot 0007088 \end{array} \qquad \cdot 0008940 : \cdot 0007088 :: 60'' : x'';$$

this gives $x = 48$; therefore $\frac{1}{2}(B-C) = 8^0\,12'\,48''$.

And $\frac{1}{2}(B+C)=30^0$; therefore $B=38^0\,12'\,48''$ and $C=21^0\,47'\,12''$.

28. Let $ABCD$ denote the square base, P the vertex. From P suppose a perpendicular PQ drawn to the ground, and from Q draw QR perpendicular to AB. Let ϕ denote the required inclination; then $\tan\phi=\frac{PQ}{QR}$.

Now $QR=100$. Also $PQ^2+QR^2=PR^2$, and $PR^2+AR^2=AP^2$; thus

$$PQ^2=PR^2-QR^2=AP^2-AR^2-QR^2=(150)^2-(100)^2-(100)^2=2500;$$

therefore $PQ=50$. Therefore $\tan\phi=\frac{50}{100}=\frac{1}{2}$.

Hence $$L\tan\phi=10+\log\frac{1}{2}=10-\log 2=9{\cdot}69897.$$

$$\begin{array}{r} 9{\cdot}69900 \\ 9{\cdot}69868 \\ \hline {\cdot}00032 \end{array} \qquad \begin{array}{r} 9{\cdot}69897 \\ 9{\cdot}69868 \\ \hline {\cdot}00029 \end{array} \qquad {\cdot}00032 : {\cdot}00029 :: 60'' : x'';$$

this gives $x=54$; therefore $\phi=26^0\,33'\,54''$.

29. $\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{1\frac{2}{9}-1}{1\frac{2}{9}+1}\cot 30^0=\frac{1}{10}\cot 30^0=\frac{\sqrt{3}}{10}$;

therefore $L\tan\frac{1}{2}(A-B)=10+\log\frac{\sqrt{3}}{10}=10+\frac{1}{2}\log 3-1=9{\cdot}2385606$.

Now $L\cot 9^0\,49'=10{\cdot}7618797$; and as $\tan\theta\times\cot\theta=1$, we have

$$\log\tan\theta+\log\cot\theta=0;$$

therefore $$L\tan\theta-10+L\cot\theta-10=0;$$

therefore $$L\tan\theta=20-L\cot\theta.$$

Thus $$L\tan 9^0\,49'=9{\cdot}2381203.$$

$$\begin{array}{r} 9{\cdot}2385606 \\ 9{\cdot}2381203 \\ \hline {\cdot}0004403 \end{array} \qquad {\cdot}0007514 : {\cdot}0004403 :: 60'' : x'';$$

this gives $x=35$; therefore $\frac{1}{2}(A-B)=9^0\,49'\,35''$.

And $\frac{1}{2}(A+B)=60^0$; therefore $A=69^0\,49'\,35''$, $B=50^0\,10'\,25''$.

30. $\sin C=\frac{c}{a}\sin A$; $L\sin C=L\sin A+\log c-\log a$

$$=L\sin A+\log 3-\log 2$$

$$=9{\cdot}5228787+{\cdot}4771213-\log 2=10-\log 2;$$

therefore $$\log \sin C = -\log 2 = \log \frac{1}{2};$$

therefore $\sin C = \frac{1}{2}$; therefore $C = 30^0$ or 150^0.

31. Let c be the given base and let h denote the given height. With the left-hand diagram of Art. 214 we have

$$\cot B = \frac{BD}{h} \text{ and } \cot C = \frac{CD}{h};$$

therefore $$\cot B + \cot C = \frac{BD + CD}{h} = \frac{c}{h} \quad \text{.......................... (1).}$$

Also $B - C$ is supposed given, so that $\cot(B - C)$ is known; call it m: thus

$$\frac{\cot B - \cot C}{1 + \cot B \cot C} = m \quad \text{.................................. (2).}$$

From (1) and (2) we can find $\cot A$ and $\cot B$.

32. Let a, b, c denote the sides; and l, m, n the perpendiculars on them respectively from the opposite angles. Then $al = bm = cn$; for each of these expressions denotes twice the area of the triangle. Hence the sides a, b, c are respectively inversely proportional to l, m, n. Thus the *ratios* of the sides are known; and hence the angles of the triangle can be calculated by Art. 217. Then the actual lengths of the sides can be found; for $l = c \sin B$, and l and B are known, so that c can be found; and then a and b can be deduced as the ratios of the sides are already known.

CHAPTER XV.

1. Take the diagram of Art. 240. The angle $PBC = 60^0$, the angle $PAC = 30^0$; therefore the angle $APB = 30^0$. Also $AB = 40$ feet.

Since the angle PAB = the angle APB, we have $BP = AB = 40$. Then

$$PC = BP \sin 60^0 = 40 \frac{\sqrt{3}}{2} = 20\sqrt{3};$$

and $$BC = BP \cos 60^0 = 40 \cdot \frac{1}{2} = 20.$$

2. Let AC produced through C meet the horizontal plane which contains B at D. Then the angle $ABD = 60^0$, and the angle $CBD = 30^0$; therefore the angle $ABC = 30^0$. The angle $ACB = 135^0$. Hence

$$\text{the angle } BAC = 180^0 - 30^0 - 135^0 = 15^0,$$

$$\frac{AB}{BC} = \frac{\sin ACB}{\sin BAC} = \frac{\sin 135^0}{\sin 15^0} = \frac{1}{\sqrt{2}} \div \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{2}{\sqrt{3} - 1};$$

therefore $$AB = \frac{2 \times 1760}{\sqrt{3} - 1} \text{ yards.}$$

The height of the mountain $= AB \sin 60^0 = AB \frac{\sqrt{3}}{2}$

$$= \frac{1760\sqrt{3}}{\sqrt{3}-1} = \frac{1760\sqrt{3}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{1760\sqrt{3}(\sqrt{3}+1)}{2} = 880(3+\sqrt{3}).$$

3. Let h denote the height of the tower in yards; then

$$\frac{h}{100} = \tan 30^0 = \frac{1}{\sqrt{3}}; \quad \text{therefore } h = \frac{100}{\sqrt{3}}.$$

4. Let h denote the height of the tower, x the distance of the foot from A, and y the distance of the foot from B. Then

$$x = h \cot 30^0, \text{ and } y = h \cot 18^0.$$

But $y^2 - x^2 = a^2$; therefore $h^2(\cot^2 18^0 - \cot^2 30^0) = a^2$;

therefore $$h^2 \left\{ \frac{10+2\sqrt{5}}{(\sqrt{5}-1)^2} - 3 \right\} = a^2;$$

therefore $$h^2 \left\{ \frac{5+\sqrt{5}}{3-\sqrt{5}} - 3 \right\} = a^2;$$

therefore $$4h^2(\sqrt{5}-1) = a^2(3-\sqrt{5});$$

therefore $$h^2 = \frac{3-\sqrt{5}}{4(\sqrt{5}-1)} a^2 = \frac{(3-\sqrt{5})(3+\sqrt{5})}{4(\sqrt{5}-1)(3+\sqrt{5})}$$

$$= \frac{4a^2}{4(2+2\sqrt{5})} = \frac{a^2}{2+2\sqrt{5}}.$$

5. Let A denote the eye of the spectator, and B the centre of the balloon. The angle α is formed by straight lines drawn from A in the vertical plane which contains B, so as to touch the balloon. Hence

$$\frac{r}{AB} = \sin \frac{\alpha}{2}; \quad \text{therefore } AB = r \operatorname{cosec} \frac{\alpha}{2}.$$

And the height of the centre of the balloon $= AB \sin \beta = r \sin \beta \operatorname{cosec} \frac{\alpha}{2}$.

6. Let O denote the station which is in the same straight line as A and B; let P be the station which is in the same straight line as A and C; and let Q be the station which is in the same straight line as B and C. Then O, P, and Q are in a straight line which is at right angles to AB. Let $OP = p$, $OQ = q$; let $APO = \alpha$, and $BQO = \beta$. Then $OA = p \tan \alpha$, and $OB = q \tan \beta$. Thus $AB = q \tan \beta - p \tan \alpha$. And the angles of the triangle ABC are known; for $ABQ = \frac{\pi}{2} - \beta$, and $OAP = \frac{\pi}{2} - \alpha$. Hence AC and BC can be found.

7. The tangent of the angle which AB subtends at E is $\frac{AB}{AE}$; and the tangent of the angle which CD subtends at E is $\frac{CD}{CE}$; therefore $\frac{AB}{AE}=\frac{CD}{CE}$;

therefore $CE=\frac{AE \,.\, CD}{AB}$; therefore $CE^2=\frac{AE^2 \,.\, CD^2}{AB^2}$;

therefore $$CA^2+AE^2=\frac{AE^2 \,.\, CD^2}{AB^2}:$$

but $CA^2=AB^2$; therefore $AE^2=\frac{AB^4}{CD^2-AB^2}$.

Again. $\cos DEA=\frac{EA}{ED}$; and $\cos BEC=\frac{EB^2+EC^2-BC^2}{2\,EB \,.\, EC}$

$$=\frac{EA^2+AB^2+EA^2+AC^2-(AB^2+AC^2)}{2EB \,.\, EC}=\frac{EA^2}{EB \,.\, EC}.$$

But by hypothesis the cosine of BEA is equal to the cosine of DEC, that is $\frac{EA}{EB}=\frac{EC}{ED}$; therefore $EA \,.\, ED=EB \,.\, EC$; therefore $\frac{EA^2}{EB \,.\, EC}=\frac{EA}{ED}$.

8. Let A be the top of the flag staff, B the top of the tower, C the foot of the tower, E the eye. From E draw a perpendicular ED on the horizontal plane which contains C. Then the angle BEC is to be equal to the angle BEA.

Now $$\frac{\sin BEC}{\sin EBC}=\frac{BC}{EC}, \text{ and } \frac{\sin BEA}{\sin EBA}=\frac{AB}{AE};$$

therefore $$\frac{BC}{EC}=\frac{AB}{AE}.$$

This coincides with Euclid VI. 3.

Let $CD=x$; then $EC=\sqrt{(h^2+x^2)}$, $EA=\sqrt{(a+b-h)^2+x^2}$;

therefore $$\frac{b}{\sqrt{h^2+x^2}}=\frac{a}{\sqrt{(a+b-h)^2+x^2}};$$

therefore $$\{(a+b-h)^2+x^2\}\,b^2=(h^2+x^2)\,a^2;$$

therefore $$x^2=\frac{b^2\,(a+b-h)^2-h^2a^2}{a^2-b^2};$$

therefore $$EC^2=\frac{h^2\,(a^2-b^2)+b^2\,(a+b-h)^2-h^2a^2}{a^2-b^2}$$

$$=\frac{b^2\,\{(a+b-h)^2-h^2\}}{a^2-b^2}=\frac{b^2\,(a+b)\,(a+b-2h)}{a^2-b^2}=\frac{b^2\,(a+b-2h)}{a-b};$$

therefore $$EC=b\left(\frac{a+b-2h}{a-b}\right)^{\frac{1}{2}}.$$

9. Let P denote the top of the tower; from P draw PQ perpendicular to the ground; then $PQ=h$. Let x denote the distance of Q from the base of the tower; $x+a$ is the distance of Q from one point of observation, and $x+b$ is the distance of Q from the other point of observation.

Thus $$\cot\theta=\frac{x}{h},\quad \cot\alpha=\frac{x+a}{h},\quad \cot\beta=\frac{x+b}{h};$$

therefore $$h\cot\alpha=x+a,\quad h\cot\beta=x+b;$$

therefore $$h=\frac{b-a}{\cot\beta-\cot\alpha};$$

and $$x=h\cot\alpha-a=\frac{(b-a)\cot\alpha}{\cot\beta-\cot\alpha}-a=\frac{b\cot\alpha-a\cot\beta}{\cot\beta-\cot\alpha}.$$

Thus $$\tan\theta=\frac{h}{x}=\frac{b-a}{b\cot\alpha-a\cot\beta}.$$

10. Let x denote the required height; and suppose θ the angle which the tower subtends: then

$$x=b\tan\theta,\quad x+a=b\tan(\theta+\gamma);$$

therefore $$x+a=\frac{b(\tan\theta+\tan\gamma)}{1-\tan\theta\tan\gamma}=\frac{x+b\tan\gamma}{1-\frac{x\tan\gamma}{b}};$$

thus we have a quadratic equation for finding x.

11. Let x denote the breadth of the river in feet; let α denote the angle subtended by the column, and β the angle subtended by the column and statue.

Thus $$\tan\alpha=\frac{200}{x},\ \text{and}\ \tan\beta=\frac{230}{x};$$

therefore $$\tan(\beta-\alpha)=\frac{\frac{230}{x}-\frac{200}{x}}{1+\frac{200\times230}{x^2}}=\frac{30x}{x^2+46000}.$$

But, by hypothesis, $\tan(\beta-\alpha)=\frac{6}{x}$; therefore

$$\frac{6}{x}=\frac{30x}{x^2+46000};\ \text{therefore}\ x^2+46000=5x^2;$$

$$\text{therefore}\ x^2=11500;\ \text{therefore}\ x=10\sqrt{115}.$$

12. The part of the house above the horizontal straight line subtends an angle of 60^0, and thus the height of the top of the house above the window is $30\tan 60^0$ feet. The part of the house below the horizontal straight line subtends an angle of 30^0, and thus the depth of the foot of the house below the window is $30\tan 30^0$ feet. Hence the distance from the foot of the house to the top of the house in feet

$$=30(\tan 60^0+\tan 30^0)=30\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)=\frac{4}{\sqrt{3}}30=40\sqrt{3}.$$

13. Let x denote the height of each chimney in feet, and y the distance between them. The distance of the first point of observation from the nearer chimney is $x \cot 60^0$, and therefore the distance of the second point of observation is $\sqrt{(80)^2 + x^2 \cot^2 60^0}$. Thus

$$\frac{x}{\sqrt{(80)^2 + x^2 \cot^2 60^0}} = \tan 45^0 = 1;$$

therefore $x^2 = (80)^2 + x^2 \cot^2 60^0 = (80)^2 + \frac{x^2}{3}$; therefore $2x^2 = 3\,(80)^2$;

therefore $x^2 = 6\,(40)^2$; therefore $x = 40\sqrt{6}$.

The distance of the first point of observation from the further chimney is $y - x\cot 60^0$, and therefore the distance of the second point of observation is $\sqrt{(80)^2 + (y - x\cot 60^0)^2}$. Thus

$$\frac{x}{\sqrt{(80)^2 + (y - x\cot 60^0)^2}} = \tan 30^0 = \frac{1}{\sqrt{3}};$$

therefore $3x^2 = (80)^2 + (y - x\cot 60^0)^2$; therefore $14\,(40)^2 = (y - x\cot 60^0)^2$;

therefore $y = x\cot 60^0 + 40\sqrt{14} = 40\,(\sqrt{2} + \sqrt{14})$.

14. Let P be the object, PQ the perpendicular from P on the horizontal plane which contains A, B, and C.

Let $PQ = x$, $CQ = y$. Suppose θ the angle PAQ, then $PBQ = 2\theta$, and $PCQ = 3\theta$. Thus

$$\tan\theta = \frac{x}{y+a+b}, \qquad \tan 2\theta = \frac{x}{y+b}, \qquad \tan 3\theta = \frac{x}{y};$$

therefore $y + a + b = x\cot\theta, \quad y + b = x\cot 2\theta, \quad y = x\cot 3\theta;$

therefore $a = x(\cot\theta - \cot 2\theta), \quad b = x(\cot 2\theta - \cot 3\theta);$

therefore $$a = x\left(\frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{\sin 2\theta}\right) = \frac{x\sin(2\theta - \theta)}{\sin\theta\sin 2\theta} = \frac{x}{\sin 2\theta},$$

and $$b = x\left(\frac{\cos 2\theta}{\sin 2\theta} - \frac{\cos 3\theta}{\sin 3\theta}\right) = \frac{x\sin(3\theta - 2\theta)}{\sin 2\theta\sin 3\theta} = \frac{x\sin\theta}{\sin 2\theta\sin 3\theta}$$

$$= \frac{x}{\sin 2\theta\,(3 - 4\sin^2\theta)}.$$

Thus $\sin 2\theta = \frac{x}{a}$, and $3 - 4\sin^2\theta = \frac{x}{b\sin 2\theta} = \frac{a}{b}$;

therefore $3 - 2(1 - \cos 2\theta) = \frac{a}{b}$; therefore $\cos 2\theta = \frac{1}{2}\left(\frac{a}{b} - 1\right)$.

Hence $$\frac{x^2}{a^2} + \frac{1}{4}\left(\frac{a}{b} - 1\right)^2 = 1;$$

therefore $$\frac{x^2}{a^2} = 1 - \frac{1}{4}\left(\frac{a}{b} - 1\right)^2 = \frac{4b^2 - (a-b)^2}{4b^2} = \frac{3b^2 + 2ab - a^2}{4b^2} = \frac{(3b-a)(a+b)}{4b^2};$$

therefore $$x = \frac{a}{2b}\sqrt{(a+b)(3b-a)}.$$

If $\tan\theta=\frac{1}{3}$, then $\sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}=\frac{\frac{2}{3}}{1+\frac{1}{9}}=\frac{3}{5}$, and $\sin 2\theta=\frac{x}{a}$; thus

$$\frac{3}{5}=\frac{\sqrt{(a+b)(3b-a)}}{2b};$$

therefore $$36b^2=25(a+b)(3b-a)=25(3b^2+2ab-a^2);$$

therefore $$39b^2+50ab-25a^2=0;$$

therefore $$(13b-5a)(3b+5a)=0;\quad \text{therefore } 13b-5a=0.$$

15. Let x denote the height of the tower in yards; then the distance from A to the foot of the tower is $x\cot 15^0$. The observer moves so that the tower always subtends the same angle, hence he must describe the arc of a circle having its centre at the foot of the tower; and as the bearing of the tower changes from north to north-east he must describe one-eighth part of the circumference; therefore

$$\frac{2\pi x\cot 15^0}{8}=100;\quad \text{therefore } x=\frac{400\tan 15^0}{\pi}.$$

16. Let A denote the object which is further from the road, B that which is nearer to the road, C the point where AB subtends the greatest angle, D the second point of observation.

It is known that the point C is such that a circle described round A, B, and C will touch CD at C; see *Notes on Euclid*, page 308. Therefore the angle BCD is equal to the angle BAC; denote it by θ. Then the angle $ABC=\theta+\beta$, and also $=\pi-\theta-\alpha$; therefore $2\theta=\pi-\alpha-\beta$.

Now $\frac{BC}{CD}=\frac{\sin\beta}{\sin(\theta+\beta)}$, therefore $BC=\frac{c\sin\beta}{\sin(\theta+\beta)}$, and $\frac{AB}{BC}=\frac{\sin\alpha}{\sin\theta}$;

therefore $$AB=\frac{c\sin\alpha\sin\beta}{\sin\theta\sin(\theta+\beta)}=\frac{2c\sin\alpha\sin\beta}{\cos\beta-\cos(2\theta+\beta)}$$

$$=\frac{2c\sin\alpha\sin\beta}{\cos\beta-\cos(\pi-\alpha)}=\frac{2c\sin\alpha\sin\beta}{\cos\beta+\cos\alpha}.$$

17. Let A denote the fortress, B the first position of the ship, C the second; produce BC through C to any point E. Then the angle $ABC=22\frac{1}{2}^0$, and the angle $ACE=67\frac{1}{2}^0$; therefore the angle $BAC=45^0$.

$$\frac{AB}{BC}=\frac{\sin ACB}{\sin BAC}=\frac{\sin(180^0-67\frac{1}{2}^0)}{\sin 45^0}=\frac{\sqrt{(2+\sqrt{2})}}{2}\div\frac{1}{\sqrt{2}}=\frac{\sqrt{(2+\sqrt{2})}}{\sqrt{2}}=\sqrt{\frac{2+\sqrt{2}}{2}};$$

therefore $$AB=4\sqrt{\frac{2+\sqrt{2}}{2}}=\sqrt{(16+8\sqrt{2})}.$$

And $\frac{AC}{BC}=\frac{\sin ABC}{\sin BAC}=\frac{\sin 22\frac{1}{2}^0}{\sin 45^0}=\frac{\sqrt{(2-\sqrt{2})}}{2}\div\frac{1}{\sqrt{2}}=\sqrt{\frac{2-\sqrt{2}}{2}}$;

therefore $AC=4\sqrt{\frac{2-\sqrt{2}}{2}}=\sqrt{(16-8\sqrt{2})}$. See Example VII. 18.

18. Let P be the first position of the ship, A the nearer lighthouse, and B the further lighthouse; let Q be the second position of the ship. Then the angle $BQP=45^0$, and the angle $AQP=22\frac{1}{2}^0$; therefore the angle $QAP=67\frac{1}{2}^0$.

$$\frac{BQ}{BA}=\frac{\sin BAQ}{\sin BQA}=\frac{\sin(180^0-67\frac{1}{2}^0)}{\sin 22\frac{1}{2}^0}=\frac{\sin 67\frac{1}{2}^0}{\sin 22\frac{1}{2}^0}=\frac{\cos 22\frac{1}{2}^0}{\sin 22\frac{1}{2}^0}=\cot 22\frac{1}{2}^0$$

$=\sqrt{2}+1$, by Example VII. 18; therefore $BQ=8(\sqrt{2}+1)$.

And $$PQ=BQ\sin 45^0=\frac{8(\sqrt{2}+1)}{\sqrt{2}}=8+4\sqrt{2}.$$

19. Let A denote the top of the lighthouse, P the top of the mast at the first observation, C the centre of the earth. Draw a straight line from P to A and let it touch the earth at B.

Let r denote the radius of the earth in feet; then

$PB=\sqrt{PC^2-BC^2}=\sqrt{(r+64)^2-r^2}=\sqrt{2r\times 64+(64)^2}=\sqrt{2r\times 64}$ very nearly,

for r is very large compared with $(64)^2$.

In precisely the same manner if Q denote the deck of the ship at the second observation, $QB=\sqrt{2r\times 16}$.

Now, since PCB is a very small angle, we may, by the principle that $\tan\theta$ is nearly equal to θ when θ is very small, consider the straight line PB to be equal to the arc which measures the distance of the ship from B at the first observation; and similarly we may consider QB to be equal to the arc which measures the distance of the ship from B at the second observation. Thus between the two observations the ship has sailed over $\sqrt{2r\times 64}-\sqrt{2r\times 16}$, that is, $4\sqrt{2r}$; that is, in half-an-hour it has sailed over $4\sqrt{8000\times 5280}$ feet, so that the rate is $8\sqrt{8000\times 5280}$ feet per hour, that is, $\frac{8\sqrt{8000\times 5280}}{5280}$ miles per hour, that is, $8\sqrt{\frac{800}{528}}$ miles per hour, that is, $8\sqrt{\frac{50}{33}}$ miles per hour; this is very nearly $8\sqrt{\frac{3}{2}}$ miles per hour.

20. Let A denote the summit of the mountain, B the base, BC the first part of the path, CA the second part. From A draw AE perpendicular to the horizontal plane which contains B; then $AE=n$.

The following are the angles:

$$BAE=\frac{\pi}{2}-\gamma,\qquad CBE=\alpha,\qquad CAE=\frac{\pi}{2}-\beta;$$

therefore $BAC=\beta-\gamma,\quad ABC=\gamma-\alpha,\quad ACB=\pi+\alpha-\beta.$

$$AB = \frac{AE}{\sin\gamma} = \frac{n}{\sin\gamma},$$

$$\frac{BC}{AB} = \frac{\sin BAC}{\sin ACB} = \frac{\sin(\beta-\gamma)}{\sin(\beta-\alpha)},$$

$$\frac{AC}{AB} = \frac{\sin ABC}{\sin ACB} = \frac{\sin(\gamma-\alpha)}{\sin(\beta-\alpha)};$$

therefore $$\frac{BC+AC}{AB} = \frac{\sin(\beta-\gamma)+\sin(\gamma-\alpha)}{\sin(\beta-\alpha)}$$

$$= \frac{2\sin\dfrac{\beta-\alpha}{2}\cos\left(\dfrac{\alpha+\beta}{2}-\gamma\right)}{\sin(\beta-\alpha)} = \frac{\cos\left(\dfrac{\alpha+\beta}{2}-\gamma\right)}{\cos\dfrac{\beta-\alpha}{2}};$$

therefore $$BC+AC = \frac{n}{\sin\gamma}\cdot\frac{\cos\left(\dfrac{\alpha+\beta}{2}-\gamma\right)}{\cos\dfrac{\beta-\alpha}{2}}.$$

21. Let O denote the foot of the object; and let A, B, and C denote the three points of observation. Let x denote the height of the object; then $OA = x\cot\alpha$, $OB = x\cot\beta$, and $OC = x\cot\gamma$.

From the triangle AOC we have

$$x^2\cot^2\alpha = x^2\cot^2\gamma + a^2 - 2ax\cot\gamma\cos ACO,$$

and from the triangle BOC we have

$$x^2\cot^2\beta = x^2\cot^2\gamma + b^2 - 2bx\cot\gamma\cos BCO.$$

Multiply the first equation by b and the second by a, and add; thus

$$x^2(b\cot^2\alpha + a\cot^2\beta) = ab(a+b) + x^2(a+b)\cot^2\gamma;$$

therefore

$$x^2 = \frac{ab(a+b)\sin^2\alpha\sin^2\beta\sin^2\gamma}{a(\cos^2\beta\sin^2\gamma - \cos^2\gamma\sin^2\beta)\sin^2\alpha + b(\cos^2\alpha\sin^2\gamma - \cos^2\gamma\sin^2\alpha)\sin^2\beta}$$

$$= \frac{ab(a+b)\sin^2\alpha\sin^2\beta\sin^2\gamma}{a(\sin^2\gamma - \sin^2\beta)\sin^2\alpha + b(\sin^2\gamma - \sin^2\alpha)\sin^2\beta}.$$

22. Let P be the summit of the lower hill, Q the summit of the higher hill; let A be the first point of observation, B the second, C the third. From P and Q draw PM and QN, respectively perpendicular to the horizontal plane which contains A, B, and C.

Let $PM = h$, and $QN = h'$.

Then $AM = h\cot\alpha$, and $AM = AB + BC + CM = c + 1 + h\cot\beta$;

therefore $h\cot\alpha = c + 1 + h\cot\beta$; therefore $h(\cot\alpha - \cot\beta) = c+1$;

therefore $$h = \frac{(c+1)\sin\alpha\sin\beta}{\sin(\beta-\alpha)}.$$

And by similar triangles

$$\frac{h'}{h} = \frac{QN}{PM} = \frac{BN}{BM} = \frac{AN - AB}{AM - AB} = \frac{h'\cot\alpha' - c}{h\cot\alpha - c};$$

thus since h is known we can find h'.

23. Let h be the height of the tower in feet, α the altitude of the sun at noon. The distance between the foot of the tower and the edge of the moat is $h\cot 60^0$; hence the distance between the foot of the tower and the extremity of the shadow is $h\cot 60^0 + 45$ at noon, and $h\cot 60^0 + 120$ when the sun is due west. The directions of the shadows include a right angle;

therefore $$(h\cot 60^0 + 45)^2 + (h\cot 60^0 + 120)^2 = (375)^2.$$

Therefore $$\frac{2h^2}{3} + \frac{2h}{\sqrt{3}} . 165 + (45)^2 + (120)^2 = (375)^2;$$

therefore $$\frac{2h^2}{3} + \frac{2h}{\sqrt{3}} 165 = 124200.$$

By solving this quadratic in the usual way we obtain $h = 180\sqrt{3}$ or $-345\sqrt{3}$; only the positive value is applicable. Then $h\cot\alpha - h\cot 60^0 = 45$;

therefore $$\cot\alpha = \cot 60^0 + \frac{45}{h} = \frac{1}{\sqrt{3}} + \frac{45}{180\sqrt{3}} = \frac{1}{\sqrt{3}} + \frac{1}{4\sqrt{3}} = \frac{5}{4\sqrt{3}};$$

therefore $$\tan\alpha = \frac{4\sqrt{3}}{5}.$$

24. Let P denote the top of the tower. Then ϕ is the angle between PA and CA produced through A. Thus the angle $CPA = \phi - \alpha$, and the angle $DPC = \alpha - \beta$.

Then $$\frac{DC}{CP} = \frac{\sin DPC}{\sin CDP} = \frac{\sin(\alpha-\beta)}{\sin\beta},$$

$$\frac{CA}{CP} = \frac{\sin CPA}{\sin CAP} = \frac{\sin(\phi-\alpha)}{\sin(\pi-\phi)} = \frac{\sin(\phi-\alpha)}{\sin\phi};$$

therefore $$\frac{\sin(\alpha-\beta)}{\sin\beta} = \frac{\sin(\phi-\alpha)}{\sin\phi};$$

therefore $$\sin\alpha\cot\beta - \cos\alpha = \cos\alpha - \sin\alpha\cot\phi;$$

therefore $$\cot\phi = 2\cot\alpha - \cot\beta.$$

Now let α', β', and ϕ' correspond to observations made in another straight line $AC'D'$; then $\cot\phi' = 2\cot\alpha' - \cot\beta'$; but by supposition $2\tan\beta' = \tan\alpha'$; therefore $\cot\phi' = 0$; therefore $\phi' = \frac{\pi}{2}$. Thus $AC'D'$ makes a right angle with AP; and therefore $AC'D'$ is a horizontal straight line.

From D draw DM perpendicular to AD', and from M draw MN perpendicular to the horizontal plane which contains D; and produce PA through A to meet the same plane at Q.

Then $$\sin\theta = \frac{MN}{MD}, \ \sin\gamma = \frac{DM}{DA}, \ \cos\phi = \frac{AQ}{AD} = \frac{MN}{AD};$$

therefore $$\cos\phi = \sin\theta \sin\gamma.$$

25. $$\sin B = \frac{b}{a}\sin A = \frac{3\sqrt{3}}{3}\sin A = \sqrt{3}\sin A;$$

thus if $A = \frac{\pi}{6}$ we have $\sin B = \sqrt{3} \cdot \frac{1}{2}$; therefore $B = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

Suppose however that $A = \frac{\pi}{6} \pm h$, where h is the circular measure of $2''$; then $\sin B = \sqrt{3}\sin\left(\frac{\pi}{6} \pm h\right) = \sqrt{3}\left\{\sin\frac{\pi}{6} \pm h\cos\frac{\pi}{6}\right\}$ very nearly. Suppose that $B = \frac{\pi}{3} \pm k$; then approximately $\sin\frac{\pi}{3} \pm k\cos\frac{\pi}{3} = \sqrt{3}\left\{\sin\frac{\pi}{6} \pm h\cos\frac{\pi}{6}\right\}$;

therefore $\pm k\cos\frac{\pi}{3} = \pm h\sqrt{3}\cos\frac{\pi}{6}$; therefore $k = h\sqrt{3} \cdot \cot\frac{\pi}{6} = 3h$.

In the same way if $B = \frac{2\pi}{3} \pm k$ we find that $k = -3h$. Thus the approximate error in B is 6 seconds.

26. Let A and B be the two objects on the opposite bank of the river; and suppose P and Q two points on this bank, such that $PQ = AB$; and let P correspond to A and Q to B, so that AP is equal and parallel to BQ. Let AQ and BP intersect at C.

Then α = the angle APB, and β = the angle AQB = the angle PAQ.

Therefore $$\frac{PC}{PA} = \frac{\sin\beta}{\sin(\alpha+\beta)}, \ \frac{AC}{PA} = \frac{\sin\alpha}{\sin(\alpha+\beta)};$$

but $PQ^2 = PC^2 + QC^2 - 2PC \cdot QC \cdot \cos PCQ$; and $QC = AC$;

therefore $$c^2 = PA^2\frac{\sin^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}.$$

Let x denote the breadth of the river; then the area of the triangle $APB = \frac{1}{2}xc$; and this area is also equal to

$$\frac{1}{2}PA \cdot PB\sin APB = PA \cdot PC\sin\alpha = \frac{PA^2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$

$$= \frac{c^2\sin\alpha\sin\beta\sin(\alpha+\beta)}{\sin^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta\cos(\alpha+\beta)}.$$

27. Let AB denote a side of the fort, C the position due south of A; let D be the second position, so that $CD=a$, and the angle $ACD=90^\circ$; also A, B, D, and C will lie on the circumference of a circle. Let E be the third position, so that E is on CD produced through D, and $DE=b$; and the angle BED is a right angle.

Let ϕ be the angle between AB produced through B and CE produced through E. Then $a+b=AB\cos\phi$; therefore $AB=(a+b)\sec\phi$.

And $\qquad BE=EC\tan BCE$, and $=ED\tan BDE$;

therefore $\quad (a+b)\tan(90^\circ-\alpha)=b\tan BAC=b\tan(90^\circ-\phi)$. (Euclid III. 22.)

28. From A draw AM perpendicular to the horizontal plane which contains the road, and draw AN perpendicular to the straight road.

Then
$$\sin\alpha=\frac{AM}{AB}, \text{ and } \sin\beta=\frac{AN}{AB}.$$

Similarly from A' draw $A'M'$ perpendicular to the horizontal plane, and $A'N'$ perpendicular to the straight road.

Then
$$\sin\alpha'=\frac{A'M'}{A'B'} \text{ and } \sin\beta'=\frac{A'N'}{A'B'}.$$

Thus we have to shew that $\dfrac{AM}{AB}\cdot\dfrac{A'N'}{A'B'}=\dfrac{A'M'}{A'B'}\cdot\dfrac{AN}{AB}$,

or that $\qquad AM\cdot A'N'=A'M'\cdot AN$, or that $\dfrac{AM}{AN}=\dfrac{A'M'}{A'N'}$.

Now if A is just hidden by A' at some point of the road, the straight line $A'A$ if produced through A will intersect the road; and then AA' and the road will lie in one plane; the sine of the inclination of this plane to the horizontal plane is expressed by $\dfrac{AM}{AN}$ and also by $\dfrac{A'M'}{A'N'}$; so that these are equal.

29. There are two cases. Suppose the angles APQ and BPR to be on the same sides of AP and BP respectively; then the angle $QPR=$ the angle $APB=\alpha$. Suppose the angles APQ and BPR not to fall on the same sides of AP and BP respectively; then the angle $RPQ=\pi-\alpha$. In both cases $AB=RQ$; for the diameter of the circle which goes round the five points A, B, P, Q, and $R=\dfrac{AB}{\sin APB}$ and also $=\dfrac{RQ}{\sin RPQ}$.

In the former case $AB=\sqrt{(a^2+b^2-2ab\cos\alpha)}$, and in the latter case
$$AB=\sqrt{(a^2+b^2+2ab\cos\alpha)}.$$

30. Suppose both straight lines OC and $O'C$ to fall within the angle ACB. Let $AC=a$, $ACO=\phi$; then from the triangles ACO and BCO we get
$$OC=\frac{a\sin(\phi+\alpha)}{\sin\alpha} \text{ and } OC=\frac{a\cos(\phi-\beta)}{\sin\beta};$$

therefore $$OC \sin \alpha = a (\sin \phi \cos \alpha + \cos \phi \sin \alpha),$$
$$OC \sin \beta = a (\cos \phi \cos \beta + \sin \phi \sin \beta).$$

Hence $$a \sin \phi = \frac{OC \sin \alpha (\cos \beta - \sin \beta)}{\cos (\alpha + \beta)},$$
$$a \cos \phi = \frac{OC \sin \beta (\cos \alpha - \sin \alpha)}{\cos (\alpha + \beta)}.$$

Square and add; thus

$$a^2 \cos^2 (\alpha + \beta) = OC^2 \{\sin^2 \alpha (\cos \beta - \sin \beta)^2 + \sin^2 \beta (\cos \alpha - \sin \alpha)^2\}$$
$$= OC^2 \{\sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta \sin (\alpha + \beta)\}.$$

Thus $$OC^2 = \frac{a^2 \cos^2 (\alpha + \beta)}{\sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta \sin (\alpha + \beta)}.$$

A similar expression will be found for $O'C^2$ in terms of α' and β'. Then $O'C^2 = OC^2 + d^2$. This finds a; and then $AB = a\sqrt{2}$.

Similarly the problem may be solved for any other positions of the lines OC, $O'C$.

31. Let α denote the Sun's altitude; then $\tan \alpha = \frac{150}{75} = 2$;

therefore $$L \tan \alpha = 10 + \log 2 = 10 \cdot 3010300.$$

$$\begin{array}{r} 10 \cdot 3013153 \\ 10 \cdot 3009994 \\ \hline \cdot 0003159 \end{array} \qquad \begin{array}{r} 10 \cdot 3010300 \\ 10 \cdot 3009994 \\ \hline \cdot 0000306 \end{array} \qquad \cdot 0003159 : \cdot 0000306 :: 60'' : x'';$$

this gives $x = 6$; therefore $\alpha = 63^\circ 26' 6''$.

32. Take the diagram of Art. 240. Here $PBC = 55^\circ$, $PAC = 48^\circ$, $AB = 30$ feet.

$$\frac{PB}{BA} = \frac{\sin PAB}{\sin APB} = \frac{\sin 48^\circ}{\sin 7^\circ}; \text{ therefore } PB = \frac{30 \sin 48^\circ}{\sin 7^\circ};$$

$$BC = BP \cos PBC = BP \cos 55^\circ = BP \sin 35^\circ = \frac{30 \sin 48^\circ \sin 35^\circ}{\sin 7^\circ};$$

$$\log BC = \log 30 + L \sin 48^\circ - 10 + L \sin 35^\circ - 10 - (L \sin 7^\circ - 10)$$
$$= 1 \cdot 47712 + 9 \cdot 87107 + 9 \cdot 75859 - 9 \cdot 08589 - 10 = 2 \cdot 02089;$$

therefore $BC = 104 \cdot 93$.

33. Let α denote the inclination; then $\sin \alpha = \frac{100}{196} = \frac{100}{4 \times 49}$;

therefore $L \sin \alpha = 10 + \log 100 - \log (4 \times 49) = 12 - 2 \log 2 - 2 \log 7 = 9 \cdot 70774$.

$$\begin{array}{r} 9 \cdot 70782 \\ 9 \cdot 70761 \\ \hline \cdot 00021 \end{array} \qquad \begin{array}{r} 9 \cdot 70774 \\ 9 \cdot 70761 \\ \hline \cdot 00013 \end{array} \qquad \cdot 00021 : \cdot 00013 :: 60'' : x'';$$

this gives $x = 37$; therefore $\alpha = 30^\circ 40' 37''$.

34. Let A be the point of intersection of the hills, B the point of observation on the hill, P the top of the object, C the bottom. Produce PB through B to meet at D the horizontal straight line which contains A; produce DA through A to any point E. Then $AB=64$ feet; and the following are the given angles:

$$CAE=60^0,\ BAD=40^0,\ BDA=70^0,\ BPC=90^0-70^0=20^0.$$

Therefore $$BAC=80^0,\ BCA=20^0,\ PBC=30^0.$$

$$\frac{BC}{BA}=\frac{\sin BAC}{\sin BCA}=\frac{\sin 80^0}{\sin 20^0},$$

$$\frac{PC}{BC}=\frac{\sin PBC}{\sin BPC}=\frac{\sin 30^0}{\sin 20^0};\ \text{therefore}\ \frac{PC}{BA}=\frac{\sin 80^0 \sin 30^0}{\sin^2 20^0};$$

therefore $$PC=\frac{64\sin 80^0 \sin 30^0}{\sin^2 20^0}=\frac{64\sin 40^0\cos 40^0}{\sin^2 20^0}=\frac{128\cos 40^0}{\tan 20^0};$$

therefore $$\log PC=7\log 2+L\cos 40^0-L\tan 20^0=2{\cdot}4303981;$$

therefore $$PC=269{\cdot}40031.$$

35. Let A, B, C be the three successive positions of the ship from which the observations are made; let P, Q, R be the corresponding positions of the other ship.

Then the straight line ABC is parallel to the straight line PQR; also $AB=BC$, and $PQ=QR$.

Let θ be the angle between the North direction and the direction of sailing.

From B draw a straight line parallel to AP, meeting PQ at M; then

$$\frac{QM}{BM}=\frac{\sin QBM}{\sin BQM}=\frac{\sin QBM}{\sin BQP}=\frac{\sin(\beta-\alpha)}{\sin(\theta-\beta)}.$$

Again, from C draw a straight line parallel to AP, meeting QR at N; then

$$\frac{RN}{CN}=\frac{\sin RCN}{\sin CRN}=\frac{\sin RCN}{\sin CRP}=\frac{\sin(\gamma-\alpha)}{\sin(\theta-\gamma)}.$$

But $BM=CN$; and $RN=2QM$, for RN is the difference of the paths of the ships in two hours, and QM is the difference in one hour.

Therefore $$\frac{2\sin(\beta-\alpha)}{\sin(\theta-\beta)}=\frac{\sin(\gamma-\alpha)}{\sin(\theta-\gamma)};$$

therefore $$2\sin(\theta-\gamma)\sin(\beta-\alpha)=\sin(\gamma-\alpha)\sin(\theta-\beta),$$

therefore

$$2(\sin\theta\cos\gamma-\cos\theta\sin\gamma)\sin(\beta-\alpha)=(\sin\theta\cos\beta-\cos\theta\sin\beta)\sin(\gamma-\alpha).$$

Divide by $\cos\theta$; thus we obtain the value of $\tan\theta$.

36. If $\alpha+\beta+C=\pi$, then $x+y=\pi$; therefore $\sin x=\sin y$; therefore $\tan\phi=1$.

We might as in Art. 242 say that

$$\frac{\sin x - \sin y}{\sin x + \sin y} = \tan\left(\phi - \frac{\pi}{4}\right),$$

that is
$$\frac{2\sin\frac{x-y}{2}\cos\frac{x+y}{2}}{2\sin\frac{x+y}{2}\cos\frac{x-y}{2}} = \tan\left(\phi - \frac{\pi}{4}\right).$$

But as $\cos\frac{x+y}{2}$ is now zero we cannot divide both numerator and denominator of the last fraction by it, and thus we cannot proceed further. In fact in this case a circle would go round P, A, C, and B, and P may be at any point of the arc between A and B.

XVI.

1. Here $s=36$, $s-a=12$, $s-b=6$, $s-c=18$.

The area of the triangle $=\sqrt{36\times 12\times 6\times 18} = \sqrt{36\times 36\times 36} = 6^3 = 216$.

2. The third angle of the triangle $=180^0 - 60^0 = 120^0$.

One of the containing sides $=\frac{10\times\sin 15^0}{\sin 120^0}$, and the other $=\frac{10\times\sin 45^0}{\sin 120^0}$.

Hence the area

$$=\frac{1}{2}\frac{(10)^2\sin 15^0\sin 45^0}{\sin^2 120^0}\sin 120^0 = \frac{50\sin 15^0\sin 45^0}{\sin 120^0} = \frac{50(\sqrt{3}-1)}{2\sqrt{2}}\times\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{3}}$$

$$=\frac{25(\sqrt{3}-1)}{\sqrt{3}}.$$

3. The area of the triangle $=\frac{1}{2}\times 3\times 12\times\sin 30^0 = \frac{36}{4} = 9$.

Let x denote the hypotenuse of the right-angled triangle; then each of the equal sides is $\frac{x}{\sqrt{2}}$, and the area is $\frac{1}{2}\times\left(\frac{x}{\sqrt{2}}\right)^2$, that is $\frac{x^2}{4}$. Hence $\frac{x^2}{4}=9$; therefore $x^2=36$; therefore $x=6$.

4. From the angle C of a triangle draw a perpendicular CD to the side AB, or AB produced.

First suppose A and B acute, so that D is between A and B. Then $CD = b\sin A$, $AD = b\cos A$; thus the area of $ACD = \frac{1}{2}b^2\sin A\cos A = \frac{1}{4}b^2\sin 2A$.

Similarly the area of $BCD = \frac{1}{2}a^2\sin B\cos B = \frac{1}{4}a^2\sin 2B$.

Therefore the area of the whole triangle $=\frac{1}{4}(a^2\sin 2B + b^2\sin 2A)$.

Next suppose the angle B obtuse, so that D falls on AB produced through D. Then as before the area of $ACD = \frac{1}{4} b^2 \sin 2A$. And the area of $CBD = \frac{1}{2} a^2 \sin (180^0 - B) \cos (180^0 - B) = \frac{a^2}{4} \sin (360^0 - 2B)$.

Therefore the area of ABC

$$= \frac{1}{4} \{b^2 \sin 2A - a^2 \sin (360^0 - 2B)\} = \frac{1}{4} (b^2 \sin 2A + a^2 \sin 2B).$$

This mode of solution shews the geometrical meaning of the two parts of the expression. We may proceed more briefly thus:

$$\frac{1}{4} (a^2 \sin 2B + b^2 \sin 2A)$$

$$= \frac{1}{2} (a \sin B\, a \cos B + b \sin A\; b \cos A) = \frac{1}{2} a \sin B (a \cos B + b \cos A), \text{ by Art. 214,}$$

$$= \frac{1}{2} ac \sin B, \text{ by Art. 216, } = \text{the area of the triangle by Art. 247.}$$

5. $$\frac{a^2 - b^2}{2} \frac{\sin A \sin B}{\sin (A - B)} = \frac{\sin A \sin B}{2 \sin (A - B)} \left\{ \frac{c^2 \sin^2 A}{\sin^2 C} - \frac{c^2 \sin^2 B}{\sin^2 C} \right\}$$

$$= \frac{c^2 \sin A \sin B (\sin^2 A - \sin^2 B)}{2 \sin (A - B) \sin^2 C} = \frac{c^2 \sin A \sin B \sin (A + B) \sin (A - B)}{2 \sin (A - B) \sin^2 C}$$

$$= \frac{c^2 \sin A \sin B}{2 \sin C} = \text{area of the triangle, by Art. 247.}$$

6. $$\frac{2abc}{a+b+c} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{2abc}{2s} \sqrt{\frac{s(s-a)}{bc}} \times \sqrt{\frac{s(s-b)}{ac}} \times \sqrt{\frac{s(s-c)}{ab}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} = S = \text{the area of the triangle.}$$

7. Here $2s = gh(k^2 + l^2) + kl(g^2 + h^2) + (hk + gl)(hl - gk)$

$$= gh(k^2 + l^2) + kl(g^2 + h^2) + h^2kl + ghl^2 - hgk^2 - klg^2$$

$$= 2ghl^2 + 2h^2kl; \text{ therefore } s = ghl^2 + klh^2 = hl(gl + hk);$$

therefore $$s - a = klh^2 - ghk^2 = kh(lh - gk),$$

$$s - b = ghl^2 - klg^2 = gl(lh - gk),$$

$$s - c = hgk^2 + klg^2 = kg(hk + lg).$$

Thus $$s(s-a)(s-b)(s-c) = g^2h^2k^2l^2 (lh - gk)^2 (hk + lg)^2;$$

therefore $$S = ghkl(lh - gk)(hk + lg).$$

Therefore by Art. 218 the sines of the angles of the triangle are rational quantities; and by Art. 215 the cosines of the angles are rational quantities.

8. Let a, b, c be in Arithmetical Progression; then $2b=a+c$. Thus the perimeter$=3b$, and the side of an equilateral triangle of equal perimeter is b.

Thus $\sqrt{s(s-a)(s-b)(s-c)}=\frac{3}{5}\cdot\frac{1}{2}b^2\sin 60^0=\frac{3\sqrt{3}}{20}b^2$,

that is $$\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}=\frac{3\sqrt{3}}{5}b^2,$$

that is $$\sqrt{3b^2(b+c-a)(a+b-c)}=\frac{3\sqrt{3}}{5}b^2;$$

therefore $$\sqrt{(b+c-a)(b+a-c)}=\frac{3}{5}b,$$

therefore $$\sqrt{\frac{3c-a}{2}\times\frac{3a-c}{2}}=\frac{3}{10}(a+c);$$

therefore $$(3c-a)(3a-c)=\frac{9}{25}(a+c)^2;$$

therefore $$10ac-3(a^2+c^2)=\frac{9}{25}(a^2+2ac+c^2);$$

therefore $$84(a^2+c^2)-232ac=0,$$

therefore $$21\left(\frac{a^2}{c^2}+1\right)=58\frac{a}{c}.$$

By solving this quadratic in the usual way we obtain $\frac{a}{c}=\frac{7}{3}$ or $\frac{3}{7}$.

Take $\frac{a}{c}=\frac{7}{3}$; thus a, b, and c are proportional to 7, 5, and 3 respectively.

Then $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{5^2+3^2-7^2}{2\times 5\times 3}=-\frac{1}{2}$; therefore $A=120^0$.

9. Let A, B, C, D, E be five consecutive angles of the hexagon; draw AC, BD, CE; let AC and BD intersect at P, and let BD and CE intersect at Q. Then PQ is the side of the second regular hexagon.

The angle DBC is half of the angle which DC would subtend at the centre of the circle circumscribing the regular hexagon, and is therefore $\frac{\pi}{6}$. Similarly the angle ACB is $\frac{\pi}{6}$.

Then $\frac{PC}{BC}=\frac{\sin\frac{\pi}{6}}{\sin\left(\pi-\frac{2\pi}{6}\right)}=\frac{\sin\frac{\pi}{6}}{\sin\frac{2\pi}{6}}=\frac{1}{2\cos\frac{\pi}{6}}$; therefore $PC=\frac{BC}{2\cos\frac{\pi}{6}}$.

And $$PQ=2PC\sin\frac{1}{2}PCQ=2PC\sin\frac{\pi}{6}=BC\tan\frac{\pi}{6}.$$

Thus $PQ = \frac{BC}{\sqrt{3}}$. And the areas of similar polygons are as the squares of their homologous sides; so that if S denote the area of the first hexagon the area of the second is $\frac{S}{3}$. In like manner the area of the next hexagon is $\frac{1}{3}$ of $\frac{S}{3}$, that is $\frac{S}{9}$; and so on. Hence the sum of the areas of all the derived figures is $\frac{S}{3} + \frac{S}{9} + \frac{S}{27} + \ldots$, that is $\frac{1}{3}\frac{S}{1-\frac{1}{3}}$, that is $\frac{S}{2}$.

10. Suppose that the original figure instead of being a hexagon is a regular polygon of n sides. Proceed as before and we have

$$\frac{PC}{BC} = \frac{\sin\frac{\pi}{n}}{\sin\left(\pi - \frac{2\pi}{n}\right)} = \frac{\sin\frac{\pi}{n}}{\sin\frac{2\pi}{n}} = \frac{1}{2\cos\frac{\pi}{n}}.$$

Then $$PQ = 2PC\sin\frac{1}{2}PCQ;$$

and the angle $PCQ = (n-4)\frac{\pi}{n}$; therefore $PQ = 2PC\sin(n-4)\frac{\pi}{2n}$

$$= 2PC\sin\left(\frac{\pi}{2} - \frac{2\pi}{n}\right) = 2PC\cos\frac{2\pi}{n} = BC\frac{\cos\frac{2\pi}{n}}{\cos\frac{\pi}{n}}.$$

Thus the area of the second polygon is $\frac{S\cos^2\frac{2\pi}{n}}{\cos^2\frac{\pi}{n}}$;

and $\Sigma = S\{m + m^2 + m^3 + \ldots\}$ where m stands for $\frac{\cos^2\frac{2\pi}{n}}{\cos^2\frac{\pi}{n}}$;

thus $$\Sigma = \frac{Sm}{1-m} = \frac{S\cos^2\frac{2\pi}{n}}{\cos^2\frac{\pi}{n} - \cos^2\frac{2\pi}{n}} = \frac{S\cos^2\frac{2\pi}{n}}{\sin^2\frac{2\pi}{n} - \sin^2\frac{\pi}{n}} = \frac{S\cos^2\frac{2\pi}{n}}{\sin\frac{3\pi}{n}\sin\frac{\pi}{n}}.$$

If $n=3$ this becomes infinite; for $\sin\pi = 0$; in this case the original figure is a triangle, and the second figure is the *same* triangle, and so on: thus the sum of the areas is infinite.

If $n=4$ the expression vanishes; for $\cos\frac{2\pi}{4}=0$; in this case the original figure is a square, and the second figure is only a point, and so on: thus the sum of the areas is zero.

11. Let ABC denote the right-angled isosceles triangle where C is the right angle. Let F be the middle point of AB; let D be on BC, and E on AC, such that DE is parallel to AB, and the triangle DEF is equilateral.

Then the angle $DEC=45^0$, and the angle $DEF=60^0$; therefore the angle $AEF=75^0$. Now $\frac{FE}{FA}=\frac{\sin FAE}{\sin FEA}=\frac{\sin 45^0}{\sin 75^0}$;

therefore $FE=\frac{FA\sin 45^0}{\sin 75^0}=\frac{a}{\sqrt{2}}\cdot\frac{\sin 45^0}{\cos 15^0}=\frac{a}{2}\frac{1}{\cos 15^0}=\frac{a\sin 15^0}{2\cos 15^0\sin 15^0}$

$=\frac{a\sin 15^0}{\sin 30^0}=2a\sin 15^0$. Therefore the area of the equilateral triangle

$$=\frac{1}{2}(2a\sin 15^0)^2\sin 60^0=2a^2\sin^2 15^0\sin 60^0.$$

12. $r_1r_2r_3=\frac{S^3}{(s-a)(s-b)(s-c)}$, $\quad r^3=\frac{S^3}{s^3}$;

therefore $$\frac{r_1r_2r_3}{r^3}=\frac{s^3}{(s-a)(s-b)(s-c)};$$

and $$\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}=\frac{s(s-a)}{(s-b)(s-c)}\times\frac{s(s-b)}{(s-a)(s-c)}\times\frac{s(s-c)}{(s-a)(s-b)}$$

$$=\frac{s^3}{(s-a)(s-b)(s-c)};$$

therefore $$\frac{r_1r_2r_3}{r^3}=\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}.$$

13. Let $C'A'$ intersect AB at E and CB at F.

The angle $A'FC$ is equal to the sum of the angles $FC'C$ and FCC', that is to the sum of the angles $A'AC$ and FCC', that is to $\frac{1}{2}A+\frac{1}{2}C$; the angle $BCA'=$ the angle $BAA'=\frac{1}{2}A$.

Thus $$\frac{FA'}{CA'}=\frac{\sin\frac{1}{2}A}{\sin\frac{1}{2}(A+C)}=\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}B}.$$

Let R be the radius of the circle; then

$$A'C=2R\sin\frac{1}{2}A;\text{ therefore } FA'=\frac{2R\sin^2\frac{A}{2}}{\cos\frac{B}{2}}.$$

In the same manner $$EC' = \frac{2R\sin^2\frac{C}{2}}{\cos\frac{B}{2}}.$$

And $$A'C' = 2R\sin\frac{1}{2}(A+C) = 2R\cos\frac{B}{2}.$$

Therefore $$EF = 2R\cos\frac{B}{2} - \frac{2R\left(\sin^2\frac{A}{2}+\sin^2\frac{C}{2}\right)}{\cos\frac{B}{2}}$$

$$= \frac{2R}{\cos\frac{B}{2}}\left\{\cos^2\frac{B}{2} - \sin^2\frac{A}{2} - \sin^2\frac{C}{2}\right\} = \frac{R}{\cos\frac{B}{2}}\{1+\cos B - (1-\cos A) - (1-\cos C)\}$$

$$= \frac{R}{\cos\frac{B}{2}}(\cos A + \cos B + \cos C - 1) = \frac{2R}{\cos\frac{B}{2}} \times 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$ by Art. 114.

14. Let a denote one side of the right-angled triangle, and $a+h$ the other side; then the hypotenuse $=\sqrt{a^2+(a+h)^2}=\sqrt{h^2+2a(a+h)}$.

But S = half the product of the sides $=\frac{1}{2}a(a+h)$; therefore $4S = 2a(a+h)$. Thus the hypotenuse $=\sqrt{(h^2+4S)}$; and the hypotenuse is a diameter of the circumscribing circle.

15. $$r = \frac{S}{s}, \quad R = \frac{abc}{4S}; \quad \text{therefore } \frac{R}{r} = \frac{sabc}{4S^2}.$$

Now $s=7$, $s-a=4$, $s-b=2$, $s-c=1$; therefore $S=\sqrt{7\times 4\times 2}$; thus

$$\frac{R}{r} = \frac{7\times 3\times 5\times 6}{4\times 7\times 4\times 2} = \frac{45}{16}.$$

16. The angle ABO = the angle $BAO = \frac{\pi}{2} - C$; and therefore the angle $BOD = \pi - 2C$; the angle $OBD = \frac{\pi}{2} - A$;

therefore the angle $BDO = 2C + A - \frac{\pi}{2} = A + C + B + C - B - \frac{\pi}{2} = \frac{\pi}{2} + C - B$.

Thus $$\frac{DO}{BO} = \frac{\sin DBO}{\sin BDO} = \frac{\sin\left(\frac{\pi}{2} - A\right)}{\sin\left(\frac{\pi}{2} + C - B\right)} = \frac{\cos A}{\cos(C-B)},$$

and $BO = AO$; therefore $DO\cos(B-C) = AO\cos A$.

17. Take the diagram of Art. 248; draw FD, DE, and EF.

The angle $FDB = \frac{1}{2}(\pi - B)$, the angle $EDC = \frac{1}{2}(\pi - C)$; therefore the angle $FDE = \frac{1}{2}(B+C)$. Similarly the angle $DEF = \frac{1}{2}(C+A)$, and the angle $EFD = \frac{1}{2}(A+B)$.

Suppose A, B, C in ascending order of magnitude; then

$$\frac{1}{2}(A+B), \quad \frac{1}{2}(A+C), \quad \frac{1}{2}(B+C),$$

are in ascending order of magnitude; and

$$\frac{1}{2}(B+C) - \frac{1}{2}(A+B) = \frac{1}{2}(C-A).$$

Thus the difference between the greatest and least angles of the first derived triangle is *half* the difference between the greatest and least angles of the original triangle. In like manner the difference between the greatest and least angles of the second derived triangle is *half* the difference between the greatest and least angles of the first derived triangle, and therefore a *fourth* of the difference between the greatest and least angles of the original triangle. Proceeding in this way we see that the triangles thus formed ultimately become equilateral.

18. $a \cot A + b \cot B + c \cot C = \frac{a}{\sin A} \cos A + \frac{b}{\sin B} \cos B + \frac{c}{\sin C} \cos C$

$$= 2R(\cos A + \cos B + \cos C) = 2R + 2R(\cos A + \cos B + \cos C - 1)$$

$$= 2R + 8R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}, \text{ by Art. 114,}$$

$$= 2R + 8R \cdot \sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{(s-a)(s-c)}{ac}} \times \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= 2R + \frac{8R}{abc}\frac{S^2}{s} = 2R + \frac{2S}{s} = 2R + 2r.$$

19. From A draw AD perpendicular to BC, and produce AD to meet the circumference of the circle at L.

Then the angle ALB = the angle $ACB = C$;

$$\alpha = DL = BD \cot ALB = BD \cot C$$

$$= \frac{c \cos B \cos C}{\sin C} = \frac{a \cos B \cos C}{\sin A};$$

therefore $\frac{a}{\alpha} = \frac{\sin A}{\cos B \cos C} = \frac{\sin(B+C)}{\cos B \cos C} = \tan B + \tan C.$

Similarly $\frac{b}{\beta} = \tan A + \tan C,$ and $\frac{c}{\gamma} = \tan C + \tan A.$

Therefore $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C).$

20. The area of the inscribed circle is to the area of the triangle as πr^2 is to S, that is, as π is to $\frac{S}{r^2}$. Thus we have to shew that

$$\frac{S}{r^2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

Now

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \times \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \times \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{s\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{s^2}{S} = S \times \frac{s^2}{S^2} = \frac{S}{r^2}.$$

21. Let the triangle constructed on BC have its vertex at L, let that constructed on CA have its vertex at M, and that constructed on AB have its vertex at N.

Take the diagram of Art. 252. The triangle CLB will be equal to the triangle COB in all respects; therefore the angle BCL = the angle $OCB = \frac{\pi}{2} - A$.

In the same manner the angle $ACM = \frac{\pi}{2} - B$;

therefore the angle $LCM = \frac{\pi}{2} - A + \frac{\pi}{2} - B + C = 2C.$

Then $(LM)^2 = R^2 + R^2 - 2R^2\cos 2C = 2R^2(1 - \cos 2C) = 4R^2\sin^2 C$;

therefore $LM = 2R\sin C = c.$

In a similar manner we find that $MN = a$, and $NL = b$. Thus the triangle LMN is in all respects equal to the triangle ABC.

22. $a\cos A + b\cos B + c\cos C = 2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C$

$= R(\sin 2A + \sin 2B + \sin 2C) = 4R\sin A\sin B\sin C,$ by Art. 114.

23. $$OD^2 = R^2 \cos^2 A = \frac{a^2}{4\sin^2 A} \cos^2 A = \frac{a^2}{4} \cot^2 A,$$

$$OE^2 = R^2 \cos^2 B = \frac{b^2}{4\sin^2 B} \cos^2 B = \frac{b^2}{4} \cot^2 B,$$

$$OF^2 = R^2 \cos^2 C = \frac{a^2}{4\sin^2 C} \cos^2 C = \frac{c^2}{4} \cot^2 C;$$

therefore $4(OD^2 + OE^2 + OF^2) = a^2 \cot^2 A + b^2 \cot^2 B + c^2 \cot^2 C.$

24. Take the diagram of Art. 248. The circle which is to be drawn will have its centre, and its point of contact with the circle already drawn, on the straight line OA. Thus the length of $OA = r + r_a + r_a \operatorname{cosec} \frac{A}{2}$; and this distance also $= r \operatorname{cosec} \frac{A}{2}$; therefore

$$r_a\left(1 + \operatorname{cosec} \frac{A}{2}\right) = r\left(\operatorname{cosec} \frac{A}{2} - 1\right);$$

therefore $$r_a = \frac{r\left(1 - \sin\frac{A}{2}\right)}{1 + \sin\frac{A}{2}} = \frac{r\left(\cos\frac{A}{4} - \sin\frac{A}{4}\right)^2}{\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)^2}.$$

25. By Example 24 we have

$$r_a r_b = \frac{r^2\left(\cos\frac{A}{4} - \sin\frac{A}{4}\right)^2\left(\cos\frac{B}{4} - \sin\frac{B}{4}\right)^2}{\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)^2\left(\cos\frac{B}{4} + \sin\frac{B}{4}\right)^2};$$

therefore $$\sqrt{(r_a r_b)} = \frac{r\left(\cos\frac{A}{4} - \sin\frac{A}{4}\right)\left(\cos\frac{B}{4} - \sin\frac{B}{4}\right)}{\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)\left(\cos\frac{B}{4} + \sin\frac{B}{4}\right)}$$

$$= \frac{r\left(\cos\frac{A}{4} - \sin\frac{A}{4}\right)\left(\cos\frac{B}{4} - \sin\frac{B}{4}\right)\left(\cos\frac{C}{4} + \sin\frac{C}{4}\right)}{\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)\left(\cos\frac{B}{4} + \sin\frac{B}{4}\right)\left(\cos\frac{C}{4} + \sin\frac{C}{4}\right)}$$

$$= \frac{r\cos\frac{A+\pi}{4}\cos\frac{B+\pi}{4}\cos\frac{C-\pi}{4}}{\cos\frac{A-\pi}{4}\cos\frac{B-\pi}{4}\cos\frac{C-\pi}{4}}$$

$$= \frac{r\left(\cos\frac{A}{2} + \cos\frac{B}{2} - \cos\frac{C}{2}\right)}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}},$$ by Examples VIII. 20 and 21.

Similar expressions can be found for $\sqrt{(r_b r_c)}$ and $\sqrt{(r_c r_a)}$; and the sum of the three expressions $=r$.

26. Suppose A to be acute; then $AB'=c\cos A$, $AC'=b\cos A$, and

$$(B'C')^2 = AB'^2 + AC'^2 - 2AC'.AB'\cos A$$
$$= \cos^2 A\,(c^2+b^2-2bc\cos A)$$
$$= a^2\cos^2 A;$$

therefore $\qquad B'C' = a\cos A = 2R\sin A\cos A = R\sin 2A.$

If A is obtuse we find that $AB'=c\cos(\pi - A)$, $AC'=b\cos(\pi - A)$, and $(B'C')^2 = a^2\cos^2 A$ as before.

27. Let P denote the point of intersection of AD and BE.

Then since PEC and PDC are right angles a circle would go round $PECD$; therefore the angle $PDE=$ the angle $PCE=\frac{\pi}{2}-A$. Similarly $PDF=\frac{\pi}{2}-A$. Therefore $FDE=\pi-2A$.

$$R_1 = \frac{FE}{2\sin FDE} = \frac{FE}{2\sin 2A} = \frac{R\sin 2A}{2\sin 2A}, \text{ by Example 26, } = \frac{1}{2}R.$$

And $$r_1 = \frac{\text{area of } FDE}{\text{semiperimeter of } FDE} = \frac{FD.ED\sin 2A}{R(\sin 2A + \sin 2B + \sin 2C)}$$
$$= \frac{R\sin 2A\sin 2B\sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \text{ by Example 26,}$$
$$= \frac{R\sin 2A\sin 2B\sin 2C}{4\sin A\sin B\sin C}, \text{ by Art. 114, } = 2R\cos A\cos B\cos C.$$

28. $$\frac{rr_1}{r_2 r_3} = \frac{S^2}{s(s-a)} \div \frac{S^2}{(s-b)(s-c)} = \frac{(s-b)(s-c)}{s(s-a)} = \tan^2\frac{A}{2}.$$

29. $$\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{(\pi r^2)}} = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{r} = \frac{1}{\sqrt{\pi}}\cdot\frac{s}{S}.$$

Similarly $\frac{1}{\sqrt{A_1}} = \frac{1}{\sqrt{\pi}}\cdot\frac{s-a}{S}$, $\quad \frac{1}{\sqrt{A_2}} = \frac{1}{\sqrt{\pi}}\cdot\frac{s-b}{S}$, $\quad \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{\pi}}\cdot\frac{s-c}{S}$;

therefore

$$\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{\pi}}\left(\frac{s-a}{S} + \frac{s-b}{S} + \frac{s-c}{S}\right) = \frac{1}{\sqrt{\pi}}\cdot\frac{3s-a-b-c}{S} = \frac{1}{\sqrt{\pi}}\cdot\frac{s}{S}.$$

30. Suppose a, b, c to be in Arithmetical Progression; so that $2b=a+c$.

The perpendicular on the mean side from the opposite angle

$$= a\sin C = \frac{ab\sin C}{b} = \frac{2S}{b}.$$

The radius of the circle which touches the mean side and the other two sides produced $= \frac{S}{s-b} = \frac{2S}{a+c-b} = \frac{2S}{b}$.

The radius of the inscribed circle $= \frac{S}{s} = \frac{2S}{a+b+c} = \frac{2S}{3b}$.

The first and the second of these are each three times the third.

31. Let O denote the centre of the inscribed circle, and P the centre of the escribed circle which is opposite to the angle A. Then O and P are both on the straight line which bisects the angle A.

$$\text{The angle } OBP = \frac{1}{2}B + \frac{1}{2}(\pi - B) = \frac{\pi}{2}.$$

Thus
$$OP = \frac{OB}{\cos BOP} = \frac{OB}{\cos\frac{1}{2}(A+B)} = \frac{OB}{\sin\frac{C}{2}};$$

and
$$OB = \frac{AB\sin\frac{1}{2}A}{\sin\left(\pi - \frac{A}{2} - \frac{B}{2}\right)} = \frac{c\sin\frac{1}{2}A}{\sin\frac{A+B}{2}} = \frac{c\sin\frac{1}{2}A}{\cos\frac{C}{2}}.$$

$$\text{Therefore } OP = \frac{c\sin\frac{1}{2}A}{\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{2c\sin\frac{1}{2}A}{\sin C} = \frac{2a\sin\frac{1}{2}A}{\sin A} = \frac{a}{\cos\frac{A}{2}}.$$

32. Let a_1, b_1, c_1 be the sides of one triangle, S_1 its area; let a_2, b_2, c_2 be the sides of the other triangle, S_2 its area.

Then, by hypothesis, $\frac{S_1}{b_1+c_1-a_1} = \frac{S_2}{a_2+c_2-b_2}$; therefore

$$\frac{S_1}{S_2} = \frac{b_1+c_1-a_1}{a_2+c_2-b_2} = \frac{a_1\frac{\sin B}{\sin A} + \frac{a_1\sin C}{\sin A} - a_1}{a_2 + \frac{a_2\sin C}{\sin A} - \frac{a_2\sin B}{\sin A}}$$

$$= \frac{a_1}{a_2}\cdot\frac{\sin B + \sin C - \sin A}{\sin A + \sin C - \sin B}.$$

But the areas of similar triangles are as the squares of their homologous sides; thus $\frac{S_1}{S_2} = \frac{a_1^2}{a_2^2}$; therefore, finally,

$$\frac{a_1}{a_2} = \frac{\sin B + \sin C - \sin A}{\sin A + \sin C - \sin B}.$$

33. The points O_2, O_3, and A are in a straight line; similarly O_3, O_1, and B are in a straight line; and O_1, O_2, and C are in a straight line.

The triangle $O_1O_2O_3$ consists of four parts; namely ABC, O_1BC, O_2CA, and O_3AB.

The area of $O_1BC=\frac{1}{2}ar_1=\frac{aS}{2(s-a)}=\frac{aS}{b+c-a}$.

Similar expressions hold for the areas of O_2CA and O_3AB.

Thus the area of $O_1O_2O_3=S\left(1+\frac{a}{b+c-a}+\frac{b}{a+c-b}+\frac{c}{a+b-c}\right)$.

34. Here we have another expression for the area of the triangle considered in the preceding solution.

We have $$\frac{O_1C}{BC}=\frac{\sin\frac{1}{2}(\pi-B)}{\sin\frac{1}{2}(B+C)}=\frac{\cos\frac{1}{2}B}{\cos\frac{1}{2}A};$$

therefore $$O_1C=\frac{a\cos\frac{1}{2}B}{\cos\frac{1}{2}A}=\frac{2R\sin A\cos\frac{1}{2}B}{\cos\frac{1}{2}A}=4R\sin\frac{1}{2}A\cos\frac{1}{2}B.$$

Similarly $$O_2C=4R\sin\frac{1}{2}B\cos\frac{1}{2}A;$$

therefore $$O_1O_2=4R\left(\sin\frac{1}{2}A\cos\frac{1}{2}B+\sin\frac{1}{2}B\cos\frac{1}{2}A\right)$$

$$=4R\sin\frac{1}{2}(A+B)=4R\cos\frac{1}{2}C.$$

In like manner $O_1O_3=4R\cos\frac{1}{2}B$.

Then area of $O_1O_2O_3=\frac{1}{2}O_1O_2\times O_1O_3\times\sin O_2O_1O_3$

$$=8R^2\cos\frac{1}{2}C\cos\frac{1}{2}B\sin\frac{1}{2}(B+C)=8R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C.$$

$$=8R^2\sqrt{\frac{s(s-a)}{bc}}\times\sqrt{\frac{s(s-b)}{ac}}\times\sqrt{\frac{s(s-c)}{ab}}$$

$$=\frac{8R^2sS}{abc}=\frac{abcs}{2S}=\frac{abc}{2r}.$$

35. We have

$$r' = \frac{\text{area of } A'B'C'}{\text{semiperimeter of } A'B'C'} = \frac{\frac{abc}{2r}}{2R\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)},$$

by the solution of the preceding Example,

$$= \frac{abc}{4Rr\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)} = \frac{S}{r\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)}$$

$$= \frac{s}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}.$$

Also $$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \frac{s^2}{S} = \frac{s}{r};$$

(see the solution of Example 20),

therefore $$r' = \frac{r\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}.$$

36. We have $r' = \dfrac{\text{area of } A'B'C'}{s'}$; therefore

$$r's' = \text{area of } A'B'C' = \frac{abc}{2r}, \text{ by Example 34.}$$

Again, $$rs = \text{area of } ABC = S.$$

Therefore $$\frac{rs}{r's'} = \frac{2rS}{abc} = \frac{2S^2}{abcs}.$$

And $$2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc} \times \frac{(s-c)(s-a)}{ac} \times \frac{(s-a)(s-b)}{ab}}$$

$$= \frac{2S^2}{abcs} = \frac{rs}{r's'}.$$

37. We have $$\alpha = r\operatorname{cosec}\frac{A}{2}, \qquad \alpha_1 = r_1\operatorname{cosec}\frac{A}{2},$$

$$\beta = r\operatorname{cosec}\frac{B}{2}, \qquad \beta_1 = r_2\operatorname{cosec}\frac{B}{2},$$

$$\gamma = r\operatorname{cosec}\frac{C}{2}, \qquad \gamma_1 = r_3\operatorname{cosec}\frac{C}{2};$$

therefore $\alpha\beta\gamma\,\alpha_1\beta_1\gamma_1 = r^3 r_1 r_2 r_3 \operatorname{cosec}^2\frac{A}{2}\operatorname{cosec}^2\frac{B}{2}\operatorname{cosec}^2\frac{C}{2}$

$$=\frac{S^3}{s^3}\times\frac{S^3}{(s-a)(s-b)(s-c)}\times\frac{bc}{(s-c)(s-b)}\times\frac{ca}{(s-a)(s-c)}\times\frac{ab}{(s-a)(s-b)}$$

$$=\frac{S^6a^2b^2c^2}{S^6}=a^2b^2c^2.$$

38. $$\frac{bc}{\alpha_1^2}+\frac{ca}{\beta_1^2}+\frac{ab}{\gamma_1^2}=\frac{bc\sin^2\frac{A}{2}}{r_1^2}+\frac{ca\sin^2\frac{B}{2}}{r_2^2}+\frac{ab\sin^2\frac{C}{2}}{r_3^2}$$

$$=\frac{1}{s^2}\left\{bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}\right\},\text{ by Art. 251,}$$

$$=\frac{1}{s^2}\{s(s-a)+s(s-b)+s(s-c)\}=\frac{1}{s}(s-a+s-b+s-c)$$

$$=\frac{1}{s}(3s-a-b-c)=1.$$

39. $$\alpha^2\left(\frac{1}{c}-\frac{1}{b}\right)+\beta^2\left(\frac{1}{a}-\frac{1}{c}\right)+\gamma^2\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$=\frac{r^2(b-c)}{bc\sin^2\frac{A}{2}}+\frac{r^2(c-a)}{ca\sin^2\frac{B}{2}}+\frac{r^2(a-b)}{ab\sin^2\frac{C}{2}}$$

$$=\frac{r^2}{abc}\left\{\frac{a(b-c)}{\sin^2\frac{A}{2}}+\frac{b(c-a)}{\sin^2\frac{B}{2}}+\frac{c(a-b)}{\sin^2\frac{C}{2}}\right\}$$

$$=\frac{4Rr^2}{abc}\left\{(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}\right\}$$

$=0$, by Example XIII. 29.

40. $$\frac{b-c}{a\alpha_1^2}+\frac{c-a}{b\beta_1^2}+\frac{a-b}{c\gamma_1^2}$$

$$=\frac{b-c}{ar_1^2}\sin^2\frac{A}{2}+\frac{c-a}{br_2^2}\sin^2\frac{B}{2}+\frac{a-b}{cr_3^2}\sin^2\frac{C}{2}$$

$$=\frac{1}{s^2}\left\{\frac{b-c}{a}\cos^2\frac{A}{2}+\frac{c-a}{b}\cos^2\frac{B}{2}+\frac{a-b}{c}\cos^2\frac{C}{2}\right\},\text{ by Art. 251,}$$

$$=\frac{1}{4Rs^2}\left\{(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}\right\}.$$

$=0$, by Example XIII. 29.

41. In order that it may be possible to inscribe a circle within a quadrilateral the sum of one pair of opposite sides must be equal to the sum of the other pair. Now if we take the point O of the diagram of Art. 248, we see that the condition is satisfied for $OFAE$, $OECD$, and $ODBF$; since $OE+AF=OF+AE$, and so on. We have then to shew that no other point but O can be taken.

Take any other point P; from it draw PM perpendicular to AC and PN perpendicular to AB. The centre of a circle inscribed within $PMAN$ must be on the straight line which bisects the angle A; and also on the straight line which bisects the angle NPM; but unless P is on AO, the latter straight line will be *parallel* to AO, the former straight line, and therefore cannot meet it. Thus P must be on AO; similarly it must be on BO and on CO.

Then take the circle inscribed in $OFAE$, and draw perpendiculars from the centre on the sides of the quadrilateral. Thus we have

$$\rho_1 (AF+FO+OE+EA)=\text{twice the area of } OFAE;$$

therefore
$$\rho_1 \left\{\rho+\rho \cot\frac{A}{2}\right\}=\rho^2 \cot\frac{A}{2};$$

therefore
$$\rho_1=\frac{\rho\cot\frac{A}{2}}{1+\cot\frac{A}{2}};\quad \text{therefore } \frac{1}{\rho_1}=\frac{1+\cot\frac{A}{2}}{\rho\cot\frac{A}{2}}.$$

Similarly
$$\frac{1}{\rho_2}=\frac{1+\cot\frac{B}{2}}{\rho\cot\frac{B}{2}}.$$

Thus
$$\left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)\left(\frac{1}{\rho_2}-\frac{1}{\rho}\right)=\frac{1}{\rho^2\cot\frac{A}{2}\cot\frac{B}{2}}=\frac{1}{\rho^2}\tan\frac{A}{2}\tan\frac{B}{2}.$$

In this manner we find that the proposed expression

$$=\frac{1}{\rho^2}\left\{\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}\right\}$$

$$=\frac{1}{\rho^2},$$ by Example VIII. 25.

42. As in Example 24 we shall find that the radii of the circles successively inscribed in the angle A are lr, l^2r, l^3r,... where

$$l=\frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}.$$

Hence the sum of the areas of all these circles is $\pi(l^2r^2+l^4r^2+l^6r^2+\dots)$;

$$\text{that is } \frac{\pi l^2r^2}{1-l^2}, \quad \text{that is } \frac{\pi\left(1-\sin\frac{A}{2}\right)^2 r^2}{4\sin\frac{A}{2}},$$

$$\text{that is } \frac{\pi\left(1-\cos\frac{B+C}{2}\right)^2 r^2}{4\sin\frac{A}{2}}, \quad \text{that is } \pi r^2 \sin^4\frac{B+C}{4}\operatorname{cosec}\frac{A}{2}.$$

Similarly we find the areas of the circles inscribed within the angles B and C. Thus the sum of all the areas is

$$\pi r^2\left\{\sin^4\frac{B+C}{4}\operatorname{cosec}\frac{A}{2}+\sin^4\frac{C+A}{4}\operatorname{cosec}\frac{B}{2}+\sin^4\frac{A+B}{4}\operatorname{cosec}\frac{C}{2}\right\}.$$

43. $R_a=\dfrac{BC}{2\sin BOC}=\dfrac{a}{2\sin 2A}$;

similarly $R_b=\dfrac{b}{2\sin 2B}$, and $R_c=\dfrac{c}{2\sin 2C}$.

Thus $\dfrac{a}{R_a}+\dfrac{b}{R_b}+\dfrac{c}{R_c}=2(\sin 2A+\sin 2B+\sin 2C)$

$$=8\sin A\sin B\sin C, \text{ by Art. 114, } =\frac{a}{R}\times\frac{b}{R}\times\frac{c}{R}=\frac{abc}{R^3}.$$

Again,

$$r_a=\frac{BC}{2\sin BO'C}=\frac{a}{2\sin\left(\pi-\frac{1}{2}B-\frac{1}{2}C\right)}=\frac{a}{2\sin\frac{1}{2}(B+C)}=\frac{a}{2\cos\frac{A}{2}};$$

similarly $r_b=\dfrac{b}{2\cos\frac{B}{2}}$, and $r_c=\dfrac{c}{2\cos\frac{C}{2}}$.

Therefore

$$\frac{r_ar_br_c}{abc}=\frac{1}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{1}{2(\sin A+\sin B+\sin C)}, \text{ by Example VIII. 16,}$$

$$=\frac{R}{2R\sin A+2R\sin B+2R\sin C}=\frac{R}{a+b+c}.$$

44. Since the angles at B' and C' are right angles it will follow that A will be on the circumference of the circle which is described round $PB'C'$, and that PA is a diameter of the circle. Let O_1 denote the centre of the circle, then $PO_1=\frac{1}{2}PA$.

In a similar manner if O_2 is the centre of the circle round $PC'A'$, and O_3 the centre of the circle round $PA'B'$, we have

$$PO_2=\frac{1}{2}PB, \text{ and } PO_3=\frac{1}{2}PC.$$

Then in the triangle PO_2O_3 we have

$$O_2O_3^2=PO_2^2+PO_3^2-2PO_2PO_3\cos O_2PO_3;$$

and in the triangle PBC we have

$$BC^2=PB^2+PC^2-2PB\,.\,PC\cos BPC.$$

Hence $O_2O_3=\frac{1}{2}BC$. Or this might be obtained by Euclid VI. 2, and VI. 4.

Similarly $O_3O_1=\frac{1}{2}CA$, and $O_1O_2=\frac{1}{2}AB$. Thus the area of $O_1O_2O_3$ is one-fourth of the area of ABC.

45. Let r_1, r_2, r_3 denote the radii of the circles; then the sides of the triangle are respectively r_2+r_3, r_3+r_1, and r_1+r_2. Thus

$$s=r_1+r_2+r_3,\quad s-a=r_1,\quad s-b=r_2,\quad s-c=r_3.$$

Therefore $$S^2=(r_1+r_2+r_3)\,r_1r_2r_3.$$

46. Suppose a, b, c in Geometrical Progression, so that $b^2=ac$; let p_1, p_2, p_3 denote the perpendiculars from the opposite angles on a, b, c respectively.

Then $\frac{1}{2}p_1a=S$, so that $p_1=\frac{2S}{a}$; similarly $p_2=\frac{2S}{b}$, and $p_3=\frac{2S}{c}$.

Let A_1, B_1, C_1 be the angles opposite p_1, p_2, p_3 respectively in the new triangle.

$$\text{Then } \cos A_1=\frac{p_2^2+p_3^2-p_1^2}{2p_2p_3}=\frac{\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2}}{\frac{2}{bc}}=\frac{\frac{b^2+c^2}{b^2c^2}-\frac{1}{a^2}}{\frac{2}{bc}}$$

$$=\frac{a^2(b^2+c^2)-b^2c^2}{2a^2bc}=\frac{b^2(a^2-c^2)+a^2c^2}{2a^2bc}=\frac{a^2-c^2+ac}{2ab}$$

$$=\frac{a^2+b^2-c^2}{2ab}=\cos C.$$

Thus $A_1=C$. Similarly $C_1=A$. Therefore $B_1=B$.

47. Here $\alpha=\frac{a}{c\sin B}=\frac{\sin A}{\sin B\sin C}$.

Similarly $$\beta=\frac{\sin B}{\sin C\sin A},\quad\text{and } \gamma=\frac{\sin C}{\sin A\sin B}.$$

Therefore $2(\beta\gamma+\gamma\alpha+\alpha\beta)-\alpha^2-\beta^2-\gamma^2=$ the product of $\dfrac{1}{\sin^2 A\sin^2 B\sin^2 C}$ into $\{2\sin^2 B\sin^2 C+2\sin^2 C\sin^2 A+2\sin^2 A\sin^2 B-\sin^4 A-\sin^4 B-\sin^4 C\}$.

The expression within brackets is equal to

$(\sin A+\sin B+\sin C)(\sin A+\sin B-\sin C)(\sin A-\sin B+\sin C)(\sin B+\sin C-\sin A)$,

as we know from a similar process in Art. 218.

Then, by Examples VIII. 16 and 17, we obtain

$$4^4\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2},\text{ that is, }4\sin^2 A\sin^2 B\sin^2 C.$$

Hence $$2(\beta\gamma+\gamma\alpha+\alpha\beta)-\alpha^2-\beta^2-\gamma^2=4,$$

and therefore $$\alpha^2+\beta^2+\gamma^2-2(\beta\gamma+\gamma\alpha+\alpha\beta)+4=0.$$

48. Let P, Q, R be the centres of the equilateral triangles described on BC, CA, AB respectively.

Then $$PQ^2=PC^2+QC^2-2PC\,.\,QC\cos PCQ;$$

also $$PC=\frac{a}{\sqrt{3}},\text{ and }QC=\frac{b}{\sqrt{3}}.$$

Thus
$$\begin{aligned}3PQ^2&=a^2+b^2-2ab\cos(C+60^\circ)\\&=a^2+b^2-2ab(\cos C\cos 60^\circ-\sin C\sin 60^\circ)\\&=a^2+b^2-ab\cos C+ab\sin C\sqrt{3}\\&=a^2+b^2-\frac{a^2+b^2-c^2}{2}+ab\sin C\sqrt{3}\\&=\frac{a^2+b^2+c^2}{2}+2S\sqrt{3}.\end{aligned}$$

We shall obtain the same symmetrical expression for $3QR^2$ and $3RP^2$. Thus $PQ=QR=RP$.

49. We have $$\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2};$$

therefore $$\cot\frac{A}{2}=\frac{65+25}{65-25}\tan 30^\circ=\frac{9}{4}\cdot\frac{1}{\sqrt{3}}=\frac{3\sqrt{3}}{4};$$

therefore $$L\cot\frac{A}{2}=10+\frac{3}{2}\log 3-2\log 2=10\cdot1136219.$$

10·1137122	10·1136219	
10·1134508	10·1134508	
·0002614	·0001711	·0002614 : ·0001711 :: 60″ : x'';

this gives $x=39$; therefore $\frac{A}{2}=37^\circ\ 36'-39''=37^\circ\ 35'\ 21''$. Therefore $A=75^\circ\ 10'\ 42''$. Thus $B+C=180^\circ-75^\circ\ 10'\ 42''$; and $B-C=60^\circ$. Therefore $B=82^\circ\ 24'\ 39''$ and $C=22^\circ\ 24'\ 39''$.

50. In the solution of Example 26 it is shewn that the sides of the new triangle are $a \cos A$, $b \cos B$, and $c \cos C$ respectively.

In the solution of Example 27 it is shewn that the angles of the new triangle are $\pi - 2A$, $\pi - 2B$, and $\pi - 2C$ respectively. Then, by Art. 215,

$$\cos(\pi - 2A) = \frac{b^2 \cos^2 B + c^2 \cos^2 C - a^2 \cos^2 A}{2bc \cos B \cos C};$$

but $\cos(\pi - 2A) = -\cos 2A$. Therefore

$$\cos 2A = \frac{a^2 \cos^2 A - b^2 \cos^2 B - c^2 \cos^2 C}{2bc \cos B \cos C}.$$

51. Let ρ_1 denote the radius of the circle which touches BD, BF and the arc DF in the diagram of Art. 250. Let ρ_2 denote the radius of the circle which touches CD, CE, and the arc DE.

The angle $DBF = \pi - B$. Hence, by the method of Example 24, we have

$$\rho_1 = r_1 \frac{1 - \sin \frac{\pi - B}{2}}{1 + \sin \frac{\pi - B}{2}} = r_1 \frac{1 - \cos \frac{B}{2}}{1 + \cos \frac{B}{2}} = r_1 \tan^2 \frac{B}{4}.$$

Similarly $$\rho_2 = r_1 \tan^2 \frac{C}{4}.$$

In this way we see that the product of three of the radii

$$= r_1 \tan^2 \frac{B}{4} \times r_2 \tan^2 \frac{C}{4} \times r_3 \tan^2 \frac{A}{4};$$

and the product of the other three

$$= r_1 \tan^2 \frac{C}{4} \times r_2 \tan^2 \frac{A}{4} \times r_3 \tan^2 \frac{B}{4}.$$

The two products are equal.

52. $\frac{AB'}{AP} = \frac{\sin APB'}{\sin AB'P}$; therefore $AB' = \frac{AP \sin APB'}{\sin AB'P}$.

Similarly $$BC' = \frac{BP \sin BPC'}{\sin BC'P}, \text{ and } CA' = \frac{CP \sin CPA'}{\sin CA'P}.$$

Thus $$AB' . BC' . CA' = \frac{AP . BP . CP \sin APB' \sin BPC' \sin CPA'}{\sin AB'P . \sin BC'P . \sin CA'P}.$$

In like manner

$$AC' . BA' . CB' = \frac{AP . BP . CP \sin APC' . \sin BPA' . \sin CPB'}{\sin AC'P . \sin BA'P . \sin CB'P}.$$

The two expressions are obviously equal; for $\sin APB' = \sin BPA'$, $\sin BPC' = \sin B'PC$, and $\sin CPA' = \sin C'PA$. Also, $\sin AB'P = \sin CB'P$, and so on.

53. Let P denote the intersection of AA' and BB'; then, if CC' does not pass through P, let a straight line be drawn from C through P, and let it meet AB at C_1.

Then, by the Example, we have

$$AB' \,.\, BC_1 \,.\, CA' = AC_1 \,.\, BA' \,.\, CB'.$$

But by hypothesis,

$$AB' \,.\, BC' \,.\, CA' = AC' \,.\, BA' \,.\, CB'.$$

Therefore $$\frac{BC_1}{BC'} = \frac{AC_1}{AC'};$$

therefore $$\frac{BC' - C_1C'}{BC'} = \frac{AC' + C_1C'}{AC'},$$

therefore $$-\frac{C_1C'}{BC'} = \frac{C_1C'}{AC'};$$

therefore $$C_1C' = 0;$$

therefore C_1 must coincide with C'.

54. Let the feet of the perpendiculars from A, B, C be denoted by A', B', C' respectively. If all the angles are acute, we have

$$AB' = c\cos A, \quad BC' = a\cos B, \quad CA' = b\cos C,$$

$$AC' = b\cos A, \quad BA' = c\cos B, \quad CB' = a\cos C;$$

thus $$AB' \,.\, BC' \,.\, CA' = AC' \,.\, BA' \,.\, CB'.$$

Therefore, by Example 53, the straight lines AA', BB', and CC' meet at a point.

Next suppose one angle obtuse, say C. Then

$$CA' = b\cos(180^\circ - C), \text{ and } CB' = a\cos(180^\circ - C);$$

the other expressions remain as before, and the result holds as before.

55. Let the straight lines which bisect the angles A, B, C respectively meet the opposite sides at A', B', C' respectively. Then

$$\frac{AB'}{BB'} = \frac{\sin\frac{1}{2}B}{\sin A}, \quad \frac{BC'}{CC'} = \frac{\sin\frac{1}{2}C}{\sin B}, \quad \frac{CA'}{AA'} = \frac{\sin\frac{1}{2}A}{\sin C};$$

therefore $$AB' \,.\, BC' \,.\, CA' = AA' \,.\, BB' \,.\, CC' \frac{\sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C}{\sin A \sin B \sin C};$$

the same value may be obtained for $AC' \,.\, BA' \,.\, CB'$.

Therefore, by Example 53, the straight lines AA', BB', and CC' meet at a point.

56. Let A', B', C' denote the middle points of BC, CA, AB respectively. Then

$$AB' \cdot BC' \cdot CA' = \frac{1}{2} b \times \frac{1}{2} c \times \frac{1}{2} a = \frac{1}{8} abc.$$

Similarly $$AC' \cdot BA' \cdot CB' = \frac{1}{8} abc.$$

Therefore, by Example 53, the straight lines AA', BB', and CC' meet at a point.

57. Let the points of contact opposite to A, B, C respectively be denoted by A', B', C' respectively.

Then $$AB' = r \cot \frac{A}{2}, \quad BC' = r \cot \frac{B}{2}, \quad CA' = r \cot \frac{C}{2}.$$

Thus $$AB' \cdot BC' \cdot CA' = r^3 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

Similarly $$AC' \cdot BA' \cdot CB' = r^3 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

Therefore, by Example 53, the straight lines AA', BB', and CC' meet at a point.

58. Let the points of contact opposite to A, B, C respectively be denoted by A', B', C' respectively.

Then $$AB' = r_2 \cot \frac{1}{2}(\pi - A) = r_2 \tan \frac{1}{2} A,$$

$$BC' = r_3 \tan \frac{1}{2} B,$$

$$CA' = r_1 \tan \frac{1}{2} C;$$

therefore $$AB' \cdot BC' \cdot CA' = r_1 r_2 r_3 \tan \frac{1}{2} A \tan \frac{1}{2} B \tan \frac{1}{2} C.$$

Similarly $$AC' \cdot BA' \cdot CB' = r_1 r_2 r_3 \tan \frac{1}{2} A \tan \frac{1}{2} B \tan \frac{1}{2} C.$$

Therefore, by Example 53, the straight lines AA', BB', and CC' meet at a point.

59. Here $AE = AF$, $CE = CD$, $BD = BF$; therefore

$$AE \cdot BF \cdot CD = AF \cdot BD \cdot CE.$$

Therefore, by Example 53, the straight lines AD, BE, and CF meet at a point.

60. Let $ABCD$ be the quadrilateral figure. Then, denoting by A, B, C, and D the internal angles of the figure, we have

$$r_a\left(\cot\frac{\pi-B}{2}+\cot\frac{\pi-C}{2}\right)=BC;$$

therefore $$r_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=BC.$$

Again, in like manner we have

$$r\left(\cot\frac{A}{2}+\cot\frac{D}{2}\right)=DA,$$

that is $$r\left(\tan\frac{C}{2}+\tan\frac{B}{2}\right)=DA,$$

for $A+C=\pi$, and $B+D=\pi$, by Euclid III. 22.

Hence $$\frac{r_a}{r}=\frac{BC}{DA}.$$

In the same manner we can shew that

$$\frac{r_b}{r}=\frac{CD}{AB},\quad \frac{r_c}{r}=\frac{AD}{BC},\quad\text{and}\quad \frac{r_d}{r}=\frac{AB}{DC}.$$

Therefore $$\frac{r_ar_c}{r^2}=1,\quad\text{and}\quad \frac{r_br_d}{r^2}=1;$$

therefore $$r_ar_br_cr_d=r^4.$$

XVII.

1. Here $r=-4$, $q=6$; therefore

$$\cos 3a=-16\left(\frac{3}{24}\right)^{\frac{3}{2}}=-16\left(\frac{1}{8}\right)^{\frac{3}{2}}=-\frac{16}{8}\left(\frac{1}{8}\right)^{\frac{1}{2}}=-\frac{2}{2\sqrt{2}}=-\frac{1}{\sqrt{2}}.$$

Therefore $3a=\frac{3\pi}{4}$. Hence $a=\frac{\pi}{4}$. Therefore the roots are

$$2\left(\frac{6}{3}\right)^{\frac{1}{2}}\cos\frac{\pi}{4},\quad\text{and}\quad 2\left(\frac{6}{3}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{3}\pm\frac{\pi}{4}\right).$$

Now $$2\left(\frac{6}{3}\right)^{\frac{1}{2}}\cos\frac{\pi}{4}=2\sqrt{2}\frac{1}{\sqrt{2}}=2;$$

$$2\left(\frac{6}{3}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)=2\sqrt{2}\cos\frac{11\pi}{12}=-2\sqrt{2}\cos\frac{\pi}{12}=-(\sqrt{3}+1);$$

$$2\left(\frac{6}{3}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)=2\sqrt{2}\cos\frac{5\pi}{12}=\sqrt{3}-1.$$

2. Here $r=1$, $q=3$; therefore $\cos 3\alpha = 4\left(\frac{3}{12}\right)^{\frac{3}{2}} = 4\left(\frac{1}{4}\right)^{\frac{3}{2}} = \frac{1}{2}$;

therefore $3\alpha = 60^0$; therefore $\alpha = 20^0$. Therefore the roots are $2\cos 20^0$, and $2\cos(120^0 \pm 20^0)$.

Also $\qquad 2\cos(120^0+20^0) = 2\cos 140^0 = -2\cos 40^0$.

And $\qquad 2\cos(120^0-20^0) = 2\cos 100^0 = -2\sin 10^0$.

3. Take the equation $\qquad x^5 - px^3 + qx - r = 0$.

Put $x = ny$; thus $\qquad n^5y^5 - pn^3y^3 + qny - r = 0$;

therefore $\qquad y^5 - \frac{p}{n^2}y^3 + \frac{q}{n^4}y = \frac{r}{n^5}$.

Now by Example VIII. 59, $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$.

Thus $\qquad \cos^5\alpha - \frac{5}{4}\cos^3\alpha + \frac{5}{16}\cos\alpha = \frac{\cos 5\alpha}{16}$.

Assume $y = \cos\alpha$, $\frac{5}{4} = \frac{p}{n^2}$, $\frac{5}{16} = \frac{q}{n^4}$, then $\frac{r}{n^5} = \frac{\cos 5\alpha}{16}$.

Here $\left(\frac{5}{4}\right)^2 = \frac{p^2}{n^4}$, and $\frac{5}{16} = \frac{q}{n^4}$; so that we have $n^4 = \frac{16p^2}{25}$, and $n^4 = \frac{16q}{5}$. Thus the process will not be admissible unless $p^2 = 5q$; and this condition is satisfied by hypothesis.

Then α must be found from $\cos 5\alpha = \frac{16r}{n^5}$; put for n its value $\left(\frac{4p}{5}\right)^{\frac{1}{2}}$: thus $\cos 5\alpha = 16r \times \left(\frac{5}{4p}\right)^{\frac{5}{2}} = \frac{r}{2}\left(\frac{5}{p}\right)^{\frac{5}{2}}$. The process then will not be admissible if this expression is numerically greater than unity. Hence $\left(\frac{r}{2}\right)^2\left(\frac{5}{p}\right)^5$ must not be greater than unity; that is $\left(\frac{r}{2}\right)^2$ must not be greater than $\left(\frac{p}{5}\right)^5$.

Suppose this condition also to hold; then one root is $n\cos\alpha$, that is $2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\alpha$.

Moreover we might also suppose $y = \cos\left(\frac{2\pi}{5} \pm \alpha\right)$ or $y = \cos\left(\frac{4\pi}{5} \pm \alpha\right)$, and we shall still arrive at the same value for $\cos 5\alpha$, since

$$\cos 5\left(\frac{2\pi}{5} \pm \alpha\right) = \cos 5\alpha \text{ and } \cos 5\left(\frac{4\pi}{5} \pm \alpha\right) = \cos 5\alpha.$$

Hence we see that the other roots of the equation are

$$2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{5} \pm \alpha\right) \text{ and } 2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\left(\frac{4\pi}{5} \pm \alpha\right).$$

4. Proceed as in Example 3.

Here $r=8$, $q=20$, $p=10$.

$$\cos 5\alpha = 4\left(\frac{1}{2}\right)^{\frac{5}{2}} = \frac{1}{\sqrt{2}}; \quad \text{therefore } 5\alpha = 45^0.$$

Hence the roots are

$$2\sqrt{2}\cos 9^0, \quad 2\sqrt{2}\cos(72^0 \pm 9^0), \quad 2\sqrt{2}\cos(144^0 \pm 9^0),$$

that is

$$2\sqrt{2}\cos 9^0, \quad 2\sqrt{2}\cos 63^0, \quad 2\sqrt{2}\cos 81^0, \quad 2\sqrt{2}\cos 153^0, \quad 2\sqrt{2}\cos 135^0;$$

the last is equal to -2.

5. Let D denote the point of contact of the circle with BC. Let AC intersect the circumference of the circle at E, and let AB intersect the circumference at F. Then the four straight lines AE, ED, DF, FA can be measured. Then, by Art. 254, the diagonal AD can be determined.

Then all the angles of the triangles ADE and ADF can be found; and thus the angles of the triangles ADC and ADB are known. Thus DC and BD can be found. See Euclid III. 32.

6. Let D be the point on AC produced through C such that the angle ADB is half the angle ACB; then $CD=CB$. Thus CB is known. Again, let E be the point on BC produced through C such that the angle AEB is half the angle ACB; then $CE=CA$. Thus CA is known. Then in the triangle ACB we know AC, and CB, and the angle ACB; thus AB can be found by Art. 215.

7. Let x denote the height of the balloon, and a, b, c the sides of the triangle ABC. Let O be the point in the plane of ABC which is vertically under the balloon. Then

$$AO = x\cot 45^0 = x, \quad BO = x\cot 45^0 = x, \quad CO = x\cot 60^0 = \frac{x}{\sqrt{3}}. \quad \text{Therefore}$$

$$\cos ACO = \frac{b^2 + \frac{x^2}{3} - x^2}{\frac{2bx}{\sqrt{3}}} = \frac{3b^2 - 2x^2}{2bx\sqrt{3}}, \quad \cos BCO = \frac{a^2 + \frac{x^2}{3} - x^2}{\frac{2ax}{\sqrt{3}}} = \frac{3a^2 - 2x^2}{2ax\sqrt{3}}.$$

But ACB is a right angle, and therefore $\cos BCO = \sin ACO$; thus

$$\left(\frac{3b^2 - 2x^2}{2bx\sqrt{3}}\right)^2 + \left(\frac{3a^2 - 2x^2}{2ax\sqrt{3}}\right)^2 = 1;$$

therefore $$a^2(3b^2 - 2x^2)^2 + b^2(3a^2 - 2x^2)^2 = 12a^2b^2x^2;$$

therefore $$4x^4(a^2 + b^2) - 36a^2b^2x^2 + 9a^2b^2(a^2 + b^2) = 0;$$

therefore $$4c^2x^4 - 36a^2b^2x^2 + 9a^2b^2c^2 = 0.$$

8. Here the angle BAC – the angle BOC = the sum of the angles ABO and ACO.

Now $\dfrac{\sin ACO}{\sin AOC} = \dfrac{AO}{AC}$; therefore $\sin ACO = \dfrac{n \sin \beta}{c}$, and since ACO is very small the circular measure of it is nearly equal to the sine, so that it is nearly equal to $\dfrac{n \sin \beta}{c}$.

Again $\dfrac{\sin ABO}{\sin AOB} = \dfrac{AO}{AB}$; therefore $\sin ABO = \dfrac{n \sin (\alpha - \beta)}{b}$, therefore the circular measure of ABO is nearly equal to $\dfrac{n \sin (\alpha - \beta)}{b}$.

Thus the circular measure of $BAC - BOC$ is nearly $n \left\{ \dfrac{\sin (\alpha - \beta)}{b} + \dfrac{\sin \beta}{c} \right\}$.

9. If the distance is 50 feet and the elevation is $\dfrac{\pi}{4}$, the height in feet is $50 \tan \dfrac{\pi}{4}$, that is 50.

But suppose the distance to be $50 + h$, and the elevation to be $\dfrac{\pi}{4} + \alpha$. Then the height is $(50 + h) \tan \left(\dfrac{\pi}{4} + \alpha \right)$. If α is very small this is very nearly equal to $(50 + h) \left(\tan \dfrac{\pi}{4} + \alpha \sec^2 \dfrac{\pi}{4} \right)$, by Art. 188, that is $(50 + h)(1 + 2\alpha)$.

If h is also very small this is very nearly $50 + h + 100\alpha$. Now suppose $h = \dfrac{1}{12}$ and $\alpha = \dfrac{\pi}{180 \times 60}$; then we obtain $50 + \dfrac{1}{12} + \dfrac{\pi}{108}$. Thus the difference between this and the former value is $\dfrac{1}{12} + \dfrac{\pi}{108}$, that is about $\dfrac{1}{12} + \dfrac{1}{36}$, that is, $1\frac{1}{3}$ inches.

10. Suppose that the tower and the spire each subtend the angle α.

Then $$\tan \alpha = \frac{b}{a}, \quad \text{and} \quad \tan 2\alpha = \frac{b+c}{a}.$$

Therefore $$\frac{b+c}{a} = \frac{\dfrac{2b}{a}}{1 - \dfrac{b^2}{a^2}} = \frac{2ab}{a^2 - b^2};$$

therefore $b + c = \dfrac{2a^2 b}{a^2 - b^2}$; therefore $c = \dfrac{2a^2 b}{a^2 - b^2} - b = \dfrac{(a^2 + b^2) b}{a^2 - b^2}$.

If however the height of the tower is $b + \beta$, and the height of the spire is $c + \gamma$, we have

$$c + \gamma = \frac{a^2 (b + \beta) + (b + \beta)^3}{a^2 - (b + \beta)^2}.$$

Hence, by subtraction,

$$\gamma = \frac{a^2(b+\beta)+(b+\beta)^3}{a^2-(b+\beta)^2} - \frac{(a^2+b^2)b}{a^2-b^2}.$$

Now $$(b+\beta)^2 = b^2+2b\beta+\beta^2,$$

and if β is very small this is very nearly $b^2+2b\beta$.

And $$(b+\beta)^3 = b^3+3b^2\beta+3b\beta^2+\beta^3,$$

and if β is very small this is very nearly $b^3+3b^2\beta$.

Thus $$\gamma = \frac{a^2b+b^3+(a^2+3b^2)\beta}{a^2-b^2-2b\beta} - \frac{a^2b+b^3}{a^2-b^2}$$

$$= \frac{(a^2+3b^2)(a^2-b^2)+2(a^2+b^2)b^2}{(a^2-b^2-2b\beta)(a^2-b^2)}\beta$$

$$= \frac{a^4+4a^2b^2-b^4}{(a^2-b^2)(a^2-b^2-2b\beta)}\beta.$$

Therefore $$\frac{\gamma}{c} = \frac{(a^4+4a^2b^2-b^4)\beta}{(a^2-b^2)(a^2-b^2-2b\beta)} \div \frac{(a^2+b^2)b}{a^2-b^2}$$

$$= \frac{\beta}{b}\cdot\frac{a^4+4a^2b^2-b^4}{(a^2+b^2)(a^2-b^2-2b\beta)}.$$

But when β is very small we may put a^2-b^2 for $a^2-b^2-2b\beta$; and thus

$$\frac{\gamma}{c} = \frac{\beta}{b}\cdot\frac{a^4+4a^2b^2-b^4}{a^4-b^4}.$$

11. We have $$a^2 = b^2+c^2-2bc\cos A;$$

suppose that b is changed to $b+\beta$, and c to $c+\gamma$; thus

$$a^2 = (b+\beta)^2+(c+\gamma)^2-2(b+\beta)(c+\gamma)\cos A.$$

Therefore, by subtraction,

$$2b\beta+\beta^2+2c\gamma+\gamma^2-2(b\gamma+c\beta+\beta\gamma)\cos A = 0.$$

If β and γ are very small this becomes very nearly

$$2b\beta+2c\gamma-2(b\gamma+c\beta)\cos A = 0;$$

therefore $$\beta(b-c\cos A)+\gamma(c-b\cos A) = 0;$$

therefore $$\beta a\cos C+\gamma a\cos B = 0, \text{ by Art. 216.}$$

Therefore $$\frac{\beta}{\cos B}+\frac{\gamma}{\cos C} = 0,$$

therefore $$\beta\sec B+\gamma\sec C = 0.$$

12. Suppose h the height of the tower, r the radius, x the distance of the first place of observation from the centre. Then

$$\frac{x}{r} = \operatorname{cosec}\frac{\beta}{2}, \qquad \frac{x-a}{r} = \operatorname{cosec}\frac{\beta'}{2};$$

$$h = x\tan\alpha, \qquad h = (x-a)\tan\alpha'.$$

Hence $$\frac{a}{r} = \operatorname{cosec}\frac{\beta}{2} - \operatorname{cosec}\frac{\beta'}{2}.$$

This finds r.

Also $$h = x \tan \alpha' - a \tan \alpha' = \frac{h \tan \alpha'}{\tan \alpha} - a \tan \alpha';$$

therefore $$h = \frac{a \tan \alpha \tan \alpha'}{\tan \alpha' - \tan \alpha}.$$

This finds h.

Again, from the first and second equations,

$$\frac{x-a}{x} = \frac{\operatorname{cosec}\frac{\beta'}{2}}{\operatorname{cosec}\frac{\beta}{2}}.$$

And from the third and fourth equations,

$$\frac{x-a}{x} = \frac{\cot \alpha'}{\cot \alpha}.$$

Therefore $$\frac{\operatorname{cosec}\frac{\beta'}{2}}{\operatorname{cosec}\frac{\beta}{2}} = \frac{\cot \alpha'}{\cot \alpha}.$$

13. We have $$\frac{a}{r} = \operatorname{cosec}\frac{\beta}{2} - \operatorname{cosec}\frac{\beta'}{2} \quad\text{..............................(1).}$$

If we suppose an error δ of the *same* sign to be made in β and β' these errors will tend to compensate each other; the greatest possible error in r will be determined by supposing that errors of *opposite* signs are made in β and β'. Suppose then that instead of β we ought to have $\beta - \delta$, and instead of β' we ought to have $\beta' + \delta$. Then we have

$$\frac{a}{r-\rho} = \operatorname{cosec}\frac{\beta-\delta}{2} - \operatorname{cosec}\frac{\beta'+\delta}{2}.$$

Hence, by subtraction, $\frac{a}{r-\rho} - \frac{a}{r}$, that is $\frac{a\rho}{r(r-\rho)}$

$$= \operatorname{cosec}\frac{\beta-\delta}{2} - \operatorname{cosec}\frac{\beta}{2} - \left\{\operatorname{cosec}\frac{\beta'+\delta}{2} - \operatorname{cosec}\frac{\beta'}{2}\right\}.$$

Therefore, if δ and ρ be very small, we obtain

$$\frac{a\rho}{r^2} = \frac{\delta}{2}\left\{\frac{\cos\frac{\beta}{2}}{\sin^2\frac{\beta}{2}} + \frac{\cos\frac{\beta'}{2}}{\sin^2\frac{\beta'}{2}}\right\};$$ see Art. 194.

Thus $$\frac{a\rho}{r^2} = \frac{\delta}{2} \frac{\cos\frac{\beta}{2}\left(1 - \cos^2\frac{\beta'}{2}\right) + \cos\frac{\beta'}{2}\left(1 - \cos^2\frac{\beta}{2}\right)}{\sin^2\frac{\beta}{2}\sin^2\frac{\beta}{2}}$$

$$= \frac{\delta}{2} \frac{\left(\cos\frac{\beta}{2} + \cos\frac{\beta'}{2}\right)\left(1 - \cos\frac{\beta}{2}\cos\frac{\beta'}{2}\right)}{\sin^2\frac{\beta}{2}\sin^2\frac{\beta'}{2}}$$

$$= \frac{\delta\cos\frac{\beta+\beta'}{4}\cos\frac{\beta'-\beta}{4}\left(1 - \cos\frac{\beta}{2}\cos\frac{\beta'}{2}\right)}{\sin^2\frac{\beta}{2}\sin^2\frac{\beta'}{2}} \quad \text{..................(2).}$$

Now (1) may be put in the form

$$\frac{a}{r} = \frac{\sin\frac{\beta'}{2} - \sin\frac{\beta}{2}}{\sin\frac{\beta'}{2}\sin\frac{\beta}{2}} = \frac{2\sin\frac{\beta'-\beta}{4}\cos\frac{\beta'+\beta}{4}}{\sin\frac{\beta'}{2}\sin\frac{\beta}{2}} \quad \text{..................(3).}$$

Divide (2) by (3); then

$$\frac{\rho}{r} = \frac{\delta}{2}\cot\frac{1}{4}(\beta'-\beta) \cdot \frac{1 - \cos\frac{\beta}{2}\cos\frac{\beta'}{2}}{\sin\frac{\beta'}{2}\sin\frac{\beta}{2}}$$

$$= \frac{\delta}{2}\cot\frac{1}{4}(\beta'-\beta)\left\{\operatorname{cosec}\frac{\beta'}{2}\operatorname{cosec}\frac{\beta}{2} - \cot\frac{\beta'}{2}\cot\frac{\beta}{2}\right\}.$$

If $\beta = 60^0$ and $\beta' = 120^0$, we obtain for $\frac{2\rho}{r}$ the value

$$\cot 15^0\{\operatorname{cosec} 30^0 \operatorname{cosec} 60^0 - \cot 30^0 \cot 60^0\}\,\delta,$$

that is $$(2+\sqrt{3})\left(\frac{4}{\sqrt{3}} - 1\right)\delta, \quad \text{that is } \frac{5+2\sqrt{3}}{\sqrt{3}}\,\delta.$$

Put for δ the circular measure of $6'$, that is $\frac{\pi}{1800}$.

Hence $\frac{2\rho}{r} = \frac{5+2\sqrt{3}}{\sqrt{3}} \times \frac{\pi}{1800}$; therefore $\frac{\rho}{r} = \frac{5+2\sqrt{3}}{\sqrt{3}} \times \frac{\pi}{3600}$.

14. Let β denote the angle PSQ, and the equal angle QSR; and let ϕ denote the angle SQR.

Then $$\frac{PQ}{SQ} = \frac{\sin PSQ}{\sin SPQ} = \frac{\sin\beta}{\sin(\phi - \beta)},$$

and $$\frac{QR}{SQ} = \frac{\sin QSR}{\sin SRQ} = \frac{\sin\beta}{\sin(\phi+\beta)},$$

therefore $$\frac{PQ}{QR} = \frac{\sin(\phi+\beta)}{\sin(\phi-\beta)}.$$

Let $PQ = a$, and $QR = b$; thus

$$a\sin(\phi-\beta) = b\sin(\phi+\beta),$$

therefore $$a(\sin\phi\cos\beta - \cos\phi\sin\beta) = b(\sin\phi\cos\beta + \cos\phi\sin\beta);$$

therefore $$\tan\phi = \frac{(a+b)\sin\beta}{(a-b)\cos\beta} = \frac{a+b}{a-b}\tan\beta.$$

Also $$\frac{1}{SQ} = \frac{\sin\beta}{a\sin(\phi-\beta)}, \quad \text{and } \frac{1}{SQ} = \frac{\sin\beta}{b\sin(\phi+\beta)};$$

therefore $$\frac{1}{SQ^2} = \frac{\sin^2\beta}{ab\sin(\phi-\beta)\sin(\phi+\beta)} = \frac{\sin^2\beta}{ab(\sin^2\phi - \sin^2\beta)}$$

$$= \frac{\tan^2\beta}{ab\{\sin^2\phi(1+\tan^2\beta) - \tan^2\beta\}}.$$

But $$\sin^2\phi = \frac{(a+b)^2\tan^2\beta}{(a-b)^2 + (a+b)^2\tan^2\beta},$$

thus $$\frac{1}{SQ^2} = \frac{(a-b)^2 + (a+b)^2\tan^2\beta}{ab\{(a+b)^2 - (a-b)^2\}} = \frac{(a-b)^2 + (a+b)^2\tan^2\beta}{4a^2b^2}.$$

Suppose that instead of β we ought to have $\beta+\alpha$, and instead of SQ we ought to have $SQ+c$, where α and c are very small. Then

$$\frac{1}{(SQ+c)^2} = \frac{(a-b)^2}{4a^2b^2} + \frac{(a+b)^2}{4a^2b^2}\tan^2(\beta+\alpha).$$

Hence, by subtraction,

$$\frac{1}{(SQ+c)^2} - \frac{1}{SQ^2} = \frac{(a+b)^2}{4a^2b^2}\{\tan^2(\beta+\alpha) - \tan^2\beta\};$$

therefore $$\frac{SQ^2 - (SQ+c)^2}{SQ^2(SQ+c)^2} = \frac{(a+b)^2}{4a^2b^2}\{(\tan\beta + \alpha\sec^2\beta)^2 - \tan^2\beta\},$$

approximately, by Art. 188.

Thus $$-\frac{2c}{SQ^3} = \frac{(a+b)^2}{4a^2b^2}\,2\tan\beta\sec^2\beta\,\alpha, \text{ nearly};$$

therefore $$\frac{c}{SQ^3} = -\frac{(a+b)^2}{4a^2b^2}\frac{\sin\beta}{\cos^3\beta}\alpha, \text{ nearly}.$$

XVIII.

1. Let $\tan^{-1}\frac{1}{3}=\theta$, then $\tan\theta=\frac{1}{3}$; therefore

$$\tan 2\theta=\frac{\frac{2}{3}}{1-\frac{1}{9}}=\frac{6}{8}=\frac{3}{4};$$

therefore $2\theta=\tan^{-1}\frac{3}{4}$. Therefore $\tan^{-1}\frac{3}{4}=2\tan^{-1}\frac{1}{3}$.

2. Let $\sin^{-1}\frac{1}{2}=\theta$, and $\cos^{-1}\frac{1}{2}=\phi$;

therefore $\sin\theta=\frac{1}{2}$, and $\cos\theta=\frac{\sqrt{3}}{2}$,

and $\cos\phi=\frac{1}{2}$, and $\sin\phi=\frac{\sqrt{3}}{2}$.

Therefore $\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi=\frac{1}{4}+\frac{3}{4}=1$.

3. Let $\sin^{-1}\frac{3}{5}=\alpha$, and $\sin^{-1}\frac{8}{17}=\beta$;

then $\sin\alpha=\frac{3}{5}$, $\cos\alpha=\sqrt{\left(1-\frac{9}{25}\right)}=\frac{4}{5}$;

and $\sin\beta=\frac{8}{17}$, $\cos\beta=\sqrt{\left(1-\frac{64}{289}\right)}=\frac{15}{17}$;

therefore $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=\frac{3\times 15}{5\times 17}+\frac{4\times 8}{5\times 17}=\frac{45+32}{85}=\frac{77}{85};$$

therefore $\alpha+\beta=\sin^{-1}\frac{77}{85}$.

4. Let $\alpha=\tan^{-1}x$, and $\beta=\cot^{-1}x$;

then $\tan\alpha=x$, and $\cot\beta=x$; therefore $\tan\beta=\frac{1}{x}$,

and $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{x+\frac{1}{x}}{1-1}=\frac{x+\frac{1}{x}}{0}$.

Thus $\tan(\alpha+\beta)$ is infinite.

5. Let $\tan^{-1}\frac{1}{3}=\alpha$, $\tan^{-1}\frac{1}{5}=\beta$, $\tan^{-1}\frac{1}{7}=\gamma$, $\tan^{-1}\frac{1}{8}=\delta$.

Thus $$\tan\alpha=\frac{1}{3},\quad\text{and}\quad \tan\beta=\frac{1}{5};$$

therefore $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}}=\frac{8}{14}=\frac{4}{7}.$$

And $$\tan(\gamma+\delta)=\frac{\tan\gamma+\tan\delta}{1-\tan\gamma\tan\delta}=\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}=\frac{15}{55}=\frac{3}{11}.$$

Then $$\tan(\alpha+\beta+\gamma+\delta)=\frac{\tan(\alpha+\beta)+\tan(\gamma+\delta)}{1-\tan(\alpha+\beta)\tan(\gamma+\delta)}=\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}=\frac{65}{65}=1;$$

therefore $$\alpha+\beta+\gamma+\delta=\frac{\pi}{4}.$$

6. Let $$\tan^{-1}a=\theta,\quad\text{and}\quad \tan^{-1}b=\phi;$$

then $$a=\tan\theta,\quad\text{and}\quad b=\tan\phi;$$

and $$\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\frac{a-b}{1+ab}.$$

Thus $$\tan^{-1}\frac{a-b}{1+ab}=\tan^{-1}a-\tan^{-1}b.$$

Similarly $$\tan^{-1}\frac{b-c}{1+bc}=\tan^{-1}b-\tan^{-1}c.$$

Therefore $$\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}=\tan^{-1}a-\tan^{-1}c,$$

and $$\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}c=\tan^{-1}a.$$

7. Let $$\alpha=\tan^{-1}\frac{1}{7},\quad \beta=\tan^{-1}\frac{1}{3},\quad \gamma=\tan^{-1}\frac{1}{26}.$$

$$\tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}=\frac{\frac{3}{7}-\frac{1}{7^3}}{1-\frac{3}{7^2}}=\frac{146}{322}=\frac{73}{161}.$$

$$\tan(\beta+\gamma)=\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=\frac{\frac{1}{3}+\frac{1}{26}}{1-\frac{1}{3\times 26}}=\frac{29}{77}.$$

$$\tan(3\alpha+\beta+\gamma)=\frac{\tan 3\alpha+\tan(\beta+\gamma)}{1-\tan 3\alpha\tan(\beta+\gamma)}=\frac{\frac{73}{161}+\frac{29}{77}}{1-\frac{73}{161}\times\frac{29}{77}}$$

$$=\frac{10290}{10280}=\frac{1029}{1028}.$$

$$\tan\left(3\alpha+\beta+\gamma-\frac{\pi}{4}\right)=\frac{\tan(3\alpha+\beta+\gamma)-1}{1+\tan(3\alpha+\beta+\gamma)}=\frac{\frac{1029}{1028}-1}{1+\frac{1029}{1028}}=\frac{1}{2057}.$$

8. We see as in the solution of Example 6 that

$$\tan^{-1}\{(\sqrt{2}+1)\tan\alpha\}-\tan^{-1}\{(\sqrt{2}-1)\tan\alpha\}$$

$$=\tan^{-1}\frac{(\sqrt{2}+1)\tan\alpha-(\sqrt{2}-1)\tan\alpha}{1+(\sqrt{2}+1)(\sqrt{2}-1)\tan^2\alpha}$$

$$=\tan^{-1}\frac{2\tan\alpha}{1+\tan^2\alpha}=\tan^{-1}(\sin 2\alpha).$$

9. $\tan(\theta-\alpha)\tan(\theta-\beta)=\tan^2\theta$;

therefore $$\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\cos(\theta-\alpha)\cos(\theta-\beta)}=\frac{1-\cos 2\theta}{1+\cos 2\theta};$$

therefore $$\frac{\cos(\alpha-\beta)-\cos(2\theta-\alpha-\beta)}{\cos(\alpha-\beta)+\cos(2\theta-\alpha-\beta)}=\frac{1-\cos 2\theta}{1+\cos 2\theta};$$

therefore $$\cos(\alpha-\beta)\cos 2\theta=\cos(2\theta-\alpha-\beta)$$

$$=\cos 2\theta\cos(\alpha+\beta)+\sin 2\theta\sin(\alpha+\beta);$$

therefore $$\tan 2\theta\sin(\alpha+\beta)=\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta;$$

therefore $$\tan 2\theta=\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)};$$

therefore $$2\theta=\tan^{-1}\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}.$$

10. Let $\alpha=\cos^{-1}\frac{9}{\sqrt{(82)}}$, and $\beta=\operatorname{cosec}^{-1}\frac{\sqrt{(41)}}{4}$;

then
$$\cos\alpha=\frac{9}{\sqrt{(82)}},\quad\text{and}\quad\sin\alpha=\frac{1}{\sqrt{(82)}};$$
$$\sin\beta=\frac{4}{\sqrt{(41)}},\quad\cos\beta=\frac{5}{\sqrt{(41)}}.$$

Therefore
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$=\frac{45-4}{\sqrt{(82)}\times\sqrt{(41)}}=\frac{41}{41\sqrt{2}}=\frac{1}{\sqrt{2}}.$$

Therefore
$$\alpha+\beta=\frac{\pi}{4}.$$

11. Let $\alpha=\sin^{-1}\frac{4}{5}$, $\beta=\sin^{-1}\frac{5}{13}$, $\gamma=\sin^{-1}\frac{16}{65}$;

then
$$\sin\alpha=\frac{4}{5},\quad\sin\beta=\frac{5}{13},\quad\sin\gamma=\frac{16}{65},$$

and
$$\cos\alpha=\frac{3}{5},\quad\cos\beta=\frac{12}{13},\quad\cos\gamma=\frac{63}{65}.$$

Then
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta=\frac{48+15}{65}=\frac{63}{65};$$

thus
$$\sin(\alpha+\beta)=\cos\gamma,\quad\text{so that}\quad\alpha+\beta+\gamma=\frac{\pi}{2}.$$

12. Let $\alpha=\tan^{-1}\frac{1}{4}$, and $\beta=\tan^{-1}\frac{1}{20}$.

$$\tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}=\frac{\frac{3}{4}-\frac{1}{4^3}}{1-\frac{3}{4^2}}=\frac{47}{52}.$$

$$\tan(3\alpha+\beta)=\frac{\tan 3\alpha+\tan\beta}{1-\tan 3\alpha\tan\beta}=\frac{\frac{47}{52}+\frac{1}{20}}{1-\frac{47}{52\times 20}}=\frac{992}{993}.$$

Again, let $\gamma=\tan^{-1}\frac{1}{1985}$; then

$$\tan\left(\frac{\pi}{4}-\gamma\right)=\frac{1-\frac{1}{1985}}{1+\frac{1}{1985}}=\frac{1984}{1986}=\frac{992}{993}.$$

Therefore
$$3\alpha+\beta=\frac{\pi}{4}-\gamma.$$

13. Let $\theta=\tan^{-1}\dfrac{2a-b}{b\sqrt{3}}$, $\phi=\tan^{-1}\dfrac{2b-a}{a\sqrt{3}}$;

then
$$\tan(\theta+\phi)=\frac{\dfrac{2a-b}{b\sqrt{3}}+\dfrac{2b-a}{a\sqrt{3}}}{1-\dfrac{(2a-b)(2b-a)}{3ab}}=\frac{a(2a-b)+b(2b-a)}{3ab-(2a-b)(2b-a)}\sqrt{3}$$
$$=\frac{2(a^2+b^2)-2ab}{2(a^2+b^2)-2ab}\sqrt{3}=\sqrt{3}.$$

Thus
$$\theta+\phi=\frac{\pi}{3}.$$

14. Let $\tan^{-1}a=\theta$, and $\tan^{-1}a^3=\phi$.

Then
$$\tan\theta=a, \quad \text{and } \tan 2\theta=\frac{2a}{1-a^2}.$$

Also
$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\frac{a+a^3}{1-a^4}=\frac{a}{1-a^2}.$$

Therefore $\tan(2\tan^{-1}a)=2\tan(\tan^{-1}a+\tan^{-1}a^3)$.

15. Let $\tan^{-1}\left(\dfrac{1}{2}\tan 2A\right)=\alpha$, then $\tan\alpha=\dfrac{1}{2}\tan 2A$;

let
$$\tan^{-1}(\cot A)=\beta, \quad \text{then } \tan\beta=\cot A;$$

let
$$\tan^{-1}(\cot^3 A)=\gamma, \quad \text{then } \tan\gamma=\cot^3 A.$$

Thus
$$\tan(\beta+\gamma)=\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=\frac{\cot A+\cot^3 A}{1-\cot^4 A}=\frac{\cot A}{1-\cot^2 A}$$
$$=\frac{\dfrac{1}{\tan A}}{1-\dfrac{1}{\tan^2 A}}=\frac{\tan A}{\tan^2 A-1}=-\frac{1}{2}\tan 2A=-\tan\alpha.$$

Therefore
$$\tan(\alpha+\beta+\gamma)=\frac{\tan\alpha+\tan(\beta+\gamma)}{1-\tan\alpha\tan(\beta+\gamma)}=0.$$

Thus
$$\alpha+\beta+\gamma=0.$$

16. Let $\cos^{-1}\dfrac{a}{b}=\theta$, then $\cos\theta=\dfrac{a}{b}$.

And
$$\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{1-\tan\dfrac{\theta}{2}}{1+\tan\dfrac{\theta}{2}}+\frac{1+\tan\dfrac{\theta}{2}}{1-\tan\dfrac{\theta}{2}}$$

$$=\frac{\left(1-\tan\frac{\theta}{2}\right)^2+\left(1+\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}}=2\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=2\frac{1}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$$

$$=\frac{2}{\cos\theta}=\frac{2b}{a}.$$

17. Let $\tan^{-1}\frac{a}{b}=\theta$; then $\tan\theta=\frac{a}{b}$;

$$\operatorname{cosec}^2\frac{\theta}{2}=\frac{1}{\sin^2\frac{\theta}{2}}=\frac{2}{1-\cos\theta}=\frac{2}{1-\frac{b}{\sqrt{(a^2+b^2)}}}=\frac{2\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}-b}$$

$$=\frac{2\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}-b}\times\frac{\sqrt{(a^2+b^2)}+b}{\sqrt{(a^2+b^2)}+b}=\frac{2(a^2+b^2)+2b\sqrt{(a^2+b^2)}}{a^2};$$

therefore $$\frac{a^3}{2}\operatorname{cosec}^2\frac{\theta}{2}=a(a^2+b^2)+ab\sqrt{(a^2+b^2)}.$$

Let $\tan^{-1}\frac{b}{a}=\phi$; then $\tan\phi=\frac{b}{a}$.

$$\sec^2\frac{\phi}{2}=\frac{1}{\cos^2\frac{\phi}{2}}=\frac{2}{1+\cos\phi}=\frac{2}{1+\frac{a}{\sqrt{(a^2+b^2)}}}=\frac{2\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}+a}$$

$$=\frac{2\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}+a}\times\frac{\sqrt{(a^2+b^2)}-a}{\sqrt{(a^2+b^2)}-a}=\frac{2(a^2+b^2)-2a\sqrt{(a^2+b^2)}}{b^2};$$

therefore $$\frac{b^3}{2}\sec^2\frac{\phi}{2}=b(a^2+b^2)-ab\sqrt{(a^2+b^2)}.$$

Therefore $$\frac{a^3}{2}\operatorname{cosec}^2\frac{\theta}{2}+\frac{b^3}{2}\sec^2\frac{\phi}{2}=(a+b)(a^2+b^2).$$

18. $\sin^{-1}x+\sin^{-1}\frac{x}{2}=\frac{\pi}{4}$;

therefore $$\sin^{-1}\frac{x}{2}=\frac{\pi}{4}-\sin^{-1}x.$$

Take the sines of both sides; thus

$$\frac{x}{2}=\sin\left(\frac{\pi}{4}-\sin^{-1}x\right)=\sin\frac{\pi}{4}\cdot\sqrt{(1-x^2)}-\cos\frac{\pi}{4}\cdot x$$

$$=\frac{\sqrt{(1-x^2)}-x}{\sqrt{2}};$$

therefore $$x\left(\frac{1}{\sqrt{2}}+1\right)=\sqrt{(1-x^2)};$$

therefore $$x^2\left(\frac{1}{\sqrt{2}}+1\right)^2=1-x^2;$$

therefore $$x^2\left(\frac{5}{2}+\frac{2}{\sqrt{2}}\right)=1;$$

therefore $$x^2(5+2\sqrt{2})=2;$$

therefore $$x^2=\frac{2}{5+2\sqrt{2}}=\frac{2}{5+2\sqrt{2}}\cdot\frac{5-2\sqrt{2}}{5-2\sqrt{2}}=\frac{2}{17}(5-2\sqrt{2}).$$

19. We shall first shew that $\sin^{-1}\frac{2a}{1+a^2}=2\tan^{-1}a$.

Let $\tan^{-1}a=\theta$; then $\tan\theta=a$; and $\sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}=\frac{2a}{1+a^2}$;

therefore $$\sin^{-1}\frac{2a}{1+a^2}=2\theta=2\tan^{-1}a.$$

Similarly $$\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}b.$$

Hence the equation may be written

$$2\tan^{-1}a+2\tan^{-1}b=2\tan^{-1}x;$$

therefore $$\tan^{-1}x=\tan^{-1}a+\tan^{-1}b.$$

Take the tangents of both sides; thus

$$x=\tan(\tan^{-1}a+\tan^{-1}b)=\frac{a+b}{1-ab}.$$

20. Let $\tan^{-1}(x-1)=\alpha$, $\tan^{-1}x=\beta$, $\tan^{-1}(x+1)=\gamma$.

Thus $$\tan^{-1}3x=\alpha+\beta+\gamma.$$

Take the tangents of both sides; thus

$$3x=\tan(\alpha+\beta+\gamma)=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\beta\tan\gamma-\tan\gamma\tan\alpha-\tan\alpha\tan\beta}$$

$$=\frac{3x-x(x^2-1)}{1-x(x+1)-(x+1)(x-1)-x(x-1)}=\frac{4x-x^3}{2-3x^2}.$$

Therefore either $x=0$, or $3(2-3x^2)=4-x^2$; the latter gives $8x^2=2$; therefore $x^2=\frac{1}{4}$; therefore $x=\pm\frac{1}{2}$.

21. $\sin^{-1}2x-\sin^{-1}x\sqrt{3}=\sin^{-1}x$.

Take the sines of both sides; thus

$$2x\sqrt{(1-3x^2)}-x\sqrt{3}\times\sqrt{(1-4x^2)}=x.$$

Thus either $x=0$, or $2\sqrt{(1-3x^2)}-\sqrt{3}\times\sqrt{(1-4x^2)}=1$.

Transpose, thus $2\sqrt{(1-3x^2)}=1+\sqrt{3}\times\sqrt{(1-4x^2)}$.

Square, $4(1-3x^2)=1+2\sqrt{3}\times\sqrt{(1-4x^2)}+3(1-4x^2)$;

therefore $2\sqrt{3}\times\sqrt{(1-4x^2)}=0$;

therefore $1-4x^2=0$; therefore $x=\pm\frac{1}{2}$.

22. $\tan^{-1}\frac{1}{4}+2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{x}=\frac{\pi}{4}$.

Let $\tan^{-1}\frac{1}{4}=\alpha,\quad \tan^{-1}\frac{1}{5}=\beta,\quad \tan^{-1}\frac{1}{6}=\gamma$.

Thus the equation may be written

$$\tan^{-1}\frac{1}{x}=\frac{\pi}{4}-(\alpha+2\beta+\gamma);$$

therefore
$$\frac{1}{x}=\frac{1-\tan(\alpha+2\beta+\gamma)}{1+\tan(\alpha+2\beta+\gamma)}.$$

Now
$$\tan(\alpha+\beta)=\frac{\frac{1}{4}+\frac{1}{5}}{1-\frac{1}{4}\times\frac{1}{5}}=\frac{9}{19},$$

$$\tan(\beta+\gamma)=\frac{\frac{1}{5}+\frac{1}{6}}{1-\frac{1}{5}\times\frac{1}{6}}=\frac{11}{29};$$

therefore
$$\tan(\alpha+\beta+\beta+\gamma)=\frac{\frac{9}{19}+\frac{11}{29}}{1-\frac{9}{19}\times\frac{11}{29}}=\frac{470}{452}.$$

Hence
$$\frac{1}{x}=\frac{1-\frac{470}{452}}{1+\frac{470}{452}}=-\frac{18}{922}=-\frac{9}{461}.$$

23. Let $\tan^{-1}x=\theta$, then $\tan\theta=x$; $\cot 2\theta=\frac{1}{\tan 2\theta}=\frac{1-x^2}{2x}$.

Thus the equation may be written

$$\sin 2\cos^{-1}\frac{1-x^2}{2x}=0.$$

Now since $2\cos^{-1}\frac{1-x^2}{2x}$ has zero for its sine, the angle must be of the form $n\pi$, where n is zero or some integer.

Thus $\quad 2\cos^{-1}\frac{1-x^2}{2x}=n\pi;$ therefore $\cos^{-1}\frac{1-x^2}{2x}=\frac{n\pi}{2};$

therefore $$\frac{1-x^2}{2x}=\cos\frac{n\pi}{2}.$$

Since n is zero or an integer we have $\cos\frac{n\pi}{2}=0$, or 1, or -1.

If $\frac{1-x^2}{2x}=0$, then $x=\pm 1$.

If $\frac{1-x^2}{2x}=1$, then $x^2+2x=1$; and from this we deduce $x=-1\pm\sqrt{2}$.

If $\frac{1-x^2}{2x}=-1$, then $x^2-2x=1$; and from this we deduce $x=1\pm\sqrt{2}$.

24. $$\tan^{-1}\frac{1}{a-1}=\tan^{-1}\frac{1}{x}+\tan^{-1}\frac{1}{a^2-x+1};$$

therefore $$\tan^{-1}\frac{1}{a-1}-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{a^2-x+1}.$$

Take the tangents of both sides; thus

$$\frac{\frac{1}{a-1}-\frac{1}{x}}{1+\frac{1}{(a-1)x}}=\frac{1}{a^2-x+1};$$

therefore $$\frac{x-a+1}{ax-x+1}=\frac{1}{a^2-x+1};$$

therefore $$(x-a+1)(a^2-x+1)=ax-x+1;$$

therefore $$-x^2+x(a^2+a)-a^3+a^2-a+1=ax-x+1;$$

therefore $$x^2-x(a^2+1)+a^3-a^2+a=0.$$

By solving this quadratic in the ordinary way we obtain $x=a$ or a^2-a+1.

25. $\sec\theta-\operatorname{cosec}\theta=\frac{4}{3};$

therefore $$\frac{1}{\cos\theta}-\frac{1}{\sin\theta}=\frac{4}{3};$$

therefore $$\sin\theta-\cos\theta=\frac{4}{3}\sin\theta\cos\theta=\frac{2}{3}\sin 2\theta.$$

Square, thus $$1-\sin 2\theta=\frac{4}{9}\sin^2 2\theta.$$

By solving this quadratic in the usual way we obtain $\sin 2\theta=\frac{3}{4}$, or -3; the former value is alone applicable. Thus $\sin 2\theta=\frac{3}{4}$;

therefore $$2\theta=\sin^{-1}\frac{3}{4};\quad \text{therefore } \theta=\frac{1}{2}\sin^{-1}\frac{3}{4}.$$

26. $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$;

therefore $$\cos\left(\frac{\pi}{2}-\pi\cos\theta\right)=\cos(\pi\sin\theta).$$

Hence, by Art. 67 the solutions are comprised in

$$\frac{\pi}{2}-\pi\cos\theta=2n\pi\pm\pi\sin\theta;$$

therefore $$\cos\theta\pm\sin\theta=\frac{1}{2}-2n.$$

Square, thus $$1\pm\sin 2\theta=\left(\frac{1}{2}-2n\right)^2.$$

If we give to n any integral value, positive or negative, the value of $\sin 2\theta$ is greater than unity. Thus we must have n zero. Then $1\pm\sin 2\theta=\frac{1}{4}$; and therefore $\sin 2\theta=\pm\frac{3}{4}$; thus $2\theta=\pm\sin^{-1}\frac{3}{4}$, and $\theta=\pm\frac{1}{2}\sin^{-1}\frac{3}{4}$.

27. Let $\psi=\sin^{-1}(\sin\theta+\sin\phi)+\sin^{-1}(\sin\theta-\sin\phi)$.

Take the cosines of both sides; thus

$$\cos\psi=\sqrt{1-(\sin\theta+\sin\phi)^2}\sqrt{1-(\sin\theta-\sin\phi)^2}-(\sin\theta+\sin\phi)(\sin\theta-\sin\phi)$$

$$=\sqrt{1-\frac{1}{2}-2\sin\theta\sin\phi}\sqrt{1-\frac{1}{2}+2\sin\theta\sin\phi}-(\sin^2\theta-\sin^2\phi)$$

$$=\sqrt{\frac{1}{4}-4\sin^2\theta\sin^2\phi}-(\sin^2\theta-\sin^2\phi).$$

Now $$\frac{1}{2}=\sin^2\theta+\sin^2\phi,\quad \text{therefore } \frac{1}{4}=(\sin^2\theta+\sin^2\phi)^2;$$

therefore $$\frac{1}{4}-4\sin^2\theta\sin^2\phi=(\sin^2\theta-\sin^2\phi)^2.$$

Thus $$\cos\psi=\pm(\sin^2\theta-\sin^2\phi)-(\sin^2\theta-\sin^2\phi).$$

Taking the upper sign we have $\cos\psi=0$, and therefore $\psi=(2n+1)\frac{\pi}{2}$, where n is any integer.

28. $$3\tan^{-1}\frac{1}{2+\sqrt{3}} - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}.$$

Now let $\tan^{-1}\frac{1}{2+\sqrt{3}} = \theta$, then $\tan\theta = \frac{1}{2+\sqrt{3}}$;

therefore $$\tan 3\theta = \frac{\frac{3}{2+\sqrt{3}} - \frac{1}{(2+\sqrt{3})^3}}{1 - \frac{3}{(2+\sqrt{3})^2}} = \frac{3(2+\sqrt{3})^2 - 1}{(2+\sqrt{3})^3 - 3(2+\sqrt{3})}$$

$$= \frac{20+12\sqrt{3}}{20+12\sqrt{3}} = 1; \quad \text{therefore } 3\theta = \tan^{-1}1.$$

This might also have been inferred from the fact that

$$\frac{1}{2+\sqrt{3}} = \tan 15^0 = \tan\frac{\pi}{12}, \quad \text{so that } \theta = \frac{\pi}{12}.$$

The equation may now be written

$$\tan^{-1}1 - \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{x}.$$

Take the tangents of both sides; thus

$$\frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{1}{x};$$

therefore $\frac{1}{x} = \frac{1}{2}$, therefore $x = 2$.

29. Let $\sin^{-1}\sqrt{\left(\frac{a+b}{a+c}\right)} = \theta$, then $\sin\theta = \sqrt{\left(\frac{a+b}{a+c}\right)}$;

then $$\cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{2(a+b)}{a+c} = \frac{c-a-2b}{a+c}$$

and $$2\theta = \cos^{-1}\frac{c-a-2b}{a+c}.$$

Thus the proposed expression is

$$\sin^{-1}\frac{2b+a-c}{a+c} \pm \cos^{-1}\frac{c-a-2b}{a+c};$$

that is $$\sin^{-1}p \pm \cos^{-1}(-p);$$

when p is put for $\frac{2b+a-c}{a+c}$.

Now $\cos\{\sin^{-1}p \pm \cos^{-1}(-p)\} = -p\sqrt{(1-p^2)} \mp p\sqrt{(1-p^2)}$;
thus zero is one of the values of the cosine, and the corresponding angle is an odd multiple of $\frac{\pi}{2}$.

30. $$\tan^{-1}x + \cot^{-1}y = \tan^{-1}3,$$

therefore $$\tan^{-1}x + \tan^{-1}\frac{1}{y} = \tan^{-1}3.$$

Take the tangents of both sides; thus

$$\frac{x+\frac{1}{y}}{1-\frac{x}{y}} = 3;$$

therefore $$3(y-x) = yx + 1;$$

therefore $$x = \frac{3y-1}{y+3} = \frac{3y+9-10}{y+3} = 3 - \frac{10}{y+3}.$$

Thus if x and y are to be positive integers $y+3$ must be a divisor of 10. Try in succession the various cases, namely $y+3=1$ or 2 or 5 or 10. It will be found that the only admissible cases are $y+3=5$, and $y+3=10$. These give $y=2$ or 7, and the corresponding values of x are 1 and 2.

31. $$\tan^{-1}x + \tan^{-1}y = \tan^{-1}c.$$

Take the tangents of both sides; thus $\frac{x+y}{1-xy} = c$;

therefore $x+y = c(1-xy)$; therefore $x = \frac{c-y}{1+cy}$.

It is obvious that if c and y are positive integers x is either a positive or negative proper fraction, and cannot be a positive integer.

Next take $$\cot^{-1}x + \cot^{-1}y = \cot^{-1}c.$$

Take the cotangents of both sides; thus $\frac{xy-1}{x+y} = c$;

therefore $$xy - 1 = c(x+y),$$

therefore $$x = \frac{cy+1}{y-c} = \frac{cy - c^2 + c^2 + 1}{y-c}$$
$$= c + \frac{c^2+1}{y-c}.$$

Thus if a denote any divisor of c^2+1 we may put $y-c=a$, so that $y = c+a$; and then $x = c + \frac{c^2+1}{a}$.

Hence we see that there are as many solutions in positive integers as there are divisors of c^2+1.

32. $$\tan^{-1}\frac{c_1x-y}{c_1y+x}=\tan^{-1}\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{c_1y}}=\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1},$$

as in the solution of Example 6.

Similarly $$\tan^{-1}\frac{c_2-c_1}{c_2c_1+1}=\tan^{-1}\frac{1}{c_1}-\tan^{-1}\frac{1}{c_2},$$

$$\tan^{-1}\frac{c_3-c_2}{c_3c_2+1}=\tan^{-1}\frac{1}{c_2}-\tan^{-1}\frac{1}{c_3},$$

and so on.

Thus the sum of the terms on the right-hand side of the proposed expression is $\tan^{-1}\frac{x}{y}$.

33. Let $\sin^{-1}\frac{2ab}{a^2+b^2}=\theta$, and $\sin^{-1}\frac{2a'b'}{a'^2+b'^2}=\phi$;

then $$\sin\theta=\frac{2ab}{a^2+b^2}, \text{ and } \sin\phi=\frac{2a'b'}{a'^2+b'^2};$$

therefore $$\cos\theta=\frac{a^2-b^2}{a^2+b^2}, \text{ and } \cos\phi=\frac{a'^2-b'^2}{a'^2+b'^2};$$

therefore $$\sin(\theta+\phi)=\frac{2ab\,(a'^2-b'^2)+2a'b'\,(a^2-b^2)}{(a^2+b^2)\,(a'^2+b'^2)}$$

$$=\frac{2ab\,(a'^2-b'^2)+2a'b'\,(a^2-b^2)}{(ab'+a'b)^2+(aa'-bb')^2}=\frac{2\,(ab'+a'b)\,(aa'-bb')}{(ab'+a'b)^2+(aa'-bb')^2};$$

therefore $$\theta+\phi=\sin^{-1}\frac{2pq}{p^2+q^2},$$

where p and q are rational expressions.

Then if there be another angle $\sin^{-1}\frac{2a''b''}{a''^2+b''^2}$, we may denote it by ψ; then $\sin\{(\theta+\phi)+\psi\}$ will take the form $\frac{2rs}{r^2+s^2}$ where r and s are rational. And so on.

34. We may take for the simplest value of $\sin^{-1}\frac{(-1)^m}{2}$ the angle $(-1)^m\frac{\pi}{6}$; as is evident by supposing m first even and then odd. This will be the α of Art. 66; and the general solution is $n\pi+(-1)^n\alpha$, that is $n\pi+(-1)^{m+n}\frac{\pi}{6}$.

Or we may take the form $(m+n)\,\pi+(-1)^n\frac{\pi}{6}$.

For the sine of this angle

$$=\sin m\pi \cos \left\{n\pi+(-1)^n\frac{\pi}{6}\right\}+\cos m\pi \sin \left\{n\pi+(-1)^n\frac{\pi}{6}\right\}$$

$$=\cos m\pi \sin \left\{n\pi+(-1)^n\frac{\pi}{6}\right\}=\cos m\pi\times\frac{1}{2}=(-1)^m\times\frac{1}{2}.$$

35. If m be even the value is $\cos^{-1}\frac{1}{2}$, that is $2n\pi\pm\frac{\pi}{3}$.

If m be odd the value is $\cos^{-1}\left(-\frac{1}{2}\right)$, that is $2n\pi\pm\left(\pi+\frac{\pi}{3}\right)$.

Both forms may be comprised in $(2p+m)\pi\pm\frac{\pi}{3}$, where p is any integer.

For $2n\pi\pm\frac{\pi}{3}$ consists of an *even* multiple of π augmented by $\pm\frac{\pi}{3}$; and $2n\pi\pm\left(\pi+\frac{\pi}{3}\right)$ consists of an *odd* multiple of π augmented by $\pm\frac{\pi}{3}$.

36. If m be even the value is $\tan^{-1}1$, that is $n\pi+\frac{\pi}{4}$.

If m be odd the value is $\tan^{-1}(-1)$, that is $n\pi-\frac{\pi}{4}$.

Both forms may be comprised in $n\pi+(-1)^m\frac{\pi}{4}$.

XIX.

1. $\{\cos 4A+\sqrt{(-1)}\sin 4A\}^{\frac{1}{2}}=\pm\{\cos 2A+\sqrt{(-1)}\sin 2A\}$ by Art. 267.

2. $-1=\cos\pi=\cos\pi+\sqrt{(-1)}\sin\pi$;

therefore one value of $(-1)^{\frac{1}{3}}=\cos\frac{\pi}{3}+\sqrt{(-1)}\sin\frac{\pi}{3}$,

so we may put $-1=\cos 3\pi$, or $\cos 5\pi$, and thus we obtain two other values for $(-1)^{\frac{1}{3}}$, namely,

$$\cos\frac{3\pi}{3}+\sqrt{(-1)}\sin\frac{3\pi}{3},\quad\text{that is } -1,$$

and

$$\cos\frac{5\pi}{3}+\sqrt{(-1)}\sin\frac{5\pi}{3}.$$

3. We may put $-1=\cos\pi$, or $\cos 3\pi$, or $\cos 5\pi$, or $\cos 7\pi$, or $\cos 9\pi$, or $\cos 11\pi$; and thus

$$-1=\cos\theta+\sqrt{(-1)}\sin\theta$$

where $\theta=\pi$, or 3π, or 5π, or 7π, or 9π, or 11π.

Hence the six values of $(-1)^{\frac{1}{6}}$ are contained in

$$\cos\frac{\theta}{6}+\sqrt{(-1)}\sin\frac{\theta}{6},$$

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where θ has any of the six values just specified.

4. $$1+\sqrt{(-1)}=\sqrt{2}\left\{\frac{1}{\sqrt{2}}+\frac{\sqrt{(-1)}}{\sqrt{2}}\right\}$$
$$=\sqrt{2}\{\cos\theta+\sqrt{(-1)}\sin\theta\},$$

where for θ we may put $\frac{\pi}{4}+2n\pi$, where n is any integer.

Therefore $$\{1+\sqrt{(-1)}\}^{\frac{1}{3}}=2^{\frac{1}{6}}\left\{\cos\frac{\theta}{3}+\sqrt{(-1)}\sin\frac{\theta}{3}\right\};$$

and the three values will be obtained by putting for θ in succession $\frac{\pi}{4}$, $2\pi+\frac{\pi}{4}$, and $4\pi+\frac{\pi}{4}$.

5. Since $\frac{\sin\theta}{\theta}$ is given nearly equal to unity, we may infer that θ is a small angle. Hence we have approximately, by Art. 274,

$$\sin\theta=\theta-\frac{\theta^3}{6};$$

thus $$1-\frac{\theta^2}{6}=\frac{2165}{2166};$$

therefore $$\frac{\theta^2}{6}=\frac{1}{2166},$$

therefore $$\theta^2=\frac{1}{361},$$

therefore $$\theta=\frac{1}{19}.$$

This is the circular measure of the angle; therefore the number of degrees $=\frac{1}{19}$ of $\frac{180}{\pi}=\frac{1}{19}$ of $57^0.29\ldots=3^0$ approximately.

6. $\sin\left(\frac{\pi}{6}+\theta\right)=\cdot 51.$

As $\cdot 51$ is very nearly equal to $\sin\frac{\pi}{6}$ we may infer that θ is very small. We have

$$\sin\frac{\pi}{6}\cos\theta+\cos\frac{\pi}{6}\sin\theta=\cdot 51,$$

therefore
$$\frac{1}{2}\left(1-\frac{\theta^2}{2}\right)+\frac{\sqrt{3}}{2}\theta=\cdot 51 \text{ approximately.}$$

Hence, neglecting θ^2, we have $\frac{\sqrt{3}}{2}\theta=\frac{1}{100}$, and therefore $\theta=\frac{1}{50\sqrt{3}}$.

Then if we retain the term in θ^2 we have

$$\theta=\frac{1}{50\sqrt{3}}+\frac{\theta^2}{2\sqrt{3}};$$

and putting for θ^2 its approximate value, we have for a closer approximation

$$\theta=\frac{1}{50\sqrt{3}}+\frac{1}{2\sqrt{3}}\left(\frac{1}{50\sqrt{3}}\right)^2$$
$$=\frac{1}{50\sqrt{3}}+\frac{1}{15000\sqrt{3}}.$$

The same result will be obtained if we solve the quadratic equation $\theta=\frac{1}{50\sqrt{3}}+\frac{\theta^2}{2\sqrt{3}}$ in the usual way, select the least root, and take its approximate value. See *Algebra*, Art. 526, Example (3).

7. Suppose
$$\tan x=a_1x+\frac{a_3x^3}{\underline{|3}}+\frac{a_5x^5}{\underline{|5}}+\ldots;$$

then
$$\sin x=\cos x\left\{a_1x+\frac{a_3x^3}{\underline{|3}}+\frac{a_5x^5}{\underline{|5}}+\ldots\ldots\right\}.$$

Substitute for $\sin x$ and $\cos x$ by Art. 274; thus

$$x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\frac{x^7}{\underline{|7}}+\ldots$$
$$=\left\{1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-\frac{x^6}{\underline{|6}}+\ldots\right\}\left\{a_1x+\frac{a_3x^3}{\underline{|3}}+\frac{a_5x^5}{\underline{|5}}+\ldots\right\}.$$

Then, according to the known principles of Algebra, we may equate the coefficient of any power of x on the left-hand side to the coefficient of the same power obtained by working out the product on the right-hand side. Take, for instance, the coefficient of x^{2n+1}; thus we obtain

$$\frac{(-1)^n}{\underline{|2n+1}}=\frac{a_{2n+1}}{\underline{|2n+1}}-\frac{a_{2n-1}}{2\underline{|2n-1}}+\frac{a_{2n-3}}{\underline{|4}\,\underline{|2n-3}}\ldots+(-1)^n\frac{a_1}{\underline{|2n}}.$$

Multiply by $\underline{|2n+1}$ and transpose; thus we get

$$a_{2n+1}=\frac{(2n+1)\,2n}{2}\,a_{2n-1}-\frac{(2n+1)\,2n\,(2n-1)\,(2n-2)}{\underline{|4}}\,a_{2n-3}$$
$$+\ldots+(2n+1)\,(-1)^{n+1}a_1+(-1)^n.$$

8. Let $\theta\cot\theta=a_0+a_2\theta^2+a_4\theta^4+\ldots;$

then $$\theta\cos\theta=\sin\theta\,\{a_0+a_2\theta^2+a_4\theta^4+\ldots\}.$$

Substitute for $\cos\theta$ and $\sin\theta$ by Art. 274; thus

$$\theta\left(1-\frac{\theta^2}{\underline{|2}}+\frac{\theta^4}{\underline{|4}}-\frac{\theta^6}{\underline{|6}}+\ldots\right)$$
$$=\left\{\theta-\frac{\theta^3}{\underline{|3}}+\frac{\theta^5}{\underline{|5}}-\frac{\theta^7}{\underline{|7}}+\ldots\right\}\{a_0+a_2\theta^2+a_4\theta^4+\ldots\}.$$

Equate the coefficients of θ^{2n+1}; thus

$$\frac{(-1)^n}{\underline{|2n}}=a_{2n}-\frac{a_{2n-2}}{\underline{|3}}+\frac{a_{2n-4}}{\underline{|5}}-\ldots+\frac{(-1)^n a_0}{\underline{|2n+1}}.$$

Transpose; thus we get

$$a_{2n}=\frac{a_{2n-2}}{\underline{|3}}-\frac{a_{2n-4}}{\underline{|5}}+\ldots+\frac{(-1)^{n-1}a_0}{\underline{|2n+1}}+\frac{(-1)^n}{\underline{|2n}}.$$

To find the first four terms of $\theta\cot\theta$ we have the following equations:

$$1=a_0,$$
$$-\frac{1}{\underline{|2}}=a_2-\frac{a_0}{\underline{|3}},$$
$$\frac{1}{\underline{|4}}=a_4-\frac{a_2}{\underline{|3}}+\frac{a_0}{\underline{|5}},$$
$$-\frac{1}{\underline{|6}}=a_6-\frac{a_4}{\underline{|3}}+\frac{a_2}{\underline{|5}}-\frac{a_0}{\underline{|7}}.$$

Hence we obtain

$$a_0=1,\qquad a_2=\frac{1}{\underline{|3}}-\frac{1}{2}=\frac{1}{6}-\frac{1}{2}=-\frac{1}{3},\qquad a_4=\frac{1}{\underline{|4}}-\frac{1}{3\underline{|3}}-\frac{1}{\underline{|5}}=-\frac{1}{45};$$

$$a_6=-\frac{1}{\underline{|6}}-\frac{1}{45\underline{|3}}+\frac{1}{3\underline{|5}}+\frac{1}{\underline{|7}}=-\frac{2}{945}.$$

(9) Let $\sec\theta = a_0 + a_2\theta^2 + a_4\theta^4 + \dots;$

then $$1 = \cos\theta\,(a_0 + a_2\theta^2 + a_4\theta^4 + \dots)$$

$$= \left(1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \dots\right)(a_0 + a_2\theta^2 + a_4\theta^4 + \dots).$$

Then equating to zero the coefficient of θ^{2n} in the expression on the right-hand side we get

$$0 = a_{2n} - \frac{a_{2n-2}}{\underline{|2}} + \frac{a_{2n-4}}{\underline{|4}} - \frac{a_{2n-6}}{\underline{|6}} + \dots + \frac{(-1)^n}{\underline{|2n}}a_0.$$

Transpose; then we obtain $a_{2n} = \dfrac{a_{2n-2}}{\underline{|2}} - \dfrac{a_{2n-4}}{\underline{|4}} + \dots + \dfrac{(-1)^{n+1}a_0}{\underline{|2n}}.$

10. $$a + b = \cos 2\alpha + \cos 2\beta + \sqrt{(-1)}\,\{\sin 2\alpha + \sin 2\beta\}$$

$$= 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2\sqrt{(-1)}\sin(\alpha+\beta)\cos(\alpha-\beta)$$

$$= 2\cos(\alpha-\beta)\{\cos(\alpha+\beta) + \sqrt{(-1)}\sin(\alpha+\beta)\}.$$

Thus $\dfrac{b}{a+b} = \dfrac{\cos 2\beta + \sqrt{(-1)}\sin 2\beta}{2\cos(\alpha-\beta)\{\cos(\alpha+\beta) + \sqrt{(-1)}\sin(\alpha+\beta)\}}$; multiply both numerator and denominator by $\cos(\alpha+\beta) - \sqrt{(-1)}\sin(\alpha+\beta)$;

thus we get $$\frac{\cos(\beta-\alpha) + \sqrt{(-1)}\sin(\beta-\alpha)}{2\cos(\alpha-\beta)}.$$

Similarly $\dfrac{c}{a+c} = \dfrac{\cos(\gamma-\alpha) + \sqrt{(-1)}\sin(\gamma-\alpha)}{2\cos(\alpha-\gamma)}.$

Therefore $\dfrac{bc}{(a+b)(a+c)} = \dfrac{\cos(\beta+\gamma-2\alpha) + \sqrt{(-1)}\sin(\beta+\gamma-2\alpha)}{4\cos(\alpha-\beta)\cos(\alpha-\gamma)}.$

11. $$\{\cos\theta + \cos\phi + \sqrt{(-1)}(\sin\theta + \sin\phi)\}^n$$

$$= \left\{2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} + 2\sqrt{(-1)}\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right\}^n$$

$$= 2^n\left(\cos\frac{\theta-\phi}{2}\right)^n\left\{\cos\frac{\theta+\phi}{2} + \sqrt{(-1)}\sin\frac{\theta+\phi}{2}\right\}^n$$

$$= 2^n\left(\cos\frac{\theta-\phi}{2}\right)^n\left\{\cos\frac{n}{2}(\theta+\phi) + \sqrt{(-1)}\sin\frac{n}{2}(\theta+\phi)\right\}.$$

Similarly $$\{\cos\theta + \cos\phi - \sqrt{(-1)}(\sin\theta + \sin\phi)\}^n$$

$$= 2^n\left(\cos\frac{\theta-\phi}{2}\right)^n\left\{\cos\frac{n}{2}(\theta+\phi) - \sqrt{(-1)}\sin\frac{n}{2}(\theta+\phi)\right\}.$$

Hence by addition we get $2^{n+1}\left(\cos\dfrac{\theta-\phi}{2}\right)^n \cos\dfrac{n}{2}(\theta+\phi)$.

12. $$\frac{c}{2n}(1+nx)\left(1+\frac{n}{x}\right)=\frac{c}{2n}\left\{1+n^2+n\left(x+\frac{1}{x}\right)\right\}$$

$$=\frac{c}{2n}\{1+n^2+2n\cos\theta\}=\frac{c(1+n^2)}{2n}+c\cos\theta.$$

Now $\sqrt{(1-c^2)}=nc-1$; therefore $1-c^2=(nc-1)^2$, therefore $-c^2=n^2c^2-2nc$, therefore $2n=(n^2+1)c$.

Thus $\dfrac{c(1+n^2)}{2n}=1$, and the expression becomes $1+c\cos\theta$.

13. Let r denote the radius, and θ the circular measure of the angle; then the length of the arc is $r\theta$.

The chord of the arc is $2r\sin\dfrac{\theta}{2}$, and the chord of half the arc is $2r\sin\dfrac{\theta}{4}$.

Now let it be required to determine two numbers l and m, such that approximately

$$l\times 2r\sin\frac{\theta}{2}+m\times 2r\sin\frac{\theta}{4}=r\theta.$$

Expand $\sin\dfrac{\theta}{4}$ and $\sin\dfrac{\theta}{2}$ by Art. 274. Thus

$$2l\left\{\frac{\theta}{2}-\frac{1}{\underline{|3}}\left(\frac{\theta}{2}\right)^3+\dots\right\}+2m\left\{\frac{\theta}{4}-\frac{1}{\underline{|3}}\left(\frac{\theta}{4}\right)^3+\dots\right\}=\theta.$$

Neglect all powers of θ above θ^3; then to make this formula hold we must put

$$l+\frac{m}{2}=1,\quad \frac{l}{(2)^3}+\frac{m}{(4)^3}=0.$$

Therefore $m=-8l$; therefore $-3l=1$.

Thus $$l=-\frac{1}{3}\text{ and }m=\frac{8}{3}.$$

This establishes the rule.

14. Proceed as in Example 13.

The chord of one-fourth of the arc is $2r\sin\dfrac{\theta}{8}$.

Let it be required to determine the numbers l, m, n such that approximately

$$l\times 2r\sin\frac{\theta}{2}+m\times 2r\sin\frac{\theta}{4}+n\times 2r\sin\frac{\theta}{8}=r\theta.$$

In this case we can make the approximation closer than in Example 13; for we shall retain θ^5 and neglect only the higher powers. Thus

$$2l\left\{\frac{\theta}{2}-\frac{1}{\underline{|3}}\left(\frac{\theta}{2}\right)^3+\frac{1}{\underline{|5}}\left(\frac{\theta}{2}\right)^5\right\}+2m\left\{\frac{\theta}{4}-\frac{1}{\underline{|3}}\left(\frac{\theta}{4}\right)^3+\frac{1}{\underline{|5}}\left(\frac{\theta}{4}\right)^5\right\}$$
$$+2n\left\{\frac{\theta}{8}-\frac{1}{\underline{|3}}\left(\frac{\theta}{8}\right)^3+\frac{1}{\underline{|5}}\left(\frac{\theta}{8}\right)^5\right\}=\theta.$$

Hence we must put

$$l+\frac{m}{2}+\frac{n}{4}=1, \qquad \frac{l}{(2)^3}+\frac{m}{(4)^3}+\frac{n}{(8)^3}=0, \qquad \frac{l}{(2)^5}+\frac{m}{(4)^5}+\frac{n}{(8)^5}=0.$$

The values of l, m, n given by these equations are

$$l=\frac{1}{45}, \qquad m=-\frac{40}{45}, \qquad n=\frac{256}{45}.$$

15.
$$x-b=\cos 2\theta+\sqrt{(-1)}\sin 2\theta-\cos 2\beta-\sqrt{(-1)}\sin 2\beta$$
$$=2\sin(\beta-\theta)\{\sin(\beta+\theta)-\sqrt{(-1)}\cos(\beta+\theta)\}$$
$$=\frac{2\sin(\beta-\theta)}{\sqrt{(-1)}}\{\cos(\beta+\theta)+\sqrt{(-1)}\sin(\beta+\theta)\}.$$

In like manner

$$a-b=\frac{2\sin(\beta-a)}{\sqrt{(-1)}}\{\cos(\beta+a)+\sqrt{(-1)}\sin(\beta+a)\}.$$

Therefore $\dfrac{x-b}{a-b}=\dfrac{\sin(\beta-\theta)}{\sin(\beta-a)}\cdot\dfrac{\cos(\beta+\theta)+\sqrt{(-1)}\sin(\beta+\theta)}{\cos(\beta+a)+\sqrt{(-1)}\sin(\beta+a)}$, multiply both numerator and denominator by $\cos(\beta+a)-\sqrt{(-1)}\sin(\beta+a)$;

thus we get $$\frac{\sin(\theta-\beta)}{\sin(a-\beta)}\{\cos(\theta-a)+\sqrt{(-1)}\sin(\theta-a)\}.$$

Similarly we transform $\dfrac{x-c}{a-c}$; and thus we obtain

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}=\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(a-\beta)\sin(a-\gamma)}\{\cos 2(\theta-a)+\sqrt{(-1)}\sin 2(\theta-a)\}.$$

In like manner we transform $\dfrac{(x-c)(x-a)}{(b-c)(b-a)}$ and $\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$. Then by equating to zero the coefficient of the imaginary part we obtain

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(a-\beta)\sin(a-\gamma)}\sin 2(\theta-a)+\frac{\sin(\theta-\gamma)\sin(\theta-a)}{\sin(\beta-\gamma)\sin(\beta-a)}\sin 2(\theta-\beta)$$
$$+\frac{\sin(\theta-a)\sin(\theta-\beta)}{\sin(\gamma-a)\sin(\gamma-\beta)}\sin 2(\theta-\gamma)=0.$$

And then by equating the real parts we have

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\cos 2(\theta-\alpha)+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}\cos 2(\theta-\beta)$$

$$+\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}\cos 2(\theta-\gamma)=1.$$

XX.

1. Proceed as in Art. 282. Thus we obtain

$$-2^{4n+1}(\sin\theta)^{4n+2}=\cos(4n+2)\theta-(4n+2)\cos 4n\theta$$

$$+\frac{(4n+2)(4n+1)}{\underline{|2}}\cos(4n-2)\theta-\dots$$

$$-\frac{(4n+2)(4n+1)\dots(2n+2)}{2\,\underline{|2n+1}}.$$

2. Proceed as in Art. 283. Thus we obtain

$$2^{4n}(\sin\theta)^{4n+1}=\sin(4n+1)\theta-(4n+1)\sin(4n-1)\theta$$

$$+\frac{(4n+1)4n}{\underline{|2}}\sin(4n-3)\theta-\dots$$

$$+\frac{(4n+1)4n(4n-1)\dots(2n+2)}{\underline{|2n}}\sin\theta.$$

3. Proceed as in Art. 280. Thus we obtain

$$2^{2n-1}\cos^{2n}\theta=\cos 2n\theta+2n\cos(2n-2)\theta+\frac{2n(2n-1)}{\underline{|2}}\cos(2n-4)\theta$$

$$+\dots\frac{2n(2n-1)\dots(n+1)}{2\,\underline{|n}}.$$

4. We have

$$\frac{a^2\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)}=\frac{a^2\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}=\frac{a^2\cos\frac{1}{2}A\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}A\sin\frac{1}{2}A}$$

$$=\frac{2a^2}{\sin A}\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)=\frac{a^2}{\sin A}(\sin B+\sin C)$$

$$=\frac{a^2\sin B}{\sin A}+\frac{a^2\sin C}{\sin A}=\frac{a^2b}{a}+\frac{a^2c}{a}=ab+ac.$$

Similarly $$\frac{b^2 \cos\frac{1}{2}(C-A)}{\cos\frac{1}{2}(C+A)} = ba + bc,$$

and $$\frac{c^2 \cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} = ca + cb.$$

Then by addition we obtain the required result.

5. Suppose the triangle to have all its angles acute.

Then $$a \sin(BAD - CAD) = a(\sin BAD \cos CAD - \cos BAD \sin CAD)$$
$$= a(\cos B \sin C - \sin B \cos C).$$

Similarly $$b \sin(CBE - ABE) = b(\cos C \sin A - \sin C \cos A),$$

and $$c \sin(ACF - BCF) = c(\cos A \sin B - \sin A \cos B).$$

The sum of the three expressions
$$= \cos A(c \sin B - b \sin C) + \cos B(a \sin C - c \sin A)$$
$$+ \cos C(b \sin A - a \sin B) = 0, \text{ by Art. 214.}$$

If the triangle has an obtuse angle, let it be C; then it will be found that instead of $\cos C$ we have $\cos(180^\circ - C)$ in the preceding expressions; and the result is still zero.

6. We have $$\rho\left\{\cot\frac{1}{2}DAB + \cot\frac{1}{2}DBA\right\} = AB.$$

Now $$\cot\frac{1}{2}DAB = \cot\frac{1}{2}\left(\frac{\pi}{2} - A\right) = \frac{\cos\frac{1}{2}\left(\frac{\pi}{2} - A\right)}{\sin\frac{1}{2}\left(\frac{\pi}{2} - A\right)}$$

$$= \frac{2\cos^2\frac{1}{2}\left(\frac{\pi}{2} - A\right)}{2\sin\frac{1}{2}\left(\frac{\pi}{2} - A\right)\cos\frac{1}{2}\left(\frac{\pi}{2} - A\right)} = \frac{1 + \cos\left(\frac{\pi}{2} - A\right)}{\sin\left(\frac{\pi}{2} - A\right)}$$

$$= \frac{1 + \sin A}{\cos A} = \sec A + \tan A.$$

Similarly $$\cot\frac{1}{2}DBA = \sec B + \tan B.$$

Therefore $$\rho\{\sec A + \tan A + \sec B + \tan B\} = AB.$$

7. By joining the centres of the circles we form an equilateral triangle of which each side is $2a$; and therefore the area is $\frac{(2a)^2\sqrt{3}}{4}$, that is $a^2\sqrt{3}$. The area of each of the three sectors which are formed by the radii and arc of a circle is $\frac{a^2}{2} \times \frac{2\pi}{6}$, that is $\frac{\pi a^2}{6}$; therefore the area of the three sectors is $\frac{\pi a^2}{2}$. Hence the area of the space between the circles $= a^2\sqrt{3} - \frac{\pi a^2}{2} = a^2\left(\sqrt{3} - \frac{\pi}{2}\right)$.

8. Let R be the radius of the circle. The inscribed polygon of n sides consists of n triangles; and therefore the area of the polygon is

$$n\frac{R^2}{2}\sin\frac{2\pi}{n}.$$

The area of an inscribed figure of half the number of sides is

$$\frac{n}{2}\frac{R^2}{2}\sin\frac{4\pi}{n}.$$

The area of a circumscribed polygon of $\frac{n}{2}$ sides is $\frac{n}{2}R^2\tan\frac{2\pi}{n}$.

We have to shew that

$$\left(\frac{nR^2}{2}\sin\frac{2\pi}{n}\right)^2 = \frac{n}{4}R^2\sin\frac{4\pi}{n} \times \frac{n}{2}R^2\tan\frac{2\pi}{n},$$

or that
$$\sin^2\frac{2\pi}{n} = \frac{1}{2}\sin\frac{4\pi}{n}\tan\frac{2\pi}{n};$$

and this is obvious since $\sin\frac{4\pi}{n} = 2\sin\frac{2\pi}{n}\cos\frac{2\pi}{n}$.

9. Let R be the radius of the circle. The area of the circumscribed polygon of n sides is $nR^2\tan\frac{\pi}{n}$. The area of the inscribed polygon of n sides is $\frac{nR^2}{2}\sin\frac{2\pi}{n}$. The area of the circumscribed polygon of $\frac{n}{2}$ sides is $\frac{n}{2}R^2\tan\frac{2\pi}{n}$. We have to shew that $nR^2\tan\frac{\pi}{n}$ is an harmonic mean between $\frac{nR^2}{2}\sin\frac{2\pi}{n}$ and $\frac{nR^2}{2}\tan\frac{2\pi}{n}$; or that $2\tan\frac{\pi}{n}$ is an harmonic mean between $\sin\frac{2\pi}{n}$ and $\tan\frac{2\pi}{n}$.

Now the harmonic mean between $\sin\frac{2\pi}{n}$ and $\tan\frac{2\pi}{n}$ is

$$\frac{2\sin\frac{2\pi}{n}\tan\frac{2\pi}{n}}{\sin\frac{2\pi}{n}+\tan\frac{2\pi}{n}}, \text{ that is } \frac{2\sin\frac{2\pi}{n}}{1+\cos\frac{2\pi}{n}}, \text{ that is } 2\tan\frac{\pi}{n}.$$

10. Let r be the radius of the circle; then $c = 2r \sin \frac{2\pi}{10} = 2r \sin \frac{\pi}{5}$;

therefore $r = \dfrac{c}{2 \sin \frac{\pi}{5}} = \dfrac{2c}{\sqrt{(10 - 2\sqrt{5})}} = \dfrac{2c}{\sqrt{2}\sqrt{(5 - \sqrt{5})}}$. Multiply both numerator and denominator by $\sqrt{(5 + \sqrt{5})}$. Thus we obtain $\dfrac{2c\sqrt{(5+\sqrt{5})}}{\sqrt{2} \times \sqrt{(25-5)}}$,

that is $\dfrac{2c\sqrt{(5+\sqrt{5})}}{\sqrt{(40)}}$, that is $\dfrac{c\sqrt{(5+\sqrt{5})}}{\sqrt{(10)}}$.

11. Let A, B, C denote the centres of the three circles. Let tangents to the arc of the first circle meet at T; then the distance of T from the point of contact is $a \tan \frac{A}{2}$.

Now $$\cos A = \frac{(a+c)^2 + (a+b)^2 - (b+c)^2}{2(a+c)(a+b)} = \frac{a^2 + a(b+c) - bc}{(a+c)(a+b)};$$

therefore $$\frac{1 - \cos A}{1 + \cos A} = \frac{bc}{a(a+b+c)};$$

therefore $$\tan \frac{A}{2} = \sqrt{\frac{bc}{a(a+b+c)}}; \text{ therefore } a \tan \frac{A}{2} = \sqrt{\frac{abc}{a+b+c}}.$$

We shall obtain the same symmetrical expression for the distance of the point of intersection of any two tangents from the points of contact; and thus it follows that the three tangents meet at a common point.

12. Use Article 254. Here $s = 7$, $s - a = 4$, $s - b = 4$, $s - c = 3$, $s - d = 3$. Thus the area $= \sqrt{4 \times 4 \times 3 \times 3} = 12$.

Since the sum of a pair of opposite sides is equal to the sum of the other pair, a circle may be inscribed in the quadrilateral. Let ρ denote the radius of this inscribed circle; then $\frac{\rho}{2}(a + b + c + d) =$ the area of the quadrilateral. Thus $\rho s =$ the area.

In the present case $$\rho = \frac{12}{7}.$$

Also $$ab + cd = 25, \quad ac + bd = 24, \quad ad + bc = 24.$$

Hence the radius of the circumscribed circle

$$= \frac{1}{4}\sqrt{\frac{25 \times 24 \times 24}{3 \times 3 \times 4 \times 4}} = \frac{5 \times 24}{4 \times 12} = \frac{5}{2}.$$

13. Let n be the number of the sides, R the radius of the circle.

Then $$\frac{n}{2} R^2 \sin \frac{2\pi}{n} \text{ is to } nR^2 \tan \frac{\pi}{n} \text{ as } 3 \text{ is to } 4.$$

Thus $$\frac{\sin\frac{2\pi}{n}}{2\tan\frac{\pi}{n}}=\frac{3}{4};$$

therefore $$\cos^2\frac{\pi}{n}=\frac{3}{4};\quad\text{therefore}\quad\cos\frac{\pi}{n}=\frac{\sqrt{3}}{2}.$$

But $$\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2};\quad\text{therefore}\quad n=6.$$

14. Let A, B, C denote the centres of the three circles.

Then $$\alpha=2a\sin\frac{A}{2},\quad \beta=2b\sin\frac{B}{2},\quad \gamma=2c\sin\frac{C}{2}.$$

Now from the solution of Example 11 we see that

$$\cos A=\frac{a^2+a(b+c)-bc}{(a+b)(a+c)};\quad\text{therefore } 1-\cos A=\frac{2bc}{(a+b)(a+c)};$$

therefore $$\sin\frac{A}{2}=\sqrt{\frac{bc}{(a+b)(a+c)}};$$

therefore $$\alpha=2a\sqrt{\frac{bc}{(a+b)(a+c)}}.$$

Similar expressions hold for β and γ. Thus

$$\frac{1}{\alpha\beta\gamma}=\frac{1}{8abc}\left\{\frac{(a+b)(a+c)}{bc}\times\frac{(b+a)(b+c)}{ac}\times\frac{(c+a)(c+b)}{bc}\right\}^{\frac{1}{2}}$$

$$=\frac{(a+b)(b+c)(c+a)}{8a^2b^2c^2};$$

therefore $$\frac{8}{\alpha\beta\gamma}=\left(\frac{1}{b}+\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{c}\right).$$

15. Let $u=\left(\frac{\tan\theta}{\theta}\right)^{\frac{3}{\theta^2}}$; then $\log u=\frac{3}{\theta^2}\log\frac{\tan\theta}{\theta}$.

Now $\tan\theta=\theta+\frac{\theta^3}{3}+$ terms in θ^5 and higher powers of θ; see Example XIX. 7.

Therefore $$\frac{\tan\theta}{\theta}=1+\frac{\theta^2}{3}+\dots$$

Then $\log\left(1+\frac{\theta^2}{3}+\dots\right)=\frac{\theta^2}{3}+$ terms in θ^4 and higher powers of θ.

Therefore $\frac{3}{\theta^2}\log\frac{\tan\theta}{\theta}=1+$ terms in θ^2 and higher powers of θ.

Therefore when θ is indefinitely diminished $\frac{3}{\theta^2}\log\frac{\tan\theta}{\theta}=1$, and therefore

$$\left(\frac{\tan\theta}{\theta}\right)^{\frac{3}{\theta^2}}=e.$$

16. Equate the expressions for AC^2 and BD^2 given in Art. 254. Thus

$$\frac{(ac+bd)(ad+bc)}{ab+cd}=\frac{(ac+bd)(ab+cd)}{ad+bc}.$$

Therefore $$(ad+bc)^2=(ab+cd)^2;$$

therefore $$ad+bc=ab+cd;$$

therefore $$(a-c)(d-b)=0.$$

Therefore either $a=c$ or $b=d$.

17. Let A and B be the centres of the two circles, and C a point of intersection. The angle between the tangents at C is therefore γ.

The angle $ACB=\frac{\pi}{2}+\frac{\pi}{2}-\gamma=\pi-\gamma$.

Then $$AB^2=a^2+b^2-2ab\cos(\pi-\gamma)=a^2+b^2+2ab\cos\gamma.$$

Let x denote the length of the common chord; then the area of the triangle $ABC=\frac{1}{2}\times\frac{x}{2}\times AB$; and this area also $=\frac{1}{2}AC.CB\sin ACB$.

Thus $$x=\frac{2AC.CB\sin ACB}{AB}$$

$$=\frac{2ab\sin\gamma}{\sqrt{(a^2+b^2+2ab\cos\gamma)}}.$$

18. We have $\frac{r}{R}=\frac{S}{s}\div\frac{abc}{4S}=\frac{4S^2}{sabc}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

Now we have shewn in the solution of Example XIII. 40, that the expression $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ can never be greater than $\frac{1}{2}$. Hence r cannot be greater than $\frac{1}{2}R$.

XXI.

1. $$\frac{\sin 2\theta}{1-\cos 2\theta} = \frac{e^{2\theta\iota} - e^{-2\theta\iota}}{2\iota\left(1 - \dfrac{e^{2\theta\iota} + e^{-2\theta\iota}}{2}\right)} = \frac{e^{2\theta\iota} - e^{-2\theta\iota}}{\iota(2 - e^{2\theta\iota} - e^{-2\theta\iota})}$$

$$= \frac{\iota(e^{2\theta\iota} - e^{-2\theta\iota})}{e^{2\theta\iota} + e^{-2\theta\iota} - 2} = \frac{\iota(e^{\theta\iota} + e^{-\theta\iota})(e^{\theta\iota} - e^{-\theta\iota})}{(e^{\theta\iota} - e^{-\theta\iota})^2}$$

$$= \frac{\iota(e^{\theta\iota} + e^{-\theta\iota})}{e^{\theta\iota} - e^{-\theta\iota}} = \frac{e^{\theta\iota} + e^{-\theta\iota}}{2} \div \frac{e^{\theta\iota} - e^{-\theta\iota}}{2\iota} = \frac{\cos\theta}{\sin\theta}.$$

2. Let the angle opposite to the smaller side be $\frac{\pi}{4} - \theta$, and the angle opposite to the larger side $\frac{\pi}{4} + \theta$. Thus

$$\frac{\sin\left(\frac{\pi}{4} - \theta\right)}{\sin\left(\frac{\pi}{4} + \theta\right)} = \frac{49}{51};$$

therefore $$\frac{\sin\left(\frac{\pi}{4} + \theta\right) - \sin\left(\frac{\pi}{4} - \theta\right)}{\sin\left(\frac{\pi}{4} + \theta\right) + \sin\left(\frac{\pi}{4} - \theta\right)} = \frac{51-49}{51+49} = \frac{1}{50};$$

therefore $$\frac{2\sin\theta\cos\frac{\pi}{4}}{2\cos\theta\sin\frac{\pi}{4}} = \frac{1}{50};$$

therefore $$\tan\theta = \frac{1}{50}.$$

But by Art. 293 we have

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \ldots,$$

thus $$\theta = {\cdot}02 - \frac{1}{3}({\cdot}02)^3 + \frac{1}{5}({\cdot}02)^5 - \ldots.$$

If we stop at the first term we have $\theta = {\cdot}02$.

Then the number of degrees in the angle $= {\cdot}02 \times 57{\cdot}29577951\ldots = 1{\cdot}14591559$; and this $= 1^0\,8'\,45''$.

(3.) We have, as in Art. 229,

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

Hence by Art. 293 the circular measure of $\frac{A-B}{2}$

$$=k-\frac{k^3}{3}+\frac{k^5}{5}-\dots,$$

where k stands for $\frac{a-b}{a+b}\cot\frac{C}{2}$.

Therefore the number of degrees in $\frac{A-B}{2}$

$$=\frac{180}{\pi}\left\{k-\frac{k^3}{3}+\frac{k^5}{5}-\dots\right\}.$$

Also $\frac{A+B}{2}=90^0-\frac{C}{2}$. Thus A is found by taking the upper sign, and B by taking the lower sign in

$$90^0-\frac{C}{2}\pm\frac{180^0}{\pi}\left\{k-\frac{k^3}{3}+\frac{k^5}{5}-\dots\right\}.$$

(4.)

$$\frac{\sin A}{\sin C}=\frac{a}{c},\qquad \frac{\sin B}{\sin C}=\frac{b}{c};$$

therefore

$$\frac{\sin A-\sin B}{\sin C}=\frac{a-b}{c};$$

therefore

$$\frac{\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2}}=\frac{a-b}{c};$$

therefore

$$\sin\frac{A-B}{2}=\frac{a-b}{c}\cos\frac{C}{2}$$

$$=\frac{a-b}{c}\sin\frac{A+B}{2}$$

$$=\frac{a-b}{c}\sin\left(\frac{A-B}{2}+B\right).$$

Hence by Art. ~~298~~ 321 the circular measure of $\frac{A-B}{2}$

$$=n\sin B+\frac{n^2}{2}\sin 2B+\frac{n^3}{3}\sin 3B+\dots,$$

where n stands for $\frac{a-b}{c}$. Therefore the circular measure of $A-B$

$$=2n\sin B+n^2\sin 2B\text{ nearly.}$$

5. $$\frac{b}{a} = \frac{\sin B}{\sin A} = \frac{e^{B\iota} - e^{-B\iota}}{e^{A\iota} - e^{-A\iota}} = \frac{e^{B\iota}(1 - e^{-2B\iota})}{e^{A\iota}(1 - e^{-2A\iota})}.$$

Take the logarithms: thus

$$\log b - \log a = B\iota - A\iota + \log(1 - e^{-2B\iota}) - \log(1 - e^{-2A\iota})$$

$$= (B - A)\iota - \left\{e^{-2B\iota} + \frac{1}{2}e^{-4B\iota} + \frac{1}{3}e^{-6B\iota} + \dots\right\}$$

$$+ e^{-2A\iota} + \frac{1}{2}e^{-4A\iota} + \frac{1}{3}e^{-6A\iota} + \dots$$

Now $\quad e^{-2B\iota} = \cos 2B - \iota \sin 2B, \quad e^{-2A\iota} = \cos 2A - \iota \sin 2A,$

and so on. Then, as the real and imaginary parts of the expression must be separately equal, we have

$$\log b - \log a = \cos 2A - \cos 2B + \frac{1}{2}(\cos 4A - \cos 4B)$$

$$+ \frac{1}{3}(\cos 6A - \cos 6B) + \dots.$$

6. By Art. 294,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

$$= \frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \dots;$$

therefore $$\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \dots.$$

7. Let $$A + B\iota = \log(m + n\iota);$$

therefore $$e^{A+B\iota} = m + n\iota;$$

therefore $$m + n\iota = e^A e^{B\iota} = e^A(\cos B + \iota \sin B);$$

therefore $$m = e^A \cos B,$$

and $$n = e^A \sin B.$$

By division $$\frac{n}{m} = \tan B.$$

By squaring and adding $$m^2 + n^2 = e^{2A};$$

therefore $$2A = \log(m^2 + n^2).$$

8.
$$\cos(\theta+\phi\iota)=\cos\theta\cos\phi\iota-\sin\theta\sin\phi\iota$$
$$=\cos\theta\,\frac{e^{-\phi}+e^{\phi}}{2}-\sin\theta\,\frac{e^{-\phi}-e^{\phi}}{2\iota}$$
$$=\cos\theta\,\frac{e^{-\phi}+e^{\phi}}{2}+\iota\sin\theta\,\frac{e^{-\phi}-e^{\phi}}{2};$$

this is of the form $\alpha+\beta\iota$ where

$$\alpha=\cos\theta\,\frac{e^{-\phi}+e^{\phi}}{2},\quad\text{and}\quad\beta=\sin\theta\,\frac{e^{-\phi}-e^{\phi}}{2}.$$

9.
$$\sin(\theta+\phi\iota)=\sin\theta\cos\phi\iota+\cos\theta\sin\phi\iota$$
$$=\sin\theta\,\frac{e^{-\phi}+e^{\phi}}{2}+\cos\theta\,\frac{e^{-\phi}-e^{\phi}}{2\iota}$$
$$=\sin\theta\,\frac{e^{-\phi}+e^{\phi}}{2}-\iota\cos\theta\,\frac{e^{-\phi}-e^{\phi}}{2};$$

this is of the form $\alpha+\beta\iota$ where

$$\alpha=\sin\theta\,\frac{e^{-\phi}+e^{\phi}}{2},\quad\text{and}\quad\beta=-\cos\theta\,\frac{e^{-\phi}-e^{\phi}}{2}.$$

10.
$$\log u=(p+q\iota)\log(a+b\iota)$$
$$=(p+q\iota)\log\sqrt{(a^2+b^2)}\left\{\frac{a}{\sqrt{(a^2+b^2)}}+\frac{b\iota}{\sqrt{(a^2+b^2)}}\right\}$$
$$=(p+q\iota)\log\sqrt{(a^2+b^2)}\{\cos\gamma+\iota\sin\gamma\},$$

where $\cos\gamma=\dfrac{a}{\sqrt{(a^2+b^2)}}$, and $\sin\gamma=\dfrac{b}{\sqrt{(a^2+b^2)}}$,

$$=(p+q\iota)\log e^{\gamma\iota}\sqrt{(a^2+b^2)}$$
$$=(p+q\iota)\{\log e^{\gamma\iota}+\log\sqrt{(a^2+b^2)}\}$$
$$=(p+q\iota)\{\gamma\iota+\log\sqrt{(a^2+b^2)}\}$$
$$=p\log\sqrt{(a^2+b^2)}-q\gamma+\{p\gamma+q\log\sqrt{(a^2+b^2)}\}\iota.$$

This is of the form $\alpha+\beta\iota$, where

$$\alpha=p\log\sqrt{(a^2+b^2)}-q\gamma,\quad\text{and}\quad\beta=p\gamma+q\log\sqrt{(a^2+b^2)}.$$

11. By Example 10 we can express $\log(a+b\iota)^{p+q\iota}$ in the form $\alpha+\beta\iota$; therefore

$$(a+b\iota)^{p+q\iota}=e^{\alpha+\beta\iota}=e^{\alpha}e^{\beta\iota}=e^{\alpha}(\cos\beta+\iota\sin\beta);$$

and this is of the form $\lambda+\iota\mu$, where $\lambda=e^{\alpha}\cos\beta$, and $\mu=e^{\alpha}\sin\beta$.

12. $\{\sin(\alpha-\theta)+e^{\alpha\iota}\sin\theta\}^n=\{\sin(\alpha-\theta)+(\cos\alpha+\iota\sin\alpha)\sin\theta\}^n$
$$=(\sin\alpha\cos\theta+\iota\sin\alpha\sin\theta)^n=\sin^n\alpha(\cos\theta+\iota\sin\theta)^n$$
$$=\sin^n\alpha(\cos n\theta+\iota\sin n\theta).$$

Again
$$\sin^{n-1}\alpha\{\sin(\alpha-n\theta)+e^{\alpha\iota}\sin n\theta\}$$
$$=\sin^{n-1}\alpha\{\sin(\alpha-n\theta)+(\cos\alpha+\iota\sin\alpha)\sin n\theta\}$$
$$=\sin^{n-1}\alpha\{\sin\alpha\cos n\theta+\iota\sin\alpha\sin n\theta\}$$
$$=\sin^{n}\alpha(\cos n\theta+\iota\sin n\theta):$$

thus the two expressions agree.

In a similar way we may proceed when we take the lower sign in the expressions.

XXII.

1.
$$\sin^2\alpha=\frac{1}{2}(1-\cos 2\alpha),$$
$$\sin^2(\alpha+\beta)=\frac{1}{2}\{1-\cos 2(\alpha+\beta)\},$$
$$\sin^2(\alpha+2\beta)=\frac{1}{2}\{1-\cos 2(\alpha+2\beta)\},$$

and so on.

Hence the sum of n terms
$$=\frac{n}{2}-\frac{1}{2}\{\cos 2\alpha+\cos 2(\alpha+\beta)+\cos 2(\alpha+2\beta)+\dots\}$$
$$=\frac{n}{2}-\frac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{2\sin\beta}.$$

2.
$$\sin^3\alpha=\frac{1}{4}(3\sin\alpha-\sin 3\alpha),$$
$$\sin^3(\alpha+\beta)=\frac{1}{4}\{3\sin(\alpha+\beta)-\sin 3(\alpha+\beta)\},$$
$$\sin^3(\alpha+2\beta)=\frac{1}{4}\{3\sin(\alpha+2\beta)-\sin 3(\alpha+2\beta)\},$$

and so on.

Hence the sum of n terms
$$=\frac{3}{4}\{\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\dots\}$$
$$-\frac{1}{4}\{\sin 3\alpha+\sin 3(\alpha+\beta)+\sin 3(\alpha+2\beta)+\dots\}$$
$$=\frac{3}{4}\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta}-\frac{1}{4}\frac{\sin\left(3\alpha+\frac{n-1}{2}3\beta\right)\sin\frac{3n\beta}{2}}{\sin\frac{3}{2}\beta}.$$

3. We have $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$;

therefore $$\cos^4\theta = \frac{1}{4}(1+\cos 2\theta)^2 = \frac{1}{4}(1+2\cos 2\theta+\cos^2 2\theta)$$

$$= \frac{1}{4}\left(1+2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) = \frac{3}{8}+\frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta.$$

Apply this transformation to every term of the proposed series; thus the sum of n terms

$$= \frac{3n}{8} + \frac{1}{2}\{\cos 2\alpha + \cos 2(\alpha+\beta) + \cos 2(\alpha+2\beta) + \dots\}$$

$$+\frac{1}{8}\{\cos 4\alpha + \cos 4(\alpha+\beta) + \cos 4(\alpha+2\beta) + \dots\}$$

$$= \frac{3n}{8} + \frac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{2\sin\beta} + \frac{\cos\{4\alpha+(n-1)2\beta\}\sin 2n\beta}{8\sin 2\beta}.$$

4. $\sin\theta + \sin 3\theta + \sin 5\theta + \dots$ to n terms

$$= \frac{\sin\{\theta+(n-1)\theta\}\sin n\theta}{\sin\theta} = \frac{\sin^2 n\theta}{\sin\theta};$$

$\cos\theta + \cos 3\theta + \cos 5\theta + \dots$ to n terms

$$= \frac{\cos\{\theta+(n-1)\theta\}\sin n\theta}{\sin\theta} = \frac{\sin n\theta\cos n\theta}{\sin\theta}.$$

Divide the former result by the latter; thus we obtain $\tan n\theta$.

5. $\cos A\cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$.

Apply this transformation to every term of the proposed series; thus the sum of n terms

$$= \frac{n}{2}\cos\alpha + \frac{1}{2}\{\cos(2\theta+\alpha) + \cos(2\theta+3\alpha) + \cos(2\theta+5\alpha) + \dots\}$$

$$= \frac{n}{2}\cos\alpha + \frac{\cos\{2\theta+\alpha+(n-1)\alpha\}\sin n\alpha}{2\sin\alpha}$$

$$= \frac{n}{2}\cos\alpha + \frac{\cos(2\theta+n\alpha)\sin n\alpha}{2\sin\alpha}.$$

6. By Art. 307 we have

$\sin\theta - \sin 2\theta + \sin 3\theta - \dots$ to n terms

$$= \frac{\sin\left\{\theta + \frac{(n-1)(\theta+\pi)}{2}\right\}\sin\frac{n(\theta+\pi)}{2}}{\sin\frac{\theta+\pi}{2}}.$$

And $\cos\theta - \cos 2\theta + \cos 3\theta - \dots$ to n terms

$$= \frac{\cos\left\{\theta + \frac{(n-1)(\theta+\pi)}{2}\right\} \sin\frac{n(\theta+\pi)}{2}}{\sin\frac{\theta+\pi}{2}}.$$

Divide the former by the latter: the result

$$= \frac{\sin\left\{\theta + \frac{(n-1)(\theta+\pi)}{2}\right\}}{\cos\left\{\theta + \frac{(n-1)(\theta+\pi)}{2}\right\}} = \frac{\sin\left\{\theta + \frac{(n-1)(\theta+\pi)}{2} + \pi\right\}}{\cos\left\{\theta + \frac{(n-1)(\theta+\pi)}{2} + \pi\right\}}$$

$$= \frac{\sin\frac{n+1}{2}(\theta+\pi)}{\cos\frac{n+1}{2}(\theta+\pi)} = \tan\frac{n+1}{2}(\theta+\pi).$$

7. $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$.

Apply this transformation to every term of the proposed series; thus the sum of n terms

$$= \frac{n\sin p\theta}{2} + \frac{1}{2}\{\sin(p+2)\theta + \sin(p+4)\theta + \sin(p+6)\theta + \dots\}$$

$$= \frac{n\sin p\theta}{2} + \frac{\sin(p+1+n)\theta\sin n\theta}{2\sin\theta}.$$

8. $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$.

Apply this transformation to every term of the proposed series; thus the sum of n terms

$$= \frac{n}{2}\cos\alpha - \frac{1}{2}\{\cos 3\alpha + \cos 5\alpha + \cos 7\alpha + \dots\}$$

$$= \frac{n}{2}\cos\alpha - \frac{\cos\{3\alpha + (n-1)\alpha\}\sin n\alpha}{2\sin\alpha} = \frac{n}{2}\cos\alpha - \frac{\cos(n+2)\alpha\sin n\alpha}{2\sin\alpha}.$$

9. Suppose that in the preceding result we put for the sines of the angles their values from Art. 274; the proposed series becomes an expansion in powers of α, and it is obvious that the coefficient of α^2 is

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 \dots + n(n+1).$$

We must therefore find the coefficient of α^2 in the expansion of the expression found for the sum of the Trigonometrical Series, and equate it to the above.

Now $\frac{n}{2}\cos\alpha=\frac{n}{2}\left(1-\frac{\alpha^2}{\underline{|2}}+\frac{\alpha^4}{\underline{|4}}-\dots\right)$, so that the coefficient of α^2 in this term is $-\frac{n}{4}$;

and
$$\frac{\cos(n+2)\alpha\sin n\alpha}{2\sin\alpha}=\frac{\left\{1-\frac{(n+2)^2}{2}\alpha^2+\dots\right\}\left\{n\alpha-\frac{n^3\alpha^3}{6}+\dots\right\}}{2\left(\alpha-\frac{\alpha^3}{6}+\dots\right)}$$

$$=n\frac{\left\{1-\frac{(n+2)^2}{2}\alpha^2+\dots\right\}\left\{1-\frac{n^2\alpha^2}{6}+\dots\right\}}{2\left(1-\frac{\alpha^2}{6}+\dots\right)}$$

$$=\frac{n}{2}\left\{1-\frac{(n+2)^2}{2}\alpha^2+\dots\right\}\left\{1-\frac{n^2\alpha^2}{6}+\dots\right\}\left\{1-\frac{\alpha^2}{6}+\dots\right\}^{-1}$$

$$=\frac{n}{2}\left\{1-\frac{(n+2)^2}{2}\alpha^2+\dots\right\}\left\{1-\frac{n^2\alpha^2}{6}+\dots\right\}\left\{1+\frac{\alpha^2}{6}+\dots\right\}.$$

Multiply out and it will be found that the coefficient of α^2 is
$$\frac{n}{2}\left\{\frac{-(n+2)^2}{2}-\frac{n^2}{6}+\frac{1}{6}\right\}.$$

Hence the required sum
$$=-\frac{n}{4}-\frac{n}{2}\left\{-\frac{(n+2)^2}{2}-\frac{n^2}{6}+\frac{1}{6}\right\}$$

$$=-\frac{n}{4}+\frac{n(n+2)^2}{4}+\frac{n^3}{12}-\frac{n}{12}$$

$$=\frac{n}{12}\{3(n+2)^2+n^2-1-3\}$$

$$=\frac{n}{12}\{4n^2+12n+8\}=\frac{n}{3}(n^2+3n+2)$$

$$=\frac{n(n+1)(n+2)}{3}.$$

10. $\sin A\sin B=\frac{1}{2}\cos(A-B)-\frac{1}{2}\cos(A+B)$.

Apply this transformation to every term of the proposed series; hence the sum of n terms

$$=\frac{1}{2}(\cos 2\theta-\cos 4\theta)+\frac{1}{2}(\cos 4\theta-\cos 8\theta)+\frac{1}{2}(\cos 8\theta-\cos 16\theta)+\dots$$

$$=\frac{1}{2}(\cos 2\theta-\cos 2^{n+1}\theta).$$

11. Omitting the first term, we can find the sum of the rest of the series by Art. 311; we must put $\cos\theta$ for c, and put $\alpha=\beta=\theta$. Hence the sum of the whole series

$$=\cos\theta+e^{\cos^2\theta}\cos(\theta+\sin\theta\cos\theta)-\cos\theta=e^{\cos^2\theta}\cos(\theta+\sin\theta\cos\theta).$$

12. In the first series of Art. 311 put -1 for c, and 0 for α, and θ for β, and change the sign. Thus we obtain for the required sum

$$-e^{-\cos\theta}\sin(-\sin\theta),\quad \text{that is } e^{-\cos\theta}\sin(\sin\theta).$$

13. For $\cos 2\theta$, $\cos 4\theta$,... put the exponential values; thus denoting $e^{\iota\theta}$ by z, the proposed series becomes

$$\frac{1}{2}\left\{2-\frac{1}{\lfloor 2}(z^2+z^{-2})+\frac{1}{\lfloor 4}(z^4+z^{-4})-\ldots\right\}$$

$$=\frac{1}{2}\{\cos z+\cos z^{-1}\}$$

$$=\frac{1}{2}\{\cos(\cos\theta+\iota\sin\theta)+\cos(\cos\theta-\iota\sin\theta)\}$$

$$=\cos(\cos\theta)\cos(\iota\sin\theta)=\frac{1}{2}\cos(\cos\theta)\,(e^{-\sin\theta}+e^{\sin\theta}).$$

14. The proposed series

$$=\cos\theta+\cos^2\theta+\cos^3\theta+\cos^4\theta+\ldots$$

$$+\cos\theta+\frac{1}{2}\cos^2\theta+\frac{1}{3}\cos^3\theta+\frac{1}{4}\cos^4\theta+\ldots$$

$$=\frac{\cos\theta}{1-\cos\theta}-\log(1-\cos\theta).$$

15. In the first series of Art. 311 put 0 for α, and $\cos\theta$ for c, and θ for β. Thus the sum $=e^{\cos^2\theta}\sin(\sin\theta\cos\theta)$.

16. In the second series of Art. 311 put θ for α, and θ for β, and $\sin\theta$ for c. Thus

$$\sin\theta\cos 2\theta+\frac{\sin^2\theta}{2}\cos 3\theta+\frac{\sin^3\theta}{\lfloor 3}\cos 4\theta+\ldots$$

$$=e^{\sin\theta\cos\theta}\cos(\theta+\sin^2\theta)-\cos\theta.$$

Therefore $$\cos\theta+\sin\theta\cos 2\theta+\frac{\sin^2\theta}{2}\cos 3\theta+\frac{\sin^3\theta}{\lfloor 3}\cos 4\theta+\ldots$$

$$=e^{\sin\theta\cos\theta}\cos(\theta+\sin^2\theta).$$

17. Put the exponential values for $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$, ... Thus denoting $e^{\iota\theta}$ by z, the proposed series becomes

$$\frac{1}{2}\left\{z-\frac{1}{2}z^2+\frac{1}{3}z^3-\frac{1}{4}z^4+\ldots\right\}+\frac{1}{2}\left\{z^{-1}-\frac{1}{2}z^{-2}+\frac{1}{3}z^{-3}-\frac{1}{4}z^{-4}+\ldots\right\};$$

that is $\frac{1}{2}\log(1+z)+\frac{1}{2}\log(1+z^{-1})$, that is $\frac{1}{2}\log(1+z)(1+z^{-1})$,

that is $\frac{1}{2}\log(2+z+z^{-1})$, that is $\frac{1}{2}\log(2+2\cos\theta)$,

that is $\frac{1}{2}\log\left(4\cos^2\frac{\theta}{2}\right)$, that is $\log\left(2\cos\frac{\theta}{2}\right)$.

18. Proceed as in the solution of Example 17. Thus the proposed series becomes

$$\frac{1}{2}\left\{z^2+\frac{1}{3}z^6+\frac{1}{5}z^{10}+\ldots\right\}+\frac{1}{2}\left\{z^{-2}+\frac{1}{3}z^{-6}+\frac{1}{5}z^{-10}+\ldots\right\}$$

$$=\frac{1}{4}\log\frac{1+z^2}{1-z^2}+\frac{1}{4}\log\frac{1+z^{-2}}{1-z^{-2}}=\frac{1}{4}\log\left(\frac{1+z^2}{1-z^2}\times\frac{1+z^{-2}}{1-z^{-2}}\right)$$

$$=\frac{1}{4}\log\frac{2+z^2+z^{-2}}{2-z^2-z^{-2}}=\frac{1}{4}\log\frac{2+2\cos 2\theta}{2-2\cos 2\theta}=\frac{1}{4}\log\frac{1+\cos 2\theta}{1-\cos 2\theta}$$

$$=\frac{1}{4}\log\cot^2\theta=\frac{1}{2}\log\cot\theta.$$

19. Put the exponential values for $\sin\theta$, $\sin 2\theta$, $\sin 3\theta$, ... Thus, denoting $e^{\iota\theta}$ by z, the proposed series becomes

$$\frac{1}{2\iota}\left\{xz-\frac{1}{2}x^2z^2+\frac{1}{3}x^3z^3-\frac{1}{4}x^4z^4+\ldots\right\}$$

$$-\frac{1}{2\iota}\left\{xz^{-1}-\frac{1}{2}x^2z^{-2}+\frac{1}{3}x^3z^{-3}-\frac{1}{4}x^4z^{-4}+\ldots\right\}.$$

This $$=\frac{1}{2\iota}\log(1+xz)-\frac{1}{2\iota}\log(1+xz^{-1})$$

$$=\frac{1}{2\iota}\log\frac{1+xz}{1+xz^{-1}}=\frac{1}{2\iota}\log\frac{1+x(\cos\theta+\iota\sin\theta)}{1+x(\cos\theta-\iota\sin\theta)}.$$

Assume $\tan\phi=\frac{x\sin\theta}{1+x\cos\theta}$; thus the sum of the proposed series

$$=\frac{1}{2\iota}\log\frac{1+\iota\tan\phi}{1-\iota\tan\phi}=\frac{1}{2\iota}\log\frac{\cos\phi+\iota\sin\phi}{\cos\phi-\iota\sin\phi}$$

$$=\frac{1}{2\iota}\log\frac{e^{\iota\phi}}{e^{-\iota\phi}}=\frac{1}{2\iota}\log e^{2\iota\phi}=\phi=\cot^{-1}\frac{1+x\cos\theta}{x\sin\theta}$$

$$=\cot^{-1}\left(\frac{\operatorname{cosec}\theta}{x}+\cot\theta\right).$$

20. By Art. 129 the limit of $\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\ldots$ is $\frac{\sin\theta}{\theta}$;

therefore the limit of $\cos\theta\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\ldots$ is $\frac{\cos\theta\sin\theta}{\theta}$, that is $\frac{\sin 2\theta}{2\theta}$. Then take the logarithms of both sides.

21. $$\sin\theta\left(\sin\frac{\theta}{2}\right)^2=\frac{1}{2}\sin\theta(1-\cos\theta)=\frac{1}{2}\sin\theta-\frac{1}{4}\sin 2\theta,$$

$$2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{4}\right)^2=\sin\frac{\theta}{2}\left(1-\cos\frac{\theta}{2}\right)=\sin\frac{\theta}{2}-\frac{1}{2}\sin\theta,$$

$$4\sin\frac{\theta}{4}\left(\sin\frac{\theta}{8}\right)^2=2\sin\frac{\theta}{4}\left(1-\cos\frac{\theta}{4}\right)=2\sin\frac{\theta}{4}-\sin\frac{\theta}{2},$$

$$8\sin\frac{\theta}{8}\left(\sin\frac{\theta}{16}\right)^2=4\sin\frac{\theta}{8}\left(1-\cos\frac{\theta}{8}\right)=4\sin\frac{\theta}{8}-2\sin\frac{\theta}{4}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$2^{n-2}\sin\frac{\theta}{2^{n-1}}-\frac{1}{4}\sin 2\theta.$$

22. $$\tan\frac{\theta}{2}\sec\theta=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}\cos\theta}=\frac{\sin\left(\theta-\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}\cos\theta}=\frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}\cos\theta}$$

$$=\tan\theta-\tan\frac{\theta}{2};$$

therefore $$\tan\frac{\theta}{4}\sec\frac{\theta}{2}=\tan\frac{\theta}{2}-\tan\frac{\theta}{4},$$

$$\tan\frac{\theta}{8}\sec\frac{\theta}{4}=\tan\frac{\theta}{4}-\tan\frac{\theta}{8},$$

and so on.

Then adding the terms, we see that all cancel on the right-hand side except two, namely

$$\tan\theta-\tan\frac{\theta}{2^n}.$$

23. $\cot\theta \operatorname{cosec}\theta = \dfrac{\cos\theta}{\sin^2\theta} = \dfrac{2\cos^2\frac{\theta}{2} - 1}{\sin^2\theta}$

$$= \frac{2\cos^2\frac{\theta}{2}}{4\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}} - \frac{1}{\sin^2\theta} = \frac{1}{2\sin^2\frac{\theta}{2}} - \frac{1}{\sin^2\theta};$$

therefore
$$2\cot 2\theta \operatorname{cosec} 2\theta = \frac{1}{\sin^2\theta} - \frac{2}{\sin^2 2\theta},$$

$$4\cot 4\theta \operatorname{cosec} 4\theta = \frac{2}{\sin^2 2\theta} - \frac{4}{\sin^2 4\theta}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{1}{2\sin^2\frac{\theta}{2}} - \frac{2^{n-1}}{\sin^2 2^{n-1}\theta}.$$

24. $\dfrac{1}{\sin\theta\sin 2\theta} = \dfrac{1}{\sin\theta}\cdot\dfrac{\sin(2\theta-\theta)}{\sin\theta\sin 2\theta} = \dfrac{1}{\sin\theta}\cdot\dfrac{\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}{\sin\theta\sin 2\theta}$

$$= \frac{1}{\sin\theta}(\cot\theta - \cot 2\theta).$$

Similarly
$$\frac{1}{\sin 2\theta\sin 3\theta} = \frac{1}{\sin\theta}\frac{\sin(3\theta - 2\theta)}{\sin 2\theta\sin 3\theta}$$

$$= \frac{1}{\sin\theta}(\cot 2\theta - \cot 3\theta);$$

$$\frac{1}{\sin 3\theta\sin 4\theta} = \frac{1}{\sin\theta}(\cot 3\theta - \cot 4\theta).$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{1}{\sin\theta}\{\cot\theta - \cot(n+1)\theta\}.$$

25. Let $\phi = \theta + \frac{\pi}{2}$; thus the proposed series becomes

$$\frac{1}{\cos\phi\cos 2\phi} + \frac{1}{\cos 2\phi\cos 3\phi} + \frac{1}{\cos 3\phi\cos 4\phi} + \dots$$

Now
$$\frac{1}{\cos\phi\cos 2\phi}=\frac{1}{\sin\phi}\,\frac{\sin(2\phi-\phi)}{\cos\phi\cos 2\phi}=\frac{1}{\sin\phi}(\tan 2\phi-\tan\phi),$$
$$\frac{1}{\cos 2\phi\cos 3\phi}=\frac{1}{\sin\phi}\,\frac{\sin(3\phi-2\phi)}{\cos 2\phi\cos 3\phi}=\frac{1}{\sin\phi}(\tan 3\phi-\tan 2\phi),$$
$$\frac{1}{\cos 3\phi\cos 4\phi}=\frac{1}{\sin\phi}\,\frac{\sin(4\phi-3\phi)}{\cos 3\phi\cos 4\phi}=\frac{1}{\sin\phi}(\tan 4\phi-\tan 3\phi).$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely
$$\frac{1}{\sin\phi}\{\tan(n+1)\phi-\tan\phi\}.$$

26.
$$\operatorname{Tan}^{-1}\frac{1}{1+m+m^2}=\tan^{-1}\frac{1}{m}-\tan^{-1}\frac{1}{1+m};$$

this is obvious, for by taking the tangent of $\tan^{-1}\frac{1}{m}-\tan^{-1}\frac{1}{1+m}$ we obtain
$$\frac{\frac{1}{m}-\frac{1}{m+1}}{1+\frac{1}{m(m+1)}},\text{ that is }\frac{1}{m^2+m+1}.$$

Apply this transformation to every term of the proposed series; thus we obtain
$$\tan^{-1}\frac{1}{1}-\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{4}+\ldots,$$

that is
$$\tan^{-1}1-\tan^{-1}\frac{1}{n+1},\text{ that is }\frac{\pi}{4}-\tan^{-1}\frac{1}{n+1}.$$

27.
$$\operatorname{Tan}^{-1}\frac{x}{1+m(m+1)x^2}=\tan^{-1}(m+1)x-\tan^{-1}mx;$$

this is obvious, for by taking the tangent of $\tan^{-1}(m+1)x-\tan^{-1}mx$, we obtain $\frac{(m+1)x-mx}{1+m(m+1)x^2}$, that is $\frac{x}{1+m(m+1)x^2}$.

Apply this transformation to every term of the proposed series after the first; thus we obtain
$$\tan^{-1}x+\tan^{-1}2x-\tan^{-1}x+\tan^{-1}3x-\tan^{-1}2x+\ldots,$$

that is $\tan^{-1}nx$.

28. $$\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B).$$

Apply this transformation to every term of the proposed series; thus we obtain

$$\frac{1}{2}(\cos 2a - \cos 4a) + \frac{1}{2}(\cos a - \cos 2a) + \frac{1}{2}\left(\cos\frac{a}{2} - \cos a\right) + \ldots,$$

that is $$\frac{1}{2}\left(\cos\frac{a}{2^{n-2}} - \cos 4a\right).$$

29. $$\frac{1}{\cos\theta + \cos 3\theta} = \frac{1}{2\cos\theta\cos 2\theta} = \frac{1}{2\sin\theta}\cdot\frac{\sin(2\theta-\theta)}{\cos\theta\cos 2\theta}$$

$$= \frac{1}{2\sin\theta}(\tan 2\theta - \tan\theta),$$

$$\frac{1}{\cos\theta + \cos 5\theta} = \frac{1}{2\cos 2\theta\cos 3\theta} = \frac{1}{2\sin\theta}\cdot\frac{\sin(3\theta-2\theta)}{\cos 2\theta\cos 3\theta}$$

$$= \frac{1}{2\sin\theta}(\tan 3\theta - \tan 2\theta),$$

$$\frac{1}{\cos\theta + \cos 7\theta} = \frac{1}{2\cos 3\theta\cos 4\theta} = \frac{1}{2\sin\theta}\cdot\frac{\sin(4\theta-3\theta)}{\cos 3\theta\cos 4\theta}$$

$$= \frac{1}{2\sin\theta}(\tan 4\theta - \tan 3\theta).$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{1}{2\sin\theta}\{\tan(n+1)\theta - \tan\theta\}.$$

30. $$\frac{\sin\theta}{\cos 2\theta + \cos\theta} = \frac{\sin\theta}{2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}} = \frac{\sin\theta}{2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}}$$

$$= \frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{3\theta}{2}} - \frac{1}{\cos\frac{\theta}{2}}\right\};$$

$$\frac{\sin 2\theta}{\cos 4\theta + \cos\theta} = \frac{\sin 2\theta}{2\cos\frac{3\theta}{2}\cos\frac{5\theta}{2}} = \frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{5\theta}{2}} - \frac{1}{\cos\frac{3\theta}{2}}\right\};$$

$$\frac{\sin 3\theta}{\cos 6\theta+\cos\theta}=\frac{\sin 3\theta}{2\cos\frac{5\theta}{2}\cos\frac{7\theta}{2}}=\frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{7\theta}{2}}-\frac{1}{\cos\frac{5\theta}{2}}\right\}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right hand except two, namely

$$\frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{(2n+1)\theta}{2}}-\frac{1}{\cos\frac{\theta}{2}}\right\}.$$

31. $$\frac{\sin\theta}{1+2\cos\theta}=\frac{\sin\theta}{1+2\left(1-2\sin^2\frac{\theta}{2}\right)}=\frac{\sin\theta}{3-4\sin^2\frac{\theta}{2}}$$

$$=\frac{\sin\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}\left(3-4\sin^2\frac{\theta}{2}\right)}=\frac{\sin\theta\sin\frac{\theta}{2}}{\sin\frac{3\theta}{2}}=\frac{\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}}{2\sin\frac{3\theta}{2}}$$

$$=\frac{2\cos\frac{\theta}{2}-2\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}}=\frac{\cos\frac{\theta}{2}(1+2\cos\theta)+\cos\frac{\theta}{2}-2\cos\theta\cos\frac{\theta}{2}-2\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}}$$

$$=\frac{\cos\frac{\theta}{2}}{4\sin\frac{\theta}{2}}+\frac{\cos\frac{\theta}{2}-2\cos\theta\cos\frac{\theta}{2}-2\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}}$$

$$=\frac{\cos\frac{\theta}{2}}{4\sin\frac{\theta}{2}}-\frac{3\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}}=\frac{1}{4}\cot\frac{\theta}{2}-\frac{3}{4}\cot\frac{3\theta}{2}.$$

Similarly $$\frac{3\sin 3\theta}{1+2\cos 3\theta}=\frac{3}{4}\cot\frac{3\theta}{2}-\frac{9}{4}\cot\frac{9\theta}{2},$$

$$\frac{3^2\sin 3^2\theta}{1+2\cos 3^2\theta}=\frac{9}{4}\cot\frac{9\theta}{2}-\frac{27}{4}\cot\frac{27\theta}{2}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right hand except two, namely

$$\frac{1}{4}\cot\frac{\theta}{2}-\frac{3^n}{4}\cot\frac{3^n\theta}{2}.$$

32. $\cot^{-1}\left\{2a^{-1}+\frac{m(m+1)}{2}a\right\}=\cot^{-1}\frac{m}{2}a-\cot^{-1}\frac{m+1}{2}a.$

For if we take the cotangent of $\cot^{-1}\frac{m}{2}a-\cot^{-1}\frac{m+1}{2}a,$

we obtain
$$\frac{\frac{m}{2}a\,.\,\frac{m+1}{2}a+1}{\frac{m+1}{2}a-\frac{m}{2}a},$$

that is
$$\frac{m(m+1)}{2}a+2a^{-1}.$$

Apply this transformation to every term of the proposed series; thus we obtain

$$\cot^{-1}\frac{a}{2}-\cot^{-1}\frac{2a}{2}+\cot^{-1}\frac{2a}{2}-\cot^{-1}\frac{3a}{2}+\cot^{-1}\frac{3a}{2}-\cot^{-1}\frac{4a}{2}+\dots;$$

that is
$$\cot^{-1}\frac{a}{2}-\cot^{-1}\frac{n+1}{2}a.$$

33. $\frac{1}{2}\sec\theta=\frac{1}{2\cos\theta}=\frac{\sin\theta}{2\sin\theta\cos\theta}=\frac{\sin\theta}{\sin 2\theta}=\frac{\sin(2\theta-\theta)}{\sin 2\theta}$

$$=\cos\theta-\cot 2\theta\sin\theta=\sin\theta(\cot\theta-\cot 2\theta);$$

$$\frac{1}{2^2}\sec\theta\sec 2\theta=\frac{1}{2}\sec\theta\sin 2\theta(\cot 2\theta-\cot 4\theta)$$

$$=\sin\theta(\cot 2\theta-\cot 4\theta);$$

$$\frac{1}{2^3}\sec\theta\sec 2\theta\sec 4\theta=\frac{1}{2}\sec\theta\sin 2\theta(\cot 4\theta-\cot 8\theta)$$

$$=\sin\theta(\cot 4\theta-\cot 8\theta).$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\sin\theta(\cot\theta-\cot 2^n\theta).$$

34. $\tan 2\theta=\frac{\sin 2\theta}{\cos 2\theta}=\frac{\sin^2 2\theta}{\sin 2\theta\cos 2\theta}=\frac{2\sin^2 2\theta}{\sin 4\theta}=\frac{4\sin^2 2\theta}{2\sin 4\theta};$

therefore
$$\frac{1}{2}\log\tan 2\theta=\log 2\sin 2\theta-\frac{1}{2}\log 2\sin 4\theta,$$

$$\frac{1}{2^2}\log\tan 2^2\theta=\frac{1}{2}\log 2\sin 4\theta-\frac{1}{2^2}\log 2\sin 8\theta,$$

$$\frac{1}{2^3}\log\tan 2^3\theta=\frac{1}{2^2}\log 2\sin 8\theta-\frac{1}{2^3}\log 2\sin 16\theta.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\log 2 \sin 2\theta - \frac{1}{2^n} \log 2 \sin 2^{n+1} \theta.$$

35. $$\cos\frac{\theta}{2} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} = \frac{\sin\theta}{2 \sin\frac{\theta}{2}} = \frac{\sin\theta}{2} \frac{2\cos^2\frac{\theta}{4} - \cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{\sin\theta}{2}\left\{\frac{2\cos^2\frac{\theta}{4}}{2\sin\frac{\theta}{4}\cos\frac{\theta}{4}} - \cot\frac{\theta}{2}\right\} = \frac{\sin\theta}{2}\left\{\cot\frac{\theta}{4} - \cot\frac{\theta}{2}\right\};$$

$$2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2} = 2\cos\frac{\theta}{2}\frac{\sin\frac{\theta}{2}}{2}\left\{\cot\frac{\theta}{8} - \cot\frac{\theta}{4}\right\}$$

$$= \frac{\sin\theta}{2}\left\{\cot\frac{\theta}{8} - \cot\frac{\theta}{4}\right\};$$

$$2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} = 2\cos\frac{\theta}{2}\cdot\frac{\sin\frac{\theta}{2}}{2}\left\{\cot\frac{\theta}{16} - \cot\frac{\theta}{8}\right\}$$

$$= \frac{\sin\theta}{2}\left\{\cot\frac{\theta}{16} - \cot\frac{\theta}{8}\right\}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{\sin\theta}{2}\left\{\cot\frac{\theta}{2^{n+1}} - \cot\frac{\theta}{2}\right\}.$$

36. Let R denote the radius of the circle, n the number of sides of the polygon. Put β for $\frac{\pi}{n}$. Let 2α denote the angular distance of a fixed point in the circumference from one of the angular points; then the angular distances from the other angular points in succession will be

$$2\alpha + 2\beta,\ 2\alpha + 4\beta,\ 2\alpha + 6\beta, \ldots\ 2\alpha + 2(n-1)\beta.$$

The lengths of the successive chords will be

$$2R\sin\alpha,\ 2R\sin(\alpha+\beta),\ 2R\sin(\alpha+2\beta), \ldots\ 2R\sin\{\alpha+(n-1)\beta\}.$$

To find the sum of the squares of the chords, we have

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta);$$

and applying this transformation to every term of the proposed series, we obtain

$$2nR^2 - 2R^2 \{\cos 2\alpha + \cos(2\alpha + 2\beta) + \cos(2\alpha + 4\beta) + \dots\}.$$

The sum of the series of cosines is zero, as in Art. 304; and thus the result is $2nR^2$.

Next, to find the sum of the fourth powers of the chords. We have

$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta;$$

and applying this transformation to every term of the proposed series, we obtain

$$6nR^4 - 8R^4\{\cos 2\alpha + \cos(2\alpha + 2\beta) + \cos(2\alpha + 4\beta) + \dots\}$$
$$+ 2R^4\{\cos 4\alpha + \cos(4\alpha + 4\beta) + \cos(4\alpha + 8\beta) + \dots\},$$

that is $6nR^4$.

37. Let A be the common vertex; let B, C, ... be the successive angular points. Put β for $\frac{\pi}{n}$.

Let PQ be one of the sides of the polygon, such that the arc ABP contains m of the sides; then the angle $AQP = m\beta$, the angle $PAQ = \beta$, and the angle $APQ = \pi - (m+1)\beta$.

Let $PQ = c$, and let r_m denote the radius of the circle inscribed in APQ. Then

$$r_m\left\{\cot\frac{1}{2}APQ + \cot\frac{1}{2}AQP\right\} = c,$$

therefore
$$r_m\left\{\cot\frac{\pi - (m+1)\beta}{2} + \cot\frac{m\beta}{2}\right\} = c,$$

therefore
$$r_m\left\{\tan\frac{m+1}{2}\beta + \cot\frac{m\beta}{2}\right\} = c,$$

therefore
$$r_m\cos\frac{\beta}{2} = c\cos\frac{m+1}{2}\beta\sin\frac{m\beta}{2}$$
$$= \frac{c}{2}\left\{\sin\frac{2m+1}{2}\beta - \sin\frac{\beta}{2}\right\}.$$

Now there are $n-2$ circles in all which can be drawn; so that we have to sum up the values of

$$\frac{c}{2}\sec\frac{\beta}{2}\left\{\sin\frac{2m+1}{2}\beta - \sin\frac{\beta}{2}\right\}$$

for values of m from 1 to $n-2$ inclusive. The sum then is

$$\frac{c}{2}\sec\frac{\beta}{2}\left\{\frac{\sin\left\{\frac{3\beta}{2} + (n-3)\frac{\beta}{2}\right\}\sin\frac{n-2}{2}\beta}{\sin\frac{\beta}{2}} - (n-2)\sin\frac{\beta}{2}\right\},$$

that is $$\frac{c}{2}\sec\frac{\beta}{2}\left\{\frac{\cos\beta}{\sin\frac{\beta}{2}}-(n-2)\sin\frac{\beta}{2}\right\},$$

that is $$\frac{c}{2}\sec\frac{\beta}{2}\left\{\frac{1}{\sin\frac{\beta}{2}}-n\sin\frac{\beta}{2}\right\}.$$

But $c=2r\sin\beta$; thus we get

$$\frac{r\sin\beta}{\cos\frac{\beta}{2}\sin\frac{\beta}{2}}-\frac{rn\sin\beta\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}},\quad\text{that is }2r-2rn\sin^2\frac{\beta}{2},$$

that is $$2r\left(1-n\sin^2\frac{\pi}{2n}\right).$$

38. Use the notation of the preceding solution. The area of the m^{th} circle

$$=\frac{\pi c^2}{4}\sec^2\frac{\beta}{2}\left\{\sin\frac{2m+1}{2}\beta-\sin\frac{\beta}{2}\right\}^2$$

$$=\frac{\pi c^2}{4}\sec^2\frac{\beta}{2}\left\{\sin^2\frac{2m+1}{2}\beta-2\sin\frac{2m+1}{2}\beta\sin\frac{\beta}{2}+\sin^2\frac{\beta}{2}\right\}$$

$$=\frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{1-\cos(2m+1)\beta-4\sin\frac{2m+1}{2}\beta\sin\frac{\beta}{2}+2\sin^2\frac{\beta}{2}\right\}.$$

Then as before we have to sum this expression for the values of m from 1 to $n-2$ inclusive. Thus we obtain

$$\frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{(n-2)\left(1+2\sin^2\frac{\beta}{2}\right)-\frac{\cos\{3\beta+(n-3)\beta\}\sin(n-2)\beta}{\sin\beta}\right.$$
$$\left.-4\sin\frac{\beta}{2}\frac{\sin\left\{\frac{3\beta}{2}+\frac{n-3}{2}\beta\right\}\sin\frac{n-2}{2}\beta}{\sin\frac{\beta}{2}}\right\};$$

and this $$=\frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{(n-2)\left(1+2\sin^2\frac{\beta}{2}\right)-2\cos\beta\right\}$$

$$=\frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{n-4+2n\sin^2\frac{\beta}{2}\right\}.$$

But $c=2r\sin\beta$; therefore $c^2\sec^2\frac{\beta}{2}=\frac{4r^2\sin^2\beta}{\cos^2\frac{\beta}{2}}=16r^2\sin^2\frac{\beta}{2}$;

so that the result $$=16\pi r^2\sin^2\frac{\pi}{2n}\left\{\frac{n}{4}\sin^2\frac{\pi}{2n}+\frac{n-4}{8}\right\}.$$

39. Let S_n denote the sum of the series; so that

$$S_n = n \sin\theta + (n-1)\sin 2\theta + (n-2)\sin 3\theta + \dots + \sin n\theta.$$

In like manner let S_{n-1} denote the sum of the series formed by changing n into $n-1$, so that

$$S_{n-1} = (n-1)\sin\theta + (n-2)\sin 2\theta + \dots + \sin(n-1)\theta;$$

therefore $\quad S_n - S_{n-1} = \sin\theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta$

$$= \frac{\sin\frac{n+1}{2}\theta \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} = \frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{2n+1}{2}\theta\right\}.$$

Similarly we have

$$S_{n-1} - S_{n-2} = \frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{2n-1}{2}\theta\right\};$$

$$S_{n-2} - S_{n-3} = \frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{2n-3}{2}\theta\right\};$$

....

$$S_2 - S_1 = \frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{5\theta}{2}\right\};$$

$$S_1 = \frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right\}.$$

Hence by addition from this series of equations we obtain

$$S_n = \frac{1}{2\sin\frac{\theta}{2}}\left\{n\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} - \cos\frac{5\theta}{2} \dots - \cos\frac{2n+1}{2}\theta\right\}$$

$$= \frac{1}{2\sin\frac{\theta}{2}}\left\{n\cos\frac{\theta}{2} - \frac{\cos\left\{\frac{3\theta}{2} + (n-1)\frac{\theta}{2}\right\}\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\right\}$$

$$= \frac{n}{2}\cot\frac{\theta}{2} - \frac{\cos\frac{(n+2)}{2}\theta\sin\frac{n\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{n}{2}\cot\frac{\theta}{2} - \frac{\sin(n+1)\theta - \sin\theta}{4\sin^2\frac{\theta}{2}}$$

$$= \frac{n+1}{2}\cot\frac{\theta}{2} - \frac{\sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}.$$

40. Let S_n denote the required sum, and S_{n-1} the sum of the series when n is changed to $n-1$. Thus

$$S_n = (n+1)\,n\sin\theta + n\,(n-1)\sin 2\theta + \ldots + 2\,.\,1\sin n\theta,$$

$$S_{n-1} = n\,(n-1)\sin\theta + (n-1)\,(n-2)\sin 2\theta + \ldots + 2\,.\,1\sin(n-1)\,\theta;$$

therefore $\quad S_n - S_{n-1} = 2\,\{n\sin\theta + (n-1)\sin 2\theta + \ldots + \sin n\theta\}$;

that is, by Example 39,

$$S_n - S_{n-1} = (n+1)\cot\frac{\theta}{2} - \frac{\sin(n+1)\,\theta}{2\sin^2\frac{\theta}{2}}.$$

Similarly $\quad S_{n-1} - S_{n-2} = n\cot\frac{\theta}{2} - \dfrac{\sin n\theta}{2\sin^2\frac{\theta}{2}}$;

....

$$S_2 - S_1 = 3\cot\frac{\theta}{2} - \frac{\sin 3\theta}{2\sin^2\frac{\theta}{2}};$$

$$S_1 = 2\cot\frac{\theta}{2} - \frac{\sin 2\theta}{2\sin^2\frac{\theta}{2}}.$$

Hence by addition from this series of equations we obtain

$$S_n = \frac{n\,(n+3)}{2}\cot\frac{\theta}{2} - \frac{1}{2\sin^2\frac{\theta}{2}}\{\sin 2\theta + \sin 3\theta + \ldots + \sin(n+1)\,\theta\}$$

$$= \frac{n\,(n+3)}{2}\cot\frac{\theta}{2} - \frac{\sin\left\{2\theta + (n-1)\frac{\theta}{2}\right\}\sin\frac{n\theta}{2}}{2\sin^2\frac{\theta}{2}\sin\frac{\theta}{2}}$$

$$= \frac{n\,(n+3)}{2}\cot\frac{\theta}{2} - \frac{\cos\frac{3\theta}{2} - \cos\frac{2n+3}{2}\theta}{4\sin^3\frac{\theta}{2}}.$$

XXIII.

1. By Art. 271 we have

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{\underline{|3}} + \frac{\theta^4}{\underline{|5}} - \frac{\theta^6}{\underline{|7}} + \dots;$$

and by Art. 320 we have

$$\frac{\sin\theta}{\theta} = \left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\dots$$

Take the logarithms of the equivalent expressions; thus

$$\log\left\{1 - \frac{\theta^2}{\underline{|3}} + \frac{\theta^4}{\underline{|5}} - \frac{\theta^6}{\underline{|7}} + \dots\right\}$$

$$= \log\left(1 - \frac{\theta^2}{\pi^2}\right) + \log\left(1 - \frac{\theta^2}{2^2\pi^2}\right) + \log\left(1 - \frac{\theta^2}{3^2\pi^2}\right) + \dots$$

Expand the logarithms; then both sides become series arranged according to powers of θ; and by equating the coefficients of θ^2 we obtain

$$-\frac{\theta^2}{\underline{|3}} = -\theta^2\left(\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \dots\right);$$

therefore $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

2. Equate the coefficients of θ^4 in the two equivalent series of the preceding solution; thus since

$$\log\left\{1 - \frac{\theta^2}{\underline{|3}} + \frac{\theta^4}{\underline{|5}}\dots\right\} = -\left\{\frac{\theta^2}{\underline{|3}} - \frac{\theta^4}{\underline{|5}} + \dots\right\} - \frac{1}{2}\left\{\frac{\theta^2}{\underline{|3}} - \frac{\theta^4}{\underline{|5}} + \dots\right\}^2 - \dots$$

we have $$\frac{1}{\underline{|5}} - \frac{1}{2}\left(\frac{1}{\underline{|3}}\right)^2 = -\frac{1}{2\pi^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right);$$

therefore $$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = 2\pi^4\left(\frac{1}{72} - \frac{1}{120}\right)$$

$$= \frac{\pi^4}{12}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{\pi^4}{90}.$$

3. Let $$S = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots;$$

and let $$\Sigma = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \dots.$$

Then
$$S=\frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\dots$$
$$+\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{6^n}+\frac{1}{8^n}+\dots$$
$$=\frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\dots$$
$$+\frac{1}{2^n}\left\{\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\dots\right\}$$
$$=\Sigma+\frac{1}{2^n}S.$$

Therefore
$$\Sigma=\frac{2^n-1}{2^n}S.$$

Hence Σ can be found when S is known.

If $n=2$ we have $S=\frac{\pi^2}{6}$ by Example 1; and then $\Sigma=\frac{3}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{8}$.

4. In the preceding solution suppose $n=4$; then we have $S=\frac{\pi^4}{90}$ by Example 2, and therefore $\Sigma=\frac{15}{16}\cdot\frac{\pi^4}{90}=\frac{\pi^4}{96}$.

5. By Art. 318 we have
$$\sin n\phi=2^{n-1}\sin\phi\,\sin(2\beta+\phi)\,\sin(4\beta+\phi)\dots\sin(2n\beta-2\beta+\phi)$$
where $\beta=\frac{\pi}{2n}$.

Let $\alpha=\frac{1}{2}\beta$, and let $\phi=\alpha$; then $\sin n\phi=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$; thus
$$\frac{1}{\sqrt{2}}=2^{n-1}\sin\alpha\sin 5\alpha\sin 9\alpha\dots\sin(4n-3)\alpha;$$
therefore
$$\sin\alpha\sin 5\alpha\sin 9\alpha\dots\sin(4n-3)\alpha=2^{-n+\frac{1}{2}}.$$

6. Let r be the radius of the circle. The polygon can be resolved into n triangles; and thus the area of the polygon
$$=\frac{r^2}{2}\{\sin\alpha+\sin 2\alpha+\sin 3\alpha+\dots+\sin n\alpha\}$$
$$=\frac{r^2}{2}\cdot\frac{\sin\frac{n+1}{2}\alpha\,\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

But $$\alpha + 2\alpha + 3\alpha + \ldots + n\alpha = 2\pi;$$

that is $$\frac{n(n+1)\alpha}{2} = 2\pi,$$

so that $$\frac{n+1}{2}\alpha = \frac{2\pi}{n}.$$

Now the area of the regular polygon of n sides

$$= \frac{nr^2}{2}\sin\frac{2\pi}{n} = \frac{nr^2}{2}\sin\frac{n+1}{2}\alpha.$$

Hence the ratio of the former area to the latter $= \dfrac{\sin\frac{n\alpha}{2}}{n\sin\frac{\alpha}{2}}$.

7. Let $A, B, C, \ldots$ be the angles of the polygon. From A draw straight lines to the other angles. Let AP be the m^{th} straight line, so that AP subtends at the centre of the circle the angle $m\frac{2\pi}{n}$. Then $AP = 2a\sin m\beta$ where $\beta = \frac{\pi}{n}$.

Thus the product of all the straight lines

$$= (2a)^{n-1}\sin\beta\sin 2\beta\sin 3\beta \ldots \sin(n-1)\beta$$
$$= nx^{n-1};$$

for by Art. 318 we have

$$n = 2^{n-1}\sin\beta\sin 2\beta\sin 3\beta \ldots \sin(n-1)\beta.$$

8. Let $A, B, C, \ldots$ be the points of contact of the circle with the circumscribed polygon taken in order. Let O be the fixed point, and suppose the arc OA to subtend an angle 2ϕ at the centre of the circle. Then the angle between OA and the tangent at A is ϕ; and the length of the perpendicular from O on this tangent is $OA\sin\phi$, that is $2a\sin^2\phi$. Thus we have

$$p_1 = 2a\sin^2\phi.$$

Let $\beta = \frac{\pi}{2n}$, then we obtain in a similar way

$$p_2 = 2a\sin^2(\phi+\beta),$$
$$p_3 = 2a\sin^2(\phi+2\beta),$$
$$p_4 = 2a\sin^2(\phi+3\beta),$$

and so on.

Thus $$p_1p_3p_5 \ldots p_{2n-1} = (2a)^n\sin^2\phi\sin^2(\phi+2\beta)\ldots\sin^2\{\phi+(2n-2)\beta\}$$

$$= (2a)^n\left\{\frac{\sin n\phi}{2^{n-1}}\right\}^2, \text{ by Art. 318,}$$

$$= \frac{a^n}{2^{n-2}}\sin^2 n\phi.$$

In the same way we have

$$p_2 p_4 \cdots p_{2n} = (2a)^n \sin^2(\phi+\beta)\sin^2(\phi+3\beta)\ldots\sin^2\{\phi+(2n-1)\beta\}$$

$$= \frac{a^n}{2^{n-2}}\cos^2 n\phi.$$

Hence by addition we obtain

$$\frac{a^n}{2^{n-2}}(\sin^2 n\phi + \cos^2 n\phi), \quad \text{that is } \frac{a^n}{2^{n-2}}.$$

9. Let $A, B, C, D, \ldots$ be the angular points of the inscribed polygon. Let O be the fixed point from which the perpendiculars are drawn. Let the arc OA subtend an angle 2α at the centre of the circle, let the arc OB subtend an angle 2β, the arc OC an angle 2γ, and so on.

Let $p_1, p_2, p_3, \ldots$ denote the perpendiculars from O on the sides of the circumscribed polygon which touch the circle at $A, B, C, \ldots$ respectively. Then

$$p_1 = OA \sin\alpha, \quad p_2 = OB\sin\beta, \quad p_3 = OC\sin\gamma, \ldots$$

Again, let $q_1, q_2, q_3, \ldots$ denote the perpendiculars from O on the sides of the inscribed polygon AB, BC, CD, ... respectively. Then

$$q_1 = OA \sin OAB = OA\sin(\pi-\beta) = OA\sin\beta;$$

similarly $$q_2 = OB\sin\gamma, \quad q_3 = OC\sin\delta, \ldots$$

Thus $p_1 p_2 p_3 \ldots$ and $q_1 q_2 q_3 \ldots$ are equal, for each is equal to the product of the same series of lengths into the same series of sines.

10. By Art. 318 we have

$$\cos 5A = 16\sin(A+18^0)\sin(A+54^0)\sin(A+90^0)\sin(A+126^0)\sin(A+162^0)$$

$$= 16\cos(72^0-A)\cos(36^0-A)\cos A\cos(A+36^0)\cos(A+72^0);$$

and $$\cos(36^0-A) = -\cos(144^0+A), \quad \cos(A+36^0) = -\cos(144^0-A),$$

therefore

$$\cos 5A = 16\cos(72^0-A)\cos(72^0+A)\cos A\cos(144^0-A)\cos(144^0+A).$$

11. Put $\frac{\pi}{6}$ for θ in the expression for $\sin\theta$ in Art. 320; thus

$$\frac{1}{2} = \frac{\pi}{6}\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{2^2 6^2}\right)\left(1-\frac{1}{3^2 6^2}\right)\ldots;$$

therefore $$3 = \pi\,\frac{35}{36}\cdot\frac{143}{144}\cdot\frac{323}{324}\cdot\frac{575}{576}\cdots;$$

therefore $$\pi = 3\cdot\frac{36}{35}\cdot\frac{144}{143}\cdot\frac{324}{323}\cdot\frac{576}{575}\cdots$$

12. In the general result of Art. 321 put $\frac{\pi}{2}$ for θ, thus

$$e^z+e^{-z}=2\left(1+\frac{4z^2}{\pi^2}\right)\left(1+\frac{4z^2}{3^2\pi^2}\right)\left(1+\frac{4z^2}{5^2\pi^2}\right)\left(1+\frac{4z^2}{7^2\pi^2}\right)\dots$$

13. It is shewn in Art. 321 that

$$e^z-2\cos\theta+e^{-z}=4\sin^2\frac{\theta}{2}\left(1+\frac{z^2}{\theta^2}\right)\left\{1+\frac{z^2}{(2\pi+\theta)^2}\right\}\left\{1+\frac{z^2}{(2\pi-\theta)^2}\right\}\dots$$

The product of the first two factors on the right-hand side

$$=4\sin^2\frac{\theta}{2}+z^2\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2,$$

and this is equal to z^2 when θ vanishes.

Thus, by supposing $\theta=0$, we obtain

$$e^z-2+e^{-z}=z^2\left(1+\frac{z^2}{2^2\pi^2}\right)^2\left(1+\frac{z^2}{4^2\pi^2}\right)^2\left(1+\frac{z^2}{6^2\pi^2}\right)^2\dots$$

Extract the square root and put $2x$ for z; thus

$$e^x-e^{-x}=2x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{2^2\pi^2}\right)\left(1+\frac{x^2}{3^2\pi^2}\right)\dots$$

14. Let s denote the series of which we require the sum,

then $$\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right)^2=2s+\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots$$

Hence by Examples 1 and 2 we have

$$\left(\frac{\pi^2}{6}\right)^2=2s+\frac{\pi^4}{90};$$

therefore $$s=\frac{\pi^4}{2}\left(\frac{1}{36}-\frac{1}{90}\right)=\frac{\pi^4}{36}\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{\pi^4}{120}.$$

15. By Art. 318 we have

$$\sin n\phi=2^{n-1}\sin\phi\sin\left(\phi+\frac{\pi}{n}\right)\sin\left(\phi+\frac{2\pi}{n}\right)\dots\sin\left(\phi+\frac{n-1}{n}\pi\right).$$

Change ϕ into $\phi+\frac{\pi}{2}$; then since n is even we have

$$\sin n\left(\phi+\frac{\pi}{2}\right)=\sin\left(n\phi+\frac{n\pi}{2}\right)=\sin n\phi\cos\frac{n\pi}{2};$$

thus

$$\sin n\phi \cos\frac{n\pi}{2} = 2^{n-1}\cos\phi\cos\left(\phi+\frac{\pi}{n}\right)\cos\left(\phi+\frac{2\pi}{n}\right)\dots\cos\left(\phi+\frac{n-1}{n}\pi\right).$$

Divide the former result by this; then we obtain

$$\sec\frac{n\pi}{2} = \tan\phi\tan\left(\phi+\frac{\pi}{n}\right)\tan\left(\phi+\frac{2\pi}{n}\right)\dots\tan\left(\phi+\frac{n-1}{n}\pi\right).$$

And
$$\sec\frac{n\pi}{2} = \frac{1}{\cos\frac{n\pi}{2}} = \frac{1}{(-1)^{\frac{n}{2}}} = (-1)^{\frac{n}{2}}.$$

16. $\sin 5A - \cos 5A = \sqrt{2}\sin(5A - 45^0) = \sqrt{2}\sin 5(A - 9^0)$.

And by Art. 318 we have $\sin 5(A - 9^0)$

$= 2^4 \sin B \sin(B + 36^0)\sin(B + 72^0)\sin(B + 108^0)\sin(B + 144^0)$,
where $B = A - 9^0$,

$= 2^4 \sin(A - 9^0)\sin(A + 27^0)\cos(27^0 - A)\cos(A + 9^0)\sin(A + 135^0)$

$= 2^4 \sin(A - 9^0)\sin(A + 27^0)\cos(27^0 - A)\cos(A + 9^0)(\cos A - \sin A)\dfrac{1}{\sqrt{2}}$.

Therefore $\sin 5A - \cos 5A$

$= 2^4 \sin(A - 9^0)\sin(A + 27^0)\cos(A - 27^0)\cos(A + 9^0)(\cos A - \sin A)$.

17. By Art. 320 we have

$$\frac{\sin\theta}{\theta} = \left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\dots$$

Put $\frac{\pi}{2}$ for θ: thus

$$\frac{2}{\pi} = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\dots;$$

therefore
$$\frac{\pi}{2} = \frac{2 . 2 . 4 . 4 . 6 . 6 . 8 . 8 \dots}{1 . 3 . 3 . 5 . 5 . 7 . 7 . 9 \dots}.$$

18. We have

$$\sin 2\theta = 2\theta\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{2^2\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{4^2\pi^2}\right)\dots$$

$$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\left(1-\frac{\theta^2}{4^2\pi^2}\right).$$

Divide the first by the second: thus

$$\frac{\sin 2\theta}{\sin\theta} = 2\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\dots;$$

therefore
$$\cos\theta = \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\dots$$

19. In the formula for $\cos\theta$ in Art. 320 put $\frac{\pi}{4}$ for θ; thus

$$\frac{1}{\sqrt{2}} = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^2.3^2}\right)\left(1-\frac{1}{2^2.5^2}\right)\left(1-\frac{1}{2^2.7^2}\right)\dots$$

$$= \frac{3.35.\ 99.195\dots}{4.36.100.196\dots};$$

therefore
$$\sqrt{2} = \frac{4.36.100.196\dots}{3.35.\ 99.195\dots}.$$

20. In the formula for $\cos\theta$ in Art. 320 put $\frac{\pi}{6}$ for θ; thus

$$\frac{\sqrt{3}}{2} = \left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{3^2.3^2}\right)\left(1-\frac{1}{3^2.5^2}\right)\left(1-\frac{1}{3^2.7^2}\right)\dots$$

$$= \frac{8.80.224.440\dots}{9.81.225.441\dots}.$$

21.
$$\cos x + \tan\frac{y}{2}\sin x = \frac{\cos x\cos\frac{y}{2} + \sin x\sin\frac{y}{2}}{\cos\frac{y}{2}} = \frac{\cos\left(x-\frac{y}{2}\right)}{\cos\frac{y}{2}}.$$

Now by Art. 320

$$\cos\left(x-\frac{y}{2}\right) = \left\{1-\frac{(2x-y)^2}{\pi^2}\right\}\left\{1-\frac{(2x-y)^2}{3^2\pi^2}\right\}\left\{1-\frac{(2x-y)^2}{5^2\pi^2}\right\}\dots$$

$$\cos\frac{y}{2} = \left(1-\frac{y^2}{\pi^2}\right)\left(1-\frac{y^2}{3^2\pi^2}\right)\left(1-\frac{y^2}{5^2\pi^2}\right)\dots$$

Divide the former by the latter. Then

$$\frac{1-\frac{(2x-y)^2}{\pi^2}}{1-\frac{y^2}{\pi^2}} = \frac{\pi^2-(2x-y)^2}{\pi^2-y^2} = \frac{\pi^2-y^2-4x^2+4xy}{\pi^2-y^2} = 1-\frac{4x^2}{\pi^2-y^2}+\frac{4xy}{\pi^2-y^2}$$

$$= \left(1+\frac{2x}{\pi-y}\right)\left(1-\frac{2x}{\pi+y}\right).$$

Similarly $$\frac{1-\dfrac{(2x-y)^2}{3^2\pi^2}}{1-\dfrac{y^2}{3^2\pi^2}}=\left(1+\frac{2x}{3\pi-y}\right)\left(1-\frac{2x}{3\pi+y}\right).$$

And so on. Thus the required result is obtained.

22. $$\cos x-\cot\frac{y}{2}\sin x=\frac{\cos x\sin\dfrac{y}{2}-\sin x\cos\dfrac{y}{2}}{\sin\dfrac{y}{2}}=\frac{\sin\left(\dfrac{y}{2}-x\right)}{\sin\dfrac{y}{2}}.$$

Now by Art. 320

$$\sin\left(\frac{y}{2}-x\right)=\left(\frac{y}{2}-x\right)\left\{1-\frac{(y-2x)^2}{4\,.\,\pi^2}\right\}\left\{1-\frac{(y-2x)^2}{4\,.\,2^2\pi^2}\right\}\left\{1-\frac{(y-2x)^2}{4\,.\,3^2\pi^2}\right\}\dots$$

$$\sin\frac{y}{2}=\frac{y}{2}\left(1-\frac{y^2}{4\,.\,\pi^2}\right)\left(1-\frac{y^2}{4\,.\,2^2\pi^2}\right)\left(1-\frac{y^2}{4\,.\,3^2\pi^2}\right)\dots$$

Divide the former by the latter. Then

$$\frac{\dfrac{y}{2}-x}{\dfrac{y}{2}}=1-\frac{2x}{y}.$$

$$\frac{1-\dfrac{(y-2x)^2}{4\,.\,\pi^2}}{1-\dfrac{y^2}{4\,.\,\pi^2}}=\frac{4\pi^2-(y-2x)^2}{4\pi^2-y^2}=\frac{4\pi^2-y^2-4x^2+4xy}{4\pi^2-y^2}$$

$$=1-\frac{4x^2}{4\pi^2-y^2}+\frac{4xy}{4\pi^2-y^2}=\left(1+\frac{2x}{2\pi-y}\right)\left(1-\frac{2x}{2\pi+y}\right).$$

Similarly $$\frac{1-\dfrac{(y-2x)^2}{4\,.\,2^2\pi^2}}{1-\dfrac{y^2}{4\,.\,2^2\pi^2}}=\left(1+\frac{2x}{4\pi-y}\right)\left(1-\frac{2x}{4\pi+y}\right).$$

And so on. Thus the required result is obtained.

23. $$\frac{\cos x-\cos y}{1-\cos y}=\frac{2\sin\dfrac{1}{2}(y-x)\sin\dfrac{1}{2}(y+x)}{2\sin^2\dfrac{y}{2}}.$$

Now by Art. 320

$$\sin\frac{1}{2}(y-x)=\frac{1}{2}(y-x)\left\{1-\frac{(y-x)^2}{4\pi^2}\right\}\left\{1-\frac{(y-x)^2}{4\,.\,2^2\pi^2}\right\}\cdots$$

$$\sin\frac{1}{2}(y+x)=\frac{1}{2}(y+x)\left\{1-\frac{(y+x)^2}{4\pi^2}\right\}\left\{1-\frac{(y+x)^2}{4\,.\,2^2\pi^2}\right\}\cdots$$

$$\sin\frac{1}{2}y=\frac{1}{2}y\left(1-\frac{y^2}{4\pi^2}\right)\left(1-\frac{y^2}{4\,.\,2^2\pi^2}\right)\cdots$$

Divide the first by the third, and divide the second by the third, and multiply the results together.

Then $\dfrac{\frac{1}{2}(y-x)}{\frac{1}{2}y}=1-\dfrac{x}{y}$; $\dfrac{\frac{1}{2}(x+y)}{\frac{1}{2}y}=1+\dfrac{x}{y}$; $\left(1-\dfrac{x}{y}\right)\left(1+\dfrac{x}{y}\right)=1-\dfrac{x^2}{y^2}$.

And as in the solution of Example 22,

$$\frac{1-\frac{(y-x)^2}{4\pi^2}}{1-\frac{y^2}{4\pi^2}}=\left(1+\frac{x}{2\pi-y}\right)\left(1-\frac{x}{2\pi+y}\right);$$

$$\frac{1-\frac{(y+x)^2}{4\pi^2}}{1-\frac{y^2}{4\pi^2}}=\left(1-\frac{x}{2\pi-y}\right)\left(1+\frac{x}{2\pi+y}\right);$$

$$\left(1+\frac{x}{2\pi-y}\right)\left(1-\frac{x}{2\pi+y}\right)\left(1-\frac{x}{2\pi-y}\right)\left(1+\frac{x}{2\pi+y}\right)$$

$$=\left\{1-\frac{x^2}{(2\pi-y)^2}\right\}\left\{1-\frac{x^2}{(2\pi+y)^2}\right\}.$$

Similarly $\dfrac{1-\frac{(y-x)^2}{4\,.\,2^2\pi^2}}{1-\frac{y^2}{4\,.\,2^2\pi^2}}\cdot\dfrac{1-\frac{(y+x)^2}{4\,.\,2^2\pi^2}}{1-\frac{y^2}{4\,.\,2^2\pi^2}}=\left\{1-\dfrac{x^2}{(4\pi-y)^2}\right\}\left\{1-\dfrac{x^2}{(4\pi+y)^2}\right\}.$

And so on. Thus the required result is obtained.

24. $$\frac{\cos x + \cos y}{1+\cos y} = \frac{2\cos\frac{1}{2}(y-x)\cos\frac{1}{2}(y+x)}{2\cos^2\frac{y}{2}}.$$

Now by Art. 320

$$\cos\frac{1}{2}(y-x) = \left\{1-\frac{(y-x)^2}{\pi^2}\right\}\left\{1-\frac{(y-x)^2}{3^2\pi^2}\right\}\left\{1-\frac{(y-x)^2}{5^2\pi^2}\right\}\cdots$$

$$\cos\frac{1}{2}(y+x) = \left\{1-\frac{(y+x)^2}{\pi^2}\right\}\left\{1-\frac{(y+x)^2}{3^2\pi^2}\right\}\left\{1-\frac{(y+x)^2}{5^2\pi^2}\right\}\cdots$$

$$\cos\frac{y}{2} = \left(1-\frac{y^2}{\pi^2}\right)\left(1-\frac{y^2}{3^2\pi^2}\right)\left(1-\frac{y^2}{5^2\pi^2}\right)\cdots$$

Divide the first by the third, and divide the second by the third, and multiply the two results together; the reductions will be similar to those in the preceding solution.

Thus $$\frac{1-\frac{(y-x)^2}{\pi^2}}{1-\frac{y^2}{\pi^2}}\cdot\frac{1-\frac{(y+x)^2}{\pi^2}}{1-\frac{y^2}{\pi^2}} = \left\{1-\frac{x^2}{(\pi-y)^2}\right\}\left\{1-\frac{x^2}{(\pi+y)^2}\right\};$$

$$\frac{1-\frac{(y-x)^2}{3^2\pi^2}}{1-\frac{y^2}{3^2\pi^2}}\cdot\frac{1-\frac{(y+x)^2}{3^2\pi^2}}{1-\frac{y^2}{3^2\pi^2}} = \left\{1-\frac{x^2}{(3\pi-y)^2}\right\}\left\{1-\frac{x^2}{(3\pi+y)^2}\right\}.$$

And so on. Thus the required result is obtained.

Or we may obtain the result in Example 24 by changing y into $\pi - y$ in the result of Example 23.

25. $$\frac{\sin x + \sin y}{\sin y} = \frac{2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)}{2\sin\frac{y}{2}\cos\frac{y}{2}}.$$

Now in the course of the solution of Example 23 we see that

$$\frac{\sin\frac{1}{2}(x+y)}{\sin\frac{1}{2}y} = \left(1+\frac{x}{y}\right)\left(1-\frac{x}{2\pi-y}\right)\left(1+\frac{x}{2\pi+y}\right)\left(1-\frac{x}{4\pi-y}\right)\left(1+\frac{x}{4\pi+y}\right)\cdots$$

And by changing y into $\pi - y$ we see that

$$\frac{\cos\frac{1}{2}(x-y)}{\cos\frac{1}{2}y} = \left(1+\frac{x}{\pi-y}\right)\left(1-\frac{x}{\pi+y}\right)\left(1+\frac{x}{3\pi-y}\right)\left(1-\frac{x}{3\pi+y}\right)\cdots$$

Hence by multiplication the required result is obtained.

26. We have $\cos x + \tan\frac{y}{2}\sin x$

$$= 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \dots + \tan\frac{y}{2}\left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \dots\right);$$

thus the coefficient of x is $\tan\frac{y}{2}$.

Now conceive the factors on the right-hand side of the formula of Example 21 multiplied together, and the product arranged according to powers of x. The first term will be unity; the second term will involve x, and the coefficient will be

$$\frac{2}{\pi - y} - \frac{2}{\pi + y} + \frac{2}{3\pi - y} - \frac{2}{3\pi + y} + \dots$$

Hence by equating the coefficients we obtain the required result.

27. Proceed as in the solution of Example 26. Then on the left-hand side the coefficient of x will be $-\cot\frac{y}{2}$, and on the right-hand side

$$-\frac{2}{y} + \frac{2}{2\pi - y} - \frac{2}{2\pi + y} + \frac{2}{4\pi - y} - \frac{2}{4\pi + y} + \dots$$

Equate the coefficients, and then change the signs of both sides; thus we obtain the required result.

28. In the formula of Example 26 put $\frac{\pi}{3}$ for y; then

$$\tan\frac{y}{2} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}};$$

thus $$\frac{1}{\sqrt{3}} = \frac{1}{\pi}\left\{\frac{6}{2} - \frac{6}{4} + \frac{6}{8} - \frac{6}{10} + \frac{6}{14} - \dots\right\};$$

that is $$\frac{1}{\sqrt{3}} = \frac{6}{\pi}\left\{\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \dots\right\};$$

therefore $$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \dots$$

29. In the formula of Example 27 put $\frac{\pi}{3}$ for y; then

$$\cot\frac{y}{2} = \cot\frac{\pi}{6} = \sqrt{3};$$

thus $$\sqrt{3} = \frac{6}{\pi}\left\{1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \dots\right\};$$

therefore $$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \dots.$$

30. Add together the results given in Examples 26 and 27; then

$$\tan\frac{y}{2}+\cot\frac{y}{2}=\frac{\sin\frac{y}{2}}{\cos\frac{y}{2}}+\frac{\cos\frac{y}{2}}{\sin\frac{y}{2}}=\frac{1}{\sin\frac{y}{2}\cos\frac{y}{2}}=\frac{2}{\sin y}.$$

Thus

$$\frac{2}{\sin y}=\frac{2}{y}+\frac{2}{\pi-y}-\frac{2}{2\pi-y}-\frac{2}{\pi+y}+\frac{2}{2\pi+y}+\frac{2}{3\pi-y}-\frac{2}{4\pi-y}-\frac{2}{3\pi+y}+\dots$$

Then divide both sides by 2.

Or we may equate the coefficients of x in Example 25.

XXIV.

1. $\sin\theta\cos\frac{\theta}{2}=2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}=2\sin\frac{\theta}{2}\sin^2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$

$$=8\sin\frac{\theta}{2}\sin^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)\cos^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)=8\sin\frac{\theta}{2}\sin^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)\sin^2\left(\frac{\pi}{4}+\frac{\theta}{4}\right).$$

2. $\left(\operatorname{cosec}^2\frac{\theta}{6}-\sec^2\frac{\theta}{2}\right)\tan\frac{\theta}{3}=\left(\frac{1}{\sin^2\frac{\theta}{6}}-\frac{1}{\cos^2\frac{\theta}{2}}\right)\tan\frac{\theta}{3}$

$$=\frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{6}}{\sin^2\frac{\theta}{6}\cos^2\frac{\theta}{2}}\tan\frac{\theta}{3}=\frac{\cos\left(\frac{\theta}{2}-\frac{\theta}{6}\right)\cos\left(\frac{\theta}{2}+\frac{\theta}{6}\right)}{\sin^2\frac{\theta}{6}\cos^2\frac{\theta}{2}}\tan\frac{\theta}{3}$$

$$=\frac{\cos\frac{\theta}{3}\cos\frac{2\theta}{3}\tan\frac{\theta}{3}}{\sin^2\frac{\theta}{6}\cos^2\frac{\theta}{2}}=\frac{\sin\frac{\theta}{3}\cos\frac{2\theta}{3}}{\sin^2\frac{\theta}{6}\cos^2\frac{\theta}{2}}.$$

Again $\left(\tan^2\frac{\theta}{2}\operatorname{cosec}^2\frac{\theta}{6}-\sec^2\frac{\theta}{2}\right)\cot\frac{2\theta}{3}=\left(\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{6}}-\frac{1}{\cos^2\frac{\theta}{2}}\right)\cot\frac{2\theta}{3}$

$$=\frac{\sin^2\frac{\theta}{2}-\sin^2\frac{\theta}{6}}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{6}}\cot\frac{2\theta}{3}=\frac{\sin\left(\frac{\theta}{2}-\frac{\theta}{6}\right)\sin\left(\frac{\theta}{2}+\frac{\theta}{6}\right)}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{6}}\cot\frac{2\theta}{3}$$

$$=\frac{\sin\frac{\theta}{3}\sin\frac{2\theta}{3}\cot\frac{2\theta}{3}}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{6}}=\frac{\sin\frac{\theta}{3}\cos\frac{2\theta}{3}}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{6}}.$$

3. $\tan 3\theta - \tan 2\theta - \tan\theta$

$$=\frac{\sin 3\theta}{\cos 3\theta}-\frac{\sin 2\theta}{\cos 2\theta}-\tan\theta$$

$$=\frac{\sin 3\theta\cos 2\theta-\sin 2\theta\cos 3\theta}{\cos 3\theta\cos 2\theta}-\tan\theta$$

$$=\frac{\sin\theta}{\cos 3\theta\cos 2\theta}-\frac{\sin\theta}{\cos\theta}$$

$$=\frac{\sin\theta}{\cos\theta\cos 2\theta\cos 3\theta}\{\cos\theta-\cos 3\theta\cos 2\theta\}$$

$$=\frac{\sin\theta}{\cos\theta\cos 2\theta\cos 3\theta}\{\cos(3\theta-2\theta)-\cos 3\theta\cos 2\theta\}$$

$$=\frac{\sin\theta\sin 2\theta\sin 3\theta}{\cos\theta\cos 2\theta\cos 3\theta}=\tan\theta\tan 2\theta\tan 3\theta.$$

4. $\tan^3 x+\cot^3 x=m^3-3m$;

therefore $(\tan x+\cot x)^3-3(\tan^2 x\cot x+\cot^2 x\tan x)=m^3-3m$;

therefore $(\tan x+\cot x)^3-3(\tan x+\cot x)=m^3-3m$;

therefore $(\tan x+\cot x)^3-m^3=3(\tan x+\cot x)-3m.$

Put y for $\tan x+\cot x$; thus

$$y^3-m^3=3(y-m);$$

therefore $$(y-m)(y^2+ym+m^2)=3(y-m).$$

Therefore either $y-m=0$, or $y^2+ym+m^2=3$.

Take $y-m=0$; thus $\tan x+\cot x=m$,

therefore $$\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=m;$$

therefore $$\frac{1}{\sin x\cos x}=m;$$

therefore $$\sin 2x=\frac{2}{m}.$$

Again, take $y^2+ym+m^2=3$. By solving this quadratic in the usual way we obtain

$$y=\frac{-m\pm\sqrt{(12-3m^2)}}{2},$$

and thus we obtain two other values for $\sin 2x$.

5. Let T denote the m^{th} point, counting from A as the first, so that the angle $TOA = m\beta$, where $\beta = \frac{\pi}{2n}$. Then, since the angle OTA is a right angle, we have $OT = OA \cos TOA = 2r \cos m\beta$, where r is the radius of the circle. The angle between OT and the tangent at T is equal to the angle TAO, and is therefore $\frac{\pi}{2} - m\beta$. Thus the perpendicular from O on the tangent at $T = OT \sin\left(\frac{\pi}{2} - m\beta\right) = OT \cos m\beta = 2r \cos^2 m\beta$. The square of the perpendicular is $4r^2 \cos^4 m\beta$.

Thus the sum of the squares of the perpendiculars

$$= 4r^2 \{1 + \cos^4 \beta + \cos^4 2\beta + \ldots + \cos^4 (2n-1)\beta\}.$$

Now $$\cos^4 \theta = \frac{3}{8} + \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta.$$

Apply this transformation to every term in the series, observing that 1 may be considered as $\cos 0$.

Thus the sum of the squares of the perpendiculars

$$= 4r^2 . \frac{3}{8} . 2n + 2r^2 \{\cos 0 + \cos 2\beta + \cos 4\beta + \ldots + \cos 2(2n-1)\beta\}$$

$$+ \frac{r^2}{2} \{\cos 0 + \cos 4\beta + \cos 8\beta + \ldots + \cos 4(2n-1)\beta\}.$$

Each series of cosines will vanish as in Art. 305; thus we find that the sum of the squares of the perpendiculars $= 3nr^2$.

6. Let $ACP = \theta$, $ACQ = \phi$, $ACR = \psi$; let $AC = \rho$. Then $\theta + \phi + \psi = \pi$. Now, in order that a triangle may exist with Ap, Aq, and Ar as sides we must have $Ap + Aq$ greater than Ar; thus $\tan\theta + \tan\phi$ must be greater than $\tan\psi$.

But by Art. 114 we have

$$\tan\theta + \tan\phi + \tan\psi = \tan\theta \tan\phi \tan\psi.$$

Hence $\tan\theta \tan\phi \tan\psi$ must be greater than $2\tan\psi$, and therefore $\tan\theta \tan\phi$ greater than 2. Therefore *a fortiori* $\tan^2\phi$ must be greater than 2.

Thus the inferior limit of Aq is when $\tan QCA = \sqrt{2}$.

Again, since $\theta + \phi + \psi = \pi$, the superior limit of θ is the value of θ when θ, ϕ, and ψ are all equal; that is, when $\theta = \frac{\pi}{3}$.

When the triangle is formed the radius of the inscribed circle, by Art. 248, is $\frac{2S}{\rho(\tan\theta+\tan\phi+\tan\psi)}$; and the radius of the circumscribed circle by Art. 252 is $\frac{\rho^3\tan\theta\tan\phi\tan\psi}{4S}$. The product of these radii is $\frac{\rho^2}{2}\cdot\frac{\tan\theta\tan\phi\tan\psi}{\tan\theta+\tan\phi+\tan\psi}$, that is $\frac{\rho^2}{2}$ by Art. 114.

Hence the product of the radii is constant, whatever θ, ϕ, and ψ may be; and therefore one radius varies inversely as the other.

7. As in Art. 250 we see that $CF=s$; also $CE=r$; hence

$$\text{the area of } CEF=\frac{1}{2}rs.$$

And, as in Art. 248 the area of $ABC=rs$; therefore CEF is half ABC.

8. The straight lines bisecting the external angles are the same as those which join the centres of the escribed circles.

Thus, by Example XVI. 34, we have $S'=\frac{abc}{2r}$;

therefore $$\frac{S'}{S}=\frac{abc}{2rS}=\frac{sabc}{2S^2}=\frac{abc}{2(s-a)(s-b)(s-c)}$$

$$=\frac{1}{2}\operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}.$$

9. Let P denote the point above D. Then $PD\times AC=$ twice the area of the triangle $APC=AP\,.\,PC\sin 2\alpha$; therefore

$$PD=\frac{AP\,.\,PC\,.\sin 2\alpha}{AC}=\frac{(a+b)\,AP\,.\,PC\,.\sin 2\alpha}{AC^2}=\frac{(a+b)\,AP\,.\,PC\,.\sin 2\alpha}{AP^2+PC^2-2AP\,.\,PC\cos 2\alpha}$$

$$=\frac{(a+b)\,ab\sin 2\alpha}{a^2+b^2-2ab\cos 2\alpha},\text{ for }\frac{AP}{PC}=\frac{a}{b}\text{ by Euclid VI. 3,}$$

$$=\frac{2(a+b)\,ab\sin\alpha\cos\alpha}{(a^2+b^2)(\sin^2\alpha+\cos^2\alpha)-2ab(\cos^2\alpha-\sin^2\alpha)}$$

$$=\frac{2(a+b)\,ab\tan\alpha}{(a-b)^2+(a+b)^2\tan^2\alpha}.$$

10. Let AB be the arc, and C the centre of the circle; in AB take any point P, join PA, and draw PM perpendicular to AB.

Let $BCA=2\gamma$, $PCA=2\theta$, and $AC=r$.

Then $AB = 2r \sin \gamma$, $AP = 2r \sin \theta$, $PM = AP \sin PAB = AP \sin (\gamma - \theta)$.

We have then to shew that $2r \sin \theta \{1 + \sin (\gamma - \theta)\}$ is less than $2r \sin \gamma$, or that $\sin (\gamma - \theta) \sin \theta$ is less than $\sin \gamma - \sin \theta$, or that

$$2 \sin \frac{1}{2}(\gamma - \theta) \cos \frac{1}{2}(\gamma - \theta) \sin \theta \text{ is less than } 2 \sin \frac{1}{2}(\gamma - \theta) \cos \frac{1}{2}(\gamma + \theta),$$

or that
$$\cos \frac{1}{2}(\gamma - \theta) \sin \theta \text{ is less than } \cos \frac{1}{2}(\gamma + \theta).$$

Now this is the case, for $\cos \frac{1}{2}(\gamma - \theta) \sin \theta$ is less than $\sin \theta$, and therefore less than $\sin \gamma$; and $\cos \frac{1}{2}(\gamma + \theta)$ is greater than $\cos \frac{1}{2}(\gamma + \gamma)$, that is greater than $\cos \gamma$; and $\cos \gamma$ is greater than $\sin \gamma$, since γ is less than $\frac{\pi}{4}$.

11. Let A be the position of the observer's eye, C the cloud, B the image of the cloud formed by the lake. Draw the horizontal straight line AH. Then $HAC = \alpha$, and $HAB = \beta$.

The straight lines CB and AB are equally inclined to the surface of the water by the Laws of Optics, and thus the angle between CB and AH is equal to β.

Now
$$\frac{CB}{AB} = \frac{\sin CAB}{\sin ACB} = \frac{\sin (\beta + \alpha)}{\sin (\beta - \alpha)};$$

therefore
$$CB = \frac{AB \sin (\beta + \alpha)}{\sin (\beta - \alpha)}.$$

The height of the cloud
$$= CB \sin \beta = \frac{\sin (\beta + \alpha)}{\sin (\beta - \alpha)} AB \sin \beta$$
$$= h \frac{\sin (\beta + \alpha)}{\sin (\beta - \alpha)}.$$

12. Let A be the position of the observer's eye, C the cloud, B the shadow of the cloud on the sea. Then the position of the sun is on BC produced through C. Draw from A a horizontal straight line meeting BC at H. Then $HAC = \alpha$, and $HAB = \beta$. On account of the enormous distance of the sun, the straight line HC may be considered as parallel to the straight line drawn from A to the sun; so that $CHA = \gamma$.

Now
$$\frac{BC}{BA} = \frac{\sin BAC}{\sin BCA} = \frac{\sin (\alpha + \beta)}{\sin (\pi - \alpha - \gamma)} = \frac{\sin (\alpha + \beta)}{\sin (\alpha + \gamma)}.$$

The height of the cloud above the surface of the sea

$$= BC \sin \gamma = \frac{\sin \gamma}{\sin \beta} BC \sin \beta = \frac{\sin \gamma}{\sin \beta} \frac{\sin (\alpha + \beta)}{\sin (\alpha + \gamma)} BA \sin \beta$$
$$= h \frac{\sin \gamma}{\sin \beta} \cdot \frac{\sin (\alpha + \beta)}{\sin (\alpha + \gamma)}.$$

13. *Assume* that

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta + \dots$$
$$+ \frac{n(n-1)\dots(n-r+1)}{\underline{|r}}\cos(n-2r)\theta + \dots$$

Multiply both sides by $2\cos\theta$. Thus

$$2^n\cos^{n+1}\theta = 2\cos n\theta\cos\theta + 2n\cos(n-2)\theta\cos\theta$$
$$+ 2\frac{n(n-1)}{1.2}\cos(n-4)\theta\cos\theta + \dots$$

Now use the formula $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ for the terms on the right-hand side. Thus

$$2^n\cos^{n+1}\theta = \cos(n+1)\theta + \cos(n-1)\theta$$
$$+ n\{\cos(n-1)\theta + \cos(n-3)\theta\}$$
$$+ \frac{n(n-1)}{1.2}\{\cos(n-3)\theta + \cos(n-5)\theta\}$$
$$+ \frac{n(n-1)\dots(n-r+1)}{\underline{|r}}\{\cos(n-2r+1)\theta + \cos(n-2r-1)\theta\}$$
$$+ \dots$$

Then re-arrange the terms on the right-hand side, and we obtain a series like that with which we started except that we have $n+1$ instead of n.

For instance, the term involving $\cos(n-3)\theta$ is

$$\left\{n + \frac{n(n-1)}{1.2}\right\}\cos(n-3)\theta;$$

that is $$\frac{(n+1)n}{1.2}\cos(n+1-4)\theta.$$

And generally the term involving $\cos(n+1-2r)\theta$ is

$$\left\{\frac{n(n-1)\dots(n-r+2)}{\underline{|r-1}} + \frac{n(n-1)\dots(n-r+1)}{\underline{|r}}\right\}\cos(n+1-2r)\theta;$$

that is $$\frac{n(n-1)\dots(n-r+2)}{\underline{|r-1}}\left(1 + \frac{n-r+1}{r}\right)\cos(n+1-2r)\theta;$$

that is $$\frac{(n+1)n\dots(n+1-r+1)}{\underline{|r}}\cos(n+1-2r)\theta.$$

This shews that if the formula holds for an assigned value of n it holds also when n is changed into $n+1$. Moreover the formula evidently holds when $n=1$.

We have not paid special attention to the *last term* in the expansion, but it is easy to do this if required.

14. Take the formula of Art. 280, and suppose n *even;* thus

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta + \dots$$

$$+\frac{n(n-1)(n-r+1)}{\underline{|r}}\cos(n-2r)\theta \dots + \frac{n(n-1)\dots\left(\frac{1}{2}n+1\right)}{2\,\underline{|\tfrac{1}{2}n}}.$$

Change θ into $\frac{\pi}{2}-\theta$; thus

$\cos^n\theta$ becomes $\cos^n\left(\frac{\pi}{2}-\theta\right)$, that is $\sin^n\theta$;

$\cos n\theta$ becomes $\cos n\left(\frac{\pi}{2}-\theta\right)$, that is $\cos n\frac{\pi}{2}\cos n\theta$, that is $(-1)^{\frac{n}{2}}\cos n\theta$;

$\cos(n-2)\theta$ becomes $\cos(n-2)\left(\frac{\pi}{2}-\theta\right)$, that is $(-1)^{\frac{n-2}{2}}\cos(n-2)\theta$;

and so on.

Thus
$$2^{n-1}\sin^n\theta = (-1)^{\frac{n}{2}}\cos n\theta + n(-1)^{\frac{n-2}{2}}\cos(n-2)\theta$$
$$+\frac{n(n-1)}{1.2}(-1)^{\frac{n-4}{2}}\cos(n-4)\theta + \dots$$

Multiply both sides by $(-1)^{\frac{n}{2}}$; thus we obtain the formula of Art. 282.

Next suppose n *odd.* Then in the same manner we deduce the formula of Art. 283 from that of Art. 280; we observe now that

$$\cos n\left(\frac{\pi}{2}-\theta\right) = \sin n\frac{\pi}{2}\sin n\theta = (-1)^{\frac{n-1}{2}}\sin n\theta,$$

$$\cos(n-2)\left(\frac{\pi}{2}-\theta\right) = (-1)^{\frac{n-3}{2}}\sin(n-2)\theta,$$

and so on.

15. Let $\tan^{-1}x=\theta$, so that $\tan\theta = x$; then

$$\cos\theta = \frac{1}{\sqrt{(1+x^2)}} \text{ and } \sin\theta = \frac{x}{\sqrt{(1+x^2)}}.$$

And by Art. 270

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$=\frac{1}{(1+x^2)^3}\{1-15x^2+15x^4-x^6\}.$$

16. Since the quadrilateral can be inscribed in a circle, the area by Art. 254 is $\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}$. But when a quadrilateral can be described about a circle it may be shewn by geometry that the sum of two opposite sides is equal to the sum of the other two. Thus in the present case $a+c=b+d$.

Now $$s=\frac{1}{2}(a+b+c+d)=a+c=b+d;$$

therefore $$s-a=c, \quad s-b=d, \quad s-c=a, \quad s-d=b;$$

therefore $$\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}=\sqrt{abcd}.$$

Thus the area $=\sqrt{abcd}$.

17. Let a, b, c, d denote the sides taken in succession; let B denote the angle between the first two, and D the angle between the last two. Thus $B+D=\theta$.

Then dividing the quadrilateral into two triangles, as in Art. 254, we have

$$S=\frac{1}{2}ab\sin B+\frac{1}{2}cd\sin D \quad \text{..........(1)}.$$

And from two values which can be obtained for the square of the diagonal opposite B and D we have

$$a^2+b^2-2ab\cos B=c^2+d^2-2cd\cos D,$$

therefore $$\frac{a^2+b^2-c^2-d^2}{4}=\frac{1}{2}ab\cos B-\frac{1}{2}cd\cos D \quad \text{..........(2)}.$$

Square and add (1) and (2): thus

$$S^2+\left(\frac{a^2+b^2-c^2-d^2}{4}\right)^2=\frac{1}{4}(a^2b^2+c^2d^2)-\frac{1}{2}abcd\cos(B+D)$$

$$=\frac{1}{4}(a^2b^2+c^2d^2)-\frac{1}{2}abcd\cos\theta=\frac{1}{4}(a^2b^2+c^2d^2)-\frac{1}{2}abcd\left(2\cos^2\frac{\theta}{2}-1\right);$$

therefore $$S^2=\frac{1}{4}(ab+cd)^2-\left(\frac{a^2+b^2-c^2-d^2}{4}\right)^2-abcd\cos^2\frac{\theta}{2}.$$

Now we know that if $\theta=\pi$ the expression for S^2 must reduce to

$$(s-a)(s-b)(s-c)(s-d).$$

Hence we are sure that $\frac{1}{4}(ab+cd)^2-\left(\frac{a^2+b^2-c^2-d^2}{4}\right)^2$ must take the form just given; and this is easily verified. For this expression

$$=\left\{\frac{1}{2}(ab+cd)+\frac{a^2+b^2-c^2-d^2}{4}\right\}\left\{\frac{1}{2}(ab+cd)-\frac{a^2+b^2-c^2-d^2}{4}\right\}$$

$$=\frac{1}{16}\{(a+b)^2-(c-d)^2\}\{(c+d)^2-(a-b)^2\}$$

$$=\frac{1}{16}(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b).$$

Thus $$S^2=(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\frac{\theta}{2}.$$

18. Let t stand for $\tan\theta$. By Art. 270 we have

$$\cos n\theta=\cos^n\theta\left\{1-\frac{n(n-1)}{\lfloor 2}t^2+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}t^4-\dots\right\}.$$

Put ι for $\sqrt{-1}$; then we may write the formula thus

$$2\cos n\theta=\cos^n\theta\{(1+\iota t)^n+(1-\iota t)^n\}.$$

Therefore $$2\cos^n\theta\cos n\theta=\cos^{2n}\theta\{(1+\iota t)^n+(1-\iota t)^n\}$$

$$=\frac{1}{(1+t^2)^n}\{(1+\iota t)^n+(1-\iota t)^n\}=\frac{1}{(1+\iota t)^n(1-\iota t)^n}\{(1+\iota t)^n+(1-\iota t)^n\}$$

$$=(1-\iota t)^{-n}+(1+\iota t)^{-n}.$$

Expand the two terms on the right-hand side by the Binomial Theorem, and the required result is obtained.

19. Proceed as in the solution of Example 18. Thus we obtain

$$2\sin n\theta=\frac{\cos^n\theta}{\iota}\{(1+\iota t)^n-(1-\iota t)^n\};$$

therefore $$2\cos^n\theta\sin n\theta=\frac{\cos^{2n}\theta}{\iota}\{(1+\iota t)^n-(1-\iota t)^n\}$$

$$=\frac{1}{\iota(1+t^2)^n}\{(1+\iota t)^n-(1-\iota t)^n\}=\frac{1}{\iota(1+\iota t)^n(1-\iota t)^n}\{(1+\iota t)^n-(1-\iota t)^n\}$$

$$=\frac{1}{\iota}\{(1-\iota t)^{-n}-(1+\iota t)^{-n}\}.$$

Expand the two terms on the right-hand side by the Binomial Theorem, and the required result is obtained.

20. Let θ have any value between 0 and $\frac{\pi}{2}$; let h be a small positive quantity. We have then to shew that $\frac{\theta+h}{\sin(\theta+h)}$ is greater than $\frac{\theta}{\sin\theta}$, that is we must shew that $\frac{\theta+h}{\sin(\theta+h)}-\frac{\theta}{\sin\theta}$ is positive.

Now the sign of the last expression is the same as the sign of

$$(\theta+h)\sin\theta-\theta\sin(\theta+h),$$

and is therefore the same as the sign of

$$\theta \sin\theta\,(1-\cos h)+h\sin\theta-\theta\cos\theta\sin h,$$

or as the sign of

$$\theta\sin\theta\,(1-\cos h)+\sin\theta\sin h\left(\frac{h}{\sin h}-\frac{\theta}{\tan\theta}\right).$$

Now $1-\cos h$ is positive; and $\frac{h}{\sin h}$ is greater than unity while $\frac{\theta}{\tan\theta}$ is less than unity, by Art. 118; thus the expression is positive.

21. Let θ have any value between 0 and $\frac{\pi}{2}$; let h be a small positive quantity. We have then to shew that $\frac{\theta}{\tan\theta}$ is greater than $\frac{\theta+h}{\tan(\theta+h)}$, that is we must shew that $\frac{\theta}{\tan\theta}-\frac{\theta+h}{\tan(\theta+h)}$ is positive.

Now the sign of the last expression is the same as the sign of

$$\theta\cos\theta\sin(\theta+h)-(\theta+h)\cos(\theta+h)\sin\theta,$$

that is of

$$\theta\sin h-h\cos(\theta+h)\sin\theta,$$

that is of

$$\frac{\theta}{\sin\theta}-\frac{h}{\sin h}\cos(\theta+h).$$

But as we may suppose h less than θ, we know by Example 20 that $\frac{\theta}{\sin\theta}$ is greater than $\frac{h}{\sin h}$, and therefore $\frac{\theta}{\sin\theta}$ is greater than $\frac{h}{\sin h}\cos(\theta+h)$.

22. Let PO intersect FD at K; then by similar triangles, since $PA=2PF$, we have $OA=2FK$; but $DF=OA$; therefore $DF=2FK$; therefore DF is bisected at K.

Also since PA is bisected at F, it follows by similar triangles that PO is bisected at K.

23. Let PO intersect AD at L. Then the triangles ALP and DLO are similar; therefore $\frac{LD}{OD}=\frac{LA}{PA}$; therefore $\frac{LD}{LA}=\frac{OD}{PA}=\frac{FA}{PA}=\frac{1}{2}$.

24. The point K in the solution of Example 22 being the middle point of DF is the centre of the nine points circle. Thus P, K, and O are in one straight line. Also this straight line cuts AD at a point L, such that $\frac{LD}{LA}=\frac{1}{2}$; and OP is divided at L so that $\frac{OL}{LP}=\frac{1}{2}$.

In like manner the straight line from B to the middle point of AC cuts OP at the same point as AD does; and so also does the straight line from C to the middle point of AB. Hence the point L is the intersection of the three straight lines from the angles of ABC to the middle points of the opposite sides.

25. The centre of the nine points circle is the middle point of OP, hence the perpendicular from it on $BC=\frac{1}{2}(OD+PG)$. Now $OD=R\cos A$, and

$$PG=BP\sin PBG=BP\cos C=\frac{BG\cos C}{\cos PBG}=\frac{BG\cos C}{\sin C}=\frac{c\cos B\cos C}{\sin C}$$
$$=2R\cos B\cos C.$$

Hence the perpendicular required

$$=\frac{1}{2}(R\cos A+2R\cos B\cos C)=\frac{R}{2}\{2\cos B\cos C-\cos(B+C)\}$$
$$=\frac{R}{2}(\cos B\cos C-\sin C\sin B)=\frac{R}{2}\cos(C-B).$$

Or thus: the required perpendicular

$$=\frac{R}{2}\sin HDG=\frac{R}{2}\cos OAG=\frac{R}{2}\cos(BAG-OAB)$$
$$=\frac{R}{2}\cos\{90^0-B-(90^0-C)\}=\frac{R}{2}\cos(C-B).$$

26. The perpendicular from the centre of the nine points circle on AG

$$=\frac{1}{2}DG=\frac{1}{2}(CD-CG)=\frac{1}{2}\left(\frac{a}{2}-b\cos C\right)=\frac{1}{2}(R\sin A-2R\sin B\cos C)$$
$$=\frac{R}{2}\{\sin(B+C)-2\sin B\cos C\}=\frac{R}{2}\sin(C-B).$$

Or thus: the required perpendicular

$$=\frac{R}{2}\cos HDG=\frac{R}{2}\sin OAG=\frac{R}{2}\sin(C-B).$$

27. We have $AP=2AF=2OD=2R\cos A$; and, as shewn in the solution of Example 25, the angle $OAP=C-B$.

Then, from the triangle OAP,

$$OP^2=OA^2+PA^2-2OA\,.\,PA\cos(C-B)$$
$$=R^2\{1+4\cos^2 A-4\cos A\cos(C-B)\}$$
$$=R^2+4R^2\cos A\{\cos A-\cos(C-B)\}$$
$$=R^2+4R^2\cos A\{-\cos(B+C)-\cos(C-B)\}$$
$$=R^2-8R^2\cos A\cos B\cos C.$$

28. Denote the centre of the nine points circle by K; then K is the middle point of OP.

Now $$OA^2 = OK^2 + KA^2 - 2OK \cdot KA \cos OKA,$$

$$PA^2 = PK^2 + KA^2 - 2PK \cdot KA \cos PKA;$$

therefore, by addition,

$$OA^2 + PA^2 = 2OK^2 + 2KA^2;$$

thus $$2KA^2 = R^2 + 4R^2 \cos^2 A - 2OK^2 = R^2 + 4R^2 \cos^2 A - \frac{1}{2} PO^2.$$

Therefore, by the aid of Example 27,

$$4KA^2 = 2R^2 + 8R^2 \cos^2 A - R^2 (1 - 8 \cos A \cos B \cos C)$$

$$= R^2 + 8R^2 \cos A (\cos A + \cos B \cos C)$$

$$= R^2 + 8R^2 \cos A \{-\cos (B + C) + \cos B \cos C\}$$

$$= R^2 + 8R^2 \cos A \sin B \sin C;$$

therefore $$KA = \frac{R}{2} \sqrt{(1 + 8 \cos A \sin B \sin C)}.$$

29. Take the diagram of Art. 332. The centre of the nine points circle is on a straight line which bisects DG at right angles; and so it cannot be at O unless D and G coincide, that is unless the perpendicular AG bisects BC, that is unless $B = C$. Similarly we see, by considering the side AC instead of BC, that it will be necessary to have $A = C$. Thus the triangle must be equilateral.

30. Take the diagram of Art. 332. The centre of the inscribed circle is on the straight line AE, which bisects the angle A; and the centre of the nine points circle is on DF: hence when the two coincide it must be at the point H. Thus, by Example 22, we must have H at the middle point of OP. Then from the similar triangles OHE and AHP we find that OE must be equal to AP, that is to twice AF. Thus $R = 2R \cos A$; therefore $A = 60^\circ$. Similarly we see by considering the side AC instead of BC, that $B = 60^\circ$. Hence the triangle must be equilateral.

Or we may use Example 25; thus we must have

$$r = \frac{R}{2} \cos (B - C) = \frac{R}{2} \cos (C - A) = \frac{R}{2} \cos (A - B);$$

so that $$\cos (B - C) = \cos (C - A) = \cos (A - B):$$

these lead to $A = B = C$.

MISCELLANEOUS EXAMPLES.

1. Let x denote the number of degrees in the unit. Then $3 : x :: \cdot 15 : 1$. Hence $x = \frac{3}{\cdot 15} = 20$. The measure of a right angle will be $\frac{90}{20}$, that is $4\frac{1}{2}$.

2. Let x denote the circular measure of the larger angle, y that of the smaller angle. Then, since the circular measure of 1^0 is $\frac{\pi}{180}$, we have $x - y = \frac{\pi}{180}$, $x + y = 1$. Hence $x = \frac{1}{2}\left(1 + \frac{\pi}{180}\right)$, $y = \frac{1}{2}\left(1 - \frac{\pi}{180}\right)$.

3. Here $\tan x + \frac{ab}{\tan x} = a + b$; therefore $\tan^2 x - (a+b) \tan x + ab = 0$.

By solving this quadratic equation we obtain $\tan x = a$, or $\tan x = b$.

4. Here $\sin(2\theta + \theta) = \sin\theta \cos 2\theta$, that is

$$\sin 2\theta \cos\theta + \cos 2\theta \sin\theta = \sin\theta \cos 2\theta;$$

therefore $\sin 2\theta \cos\theta = 0$, that is $2 \sin\theta \cos^2\theta = 0$.

If $\cos\theta = 0$ we have θ an odd multiple of $\frac{\pi}{2}$; and if $\sin\theta = 0$ we have θ an even multiple of $\frac{\pi}{2}$: hence all the solutions are comprised in $\theta = n\frac{\pi}{2}$, where n is zero or an integer.

5. Let $2A$ denote the whole angle, and $A + x$ one of the two unequal parts; then $A - x$ denotes the other. Hence we have to shew that

$$\sin(A + x) \sin(A - x) + \sin^2 x = \sin^2 A;$$

and this is obvious by Art. 83.

6. $(\sec\theta \sec\phi + \tan\theta \tan\phi)^2 - (\tan\theta \sec\phi + \sec\theta \tan\phi)^2$

$$= \sec^2\theta \sec^2\phi + \tan^2\theta \tan^2\phi - \tan^2\theta \sec^2\phi - \sec^2\theta \tan^2\phi$$

$$= \sec^2\phi (\sec^2\theta - \tan^2\theta) - \tan^2\phi (\sec^2\theta - \tan^2\theta)$$

$$= \sec^2\phi - \tan^2\phi = 1.$$

$$2(1 + \tan^2\theta \tan^2\phi) - \sec^2\theta \sec^2\phi = \frac{2(\cos^2\theta \cos^2\phi + \sin^2\phi \sin^2\theta) - 1}{\cos^2\theta \cos^2\phi}$$

$$= \frac{(1 + \cos 2\theta)\cos^2\phi + (1 - \cos 2\theta)\sin^2\phi - 1}{\cos^2\theta \cos^2\phi}$$

$$= \frac{\cos 2\theta (\cos^2\phi - \sin^2\phi)}{\cos^2\theta \cos^2\phi} = \frac{\cos 2\theta \cos 2\phi}{\cos^2\theta \cos^2\phi}.$$

And $$1 \div \frac{\cos 2\theta \cos 2\phi}{\cos^2\theta \cos^2\phi} = \frac{\cos^2\theta \cos^2\phi}{\cos 2\theta \cos 2\phi} = \frac{\sec 2\theta \sec 2\phi}{\sec^2\theta \sec^2\phi}.$$

7. Since $A+B+C=360^0$, we have $\cos C=\cos(A+B)$.

Thus $1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C$

$$=1-\cos^2 A-\cos^2 B+\cos C(2\cos A\cos B-\cos C)$$

$$=1-\cos^2 A-\cos^2 B+\cos(A+B)(\cos A\cos B+\sin A\sin B)$$

$$=1-\cos^2 A-\cos^2 B+(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)$$

$$=1-\cos^2 A-\cos^2 B+\cos^2 A\cos^2 B-\sin^2 A\sin^2 B$$

$$=1-\cos^2 A-\cos^2 B+\cos^2 A\cos^2 B-(1-\cos^2 A)(1-\cos^2 B)$$

$$=0.$$

8. $$\sin A=\frac{3}{5};\quad \text{therefore } \cos A=\frac{4}{5}.$$

$$\sin B=\frac{12}{13};\quad \text{therefore } \cos B=\frac{5}{13}.$$

$$\sin C=\frac{7}{25};\quad \text{therefore } \cos C=\frac{24}{25}.$$

Hence we obtain $\sin(A+B)=\frac{63}{65}$, $\cos(A+B)=-\frac{16}{65}$;

then $$\sin(A+B+C)=\frac{63\times 24-7\times 16}{25\times 65}=\frac{1400}{25\times 65}=\frac{56}{65}.$$

9. $x=r\left(\sin\frac{\theta}{2}\cos\frac{\alpha}{2}-\cos\frac{\theta}{2}\sin\frac{\alpha}{2}\right)$, $y=r\left(\sin\frac{\theta}{2}\cos\frac{\alpha}{2}+\cos\frac{\theta}{2}\sin\frac{\alpha}{2}\right)$.

From these we obtain $\sin\frac{\theta}{2}=\dfrac{x+y}{2r\cos\frac{\alpha}{2}}$, $\cos\frac{\theta}{2}=\dfrac{y-x}{2r\sin\frac{\alpha}{2}}$.

Square and add; thus $1=\dfrac{1}{4r^2}\left\{\dfrac{(x+y)^2}{\cos^2\frac{\alpha}{2}}+\dfrac{(y-x)^2}{\sin^2\frac{\alpha}{2}}\right\}$;

therefore $$4r^2\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}=(x+y)^2\sin^2\frac{\alpha}{2}+(y-x)^2\cos^2\frac{\alpha}{2};$$

that is $r^2\sin^2\alpha=x^2+y^2-2xy\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)=x^2+y^2-2xy\cos\alpha.$

10. Here $\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}=\dfrac{a+b}{a-b}$, therefore $\dfrac{\sin(\theta+\phi)+\sin(\theta-\phi)}{\sin(\theta+\phi)-\sin(\theta-\phi)}=\dfrac{a}{b}$,

that is $\dfrac{\sin\theta\cos\phi}{\cos\theta\sin\phi}=\dfrac{a}{b}$; so that $a\tan\phi=b\tan\theta$.

Hence $$\frac{a\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}=\frac{b\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}};$$

therefore $$a \tan\frac{\phi}{2} - b\tan\frac{\theta}{2} = \tan\frac{\theta}{2}\tan\frac{\phi}{2}\left(a\tan\frac{\theta}{2} - b\tan\frac{\phi}{2}\right)$$

$$= c\tan\frac{\theta}{2}\tan\frac{\phi}{2};$$

therefore $$a\tan\frac{\phi}{2} = \tan\frac{\theta}{2}\left(b + c\tan\frac{\phi}{2}\right).$$

Substitute for $\tan\frac{\theta}{2}$ from the second of the given equations, and we obtain

$$a^2\tan\frac{\phi}{2} = \left(b + c\tan\frac{\phi}{2}\right)\left(c + b\tan\frac{\phi}{2}\right).$$

11. Let x denote the number of degrees in one angle; then $90 - x$ denotes the number of degrees in the other angle, and consequently $\frac{10}{9}(90 - x)$ the number of grades. Hence $x = \frac{3}{10} \times \frac{10}{9}(90 - x) = \frac{1}{3}(90 - x)$. Therefore $4x = 90$, and $x = 22\frac{1}{2}$.

12. Let the circular measure of an angle be $\frac{n\pi}{20}$; then the number of degrees in it is $\frac{n\pi}{20} \cdot \frac{180}{\pi}$, that is $9n$; and the number of grades is $\frac{n\pi}{20} \cdot \frac{200}{\pi}$, that is $10n$.

13. Here $(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\cos\gamma = (\sin\alpha\cos\gamma + \cos\alpha\sin\gamma)\cos\beta$;

therefore $$\cos\alpha(\sin\beta\cos\gamma - \sin\gamma\cos\beta) = 0,$$

that is $$\cos\alpha\sin(\beta - \gamma) = 0.$$

Either then $\cos\alpha = 0$, so that α is an odd multiple of $\frac{\pi}{2}$; or $\sin(\beta - \gamma) = 0$, so that $\beta - \gamma$ is a multiple of π.

14. $\sin 4A(\tan^4 A + 2\tan^2 A + 1) = \sin 4A(\tan^2 A + 1)^2$

$$= \frac{\sin 4A}{\cos^4 A} = \frac{2\sin 2A\cos 2A}{\cos^4 A} = \frac{4\sin A\cos 2A}{\cos^3 A}$$

$$= \frac{4\sin A(\cos^2 A - \sin^2 A)}{\cos^3 A}.$$

And $$4\tan^3 A - 4\tan A = 4\tan A(\tan^2 A - 1)$$

$$= \frac{4\sin A(\sin^2 A - \cos^2 A)}{\cos^3 A}.$$

Therefore $$\sin 4A(\tan^4 A + 2\tan^2 A + 1) + 4\tan^3 A - 4\tan A = 0.$$

15. By Art. 83; $\sin^2 24^0 - \sin^2 6^0 = \sin(24^0 + 6^0)\sin(24^0 - 6^0)$
$= \sin 30^0 \sin 18^0.$

Also $\sin 30^0 = \frac{1}{2}$, and $\sin 18^0 = \frac{\sqrt{5}-1}{4}$.

16. The given expression is

$$\sin A(\sin A + \cos B \sin C + \cos C \sin B)$$
$$+ \sin B(\sin B + \cos C \sin A + \cos A \sin C)$$
$$+ \sin C(\sin C + \cos A \sin B + \cos B \sin A),$$

that is $\sin A\{\sin A + \sin(B+C)\} + \sin B\{\sin B + \sin(C+A)\}$
$+ \sin C\{\sin C + \sin(A+B)\}$.

Now since $A + B + C = 360^0$, we have

$$\sin(B+C) = -\sin A, \quad \sin(C+A) = -\sin B, \quad \sin(A+B) = -\sin C:$$

thus the whole expression vanishes.

17. We have

$$\frac{\cos\alpha\cos\gamma}{a} + \frac{\sin\alpha\sin\gamma}{b} = \frac{1}{c}, \quad \text{and} \quad \frac{\cos\beta\cos\gamma}{a} + \frac{\sin\beta\sin\gamma}{b} = \frac{1}{c}.$$

From these equations we find $\cos\gamma$ and $\sin\gamma$. We get

$$\cos\gamma = \frac{a(\sin\beta - \sin\alpha)}{c(\cos\alpha\sin\beta - \cos\beta\sin\alpha)} = \frac{2a\sin\frac{1}{2}(\beta-\alpha)\cos\frac{1}{2}(\beta+\alpha)}{c\sin(\beta-\alpha)}$$
$$= \frac{a\cos\frac{1}{2}(\beta+\alpha)}{c\cos\frac{1}{2}(\beta-\alpha)};$$

$$\sin\gamma = \frac{b(\cos\alpha - \cos\beta)}{c(\cos\alpha\sin\beta - \cos\beta\sin\alpha)} = \frac{2b\sin\frac{1}{2}(\beta+\alpha)\sin\frac{1}{2}(\beta-\alpha)}{c\sin(\beta-\alpha)}$$
$$= \frac{b\sin\frac{1}{2}(\beta+\alpha)}{c\cos\frac{1}{2}(\beta-\alpha)}.$$

Square, and add; thus

$$1 = \frac{a^2\cos^2\frac{1}{2}(\beta+\alpha) + b^2\sin^2\frac{1}{2}(\beta+\alpha)}{c^2\cos^2\frac{1}{2}(\beta-\alpha)};$$

therefore $c^2\{1 + \cos(\beta-\alpha)\} = a^2\{1 + \cos(\beta+\alpha)\} + b^2\{1 - \cos(\beta+\alpha)\}$;

therefore $(b^2 + c^2 - a^2)\cos\alpha\cos\beta + (a^2 + c^2 - b^2)\sin\alpha\sin\beta = a^2 + b^2 - c^2$.

18. $$\sin A = \frac{1}{3}; \text{ therefore } \cos 2A = 1 - \frac{2}{9} = \frac{7}{9},$$

and $$\sin 2A = \sqrt{\left(1 - \frac{49}{81}\right)} = \frac{\sqrt{32}}{9}.$$

$$\sin B = \frac{1}{2}; \text{ therefore } \cos 2B = 1 - \frac{1}{2} = \frac{1}{2},$$

and $$\sin 2B = \sqrt{\left(1 - \frac{1}{4}\right)} = \frac{\sqrt{3}}{2}.$$

Hence $$\sin(2A + 2B) = \frac{\sqrt{32} + 7\sqrt{3}}{18} = \frac{4\sqrt{2} + 7\sqrt{3}}{18}.$$

19. $$\cos 4x + \cos 2x + \cos x = 0;$$

therefore $$2\cos 3x \cos x + \cos x = 0;$$

therefore either $\cos x = 0$ or $2\cos 3x + 1 = 0$.

If $\cos x = 0$, then $x = (2n+1)\frac{\pi}{2}$.

If $\cos 3x = -\frac{1}{2}$, then $3x = 2n\pi \pm \frac{2\pi}{3}$.

20. $$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2};$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$= -2\cos\frac{A+B}{2}\cos\frac{C-D}{2}, \text{ by Art. 48.}$$

Hence, by addition,

$$\cos A + \cos B + \cos C + \cos D = 2\cos\frac{A+B}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{C-D}{2}\right\}$$

$$= 4\cos\frac{A+B}{2}\sin\frac{A+C-B-D}{4}\sin\frac{C+B-A-D}{4}.$$

Also $$\sin\frac{A+C-B-D}{4} = \sin\frac{2A+2C-360^\circ}{4} = \sin\left(\frac{A+C}{2} - 90^\circ\right) = -\cos\frac{A+C}{2};$$

and in like manner $$\sin\frac{C+B-A-D}{4} = -\cos\frac{B+C}{2}.$$

Thus we obtain finally $4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}$.

21. Suppose that the smaller unit contains x degrees, and therefore the larger unit $x+10$ degrees. Let n denote the number of degrees in the angle measured; then $\frac{n}{x}$ is to $\frac{n}{x+10}$ as 3 is to 2. Therefore $\frac{2}{x}=\frac{3}{x+10}$; whence $x=20$.

22. $\sin^2 x+\sin x=1$. Solving this quadratic in the ordinary way we obtain $\sin x=\frac{-1\pm\sqrt{5}}{2}$; the upper sign must be taken, as the lower would make $\sin x$ numerically greater than unity.

Thus $$\sin^2 x=\frac{6-2\sqrt{5}}{4};$$

therefore $$\cos^2 x=1-\frac{6-2\sqrt{5}}{4}=\frac{-2+2\sqrt{5}}{4}=\frac{-1+\sqrt{5}}{2};$$

therefore $$\cos^4 x=\frac{6-2\sqrt{5}}{4}=\frac{3-\sqrt{5}}{2};$$

therefore $$\cos^2 x+\cos^4 x=1.$$

Or thus: $$\sin x=1-\sin^2 x=\cos^2 x;\ \text{square};$$

therefore $$\sin^2 x=\cos^4 x,\ \text{that is}\ 1-\cos^2 x=\cos^4 x;$$

therefore $$1=\cos^2 x+\cos^4 x.$$

23. Here $$\tan^2 x+\frac{1}{\tan^2 x}=2;$$

therefore $$\tan^4 x-2\tan^2 x+1=0;$$

hence $$\tan^2 x=1;\ \text{therefore}\ \tan x=\pm 1;$$

therefore $$x=n\pi\pm\frac{\pi}{4}.$$

24. We have $$a\sin\theta+b\cos\theta=c,\qquad \frac{a\cos\theta+b\sin\theta}{\sin\theta\cos\theta}=c;$$

hence $$(a\sin\theta+b\cos\theta)(a\cos\theta+b\sin\theta)=c^2\sin\theta\cos\theta;$$

therefore $$(a^2+b^2)\sin\theta\cos\theta+ab=c^2\sin\theta\cos\theta;$$

therefore $$\sin 2\theta\,(c^2-a^2-b^2)=2ab.$$

25. $$\cos^2(A+B)+\cos^2(A-B)=\frac{1+\cos(2A+2B)}{2}+\frac{1+\cos(2A-2B)}{2}$$
$$=1+\cos 2A\cos 2B;$$

therefore $$\cos^2(A+B)+\cos^2(A-B)-\cos 2A\cos 2B=1.$$

26. $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}=\frac{\frac{3}{2}\tan B-\tan B}{1+\frac{3}{2}\tan^2 B}$

$$=\frac{\tan B}{2+3\tan^2 B}=\frac{\sin B\cos B}{2\cos^2 B+3\sin^2 B}=\frac{\sin 2B}{2(1+\cos 2B)+3(1-\cos 2B)}$$

$$=\frac{\sin 2B}{5-\cos 2B}.$$

27. $$\sin\frac{n+1}{2}\theta+\sin\frac{n-1}{2}\theta=\sin\theta,$$

therefore $$2\sin\frac{n\theta}{2}\cos\frac{\theta}{2}=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

Thus either $\cos\frac{\theta}{2}=0$, or $\sin\frac{n\theta}{2}=\sin\frac{\theta}{2}$.

From the former we have $\frac{\theta}{2}=(2m+1)\frac{\pi}{2}$. All the solutions of the latter are comprised in $\frac{n\theta}{2}=m\pi+(-1)^m\frac{\theta}{2}$, where m is zero or an integer.

28. Here $$\tan(2\alpha-3\beta)=\tan\left(\frac{\pi}{2}-3\alpha+2\beta\right),$$

and $$\tan(2\alpha+3\beta)=\tan\left(\frac{\pi}{2}-3\alpha-2\beta\right).$$

Hence all possible solutions are comprised in

$$2\alpha-3\beta=m\pi+\frac{\pi}{2}-3\alpha+2\beta,\quad\text{and}\quad 2\alpha+3\beta=n\pi+\frac{\pi}{2}-3\alpha-2\beta,$$

where m and n are zero or integers.

From these we obtain $\alpha=(m+n+1)\frac{\pi}{10}$, $\beta=(n-m)\frac{\pi}{10}$,

so that α and β are multiples of $\frac{\pi}{10}$.

29. Here $$\frac{\sin(\alpha+x)\sin(\alpha-x)}{\cos(\alpha+x)\cos(\alpha-x)}=\frac{1-2\cos 2\alpha}{1+2\cos 2\alpha};$$

therefore, by Art. 83, $$\frac{\sin^2\alpha-\sin^2 x}{\cos^2\alpha-\sin^2 x}=\frac{1-2\cos 2\alpha}{1+2\cos 2\alpha};$$

therefore $$4\cos 2\alpha\sin^2 x=\sin^2\alpha(1+2\cos 2\alpha)-\cos^2\alpha(1-2\cos 2\alpha)$$
$$=-\cos 2\alpha+2\cos 2\alpha=\cos 2\alpha;$$

therefore $\sin^2 x=\frac{1}{4}$; therefore $\sin x=\pm\frac{1}{2}$; therefore $x=n\pi\pm\frac{\pi}{6}$.

30. $$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2};$$

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$= 2 \sin \frac{A+B}{2} \cos \frac{C-D}{2}, \text{ by Art. 48.}$$

Hence, by addition,

$$\sin A + \sin B + \sin C + \sin D = 2 \sin \frac{A+B}{2} \left\{ \cos \frac{A-B}{2} + \cos \frac{C-D}{2} \right\}$$

$$= 4 \sin \frac{A+B}{2} \cos \frac{A+C-B-D}{4} \cos \frac{A+D-B-C}{4}.$$

Then, as in Example 20, we can shew that

$$\cos \frac{A+C-B-D}{4} = \sin \frac{A+C}{2}, \text{ and } \cos \frac{A+D-B-C}{4} = \sin \frac{B+C}{2}.$$

31. The first angle contains 60 degrees; the second angle contains $\frac{9}{10} \times 50$ degrees, that is 45 degrees; the third angle contains $\frac{3\pi}{4} \times \frac{180}{\pi}$ degrees, that is 135 degrees. Therefore the fourth angle must contain $360 - 60 - 45 - 135$ degrees, that is 120 degrees.

32. $$\sin A = \sqrt{1 - \left(\frac{40}{41}\right)^2} = \frac{\sqrt{(41-40)(41+40)}}{41} = \frac{\sqrt{81}}{41} = \frac{9}{41};$$

$$\sin B = \sqrt{1 - \left(\frac{60}{61}\right)^2} = \frac{\sqrt{(61-60)(61+60)}}{61} = \frac{\sqrt{121}}{61} = \frac{11}{61}.$$

$$\cos (A-B) = \cos A \cos B + \sin A \sin B = \frac{40 \times 60 + 9 \times 11}{41 \times 61} = \frac{2499}{2501};$$

thus $$1 - 2 \sin^2 \frac{1}{2}(A-B) = \frac{2499}{2501};$$

therefore $$2 \sin^2 \frac{1}{2}(A-B) = \frac{2}{41 \times 61};$$

therefore $$\sin^2 \frac{1}{2}(A-B) = \frac{1}{41 \times 61}.$$

33. Here $$3 \sin \theta - 4 \sin^3 \theta = 8 \sin^3 \theta;$$

therefore $$3 \sin \theta = 12 \sin^3 \theta;$$

therefore either $$\sin \theta = 0, \text{ or } \sin^2 \theta = \frac{1}{4}:$$

the former gives $\theta = n\pi$; the latter gives $\theta = n\pi \pm \frac{\pi}{6}$.

34. Divide the first by the second; thus we get

$$a \cos \theta - b \cos \phi = \frac{c^2}{r};$$

therefore $\cos \theta = \frac{1}{2a}\left(r + \frac{c^2}{r}\right), \quad \cos \phi = \frac{1}{2b}\left(r - \frac{c^2}{r}\right).$

Now $\frac{a^2 \sin^2 \theta}{\cos^2 \theta} = \frac{b^2 \sin^2 \phi}{\cos^2 \phi}$; therefore $a^2 (\sec^2 \theta - 1) = b^2 (\sec^2 \phi - 1)$;

therefore $a^2 \left\{\frac{4r^2 a^2}{(r^2 + c^2)^2} - 1\right\} = b^2 \left\{\frac{4r^2 b^2}{(r^2 - c^2)^2} - 1\right\}.$

35. Here $\tan A + \tan C = 2 \tan B$, and $\frac{1}{\tan A} + \frac{1}{\tan D} = \frac{2}{\tan B}$;

therefore $\frac{\tan C}{\tan D} = (2 \tan B - \tan A)\left(\frac{2}{\tan B} - \frac{1}{\tan A}\right) = 5 - 2\left(\frac{\tan B}{\tan A} + \frac{\tan A}{\tan B}\right)$

$$= 5 - 2\left(\frac{\sin B \cos A}{\cos B \sin A} + \frac{\sin A \cos B}{\cos A \sin B}\right) = 5 - 2\,\frac{\sin^2 B \cos^2 A + \sin^2 A \cos^2 B}{\sin A \cos A \sin B \cos B}$$

$$= 1 - \frac{2 (\sin A \cos B - \cos A \sin B)^2}{\sin A \cos A \sin B \cos B} = 1 - \frac{8 \sin^2 (A - B)}{\sin 2A \sin 2B}.$$

36. Here $\cos x (1 - \cos^2 x) = \sin \alpha (1 - \sin^2 \alpha)$; therefore

$\cos x - \sin \alpha = \cos^3 x - \sin^3 \alpha = (\cos x - \sin \alpha)(\cos^2 x + \cos x \sin \alpha + \sin^2 \alpha)$;

therefore either $\cos x - \sin \alpha = 0$, or $1 = \cos^2 x + \cos x \sin \alpha + \sin^2 \alpha$.

The latter gives $\cos^2 x + \cos x \sin \alpha = \cos^2 \alpha$;

by solving this quadratic equation we obtain

$$\cos x = \frac{-\sin \alpha \pm \sqrt{(\sin^2 \alpha + 4 \cos^2 \alpha)}}{2};$$

it will be found that only one of these values is numerically less than unity, namely, the numerically less of the two.

37. We have $2 \cos \theta - 1 = \frac{4 \cos^2 \theta - 1}{2 \cos \theta + 1} = \frac{2 \cos 2\theta + 1}{2 \cos \theta + 1}$,

$$2 \cos 2\theta - 1 = \frac{4 \cos^2 2\theta - 1}{2 \cos 2\theta + 1} = \frac{2 \cos 4\theta + 1}{2 \cos 2\theta + 1},$$

and so on, which we use down to

$$2 \cos 2^{n-1} \theta - 1 = \frac{4 \cos^2 2^{n-1} \theta - 1}{2 \cos 2^{n-1} \theta + 1} = \frac{2 \cos 2^n \theta + 1}{2 \cos 2^{n-1} \theta + 1}.$$

Multiply these expressions together, then by cancelling we obtain the required result.

38. Here $$\tan(\pi \cot\theta) = \tan\left(\frac{\pi}{2} - \pi\tan\theta\right);$$

hence all possible solutions are comprised in the formula

$$\pi\cot\theta = n\pi + \frac{\pi}{2} - \pi\tan\theta;$$

thus $$\tan^2\theta - \left(n + \frac{1}{2}\right)\tan\theta + 1 = 0;$$

by solving this quadratic equation we obtain the value of $\tan\theta$.

39. $\cos^2 A + \cos^2 B + \cos^2 C - 2\cos A\cos B\cos C - 1$

$$= (\cos A - \cos B\cos C)^2 + \cos^2 B + \cos^2 C - 1 - \cos^2 B\cos^2 C$$
$$= (\cos A - \cos B\cos C)^2 - (1 - \cos^2 B)(1 - \cos^2 C)$$
$$= (\cos A - \cos B\cos C)^2 - \sin^2 B\sin^2 C$$
$$= (\cos A - \cos B\cos C + \sin B\sin C)(\cos A - \cos B\cos C - \sin B\sin C)$$
$$= \{\cos A - \cos(B + C)\}\{\cos A - \cos(B - C)\}$$
$$= 4\sin\frac{A+B+C}{2}\sin\frac{B+C-A}{2}\sin\frac{A+B-C}{2}\sin\frac{B-C-A}{2}$$
$$= -4\sin\frac{A+B+C}{2}\sin\frac{B+C-A}{2}\sin\frac{A+C-B}{2}\sin\frac{A+B-C}{2}.$$

40. $$\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2};$$

$$\sin C - \sin D = 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}$$
$$= -2\sin\frac{C-D}{2}\cos\frac{A+B}{2}, \text{ by Art. 48.}$$

Hence, by addition,

$$\sin A - \sin B + \sin C - \sin D = 2\cos\frac{A+B}{2}\left\{\sin\frac{A-B}{2} - \sin\frac{C-D}{2}\right\}$$
$$= 4\cos\frac{A+B}{2}\sin\frac{A+D-B-C}{4}\cos\frac{A+C-B-D}{4}.$$

Then, as in Example 20, we can shew that

$$\sin\frac{A+D-B-C}{4} = \cos\frac{B+C}{2}, \text{ and } \cos\frac{A+C-B-D}{4} = \sin\frac{A+C}{2}.$$

41. The angle described is $\frac{25}{60}$ of four right angles; the number of degrees $= \frac{5}{12} \times 360 = 150$; the number of grades $= \frac{5}{12} \times 400 = 166\frac{2}{3}$; the circular measure $= \frac{5}{12} \times 2\pi = \frac{5\pi}{6}$.

42. $$\cos\theta+\sin\theta=\sqrt{2}\left(\frac{\cos\theta}{\sqrt{2}}+\frac{\sin\theta}{\sqrt{2}}\right)=\sqrt{2}\,.\cos\left(\theta-\frac{\pi}{4}\right);$$

similarly $$\cos 2\theta+\sin 2\theta=\sqrt{2}\cos\left(2\theta-\frac{\pi}{4}\right):$$

$$\text{the product}=2\cos\left(\theta-\frac{\pi}{4}\right)\cos\left(2\theta-\frac{\pi}{4}\right)=\cos\theta+\cos\left(3\theta-\frac{\pi}{2}\right).$$

43. $$\operatorname{cosec}2A\,(\operatorname{cosec}A+\operatorname{cosec}3A)=\frac{1}{\sin 2A}\cdot\frac{\sin A+\sin 3A}{\sin A\sin 3A}$$

$$=\frac{1}{\sin 2A}\cdot\frac{2\sin 2A\cos A}{\sin A\sin 3A}=\frac{2\cos A}{\sin A\sin 3A};$$

and $$\operatorname{cosec}A\,(\cot A-\cot 3A)=\frac{1}{\sin A}\left(\frac{\cos A}{\sin A}-\frac{\cos 3A}{\sin 3A}\right)$$

$$=\frac{1}{\sin A}\cdot\frac{\sin 3A\cos A-\cos 3A\sin A}{\sin A\sin 3A}=\frac{1}{\sin A}\cdot\frac{\sin(3A-A)}{\sin A\sin 3A}$$

$$=\frac{\sin 2A}{\sin^2 A\sin 3A}=\frac{2\sin A\cos A}{\sin^2 A\sin 3A}=\frac{2\cos A}{\sin A\sin 3A}:$$

thus the proposed expressions are equal.

44. $$\sec^2\frac{1}{2}A\sec A\,\frac{\cot^2\frac{1}{2}A-\cot^2\frac{3}{2}A}{1+\cot^2\frac{3}{2}A}$$

$$=\frac{1}{\cos^2\frac{1}{2}A}\,\frac{1}{\cos A}\,\frac{\cos^2\frac{1}{2}A\sin^2\frac{3}{2}A-\cos^2\frac{3}{2}A\sin^2\frac{1}{2}A}{\sin^2\frac{1}{2}A\left(\cos^2\frac{3A}{2}+\sin^2\frac{3A}{2}\right)}$$

$$=\frac{\left(\cos\frac{1}{2}A\sin\frac{3}{2}A-\cos\frac{3}{2}A\sin\frac{1}{2}A\right)\left(\cos\frac{1}{2}A\sin\frac{3}{2}A+\cos\frac{3}{2}A\sin\frac{1}{2}A\right)}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A\cos A}$$

$$=\frac{\sin\left(\frac{3A}{2}-\frac{A}{2}\right)\sin\left(\frac{3A}{2}+\frac{A}{2}\right)}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A\cos A}=\frac{\sin A\sin 2A}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A\cos A}$$

$$=\frac{2\sin^2 A}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A}=\frac{2\left(2\sin\frac{1}{2}A\cos\frac{1}{2}A\right)^2}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A}=8.$$

45. $\sec A + \operatorname{cosec} A\,(1+\sec A) = \dfrac{1}{\cos A} + \dfrac{1}{\sin A}\left(1+\dfrac{1}{\cos A}\right)$

$$= \frac{1+\cos A+\sin A}{\cos A \sin A} = \frac{2\cos^2\frac{1}{2}A + 2\sin\frac{1}{2}A\cos\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A\cos A}$$

$$= \frac{\cos\frac{1}{2}A+\sin\frac{1}{2}A}{\sin\frac{1}{2}A\cos A} = \frac{\cos\frac{1}{2}A}{\cos A}\left(\sec\frac{1}{2}A + \operatorname{cosec}\frac{1}{2}A\right);$$

$$1-\tan^2\frac{1}{2}A = \frac{\cos^2\frac{1}{2}A - \sin^2\frac{1}{2}A}{\cos^2\frac{1}{2}A} = \frac{\cos A}{\cos^2\frac{1}{2}A};$$

$$1-\tan^2\frac{1}{4}A = \frac{\cos^2\frac{1}{4}A - \sin^2\frac{1}{4}A}{\cos^2\frac{1}{4}A} = \frac{\cos\frac{1}{2}A}{\cos^2\frac{1}{4}A}.$$

Hence by multiplication we obtain the required result.

46. Put c^2 for $a^4 + \dfrac{a^2b^2}{a^2-1}$: thus

$$\cos\theta = \frac{a-b}{c}, \quad \text{and} \quad \cos\phi = \frac{a+b}{c}:$$

from these we obtain

$$\sin\theta = \frac{a(a^2-1)+b}{c\sqrt{(a^2-1)}}, \quad \text{and} \quad \sin\phi = \frac{a(a^2-1)-b}{c\sqrt{(a^2-1)}}.$$

Next we obtain $\qquad \cos(\theta-\phi) = \dfrac{a^4-a^2-b^2}{a^4-a^2+b^2};$

and thus $\qquad \tan^2\frac{1}{2}(\theta-\phi) = \dfrac{1-\cos(\theta-\phi)}{1+\cos(\theta-\phi)} = \dfrac{b^2}{a^4-a^2}.$

47. $\dfrac{a}{b} = \dfrac{\sin(\phi-\theta)}{\sin(\phi+\theta)}$ (1), $\quad \dfrac{c}{x} = \cos(\phi-\theta)$ (2), $\quad \dfrac{b}{x} = \dfrac{\sin\theta}{\sin\phi}$ (3).

From (1) we get $\qquad \dfrac{b+a}{b-a} = \dfrac{\sin\phi\cos\theta}{\sin\theta\cos\phi} = \dfrac{x\cos\theta}{b\cos\phi}$, by (3).

By (2) we have $\qquad \dfrac{c}{x} = \cos\phi\cos\theta + \sin\phi\sin\theta$

$$= \frac{b}{x}\cdot\frac{b+a}{b-a}\cos^2\phi + \frac{b}{x}\sin^2\phi;$$

hence we get $$\cos^2\phi = \frac{(c-b)(b-a)}{2ab};$$

therefore $$\cos^2\theta = \frac{(c-b)(a+b)^2}{2ab(b-a)}\frac{b^2}{x^2}.$$

But from (3) we have $\frac{b^2}{x^2}(1-\cos^2\phi) = 1-\cos^2\theta$,

so that $$\frac{b^2}{x^2} - 1 = \frac{b^2}{x^2}\left\{\frac{(c-b)(b-a)}{2ab} - \frac{(c-b)(a+b)^2}{2ab(b-a)}\right\},$$

or $$x^2 - b^2 = \frac{b^2(c-b)\,4ab}{2ab(b-a)} = \frac{2b^2(c-b)}{b-a};$$

therefore $$x^2(b-a) = b^2(2c-a-b).$$

48. Put $2\cos^2\frac{x}{2} - 1$ for $\cos x$; thus we obtain the quadratic equation

$$2\cos^2\frac{x}{2}\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right) - \sin\beta\cos\frac{x}{2} = \cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right).$$

By solving this we have

$$\cos\frac{x}{2} = \frac{\sin\beta \pm \sqrt{\left\{\sin^2\beta + 8\cos^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}}}{4\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)};$$

and $$\sin^2\beta + 8\cos^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right) = \sin^2\beta + 4\left\{1+\cos\left(\frac{\pi}{2}-\beta\right)\right\}$$

$$= \sin^2\beta + 4 + 4\sin\beta.$$

Hence $$\cos\frac{x}{2} = \frac{\sin\beta \pm (2+\sin\beta)}{4\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}.$$

Take the upper sign; then $\cos\frac{x}{2} = \dfrac{1+\sin\beta}{2\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}$

$$= \frac{1+\cos\left(\frac{\pi}{2}-\beta\right)}{2\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)} = \frac{2\cos^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{2\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)} = \cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right).$$

Take the lower sign; then $\cos x = -\dfrac{1}{2\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}$.

49. Write the first equation in the form

$$\cos\left(\frac{\pi}{2} - 2\theta - 2\phi\right) = \cos(\theta + 3\phi);$$

hence all possible solutions are comprised in

$$\frac{\pi}{2} - 2\theta - 2\phi = 2m\pi \pm (\theta + 3\phi).$$

If we take the upper sign we have

$$3\theta + 5\phi = \frac{\pi}{2} - 2m\pi \text{ .... } (1).$$

If we take the lower sign we have

$$\phi - \theta = 2m\pi - \frac{\pi}{2} \text{ .... } (2).$$

Again, write the second equation in the form

$$\cos\left(\frac{\pi}{2} - \phi - 3\theta\right) = \cos(2\theta + 2\phi);$$

hence all possible solutions are comprised in

$$\frac{\pi}{2} - \phi - 3\theta = 2n\pi \pm (2\theta + 2\phi).$$

If we take the upper sign we have

$$5\theta + 3\phi = \frac{\pi}{2} - 2n\pi \text{ .... } (3).$$

If we take the lower sign we have

$$\phi - \theta = 2n\pi - \frac{\pi}{2};$$

this agrees with (2).

Thus either (2) holds, or both (1) and (3) hold. From (1) and (3) we obtain

$$16\theta = (3m - 5n)\,2\pi + \pi, \quad 16\phi = (3n - 5m)\,2\pi + \pi.$$

50. We have $1 + \sec 2\theta = \dfrac{1 + \cos 2\theta}{\cos 2\theta} = \dfrac{\sin 2\theta}{\cos 2\theta} \cdot \dfrac{1 + \cos 2\theta}{\sin 2\theta}$

$$= \tan 2\theta \,.\, \cot\theta = \frac{\tan 2\theta}{\tan\theta}.$$

Similarly $$1 + \sec 4\theta = \frac{\tan 4\theta}{\tan 2\theta},$$

and so on, which we use down to

$$1 + \sec 2^n\theta = \frac{\tan 2^n\theta}{\tan 2^{n-1}\theta}.$$

Multiply these expressions together, then by cancelling we obtain the required result.

51. Here the circular measure of an angle is given equal to $\frac{9}{10}$; hence the number of degrees in it is $\frac{9}{10} \cdot \frac{180}{\pi}$, that is $\frac{162}{\pi}$.

52. $$\sin (A - B) + \sin (A + 3B) = 2 \sin (A + B) \cos 2B;$$

therefore $$\{\sin (A - B) + \sin (A + 3B)\} \sec 2B = 2 \sin (A + B).$$

$$\cos 2B - \cos 2A = 2 \sin (A - B) \sin (A + B);$$

therefore $$(\cos 2B - \cos 2A) \operatorname{cosec} (A - B) = 2 \sin (A + B):$$

thus the proposed expressions are equal.

53. Here $$\frac{\sin \theta}{\cos \theta}(1 + \sin^2 \theta) = \frac{\sin \alpha}{\cos \alpha}(1 + \cos^2 \theta);$$

therefore $$\cos \alpha (\sin \theta + \sin^3 \theta) = \sin \alpha (\cos \theta + \cos^3 \theta);$$

therefore $$\cos \alpha \sin \theta + \cos \alpha \frac{3 \sin \theta - \sin 3\theta}{4} = \sin \alpha \cos \theta + \sin \alpha \frac{3 \cos \theta + \cos 3\theta}{4};$$

therefore $$7 (\sin \theta \cos \alpha - \cos \theta \sin \alpha) = \sin 3\theta \cos \alpha + \cos 3\theta \sin \alpha;$$

that is $$7 \sin (\theta - \alpha) = \sin (3\theta + \alpha).$$

54. Here $$2 \cos 4\theta \cos \theta + \sqrt{2} (\cos \theta + \sin \theta) \cos \theta = 0;$$

therefore either $$\cos \theta = 0 \quad \text{or} \quad \cos 4\theta = -\frac{1}{\sqrt{2}} (\cos \theta + \sin \theta).$$

Take the former; then $\theta = (2n + 1)\frac{\pi}{2}$.

Take the latter; thus $\cos 4\theta = \cos \left(\frac{3\pi}{4} + \theta\right)$;

therefore $$4\theta = 2n\pi \pm \left(\frac{3\pi}{4} + \theta\right).$$

55. We have $$\sin \phi = \frac{n \sin \theta - m \cos \theta}{2m};$$

and $$n \sin 2\theta - m (1 - 2 \sin^2 \phi) = n;$$

therefore $$n \sin 2\theta + 2m \left(\frac{n \sin \theta - m \cos \theta}{2m}\right)^2 = m + n;$$

therefore $$2mn \sin 2\theta + (n \sin \theta - m \cos \theta)^2 = 2m (m + n);$$

therefore $$(n \sin \theta + m \cos \theta)^2 = 2m (m + n).$$

56. Substitute $\frac{3\cos\theta+\cos 3\theta}{4}$ for $\cos^3\theta$ and $\frac{3\sin\theta-\sin 3\theta}{4}$ for $\sin^3\theta$; thus the equation becomes

$$2\sin\left(\theta-\frac{\pi}{3}\right)\{3\cos\theta+\cos 3\theta\}+2\cos\left(\theta-\frac{\pi}{3}\right)\{3\sin\theta-\sin 3\theta\}$$

$$-6\sin\left(2\theta-\frac{\pi}{3}\right)=\sqrt{3},$$

that is $$2\sin\left(\theta-\frac{\pi}{3}\right)\cos 3\theta-2\cos\left(\theta-\frac{\pi}{3}\right)\sin 3\theta=\sqrt{3},$$

that is $$-\sin\left(3\theta-\theta+\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2};$$

thus $$\sin\left(2\theta+\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2};$$

therefore $$2\theta+\frac{\pi}{3}=n\pi+(-1)^n\frac{4\pi}{3}.$$

57. $$\sin\theta\cos(\beta-\theta)=\frac{1}{2}\{\sin\beta+\sin(2\theta-\beta)\}.$$

The greatest value of $\sin(2\theta-\beta)$ is obviously when $2\theta-\beta=\frac{\pi}{2}$, so that $\theta=\frac{\beta}{2}+\frac{\pi}{4}$.

58. $$\cos 55^0+\cos 65^0=2\cos 60^0\cos 5^0=\cos 5^0=-\cos 175^0;$$

therefore $$\cos 55^0+\cos 65^0+\cos 175^0=0.$$

$$\cos 55^0\cos 65^0=\frac{1}{2}(\cos 10^0+\cos 120^0),$$

$$\cos 65^0\cos 175^0=\frac{1}{2}(\cos 110^0+\cos 240^0),$$

$$\cos 55^0\cos 175^0=\frac{1}{2}(\cos 120^0+\cos 230^0);$$

hence by addition we obtain $-\frac{3}{4}+\frac{1}{2}(\cos 10^0+\cos 110^0+\cos 230^0)$,

that is $$-\frac{3}{4}+\frac{1}{2}(\cos 10^0+\cos 110^0-\cos 50^0),$$

that is $$-\frac{3}{4}+\frac{1}{2}(2\cos 60^0\cos 50^0-\cos 50^0),\qquad \text{that is } -\frac{3}{4}.$$

$$\cos 55^0 \cos 65^0 \cos 175^0 = \frac{1}{2}(\cos 10^0 + \cos 120^0)\cos 175^0$$

$$= \frac{1}{4}(\cos 165^0 + \cos 185^0) + \frac{1}{4}(\cos 55^0 + \cos 295^0)$$

$$= \frac{1}{4}(-\cos 15^0 + \cos 175^0 + \cos 55^0 + \cos 65^0)$$

$$= -\frac{1}{4}\cos 15^0, \text{ by what has been already shewn,}$$

$$= -\frac{1}{4}\cdot\frac{\sqrt{3}+1}{2\sqrt{2}} = -\frac{\sqrt{3}+1}{8\sqrt{2}}.$$

59. From $x\cos(\alpha+\beta)+\cos(\alpha-\beta)=x\cos(\beta+\gamma)+\cos(\beta-\gamma)$

we obtain
$$x=\frac{\cos(\beta-\gamma)-\cos(\alpha-\beta)}{\cos(\alpha+\beta)-\cos(\beta+\gamma)}=-\frac{\sin\left(\frac{\alpha+\gamma}{2}-\beta\right)}{\sin\left(\frac{\alpha+\gamma}{2}+\beta\right)}.$$

Two other expressions for the value of x may be obtained; and thus we have

$$\frac{\sin\left(\frac{\alpha+\gamma}{2}-\beta\right)}{\sin\left(\frac{\alpha+\gamma}{2}+\beta\right)}=\frac{\sin\left(\frac{\beta+\alpha}{2}-\gamma\right)}{\sin\left(\frac{\beta+\alpha}{2}+\gamma\right)}=\frac{\sin\left(\frac{\gamma+\beta}{2}-\alpha\right)}{\sin\left(\frac{\gamma+\beta}{2}+\alpha\right)}=-x;$$

hence
$$\frac{\sin\frac{\alpha+\gamma}{2}\cos\beta}{\cos\frac{\alpha+\gamma}{2}\sin\beta}=\frac{1-x}{1+x}, \text{ that is } \frac{\tan\frac{\alpha+\gamma}{2}}{\tan\beta}=\frac{1-x}{1+x}.$$

Similarly $\dfrac{\tan\frac{\beta+\alpha}{2}}{\tan\gamma}$ and $\dfrac{\tan\frac{\gamma+\beta}{2}}{\tan\alpha}$ are also equal to $\dfrac{1-x}{1+x}$.

60.
$$\cos A-\cos B=2\sin\frac{A+B}{2}\sin\frac{B-A}{2};$$

$$\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$

$$=2\sin\frac{A+B}{2}\sin\frac{D-C}{2}, \text{ by Art. 48.}$$

Hence by addition,

$$\cos A - \cos B + \cos C - \cos D = 2\sin\frac{A+B}{2}\left\{\sin\frac{B-A}{2} + \sin\frac{D-C}{2}\right\}$$

$$= 4\sin\frac{A+B}{2}\sin\frac{B+D-A-C}{4}\cos\frac{B+C-A-D}{4}.$$

Then, as in Example 20, we can shew that

$$\sin\frac{B+D-A-C}{4} = \cos\frac{A+C}{2}, \text{ and } \cos\frac{B+C-A-D}{4} = \sin\frac{B+C}{2}.$$

61. Let x denote the number of sides in the first regular polygon, and y the number of sides in the second. All the angles of the first polygon are equal to $2x - 4$ right angles; therefore each angle is equal to $\frac{2x-4}{x}$ right angles, and therefore contains $\frac{2x-4}{x}90$ degrees. In the same way each angle of the second polygon contains $\frac{2y-4}{y}100$ grades. Then, by supposition, we have $\frac{2x-4}{x}90 : \frac{2y-4}{y}100 :: 3 : 5$;

therefore $$5\frac{2x-4}{x}90 = 3\frac{2y-4}{y}100;$$

therefore $$\frac{3(x-2)}{x} = \frac{2(y-2)}{y};$$

therefore $3y(x-2) = 2x(y-2)$; therefore $y(6-x) = 4x$. This formula shews that x cannot be greater than 5; for if $x=6$ we should have $y \times 0 = 24$, which is absurd; and if x is greater than 6 we should have a negative value for y, which is also absurd. And x cannot be less than 3. Thus the only possible solutions are $x=3$, $x=4$, and $x=5$; which give respectively $y=4$, $y=8$, and $y=20$.

62. Here $$\frac{1}{\cos^2\frac{x}{2}} + \frac{1}{\sin^2\frac{x}{2}} = \frac{16\cos x}{\sin x};$$

therefore $$\sin^2\frac{x}{2} + \cos^2\frac{x}{2} = \frac{16\cos x\cos^2\frac{x}{2}\sin^2\frac{x}{2}}{\sin x} = 8\cos x\cos\frac{x}{2}\sin\frac{x}{2};$$

therefore $$1 = 4\cos x\sin x = 2\sin 2x;$$

therefore $$\sin 2x = \frac{1}{2}; \quad \text{therefore } 2x = n\pi + (-1)^n\frac{\pi}{6}.$$

63. Here $$m\sin 2\theta = n\sin\theta, \quad p\cos 2\theta = q\cos\theta;$$

from the first equation $2m\sin\theta\cos\theta = n\sin\theta$; therefore $\cos\theta = \frac{n}{2m}$.

Substitute in the second equation, that is in $p(2\cos^2\theta - 1) = q\cos\theta$;

thus $$p\left\{2\left(\frac{n}{2m}\right)^2 - 1\right\} = \frac{qn}{2m}; \quad \text{therefore } p(n^2 - 2m^2) = qmn.$$

64. We may write the equation thus:

$$\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{\sqrt{2}}\cos\alpha - \frac{1}{\sqrt{2}}\sin\alpha\,;$$

therefore $$\cos\left(\theta+\frac{\pi}{4}\right) = \cos\left(\alpha+\frac{\pi}{4}\right);$$

therefore $$\theta+\frac{\pi}{4} = 2n\pi \pm \left(\alpha+\frac{\pi}{4}\right).$$

65. $\sin(A+C)\sin(A+D) - \sin(B+C)\sin(B+D)$

$$=\frac{1}{2}\{\cos(C-D)-\cos(2A+C+D)\} - \frac{1}{2}\{\cos(C-D)-\cos(2B+C+D)\}$$

$$=\frac{1}{2}\{\cos(2B+C+D)-\cos(2A+C+D)\}$$

$$=\sin(A-B)\sin(A+B+C+D).$$

Thus, if $\sin(A+B+C+D)$ vanishes, the difference between the two proposed expressions vanishes; and therefore the two expressions are equal.

66. We have $\sin\phi = p - \sin\theta$, $\cos\phi = q - \cos\theta$; square and add,

thus $$1 = p^2+q^2-2p\sin\theta-2q\cos\theta+1\,;$$

therefore $$2p\sin\theta+2q\cos\theta = p^2+q^2.$$

Now assume that $\tan\alpha = \frac{q}{p}$, so that

$$\sin\alpha = \frac{q}{\sqrt{(p^2+q^2)}} \quad\text{and}\quad \cos\alpha = \frac{p}{\sqrt{(p^2+q^2)}}\,;$$

thus $$2\sqrt{(p^2+q^2)}\{\sin\theta\cos\alpha+\cos\theta\sin\alpha\} = p^2+q^2,$$

therefore $$\sin(\theta+\alpha) = \frac{\sqrt{(p^2+q^2)}}{2} = \sin\beta \text{ say};$$

therefore $$\theta+\alpha = n\pi + (-1)^n\beta.$$

67. $$\cos\frac{\pi}{15}\cos\frac{4\pi}{15} = \frac{1}{2}\left(\cos\frac{\pi}{3}+\cos\frac{\pi}{5}\right) = \frac{1}{2}\left(\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right) = \frac{3+\sqrt{5}}{8};$$

$$\cos\frac{2\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{2}\left(\cos\frac{\pi}{3}+\cos\frac{3\pi}{5}\right) = \frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right) = \frac{3-\sqrt{5}}{8};$$

$$\cos\frac{3\pi}{15}\cos\frac{6\pi}{15} = \frac{1}{2}\left(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}\right) = \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}\right) = \frac{1}{4};$$

$$\cos\frac{5\pi}{15} = \frac{1}{2};$$

therefore $$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$

$$=\frac{3+\sqrt{5}}{8}\cdot\frac{3-\sqrt{5}}{8}\cdot\frac{1}{8}=\frac{4}{8^3}=\frac{1}{2^7}.$$

68. $a\sin^2\theta+b\sin\theta\cos\theta+c\cos^2\theta$

$$=\frac{1}{2}\{a(1-\cos 2\theta)+b\sin 2\theta+c(1+\cos 2\theta)\}$$

$$=\frac{1}{2}\{a+c+b\sin 2\theta-(a-c)\cos 2\theta\}.$$

Now let α be an angle such that $\tan\alpha=\frac{a-c}{b}$, so that $\cos\alpha=\frac{b}{\sqrt{b^2+(a-c)^2}}$, and $\sin\alpha=\frac{a-c}{\sqrt{b^2+(a-c)^2}}$. Then the above expression

$$=\frac{1}{2}(a+c)+\frac{1}{2}\sqrt{b^2+(a-c)^2}\{\sin 2\theta\cos\alpha-\cos 2\theta\sin\alpha\}$$

$$=\frac{1}{2}(a+c)+\frac{1}{2}\sqrt{b^2+(a-c)^2}\sin(2\theta-\alpha).$$

Then as $\sin(2\theta-\alpha)$ must lie between -1 and $+1$, we obtain the required result.

69. $$\cos\left(\frac{2\pi}{3}+\alpha\right)+\cos\left(\frac{2\pi}{3}-\alpha\right)=2\cos\frac{2\pi}{3}\cos\alpha=-\cos\alpha;$$

therefore $$\cos\alpha+\cos\left(\frac{2\pi}{3}+\alpha\right)+\cos\left(\frac{2\pi}{3}-\alpha\right)=0.$$

$$\cos\alpha\cos\left(\frac{2\pi}{3}+\alpha\right)=\frac{1}{2}\left\{\cos\frac{2\pi}{3}+\cos\left(\frac{2\pi}{3}+2\alpha\right)\right\}$$

$$\cos\alpha\cos\left(\frac{2\pi}{3}-\alpha\right)=\frac{1}{2}\left\{\cos\frac{2\pi}{3}+\cos\left(\frac{2\pi}{3}-2\alpha\right)\right\}$$

$$\cos\left(\frac{2\pi}{3}+\alpha\right)\cos\left(\frac{2\pi}{3}-\alpha\right)=\frac{1}{2}\left\{\cos\frac{4\pi}{3}+\cos 2\alpha\right\}.$$

Now $\cos 2\alpha+\cos\left(\frac{2\pi}{3}+2\alpha\right)+\cos\left(\frac{2\pi}{3}-2\alpha\right)$ is zero, in the manner already shewn; and $\cos\frac{2\pi}{3}$ and $\cos\frac{4\pi}{3}$ are each $-\frac{1}{2}$: thus the sum is $-\frac{3}{4}$.

$$\cos\alpha\cos\left(\frac{2\pi}{3}+\alpha\right)\cos\left(\frac{2\pi}{3}-\alpha\right)=\cos\alpha\left(\cos^2\alpha-\sin^2\frac{2\pi}{3}\right),\text{ by Art. 83,}$$

$$=\cos\alpha\left(\cos^2\alpha-\frac{3}{4}\right)=\frac{1}{4}\cos\alpha\,(4\cos^2\alpha-3)=\frac{\cos 3\alpha}{4}.$$

70. $$\cos\frac{\alpha}{2^n}+\cos\frac{\beta}{2^n}=\frac{\cos^2\frac{\alpha}{2^n}-\cos^2\frac{\beta}{2^n}}{\cos\frac{\alpha}{2^n}-\cos\frac{\beta}{2^n}}=\frac{1}{2}\,\frac{\cos\frac{\alpha}{2^{n-1}}-\cos\frac{\beta}{2^{n-1}}}{\cos\frac{\alpha}{2^n}-\cos\frac{\beta}{2^n}};$$

similarly $$\cos\frac{\alpha}{2^{n-1}}+\cos\frac{\beta}{2^{n-1}}=\frac{1}{2}\,\frac{\cos\frac{\alpha}{2^{n-2}}-\cos\frac{\beta}{2^{n-2}}}{\cos\frac{\alpha}{2^{n-1}}-\cos\frac{\beta}{2^{n-1}}};$$

and we use a series of these transformations down to

$$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}=\frac{1}{2}\,\frac{\cos\alpha-\cos\beta}{\cos\frac{1}{2}\alpha-\cos\frac{1}{2}\beta}.$$

Then by multiplication we obtain for the product

$$\frac{1}{2^n}\,\frac{\cos\alpha-\cos\beta}{\cos\frac{\alpha}{2^n}-\cos\frac{\beta}{2^n}}.$$

71. The angle a^0b' is $\frac{60a+b}{60\times 90}$ of a right angle; the angle $p^g\,q^`$ is $\frac{100p+q}{100\times 100}$ of a right angle. Hence the excess of the former above the latter is $\left\{\frac{60a+b}{60\times 90}-\frac{100p+q}{100\times 100}\right\}$ of a right angle.

72. Here $$1-\cos 2x+\sin^2 2x=2;$$

therefore $$1-\cos 2x+1-\cos^2 2x=2;$$

therefore $$\cos 2x\,(1+\cos 2x)=0.$$

Therefore either $\cos 2x=0$, or $1+\cos 2x=0$.

If $\cos 2x=0$, we have $2x=2n\pi\pm\frac{\pi}{2}$, which may be written more simply as $2x=(2m+1)\frac{\pi}{2}$.

If $1+\cos 2x=0$ we have $\cos 2x=-1$, and therefore $2x=2n\pi\pm\pi$ which may be written more simply as $2x=(2m+1)\pi$.

73. $\tan A - \cot A = \frac{\sin A}{\cos A} - \frac{\cos A}{\sin A} = \frac{\sin^2 A - \cos^2 A}{\sin A \cos A} = \frac{2(\sin^2 A - \cos^2 A)}{2 \sin A \cos A}$

$$= -\frac{2 \cos 2A}{\sin 2A} = -2 \cot 2A.$$

Similarly $\quad 2 \tan 2A - 2 \cot 2A = -4 \cot 4A,$

and $\quad 4 \tan 4A - 4 \cot 4A = -8 \cot 8A.$

Therefore by addition and cancelling

$$\tan A - \cot A + 2 \tan 2A + 4 \tan 4A = -8 \cot 8A;$$

therefore $\quad \tan A + 2 \tan 2A + 4 \tan 4A + 8 \cot 8A = \cot A.$

74. Here $2 \sin 3x \sin x = \sin x$; therefore either $\sin x = 0$ or $2 \sin 3x = 1$.

If $\sin x = 0$, then $x = n\pi$. If $\sin 3x = \frac{1}{2}$, then $3x = n\pi + (-1)^n \frac{\pi}{6}$.

75. Let r denote the common ratio of the Geometrical Progression, so that $\tan B = r \tan A$, $\tan C = r \tan B$, $\tan D = r \tan C$;

therefore $\quad \tan A \tan D = \tan B \tan C.$

Now since $A + D = 360^0 - B - C$, we have $\tan (A + D) = -\tan (B + C)$;

therefore $\quad \frac{\tan A + \tan D}{1 - \tan A \tan D} = -\frac{\tan B + \tan C}{1 - \tan B \tan C}.$

Thus we must have either $\tan A \tan D = \tan B \tan C = 1$,

or else $\quad \tan A + \tan D = -(\tan B + \tan C).$

The latter gives $\quad (1 + r^3) \tan A = -(r + r^2) \tan A,$

so that $\quad 1 + r^3 + r + r^2 = 0$, that is $(1 + r)(1 + r^2) = 0$:

the only possible solution is $1 + r = 0$, so that $r = -1$.

76. Let A, B, C denote the angles; then $A + B + C = 180^0$; and since the angles are in Arithmetical Progression $A + C = 2B$; thus $3B = 180^0$; therefore $B = 60^0$.

Again we have $\frac{1}{\sin 2A} + \frac{1}{\sin 2C} = \frac{2}{\sin 2B} = \frac{4}{\sqrt{3}}$. Let x denote the common difference of the angles; so that $A = 60^0 - x$, and $C = 60^0 + x$. Then

$$\frac{\sin 2A + \sin 2C}{\sin 2A \sin 2C} = \frac{4}{\sqrt{3}}; \text{ therefore } \frac{2 \sin (A + C) \cos (A - C)}{\sin (120^0 - 2x) \sin (120^0 + 2x)} = \frac{4}{\sqrt{3}};$$

therefore $\frac{\sqrt{3} \cos 2x}{\sin^2 120^0 - \sin^2 2x} = \frac{4}{\sqrt{3}}$; therefore

$$\cos 2x = \frac{4}{3} (\sin^2 120^0 - \sin^2 2x) = \frac{4}{3} \left(\frac{3}{4} - 1 + \cos^2 2x\right) = -\frac{1}{3} + \frac{4}{3} \cos^2 2x.$$

By solving this quadratic we obtain $\cos 2x = 1$, or $-\frac{1}{4}$. The latter must be taken: then $\cos^2 x = \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$.

77. $$\cos A + \cos 2A + \cos 3A = 2 \cos 2A \cos A + \cos 2A$$

$$= \cos 2A\,(2\cos A + 1) = \cos 2A\left(2 - 4\sin^2\frac{1}{2}A + 1\right)$$

$$= \cos 2A\left(3 - 4\sin^2\frac{1}{2}A\right) = \frac{\cos 2A}{\sin\frac{1}{2}A}\left(3\sin\frac{1}{2}A - 4\sin^3\frac{1}{2}A\right)$$

$$= \frac{\cos 2A}{\sin\frac{1}{2}A}\sin\frac{3A}{2}.$$

78. Multiply the given expression out. The coefficient of x^2 is

$$-2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right);$$

by Example 77 this $$= -\frac{2\cos\frac{4\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} = -\frac{1}{\sin\frac{\pi}{7}}\left(\sin\frac{7\pi}{7} - \sin\frac{\pi}{7}\right) = 1.$$

The coefficient of x is

$$4\left(\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7}\cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\cos\frac{6\pi}{7}\right);$$

this $$= 2\left(\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{10\pi}{7}\right)$$

$$= 2\left(\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\right) + 2\left(\cos\frac{8\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{10\pi}{7}\right)$$

$$= 2\left(\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\right) + 2\left(\cos\frac{8\pi}{7} + \cos\frac{16\pi}{7} + \cos\frac{24\pi}{7}\right).$$

The former expression $= -1$, as we have already shewn. And by Example 77 the latter expression

$$= \frac{2\cos\frac{16\pi}{7}\sin\frac{12\pi}{7}}{\sin\frac{4\pi}{7}} = \frac{1}{\sin\frac{4\pi}{7}}\left(\sin 4\pi - \sin\frac{4\pi}{7}\right) = -1.$$

Hence the entire coefficient is -2.

The term independent of x is $-8\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$; this

$$=-4\cos\frac{6\pi}{7}\left(\cos\frac{2\pi}{7}+\cos\frac{6\pi}{7}\right)$$

$$=-2\left(\cos\frac{8\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{12\pi}{7}+1\right)$$

$$=-\frac{2\cos\frac{8\pi}{7}\sin\frac{6\pi}{7}}{\sin\frac{2\pi}{7}}-2=-\frac{1}{\sin\frac{2\pi}{7}}\left(\sin 2\pi-\sin\frac{2\pi}{7}\right)-2$$

$$=1-2=-1.$$

79. $\sin^3 A+\sin^3 B+\sin^3 C$

$$=\frac{1}{4}(3\sin A+3\sin B+3\sin C-\sin 3A-\sin 3B-\sin 3C).$$

Then by Example 32 of Chap. VIII. we have

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

and $$\sin 3A+\sin 3B+\sin 3C=-4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}.$$

80. By solving the quadratic equation we obtain

$$\sin x=-b\pm\sqrt{(b^2-c)}.$$

Hence b^2-c must not be negative, and $b+\sqrt{(b^2-c)}$ must not be greater than unity, in order that both values may be admissible.

81. Let mx denote the number of sides in the first regular polygon, and nx the number of sides in the second. Then, proceeding as in Example 61, we find that the number of degrees in an angle of the first polygon is $\frac{2mx-4}{mx}90$, and the number of grades in an angle of the second polygon is $\frac{2nx-4}{nx}100$. Therefore $\frac{2mx-4}{mx}90 : \frac{2nx-4}{nx}100 :: p : q$.

Therefore $9q\frac{mx-2}{m}=10p\frac{nx-2}{n}$; therefore $x=\frac{2(9qn-10pm)}{mn(9q-10p)}$.

Hence mx and nx are known.

82. Here $2\cos 3x\cos 4x=\cos 4x$. Therefore either $\cos 4x=0$ or $2\cos 3x=1$.

If $\cos 4x=0$ then $4x=(2n+1)\frac{\pi}{2}$.

If $\cos 3x=\frac{1}{2}$ then $3x=2n\pi\pm\frac{\pi}{3}$.

83. From the first equation we have

$$x \sin(\alpha-\beta)\cos(\alpha+\beta)=y\sin(\alpha+\beta)\cos(\alpha-\beta),$$

therefore $$x(\sin 2\alpha-\sin 2\beta)=y(\sin 2\alpha+\sin 2\beta),$$

therefore $$(x-y)\sin 2\alpha=(x+y)\sin 2\beta.$$

Thus $$\sin 2\alpha=\frac{(x+y)\sin 2\beta}{x-y},$$

and $$\cos 2\alpha=\frac{z-(x+y)\cos 2\beta}{x-y}.$$

Square and add; thus

$$1=\frac{(x+y)^2\sin^2 2\beta}{(x-y)^2}+\frac{\{z-(x+y)\cos 2\beta\}^2}{(x-y)^2}.$$

Therefore $$(x-y)^2=(x+y)^2\sin^2 2\beta+\{z-(x+y)\cos 2\beta\}^2$$

$$=(x+y)^2+z^2-2z(x+y)\cos 2\beta;$$

therefore $$z^2+4xy=2z(x+y)\cos 2\beta.$$

84. Let A denote the sum of x and y. Suppose $x=\frac{A}{2}+z$, then $y=\frac{A}{2}-z$; and $\sin x\sin y=\sin\left(\frac{A}{2}-z\right)\sin\left(\frac{A}{2}+z\right)=\sin^2\frac{A}{2}-\sin^2 z$. Now $\sin^2 z$ ranges between the values 0 and 1; hence $\sin x\sin y$ ranges between the values $\sin^2\frac{A}{2}$ and $-\cos^2\frac{A}{2}$: the former is always the greatest value *algebraically*.

85. We have $\sin\left(A+\frac{B}{2}\right)=\sin\left(\frac{A-C}{2}+\frac{A+B+C}{2}\right)=\cos\frac{A-C}{2}$;

similarly $$\sin\left(B+\frac{C}{2}\right)=\cos\frac{B-A}{2},\quad \sin\left(C+\frac{A}{2}\right)=\cos\frac{C-B}{2}.$$

Then $$\cos\frac{A-C}{2}+\cos\frac{B-A}{2}=2\cos\frac{B-C}{4}\cos\frac{2A-B-C}{4},$$

and $$1+\cos\frac{C-B}{2}=2\cos^2\frac{B-C}{4};$$

therefore $$\cos\frac{A-C}{2}+\cos\frac{B-A}{2}+\cos\frac{C-B}{2}+1$$

$$=2\cos\frac{B-C}{4}\left\{\cos\frac{B-C}{4}+\cos\frac{2A-B-C}{4}\right\}$$

$$=4\cos\frac{B-C}{4}\cos\frac{A-C}{4}\cos\frac{A-B}{4}=4\cos\frac{A-B}{4}\cos\frac{B-C}{4}\cos\frac{C-A}{4}.$$

86. Bring the proposed expression to a common denominator; then the numerator

$$=2\cos\alpha(1-\cos^2\alpha)\cos B\cos C+2\cos\beta(1-\cos^2\beta)\cos C\cos A$$
$$+2\cos\gamma(1-\cos^2\gamma)\cos A\cos B+2\cos\alpha\cos\beta\cos\gamma$$
$$=2\cos\alpha(\cos^2\beta+\cos^2\gamma)\cos B\cos C+2\cos\beta(\cos^2\gamma+\cos^2\alpha)\cos C\cos A$$
$$+2\cos\gamma(\cos^2\alpha+\cos^2\beta)\cos A\cos B+2\cos\alpha\cos\beta\cos\gamma$$
$$=2\cos\alpha\cos\beta(\cos\alpha\cos A+\cos\beta\cos B)\cos C$$
$$+2\cos\beta\cos\gamma(\cos\beta\cos B+\cos\gamma\cos C)\cos A$$
$$+2\cos\gamma\cos\alpha(\cos\gamma\cos C+\cos\alpha\cos A)\cos B+2\cos\alpha\cos\beta\cos\gamma$$
$$=-2\cos\alpha\cos\beta\cos\gamma\cos^2 C-2\cos\alpha\cos\beta\cos\gamma\cos^2 A-2\cos\alpha\cos\beta\cos\gamma\cos^2 B$$
$$+2\cos\alpha\cos\beta\cos\gamma$$
$$=2\cos\alpha\cos\beta\cos\gamma(1-\cos^2 C-\cos^2 A-\cos^2 B)=0.$$

87. $$\sin^2 7\tfrac{1}{2}^0=\frac{1}{2}(1-\cos 15^0)=\frac{1}{2}\{1-\cos(45^0-30^0)\}=\frac{1}{2}\left\{1-\frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$$
$$=\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}=\frac{8-2\sqrt{6}-2\sqrt{2}}{16}.$$

Now it will be found that

$$8-2\sqrt{6}-2\sqrt{2}=(2-\sqrt{2})(6+2\sqrt{2}-2\sqrt{3}-2\sqrt{6})=(2-\sqrt{2})(1+\sqrt{2}-\sqrt{3})^2;$$

therefore $$\sin 7\tfrac{1}{2}^0=\frac{1+\sqrt{2}-\sqrt{3}}{4}\sqrt{2-\sqrt{2}}.$$

88. The second equation gives $$\frac{1+\tan\frac{\phi}{2}}{1-\tan\frac{\phi}{2}}=\frac{1+c}{1-c}\,\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}};$$

therefore $$\tan\frac{\phi}{2}=\frac{c+\tan\frac{\theta}{2}}{1+c\tan\frac{\theta}{2}}.$$

The first equation gives $$\frac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}=\frac{1+2c^2}{1-c^2}\,\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}};$$

therefore $$\frac{\left(c+\tan\frac{\theta}{2}\right)\left(1+c\tan\frac{\theta}{2}\right)}{\left(1+c\tan\frac{\theta}{2}\right)^2-\left(c+\tan\frac{\theta}{2}\right)^2}=\frac{1+2c^2}{1-c^2}\,\frac{\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}},$$

therefore $$\frac{c+(1\mp c^2)\tan\frac{\theta}{2}+c\tan^2\frac{\theta}{2}}{(1-c^2)\left(1-\tan^2\frac{\theta}{2}\right)}=\frac{(1+2c^2)\tan\frac{\theta}{2}}{(1-c^2)\left(1-\tan^2\frac{\theta}{2}\right)};$$

therefore either $1-\tan^2\frac{\theta}{2}=0$, or $c\tan^2\frac{\theta}{2}-c^2\tan\frac{\theta}{2}+c=0$.

The former gives $\cos\theta=0$; the latter gives $1-c\sin\frac{\theta}{2}\cos\frac{\theta}{2}=0$, therefore $2-c\sin\theta=0$.

89. Put $2\cos^2 x-1$ for $\cos 2x$; then $2\cos^2 x+b\cos x+c-1=0$. By solving this quadratic equation we obtain

$$\cos x=\frac{-b\pm\sqrt{b^2-8(c-1)}}{4}.$$

Hence b^2+8-8c must not be negative, and $\sqrt{(b^2+8-8c)}-b$ must not be numerically greater than 4.

90. Suppose θ_2 greater than θ_1, and each between 0 and γ. Now

$$\sin\theta\,\{1+\sin(\gamma-\theta)\}=\sin\theta+\frac{1}{2}\sin\gamma\sin 2\theta-\cos\gamma\sin^2\theta.$$

Put θ_1 and θ_2 in succession for θ, and subtract the second value of the expression from the first. Thus we get

$$(\sin\theta_2-\sin\theta_1)\left\{1+\frac{1}{2}\sin\gamma\,\frac{\sin 2\theta_2-\sin 2\theta_1}{\sin\theta_2-\sin\theta_1}-(\sin\theta_2+\sin\theta_1)\cos\gamma\right\}.$$

Now $(\sin\theta_2+\sin\theta_1)\cos\gamma$ is less than $2\sin\gamma\cos\gamma$, that is less than $\sin 2\gamma$, and therefore less than 1. Hence the preceding expression is necessarily positive, and this is what was to be proved.

91. Let x be the number of sides in one regular polygon, and y the number of sides in another. Then, as in Example 61, the number of degrees in an angle of the first polygon is $\frac{2x-4}{x}90$, and the number of grades in an angle of the second polygon is $\frac{2y-4}{y}100$. Hence we must have $\frac{2x-4}{x}90=\frac{2y-4}{y}100$; therefore $9y(x-2)=10x(y-2)$; therefore

$$x(20-y)=18y.$$

We must then try in succession values of y from 3 to 19 inclusive, and ascertain in how many cases we obtain an integral value of x. The admissible values will be found to be these:

y	5	8	10	11	12	14	15	16	17	18	19
x	6	12	18	22	27	42	54	72	102	162	342.

The cases in which the angles are expressed by integers are when

$$y=5,\ 8,\ 10 \text{ or } 16.$$

92. We have $$\tan\gamma=\frac{1+\sin\alpha\sin\beta}{\cos\alpha\cos\beta},$$

and $$\cos 2\gamma=\frac{1-\tan^2\gamma}{1+\tan^2\gamma}=\frac{\cos^2\alpha\cos^2\beta-(1+\sin\alpha\sin\beta)^2}{\cos^2\alpha\cos^2\beta+(1+\sin\alpha\sin\beta)^2}.$$

The numerator

$$=(\cos\alpha\cos\beta+1+\sin\alpha\sin\beta)(\cos\alpha\cos\beta-1-\sin\alpha\sin\beta)$$
$$=-\{1+\cos(\alpha-\beta)\}\{1-\cos(\alpha+\beta)\};$$

and this cannot be positive, for $1+\cos(\alpha-\beta)$ and $1-\cos(\alpha+\beta)$ cannot be negative.

93. Denote the angle by θ; then $\dfrac{\cos\theta}{\tan\theta}=\dfrac{3}{2}$; therefore $\cos^2\theta=\dfrac{3}{2}\sin\theta$, therefore $1-\sin^2\theta=\dfrac{3}{2}\sin\theta$. By solving this quadratic equation we get $\sin\theta=\dfrac{1}{2}$, or -2; the former is the only admissible value, and hence

$$\theta=n\pi+(-1)^n\frac{\pi}{6}.$$

94. Let A denote the sum of x and y. Then

$$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=2\sin\frac{A}{2}\cos\frac{x-y}{2};$$

and as $\cos\dfrac{x-y}{2}$ ranges between -1 and $+1$ the value of $\sin x+\sin y$ ranges between $-2\sin\dfrac{A}{2}$ and $2\sin\dfrac{A}{2}$: and the positive value out of these two is algebraically and arithmetically the greatest value of $\sin x+\sin y$.

95. Here $\dfrac{\cos\theta}{\sqrt{2}}-\dfrac{\sin\theta}{\sqrt{2}}=1$; therefore $\cos\left(\theta+\dfrac{\pi}{4}\right)=1$; therefore

$$\theta+\frac{\pi}{4}=2n\pi.$$

96. $$\sin^2 A+\sin^2 B+\sin^2 C-2\sin A\sin B\sin C-1$$
$$=(\sin A-\sin B\sin C)^2+\sin^2 B+\sin^2 C-1-\sin^2 B\sin^2 C$$
$$=(\sin A-\sin B\sin C)^2-(1-\sin^2 B)(1-\sin^2 C)$$
$$=(\sin A-\sin B\sin C)^2-\cos^2 B\cos^2 C$$
$$=(\sin A-\sin B\sin C-\cos B\cos C)(\sin A-\sin B\sin C+\cos B\cos C)$$
$$=\{\sin A-\cos(B-C)\}\{\sin A+\cos(B+C)\}$$

$$=\left\{\cos\left(\frac{\pi}{2}-A\right)-\cos(B-C)\right\}\left\{\cos\left(\frac{\pi}{2}-A\right)+\cos(B+C)\right\}$$

$$=\text{the product of } 4\sin\left(\frac{B-C-A}{2}+\frac{\pi}{4}\right)\sin\left(\frac{B-C+A}{2}-\frac{\pi}{4}\right)$$

$$\text{into}\quad \cos\left(\frac{B+C-A}{2}+\frac{\pi}{4}\right)\cos\left(\frac{B+C+A}{2}-\frac{\pi}{4}\right).$$

Instead of $\sin\left(\frac{B-C-A}{2}+\frac{\pi}{4}\right)\sin\left(\frac{B-C+A}{2}-\frac{\pi}{4}\right)$ we may put

$$-\cos\left(\frac{A+C-B}{2}+\frac{\pi}{4}\right)\cos\left(\frac{A+B-C}{2}+\frac{\pi}{4}\right).$$

Thus the expression becomes the product of

$$-4\cos\left(\frac{A+B+C}{2}-\frac{\pi}{4}\right)\cos\left(\frac{B+C-A}{2}+\frac{\pi}{4}\right)$$

$$\text{into}\quad \cos\left(\frac{A+C-B}{2}+\frac{\pi}{4}\right)\cos\left(\frac{A+B-C}{2}+\frac{\pi}{4}\right).$$

97. $\cos^2 7\tfrac{1}{2}^\circ=\frac{1}{2}(1+\cos 15^\circ)=\frac{1}{2}\{1+\cos(45^\circ-30^\circ)\}=\frac{1}{2}\left\{1+\frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$

$$=\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}=\frac{8+2\sqrt{6}+2\sqrt{2}}{16}.$$

Now it will be found that

$$8+2\sqrt{6}+2\sqrt{2}=(2+\sqrt{2})(6-2\sqrt{2}-2\sqrt{3}+2\sqrt{6})=(2+\sqrt{2})(-1+\sqrt{2}+\sqrt{3})^2;$$

therefore $$\cos 7\tfrac{1}{2}^\circ=\frac{-1+\sqrt{2}+\sqrt{3}}{4}\sqrt{2+\sqrt{2}}.$$

98. By addition $2a(\sin\theta+\cos\theta)=c(1+\sin 2\theta+\cos 2\theta)$

$$=2c\cos\theta(\sin\theta+\cos\theta);$$

therefore $$a=c\cos\theta\text{..............................} \quad (1).$$

Again, by subtraction, $2b(\sin\theta-\cos\theta)=c(1-\sin 2\theta-\cos 2\theta)$

$$=2c\sin\theta(\sin\theta-\cos\theta);$$

therefore $$b=c\sin\theta\text{..............................}(2).$$

Square and add (1) and (2); thus $a^2+b^2=c^2$. This assumes that $\tan\theta$ is neither equal to 1 nor to -1.

99. We have

$$A\cot\alpha(1-\cot\beta\cot\gamma)+B\cot\beta(1-\cot\alpha\cot\gamma)+C\cot\gamma(1-\cot\beta\cot\alpha)=0,$$

and

$$A\cot\alpha(\cot\beta+\cot\gamma)+B\cot\beta(\cot\gamma+\cot\alpha)+C\cot\gamma(\cot\alpha+\cot\beta)=0.$$

These may be written

$$A\cos\alpha\cos(\beta+\gamma)+B\cos\beta\cos(\gamma+\alpha)+C\cos\gamma\cos(\alpha+\beta)=0,$$
$$A\cos\alpha\sin(\beta+\gamma)+B\cos\beta\sin(\gamma+\alpha)+C\cos\gamma\sin(\alpha+\beta)=0.$$

Hence by Algebra, Art. 385, we have

$$A=k\cos\beta\cos\gamma\{\cos(\gamma+\alpha)\sin(\alpha+\beta)-\cos(\alpha+\beta)\sin(\gamma+\alpha)\},$$
$$B=k\cos\gamma\cos\alpha\{\cos(\alpha+\beta)\sin(\beta+\gamma)-\cos(\beta+\gamma)\sin(\alpha+\beta)\},$$
$$C=k\cos\alpha\cos\beta\{\cos(\beta+\gamma)\sin(\gamma+\alpha)-\cos(\gamma+\alpha)\sin(\beta+\gamma)\},$$

where k is some constant.

Thus
$$A=k\cos\beta\cos\gamma\sin(\beta-\gamma),$$
$$B=k\cos\gamma\cos\alpha\sin(\gamma-\alpha),$$
$$C=k\cos\alpha\cos\beta\sin(\alpha-\beta).$$

Therefore
$$A\sin2\alpha+B\sin2\beta+C\sin2\gamma$$
$$=2k\cos\alpha\cos\beta\cos\gamma\{\sin\alpha\sin(\beta-\gamma)+\sin\beta\sin(\gamma-\alpha)+\sin\gamma\sin(\alpha-\beta)\}.$$

The term within the brackets will be seen to vanish, since

$$\sin\alpha\sin(\beta-\gamma)=\frac{1}{2}\{\cos(\gamma+\alpha-\beta)-\cos(\alpha+\beta-\gamma)\},$$

$$\sin\beta\sin(\gamma-\alpha)=\frac{1}{2}\{\cos(\alpha+\beta-\gamma)-\cos(\beta+\gamma-\alpha)\},$$

and
$$\sin\gamma\sin(\alpha-\beta)=\frac{1}{2}\{\cos(\beta+\gamma-\alpha)-\cos(\gamma+\alpha-\beta)\}.$$

Or we might proceed thus: let σ stand for $\alpha+\beta+\gamma$; then the two given relations may be written

$$A\cos\alpha\cos(\sigma-\alpha)+B\cos\beta\cos(\sigma-\beta)+C\cos\gamma\cos(\sigma-\gamma)=0,$$
$$A\cos\alpha\sin(\sigma-\alpha)+B\cos\beta\sin(\sigma-\beta)+C\cos\gamma\sin(\sigma-\gamma)=0;$$

therefore
$$(A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma)\cos\sigma$$
$$=-(A\sin\alpha\cos\alpha+B\sin\beta\cos\beta+C\sin\gamma\cos\gamma)\sin\sigma\ldots\ldots(1).$$

And
$$(A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma)\sin\sigma$$
$$=(A\sin\alpha\cos\alpha+B\sin\beta\cos\beta+C\sin\gamma\cos\gamma)\cos\sigma\ldots(2).$$

Multiply (1) by $\sin\sigma$ and (2) by $\cos\sigma$ and subtract: thus

$$A\sin\alpha\cos\alpha+B\sin\beta\cos\beta+C\sin\gamma\cos\gamma=0,$$

which is the required result.

Again, multiply (1) by $\cos\sigma$ and (2) by $\sin\sigma$ and add; then we obtain the additional result $A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma=0$.

100. From the first equation

$$(\cos\theta - \cos\alpha\cos\beta)^2 = \sin^2\alpha\sin^2\beta\,(1 - c^2\sin^2\theta);$$

substitute $1-\cos^2\theta$ for $\sin^2\theta$; thus

$$\cos^2\theta\,(1 - c^2\sin^2\alpha\sin^2\beta) - 2\cos\theta\cos\alpha\cos\beta$$
$$+\cos^2\alpha\cos^2\beta - (1-c^2)\sin^2\alpha\sin^2\beta = 0.$$

The second equation leads to the same quadratic for finding $\cos\phi$. Hence we infer that $\cos\theta$ is one root of the quadratic and $\cos\phi$ the other. Hence by the theory of quadratic equations, *Algebra*, Chapter XXII,

$$\cos\theta + \cos\phi = \frac{2\cos\alpha\cos\beta}{1 - c^2\sin^2\alpha\sin^2\beta},$$

$$\cos\theta\cos\phi = \frac{\cos^2\alpha\cos^2\beta - (1-c^2)\sin^2\alpha\sin^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta};$$

therefore $$1 + \cos\theta\cos\phi = \frac{1 - \sin^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta}$$

$$= \frac{\cos^2\alpha + \cos^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta}.$$

Then $$\sin^2\theta\sin^2\phi = (1 - \cos^2\theta)(1 - \cos^2\phi)$$

$$= (1 + \cos\theta\cos\phi)^2 - (\cos\theta + \cos\phi)^2$$

$$= \frac{(\cos^2\alpha - \cos^2\beta)^2}{(1 - c^2\sin^2\alpha\sin^2\beta)^2};$$

therefore $$\sin\theta\sin\phi = \pm\frac{\cos^2\alpha - \cos^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta}.$$

And $$\tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{1-\cos\theta}{\sin\theta}\cdot\frac{1-\cos\phi}{\sin\phi}$$

$$= \frac{1 - (\cos\theta + \cos\phi) + \cos\theta\cos\phi}{\sin\theta\sin\phi}$$

$$= \frac{(\cos\alpha - \cos\beta)^2}{\pm(\cos^2\alpha - \cos^2\beta)} = \pm\frac{\cos\alpha - \cos\beta}{\cos\alpha + \cos\beta}.$$

101. $$\cos 11A + \cos 5A = 2\cos 8A\cos 3A,$$

$$3\cos 9A + 3\cos 7A = 6\cos 8A\cos A;$$

hence by addition we find that the proposed expression

$$= 2\cos 8A\,(\cos 3A + 3\cos A) = 8\cos 8A\cos^3 A$$

$$= 8\cos^3 A\,(2\cos^2 4A - 1) = 16\cos^3 A\left(\cos^2 4A - \frac{1}{2}\right)$$

$$= 16\cos^3 A\left(\cos^2 4A - \sin^2\frac{\pi}{4}\right) = 16\cos^3 A\cos\left(4A + \frac{\pi}{4}\right)\cos\left(4A - \frac{\pi}{4}\right);$$

see Art. 83.

102. Let the distance be denoted by x inches; then we must have $\frac{\frac{1}{2}}{x}$ = the tangent of a quarter of a degree. As the tangent of a small angle is approximately equal to its circular measure we have approximately $\frac{1}{2x}=\frac{1}{4}\cdot\frac{\pi}{180}$; therefore $x=\frac{2\times 180}{\pi}=114\cdot 6$ nearly.

103. By Example 27 of Chapter VI. this becomes $\sin 4\theta=1$; therefore

$$4\theta=(4n+1)\frac{\pi}{2}.$$

104. Here $c\sin\theta=a\sin\theta\cos\phi+a\cos\theta\sin\phi$;

substitute for $\sin\phi$ and $\cos\phi$; thus

$$c\sin\theta=a(\cos\theta-2m)\sin\theta+a\cos\theta\times\frac{b}{a}\sin\theta;$$

therefore $$c=a(\cos\theta-2m)+b\cos\theta;$$

therefore $$\cos\theta=\frac{c+2am}{a+b}.$$

Therefore $$\cos\phi=\frac{c+2am}{a+b}-2m=\frac{c-2bm}{a+b}.$$

But $a^2\sin^2\phi=b^2\sin^2\theta$; therefore

$$a^2-b^2=a^2\cos^2\phi-b^2\cos^2\theta=\frac{a^2(c-2bm)^2-b^2(c+2am)^2}{(a+b)^2}$$

$$=\frac{(a-b)c^2-4abcm}{a+b}.$$

105. We have, by Art. 252,

$$a=2R\sin A,\quad b=2R\sin B,\quad c=2R\sin C;$$

hence the proposed expression

$$=2R\{\sin A\sin(B-C)+\sin B\sin(C-A)+\sin C\sin(A-B)\}$$

$$=2R\{\sin(B+C)\sin(B-C)+\sin(C+A)\sin(C-A)+\sin(A+B)\sin(A-B)\}$$

$$=2R\{\sin^2 B-\sin^2 C+\sin^2 C-\sin^2 A+\sin^2 A-\sin^2 B\}$$

$$=0.$$

106. By Art. 108 the proposed expression

$$=\frac{10+2\sqrt{5}}{16}\cdot\frac{10-2\sqrt{5}}{16}-\frac{\sqrt{5}+1}{4}\cdot\frac{\sqrt{5}-1}{4}=\frac{5}{16}-\frac{1}{4}=\frac{1}{16}.$$

107. From the triangle OAB we have

$$\frac{OA}{AB} = \frac{\sin OBA}{\sin AOB} = \frac{\cos A}{\sin C};$$

therefore $$x = \frac{c \cos A}{\sin C} = \frac{a \cos A}{\sin A}.$$

Similarly $$y = \frac{b \cos B}{\sin B}, \text{ and } z = \frac{c \cos C}{\sin C}.$$

Hence we have only to shew that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C;$$

and this is known to be true by Art. 114.

108. Let l denote the length of the pole. The distance of the coping from the ground is $l \sin A$, and the distance of the sill from the ground is $l \sin B$; hence the distance from the coping to the sill $= l(\sin A - \sin B)$.

The distance of the foot of the ladder from the wall is $l \cos A$ at first, and $l \cos B$ afterwards; therefore $a = l(\cos B - \cos A)$.

Substitute for l in the former expression, and we obtain

$$\frac{a(\sin A - \sin B)}{\cos B - \cos A}, \text{ that is } \frac{a \cos \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A+B)}, \text{ that is } a \cot \frac{1}{2}(A+B).$$

109. Let r denote the radius. Then the area of the sector PCB $= \frac{r^2}{2}\left(\frac{\pi}{2} - \theta\right)$, by Art. 258;

and the area of the triangle $ACP = \frac{r^2}{2} \sin\left(\frac{\pi}{2} + \theta\right)$, by Art. 247.

The sum of these two areas by supposition is equal to half the area of the semicircle; thus

$$\frac{r^2}{2}\left(\frac{\pi}{2} - \theta\right) + \frac{r^2}{2}\cos\theta = \frac{\pi r^2}{4};$$

then by simplifying we obtain $\cos\theta = \theta$.

110. By Art. 255 we have $a = 2R \sin 36^0$, and $a' = 2R \sin 18^0$;

therefore $$a^2 - a'^2 = 4R^2\left\{\frac{10 - 2\sqrt{5}}{16} - \frac{(\sqrt{5}-1)^2}{16}\right\} = \frac{4R^2 \times 4}{16} = R^2.$$

Also $$\frac{a}{r} = 2 \tan 36^0, \text{ and } \frac{a'}{r'} = 2 \tan 18^0;$$

therefore $$\frac{a}{r}+\frac{a'}{r'}=2\left(\frac{\sin 36^0}{\cos 36^0}+\frac{\sin 18^0}{\cos 18^0}\right)=\frac{2\sin(36^0+18^0)}{\cos 36^0\cos 18^0}$$

$$=\frac{2\sin 54^0}{\cos 36^0\cos 18^0}=\frac{2}{\cos 18^0}=\frac{2R}{r'}.$$

111. $\cos^2\frac{A}{2}=\frac{1}{2}(1+\cos A)$,

$$\cos^4\frac{A}{2}=\frac{1}{4}(1+\cos A)^2=\frac{1}{4}(1+2\cos A+\cos^2 A)$$

$$=\frac{1}{4}\left\{1+2\cos A+\frac{1}{2}(1+\cos 2A)\right\}=\frac{3}{8}+\frac{1}{2}\cos A+\frac{1}{8}\cos 2A.$$

In this way the proposed expression becomes

$$\frac{9}{8}+\frac{1}{2}(\cos A+\cos B+\cos C)+\frac{1}{8}(\cos 2A+\cos 2B+\cos 2C)$$

$$-\frac{1}{2}(1+\cos A)(1+\cos B)-\frac{1}{2}(1+\cos B)(1+\cos C)-\frac{1}{2}(1+\cos C)(1+\cos A)$$

$$+\frac{1}{2}(1+\cos A)(1+\cos B)(1+\cos C)$$

$$=\frac{1}{8}+\frac{1}{8}(\cos 2A+\cos 2B+\cos 2C)+\frac{1}{2}\cos A\cos B\cos C$$

$=\frac{1}{8}-\frac{1}{8}=0$; see Example VIII. 18.

112. We have $1+\tan^2\theta=\frac{1}{\cos^2\theta}=\frac{1}{1-\sin^2\theta}$; take the logarithms of both sides; thus

$$\log(1+\tan^2\theta)=\log\frac{1}{1-\sin^2\theta}=-\log(1-\sin^2\theta);$$

therefore, by Art. 146,

$$\tan^2\theta-\frac{1}{2}\tan^4\theta+\frac{1}{3}\tan^6\theta-\ldots=\sin^2\theta+\frac{1}{2}\sin^4\theta+\frac{1}{3}\sin^6\theta+\ldots$$

The series are convergent, since $\tan^2\theta$ is supposed to be less than unity.

113. Here $$2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}=2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2};$$

therefore either $$\sin\frac{3\theta}{2}=0,\quad\text{or}\ \sin\frac{\theta}{2}=\cos\frac{3\theta}{2}.$$

If $\sin\frac{3\theta}{2}=0$, then $\frac{3\theta}{2}=n\pi$.

If $\sin\frac{\theta}{2}=\cos\frac{3\theta}{2}$, or $\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\cos\frac{3\theta}{2}$, then $\frac{\pi}{2}-\frac{\theta}{2}=2n\pi\pm\frac{3\theta}{2}$.

114. We have

$$a=2R \sin A, \quad b=2R \sin B, \quad c=2R \sin C;$$

hence the proposed expression

$$=4R^2 \{\sin A \sin (B-C)+\sin B \sin (C-A)+\sin C \sin (A-B)\},$$

and this is zero, as in the solution of Example 105.

115. We have $\quad \dfrac{\cos 3\theta}{\sqrt{2}}+\dfrac{\sin 3\theta}{\sqrt{2}}=\dfrac{\cos \theta}{\sqrt{2}}+\dfrac{\sin \theta}{\sqrt{2}};$

therefore $\quad \cos\left(3\theta-\dfrac{\pi}{4}\right)=\cos\left(\theta-\dfrac{\pi}{4}\right);$

therefore $\quad 3\theta-\dfrac{\pi}{4}=2n\pi \pm \left(\theta-\dfrac{\pi}{4}\right).$

116. We have $\tan^2 x=\dfrac{\sin (\alpha+x) \sin (\alpha-x)}{\cos (\alpha+x) \cos (\alpha-x)}=\dfrac{\sin^2 \alpha-\sin^2 x}{\cos^2 x-\sin^2 \alpha},$

therefore $\quad \sin^2 x(\cos^2 x-\sin^2 \alpha)=\cos^2 x(\sin^2 \alpha-\sin^2 x),$

therefore $\quad 2 \sin^2 x \cos^2 x=\sin^2 \alpha (\sin^2 x+\cos^2 x)=\sin^2 \alpha,$

therefore $\quad 4 \sin^2 x \cos^2 x=2 \sin^2 \alpha,$

therefore $\quad 2 \sin x \cos x=\sqrt{2} . \sin \alpha,$

therefore $\quad \sin 2x=\sqrt{2} . \sin \alpha.$

117. Put x for $\tan A$, y for $\tan B$, z for $\tan C$, x' for $\tan A'$, y' for $\tan B'$, and z' for $\tan C'$, for the sake of abbreviation. Then we have given that

$$x^2x'=y^2y'=z^2z'=xyz \quad \text{.............................} (1),$$

and $$\frac{1+x^2}{2x}+\frac{1+y^2}{2y}+\frac{1+z^2}{2z}=0 \quad \text{......................} (2).$$

Now $\quad \tan (A-A')=\dfrac{x-x'}{1+xx'}=\dfrac{x-\dfrac{yz}{x}}{1+yz}$, by (1), $=\dfrac{x^2-yz}{x(1+yz)}.$

But from (2) we have

$$x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}=0,$$

therefore $\quad xyz(x+y+z)+xy+yz+zx=0,$

therefore $\quad x^2-yz=x^2+xyz(x+y+z)+xy+zx$

$$=x(x+y+z)+xyz(x+y+z)=x(1+yz)(x+y+z);$$

therefore $\quad \dfrac{x^2-yz}{x(1+yz)}=x+y+z.$

Thus $$\tan(A-A')=\tan A+\tan B+\tan C.$$

Similarly $\tan(B-B')$ and $\tan(C-C')$ may be shewn to be equal to the same expression.

118. Here $$\cos A=\frac{\cos 60^0}{\sin 36^0},$$

therefore $$\sin A=\frac{\sqrt{\sin^2 36^0-\cos^2 60^0}}{\sin 36^0},$$

and $$\sin^2 36-\cos^2 60^0=\frac{10-2\sqrt{5}}{16}-\frac{1}{4}=\frac{6-2\sqrt{5}}{16}=\left(\frac{\sqrt{5}-1}{4}\right)^2;$$

therefore $$\sin A=\frac{\sqrt{5}-1}{4\sin 36^0}.$$

Hence $$\tan A=\frac{\sqrt{5}-1}{4\cos 60^0}=\frac{\sqrt{5}-1}{2}.$$

Again, $$\cos B=\frac{\cos 36^0}{\sin 60^0};$$

therefore $$\sin B=\frac{\sqrt{\sin^2 60^0-\cos^2 36^0}}{\sin 60^0}=\frac{\sqrt{\sin^2 36^0-\cos^2 60^0}}{\sin 60^0}=\frac{\sqrt{5}-1}{4\sin 60^0}.$$

Hence $$\tan B=\frac{\sqrt{5}-1}{4\cos 36^0}=\frac{\sqrt{5}-1}{\sqrt{5}+1}=\frac{(\sqrt{5}-1)^2}{(\sqrt{5}+1)(\sqrt{5}-1)}=\frac{6-2\sqrt{5}}{4}=\frac{3-\sqrt{5}}{2}.$$

Therefore $$\tan A+\tan B=\frac{\sqrt{5}-1}{2}+\frac{3-\sqrt{5}}{2}=1;$$

therefore $\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}=1$, therefore $\sin(A+B)=\cos A\cos B=\cos C$; therefore $A+B=90^0-C$ is one solution.

119. Let θ be the sun's altitude at the first observation, and $\theta+\alpha$ that at the second observation; then

$$h=a\tan\theta,\ \text{and}\ h=b\tan(\theta+\alpha).$$

Thus $$h=\frac{b(\tan\theta+\tan\alpha)}{1-\tan\theta\tan\alpha}=\frac{\frac{hb}{a}+b\tan\alpha}{1-\frac{h}{a}\tan\alpha},$$

therefore $$h\left(1-\frac{h}{a}\tan\alpha\right)=\frac{hb}{a}+b\tan\alpha,$$

therefore $$h^2\tan\alpha+h(b-a)+ab\tan\alpha=0,$$

therefore $$h^2+h(b-a)\cot\alpha+ab=0.$$

120. Let $AP=b$, $BP=a$, $AB=c$.

The diameter of the circle which touches the semicircle and also touches AP at its middle point is $\frac{c}{2}-\frac{c}{2}\sin PAB$, that is $\frac{c}{2}-\frac{c}{2}\frac{a}{c}$, that is $\frac{c-a}{2}$; therefore the radius of this circle is $\frac{c-a}{4}$.

Similarly the radius of the circle which touches the semicircle and also touches BP at its middle point is $\frac{c-b}{4}$. We have then to shew that

$$\frac{(c-a)(c-b)}{16}=\frac{r^2}{8}, \text{ that is } \frac{(c-a)(c-b)}{2}=r^2.$$

But $r=\frac{S}{s}=\frac{ab}{a+b+c}=\frac{ab(a+b-c)}{(a+b)^2-c^2}=\frac{a+b-c}{2}$, since $c^2=a^2+b^2$; therefore

$$r^2=\frac{(a+b-c)^2}{4}=\frac{2c^2+2ab-2c(a+b)}{4}=\frac{(c-a)(c-b)}{2}.$$

121.
$$\sin\alpha\sin(\beta-\gamma)\cos(\beta+\gamma-\alpha)$$
$$=\frac{1}{2}\{\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta-\gamma)\}\cos(\beta+\gamma-\alpha)$$
$$=\frac{1}{4}\{\cos 2\gamma+\cos 2(\alpha-\beta)-\cos 2\beta-\cos 2(\alpha-\gamma)\}.$$

The other two terms may be transformed in a similar manner, and then it will be obvious that the sum is zero.

122.
$$\cot\theta=\frac{\cos\theta}{\sin\theta}=\frac{2\cos^2\theta}{2\sin\theta\cos\theta}=\frac{1+\cos 2\theta}{\sin 2\theta}$$
$$=\frac{1+\cos 2\theta}{\sqrt{(1-\cos^2 2\theta)}}=\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}.$$

Hence, taking logarithms we have

$$\log\cot\theta=\frac{1}{2}\log\frac{1+\cos 2\theta}{1-\cos 2\theta}=\cos 2\theta+\frac{1}{3}(\cos 2\theta)^3+\frac{1}{5}(\cos 2\theta)^5+\ldots$$

123. We have shewn in the solution of Example VIII. 15 that

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2};$$

hence the expression on the right hand-side

$$=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\cdot\frac{8\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{\cos A\cos B\cos C}$$
$$=\frac{8\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{\cos A\cos B\cos C}=\tan A\tan B\tan C.$$

Again $\cot A - 2\cot 2A = \cot A - \dfrac{2(\cos^2 A - \sin^2 A)}{2\cos A \sin A} = \tan A$;

similarly $\cot B - 2\cot 2B = \tan B$, and $\cot C - 2\cot 2C = \tan C$;

thus the expression on the left-hand side

$$= \tan A + \tan B + \tan C = \tan A \tan B \tan C, \text{ by Art. 114.}$$

Thus the two expressions are equivalent.

124. $\sin A \sin B \sin(A-B) = \dfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}\sin(A-B)$

$$= \frac{1}{4}\sin(2A-2B) - \frac{1}{4}(\sin 2A - \sin 2B).$$

Transform the second and third terms in the same way; then by addition we obtain the required result.

125. $\dfrac{1}{a}\cos^2\dfrac{A}{2} + \dfrac{1}{b}\cos^2\dfrac{B}{2} + \dfrac{1}{c}\cos^2\dfrac{C}{2} = \dfrac{s(s-a)+s(s-b)+s(s-c)}{abc}$

$$= \frac{3s^2 - s(a+b+c)}{abc} = \frac{3s^2 - 2s^2}{abc} = \frac{s^2}{abc} = \frac{(a+b+c)^2}{4abc}.$$

126. Here $\dfrac{1}{\cos\left(\dfrac{\pi}{4}+x\right)} + \dfrac{1}{\cos\left(\dfrac{\pi}{4}-x\right)} = 2\sqrt{2}$,

therefore $\dfrac{\sqrt{2}}{\cos x - \sin x} + \dfrac{\sqrt{2}}{\cos x + \sin x} = 2\sqrt{2}$,

therefore $\cos x = \cos^2 x - \sin^2 x = \cos 2x$,

therefore $2x = 2n\pi \pm x$.

127. Express the fractions with the common denominator

$$\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha):$$

then the numerator becomes

$$-\{\sin(\beta-\gamma)\sin(\theta-\alpha) + \sin(\gamma-\alpha)\sin(\theta-\beta) + \sin(\alpha-\beta)\sin(\theta-\gamma)\}.$$

Now $\sin(\beta-\gamma)\sin(\theta-\alpha) = \dfrac{1}{2}\cos(\theta-\alpha-\beta+\gamma) - \dfrac{1}{2}\cos(\theta-\alpha+\beta-\gamma)$,

$$\sin(\gamma-\alpha)\sin(\theta-\beta) = \frac{1}{2}\cos(\theta-\beta+\alpha-\gamma) - \frac{1}{2}\cos(\theta-\beta+\gamma-\alpha),$$

$$\sin(\alpha-\beta)\sin(\theta-\gamma) = \frac{1}{2}\cos(\theta-\gamma+\beta-\alpha) - \frac{1}{2}\cos(\theta-\gamma+\alpha-\beta);$$

thus the sum of the expressions is zero.

128. Let x denote the length of the pillar, h the height of the foot of the pillar above the horizontal plane, b the horizontal distance of the pillar from the first station. Let θ be the angle subtended by the pillar. Then

$$\frac{h}{b}=\tan(\alpha-\theta),\quad \frac{h}{b+c}=\tan(\beta-\theta),\quad \frac{h+x}{b}=\tan\alpha,\quad \frac{h+x}{b+c}=\tan\beta.$$

And from the fact that a circle would pass through the two stations and the top and the foot of the pillar we have $\alpha+\beta-\theta=\frac{\pi}{2}$. Thus

$$\frac{h}{b}=\cot\beta,\ \frac{h+x}{b}=\tan\alpha;\ \text{therefore}$$

$$\frac{x}{b}=\tan\alpha-\cot\beta=-\frac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta}.$$

Similarly $$\frac{x}{b+c}=\tan\beta-\cot\alpha=-\frac{\cos(\alpha+\beta)}{\cos\beta\sin\alpha}.$$

Therefore $$\frac{c}{x}=\frac{\cos\alpha\sin\beta-\cos\beta\sin\alpha}{\cos(\alpha+\beta)}=\frac{\sin(\beta-\alpha)}{\cos(\beta+\alpha)};$$

therefore $$x=\frac{c\cos(\beta+\alpha)}{\sin(\beta-\alpha)}.$$

129. In every *right-angled* triangle $r=\frac{1}{2}(a+b-c)$; see the solution of Example 120. In the present case $2\sqrt{R^2-2Rr}=\sqrt{c(a+b-c)}$; and $R=\frac{c}{2}$: thus $4\left(\frac{c^2}{4}-cr\right)=c(a+b-c)$; therefore $c-4r=a+b-c$; therefore $4r=2c-a-b$; therefore $2(a+b-c)=2c-a-b$; therefore $a+b=\frac{4c}{3}$; and therefore $r=\frac{c}{6}$.

130. $$\cos^2\frac{1}{2}A+\cos^2\frac{1}{2}B+\cos^2\frac{1}{2}C=\frac{1}{2}(3+\cos A+\cos B+\cos C)$$

$$=\frac{1}{2}\left(4+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right),\ \text{by Art. 114},$$

$$=2+\frac{2S^2}{sabc}=2+\frac{2rS}{abc}=2+\frac{r}{2R}.$$

131. $$\frac{\sin(x+A)}{\sqrt{\sin 2A}}=\frac{\sin(x+B)}{\sqrt{\sin 2A}};$$

therefore $$\frac{\sin x\cos A+\cos x\sin A}{\sqrt{\sin 2A}}=\frac{\sin x\cos B+\cos x\sin B}{\sqrt{\sin 2B}},$$

therefore $$\frac{\cos A(\tan x+\tan A)}{\sqrt{\sin 2A}}=\frac{\cos B(\tan x+\tan B)}{\sqrt{\sin 2B}},$$

therefore $$\frac{\tan x+\tan A}{\sqrt{\tan A}}=\frac{\tan x+\tan B}{\sqrt{\tan B}},$$

therefore $$\tan x\,(\sqrt{\tan B}-\sqrt{\tan A})=\sqrt{\tan A\tan B}\,(\sqrt{\tan B}-\sqrt{\tan A}),$$

therefore $$\tan x=\sqrt{\tan A\tan B}.$$

132. $$\sin^2 2A+\cos 2A\cos 2B\cos 2C=1-\cos^2 2A+\cos 2A\cos 2B\cos 2C$$
$$=1+\cos 2A\,\{\cos 2B\cos 2C-\cos 2A\}$$
$$=1+\cos 2A\,\{\cos 2B\cos 2C-\cos(2B+2C)\}$$
$$=1+\cos 2A\,\sin 2B\sin 2C.$$

Similarly
$$\sin^2 2B+\cos 2A\cos 2B\cos 2C=1+\cos 2B\sin 2A\sin 2C.$$

Hence the proposed expression
$$=2+\sin 2C\,\{\sin 2C+\sin 2B\cos 2A+\sin 2A\cos 2B\}$$
$$=2+\sin 2C\,\{-\sin(2A+2B)+\sin(2A+2B)\}$$
$$=2.$$

133. The series may be separated into two, namely
$$\log 2+\frac{1}{\underline{|2}}(\log 2)^2+\frac{1}{\underline{|3}}(\log 2)^3+\ldots$$
and
$$2\log 2+\frac{1}{\underline{|2}}(2\log 2)^2+\frac{1}{\underline{|3}}(2\log 2)^3+\ldots$$
and is therefore equal to $e^{\log 2}-1+e^{2\log 2}-1$, that is to $e^{\log 2}-1+e^{\log 4}-1$, that is to $2-1+4-1$, that is to 4.

134. $$\sin(A-B)\cos(C-B)\cos(A-C)$$
$$=\frac{1}{2}\sin(A-B)\,\{\cos(A+B-2C)+\cos(A-B)\}$$
$$=\frac{1}{4}\sin 2(A-C)+\frac{1}{4}\sin 2(C-B)+\frac{1}{4}\sin 2(A-B).$$

Transform the second and third terms in like manner, then by addition we obtain the required result.

135. $$\frac{\sin(A-B)}{\sin(A+B)}=\frac{\sin(A+B)\sin(A-B)}{\sin^2(A+B)}=\frac{\sin^2 A-\sin^2 B}{\sin^2 C},$$ by Art. 83,
$$=\left(\frac{\sin A}{\sin C}\right)^2-\left(\frac{\sin B}{\sin C}\right)^2=\left(\frac{a}{c}\right)^2-\left(\frac{b}{c}\right)^2=\frac{a^2-b^2}{c^2}.$$

136. Suppose the diagonal h of the quadrilateral to make an angle θ with the sides of the rectangle which pass through its extremities; then each of the other sides is equal to $h\sin\theta$. It will be seen from a diagram that

the diagonal k of the quadrilateral will make an angle $\frac{3\pi}{2}-(\theta+A)$ with the sides of the rectangle which pass through its extremities; then each of the other sides is equal to $k\sin\left(\frac{3\pi}{2}-\theta-A\right)$, that is to $-k\cos(\theta+A)$. Hence the area of the rectangle $=-hk\sin\theta\cos(\theta+A)=\frac{hk}{2}\{-\sin(2\theta+A)+\sin A\}$. The greatest value of this is when $2\theta+A=\frac{3\pi}{2}$, and is $\frac{hk}{2}(1+\sin A)$.

137. Let h denote the height of the house, x the height of the wall, y the height of the church. Then $x\cot\alpha$ is the distance of the wall from the house, and $y\cot\alpha$ is the distance of the church from the house. By similar triangles $\frac{h}{x\cot\alpha}=\frac{y}{y\cot\alpha-x\cot\alpha}$; therefore $h(y-x)=xy$.

Also $\frac{y-h}{y\cot\alpha}=\tan\beta$; therefore $y=\frac{h}{1-\cot\alpha\tan\beta}=\frac{h\tan\alpha}{\tan\alpha-\tan\beta}$.

Then
$$x=\frac{hy}{h+y}=\frac{h\tan\alpha}{2\tan\alpha-\tan\beta}.$$

138. $a\cos^2\frac{1}{2}A+b\cos^2\frac{1}{2}B+c\cos^2\frac{1}{2}C$

$$=\frac{1}{2}(a+b+c+a\cos A+b\cos B+c\cos C)$$

$$=s+\frac{1}{2}R(\sin 2A+\sin 2B+\sin 2C)$$

$$=s+2R\sin A\sin B\sin C, \text{ by Art. 114,}$$

$$=s+2R\frac{8S^3}{a^2b^2c^2}=s+\frac{4S^2}{abc}=s+\frac{S}{R}.$$

139. Through C draw a plane parallel to the horizon; from A draw AP perpendicular to the intersection of this plane with that which contains A, B, and C; from B draw BQ perpendicular to the same intersection. Let $ACP=\phi$, and $BCQ=\psi$; so that $\phi+\psi+\gamma=\pi$. Therefore

$$\cos\gamma=\sin\phi\sin\psi-\cos\phi\cos\psi.$$

Now $AP=AC\sin\phi$; thus the perpendicular from A on the plane drawn through C parallel to the horizon $=AP\sin\theta=AC\sin\theta\sin\phi$; but this perpendicular also $=AC\sin\alpha$; therefore

$$\sin\alpha=\sin\theta\sin\phi.$$

Similarly
$$\sin\beta=\sin\theta\sin\psi.$$

Hence
$$\cos\gamma=\frac{\sin\alpha\sin\beta}{\sin^2\theta}-\frac{\sqrt{(\sin^2\theta-\sin^2\alpha)(\sin^2\theta-\sin^2\beta)}}{\sin^2\theta},$$

therefore $(\cos\gamma\sin^2\theta - \sin\alpha\sin\beta)^2 = (\sin^2\theta - \sin^2\alpha)(\sin^2\theta - \sin^2\beta)$,

therefore $\cos^2\gamma\sin^2\theta - 2\cos\gamma\sin\alpha\sin\beta = \sin^2\theta - \sin^2\alpha - \sin^2\beta$,

therefore $\sin^2\theta\sin^2\gamma = \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos\gamma$.

140. With the diagram of Art. 248 we see that

$$a' = 2r\sin FOA = 2r\cos\frac{1}{2}A\,;$$

and we have similar values for b' and c'.

Thus $$a'b'c' = 8r^3\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C = 8r^3\frac{sS}{abc};$$

therefore $$\frac{a'b'c'}{abc} = \frac{8r^2S^2}{a^2b^2c^2} = \frac{8r^2}{(4R)^2} = \frac{r^2}{2R^2}.$$

141. Here $$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} + \frac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta = 0.$$

Therefore either $\tan\theta = 0$ or $\dfrac{3 - \tan^2\theta}{1 - 3\tan^2\theta} + \dfrac{2}{1 - \tan^2\theta} + 1 = 0$.

The latter gives

$$(3 - \tan^2\theta)(1 - \tan^2\theta) + 2(1 - 3\tan^2\theta) + (1 - \tan^2\theta)(1 - 3\tan^2\theta) = 0;$$

therefore $$4\tan^4\theta - 14\tan^2\theta + 6 = 0.$$

By solving this quadratic we obtain $\tan^2\theta = 3$ or $\dfrac{1}{2}$.

142. We may obtain the result by taking the values of the four cosines and raising them to the eighth power. Or we may proceed thus:

$$\cos^8\frac{\pi}{8} + \cos^8\frac{3\pi}{8} + \cos^8\frac{5\pi}{8} + \cos^8\frac{7\pi}{8}$$

$$= 2\left(\cos^8\frac{\pi}{8} + \cos^8\frac{3\pi}{8}\right) = 2\left(\cos^8\frac{\pi}{8} + \sin^8\frac{\pi}{8}\right)$$

$$= \frac{1}{32}\left(\cos\pi + 28\cos\frac{\pi}{2} + 35\right),$$ by Example IX. 13,

$$= \frac{34}{32} = \frac{17}{16}.$$

143. Here $\dfrac{b}{a} = \dfrac{\cos\phi}{\cos\theta}$; therefore $\dfrac{a+b}{a-b} = \dfrac{\cos\theta + \cos\phi}{\cos\theta - \cos\phi}$

$$= \frac{2\cos\frac{1}{2}(\phi+\theta)\cos\frac{1}{2}(\phi-\theta)}{2\sin\frac{1}{2}(\phi+\theta)\sin\frac{1}{2}(\phi-\theta)} = \cot\frac{1}{2}(\phi+\theta)\cot\frac{1}{2}(\phi-\theta).$$

144. $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)$;

and the sum of $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$, and $\frac{\pi}{2}-\frac{C}{2}$ is a fixed quantity, namely π.

Hence proceeding as in Example XIII. 40, we see that the proposed product is greatest when

$$\frac{\pi}{2}-\frac{A}{2}=\frac{\pi}{2}-\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2},$$

that is when
$$\frac{A}{2}=\frac{B}{2}=\frac{C}{2}=\frac{\pi}{6};$$

and then the product
$$-\left(\frac{\sqrt{3}}{2}\right)^3=\frac{3\sqrt{3}}{8}.$$

145. We have $\cot\frac{B}{2}=\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}=\frac{s-b}{s-a}\cot\frac{A}{2}$;

similarly
$$\cot\frac{C}{2}=\frac{s-c}{s-a}\cot\frac{A}{2}.$$

Hence $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\left(1+\frac{s-b}{s-a}+\frac{s-c}{s-a}\right)\cot\frac{A}{2}$

$$=\frac{3s-a-b-c}{s-a}\cot\frac{A}{2}=\frac{s}{s-a}\cot\frac{A}{2}=\frac{a+b+c}{b+c-a}\cot\frac{A}{2}.$$

146. Let A denote the angle between the diagonals; then $C=\frac{1}{2}hk\sin A$; and by the solution of Example 136 the area of the circumscribed rectangle is $-hk\sin\theta\cos(A+\theta)$. And since the rectangle is to be a square, we have by the solution of Example 136

$$h\sin\theta=-k\cos(A+\theta);$$

therefore
$$h=-k(\cos A\cot\theta-\sin A);$$

therefore
$$\cot\theta=\frac{k\sin A-h}{k\cos A};$$

therefore $\sin^2\theta=\frac{k^2\cos^2 A}{(k\sin A-h)^2+k^2\cos^2 A}=\frac{k^2-k^2\sin^2 A}{k^2-2kh\sin A+h^2}=\frac{k^2-\frac{4C^2}{h^2}}{h^2+k^2-4C}$.

And the area of the circumscribing square $=h^2\sin^2\theta=\frac{h^2k^2-4C^2}{h^2+k^2-4C}$.

147. We may put the proposed expression in the form

$$L\sin^2\theta+M\sin\theta\cos\theta+N\cos^2\theta,$$

where L, M, N involve the angles α, β, γ and also x, y, z: moreover x, y, z occur only in the first power. Now if we put $M=0$ and $L=N=$ the given constant, the expression is equal to the given constant whatever θ may be. So we have only to determine x, y, and z from the three simple equations $M=0$, and $L=N=$ the given constant.

As soon as we have thus shewn that such values of x, y, z as we require must exist, we can determine the values more simply. For let C denote the given constant; put α for θ, then

$$x \sin(\alpha-\beta) \sin(\alpha-\gamma)=C.$$

This finds x. Similarly, by putting β for θ we find y, and by putting γ for θ we find z.

148. Let AB denote the side of the regular pentagon, P the middle point of the arc subtended by the side adjacent to AB at B. Then the angle APB is the angle subtended at the circumference of the circle by the side of a regular pentagon inscribed in the circle, so that the angle $=\frac{\pi}{5}$. Similarly the angle $PAB=\frac{\pi}{10}$; and therefore the angle $ABP=\frac{7\pi}{10}$.

Let r denote the radius of the circle, so that

$$AB=2r\sin\frac{\pi}{5},\quad PB=2r\sin\frac{\pi}{10},\ \text{and}\ PA=2r\sin\frac{7\pi}{10}=2r\sin\frac{3\pi}{10}.$$

Hence
$$PA-PB=2r\left\{\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}\right\}=r,$$

$$PA\,.\,PB=\frac{4r^2(\sqrt{5}+1)(\sqrt{5}-1)}{16}=r^2,$$

$$PA^2+PB^2=4r^2\left\{\left(\frac{\sqrt{5}+1}{4}\right)^2+\left(\frac{\sqrt{5}-1}{4}\right)^2\right\}=3r^2.$$

149. Suppose the tower to subtend an angle ϕ at the eye of the observer; let x be the length of the flag-staff: then

$$\frac{a}{b}=\tan\phi,\quad \frac{a+x}{b}=\tan(\phi+\theta)=\frac{\tan\phi+\tan\theta}{1-\tan\phi\tan\theta}=\frac{a+b\tan\theta}{b-a\tan\theta};$$

therefore
$$\frac{x}{b}=\frac{a+b\tan\theta}{b-a\tan\theta}-\frac{a}{b}=\frac{(b^2+a^2)\tan\theta}{b(b-a\tan\theta)};$$

then if θ be very small we may put θ for $\tan\theta$, and neglect $a\tan\theta$ in comparison with b, so that $x=\frac{b^2+a^2}{b}\theta$ nearly.

150. We have by Art. 249

$$(s-a)^2\sin A+(s-b)^2\sin B+(s-c)^2\sin C$$
$$=r\left\{(s-a)\sin A\cot\frac{A}{2}+(s-b)\sin B\cot\frac{B}{2}+(s-c)\sin C\cot\frac{C}{2}\right\}$$
$$=2r\left\{(s-a)\cos^2\frac{A}{2}+(s-b)\cos^2\frac{B}{2}+(s-c)\cos^2\frac{C}{2}\right\};$$

and by Examples 130 and 138, this

$$=2r\left\{\left(2+\frac{r}{2R}\right)s-\left(s+\frac{S}{R}\right)\right\}=2r\left(s+\frac{S}{2R}-\frac{S}{R}\right)$$

$$=2r\left(s-\frac{S}{2R}\right)=2r\left(\frac{S}{r}-\frac{S}{2R}\right)=\frac{S(2R-r)}{R}.$$

And $$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{Ss}{abc}=\frac{s}{4R};$$

so that $$4r(2R-r)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=4r(2R-r)\frac{s}{4R}=\frac{S(2R-r)}{R}.$$

Thus the proposed expressions are equal.

151. $$2\sin 7A\cos A=\sin 8A+\sin 6A;$$

therefore $2\sin 7A\cos A+16\sin A\cos^3 A=\sin 6A+\sin 8A+16\sin A\cos^3 A$

$$=\sin 6A+2\sin 4A\cos 4A+8\sin 2A\cos^2 A$$

$$=\sin 6A+4\sin 2A\cos 2A\cos 4A+8\sin 2A\cos^2 A$$

$$=\sin 6A+4\sin 2A\,(2\cos^2 A+\cos 2A\cos 4A)$$

$$=\sin 6A+4\sin 2A\,\{1+\cos 2A\,(1+\cos 4A)\}$$

$$=\sin 6A+4\sin 2A\,(1+2\cos^3 2A).$$

152. Let x denote the logarithm of 32 to the base $\sqrt[3]{4}$; then $32=(\sqrt[3]{4})^x$, that is $2^5=4^{\frac{x}{3}}=2^{\frac{2x}{3}}$; therefore $5=\frac{2x}{3}$; therefore $x=\frac{15}{2}$.

Let x denote the logarithm of $81\sqrt[3]{3}$ to the base $\sqrt[3]{9}$; then $81\sqrt[3]{3}=(\sqrt[3]{9})^x$, that is $3^{4+\frac{1}{3}}=9^{\frac{x}{3}}=3^{\frac{2x}{3}}$; therefore $\frac{2x}{3}=4\frac{1}{3}=\frac{13}{3}$; therefore $x=\frac{13}{2}$.

153. Here $$\frac{\sin(A+B)}{\cos(A+B)}=\frac{3\sin A}{\cos A};$$

therefore $\sin(A+B)\cos A-\cos(A+B)\sin A=2\sin A\cos(A+B)$;

therefore $$\sin(A+B-A)=2\sin A\cos(A+B);$$

that is $$\sin B=2\sin A\cos(A+B)=\sin(2A+B)-\sin B;$$

therefore $$2\sin B=\sin(2A+B);$$

therefore $$2\sin B\cos B=\sin(2A+B)\cos B;$$

therefore $$\sin 2B=\frac{1}{2}\{\sin(2A+2B)+\sin 2A\};$$

therefore $$2\sin 2B=\sin(2A+2B)+\sin 2A.$$

154. $\frac{1}{a}\sin^2\frac{A}{2}+\frac{1}{b}\sin^2\frac{B}{2}+\frac{1}{c}\sin^2\frac{C}{2}$

$$=\frac{1}{abc}\{(s-b)(s-c)+(s-a)(s-c)+(s-a)(s-b)\}$$

$$=\frac{1}{abc}\{3s^2-2s(a+b+c)+ab+bc+ca\}$$

$$=\frac{1}{abc}\{ab+bc+ca-s^2\}$$

$$=\frac{1}{4abc}\{4ab+4bc+4ca-(a+b+c)^2\}$$

$$=\frac{1}{4abc}\{2ab+2bc+2ca-a^2-b^2-c^2\}.$$

155. Let $ABCD$ denote the quadrilateral figure. Let P, Q, R, S be taken in AB, BC, CD, DA respectively, such that

$$\frac{AP}{PB}=\frac{BQ}{QC}=\frac{CR}{RD}=\frac{DS}{SA}=\frac{m}{n}.$$

Then $$\frac{PB}{AB}=\frac{n}{m+n},\quad \frac{BQ}{BC}=\frac{m}{m+n};$$

and the area of the triangle $PBQ=\frac{1}{2}BP.\,BQ\sin B=\frac{mn}{2(m+n)^2}AB.\,BC\sin B$

$$=\frac{mn}{(m+n)^2}\text{ area of the triangle }ABC.$$

Similarly the area of the triangle $RDS=\frac{mn}{(m+n)^2}$ area of the triangle ADC.

Therefore the area of the triangles PBQ and $RDS=\frac{mnH}{(m+n)^2}$, where H denotes the area of the quadrilateral figure $ABCD$.

In the same way we shew that the area of the triangles QCR and SAP $=\frac{mnH}{(m+n)^2}$.

Thus the area of the four triangles PBQ, QCR, RDS, and $SAP=\frac{2mnH}{(m+n)^2}$. Therefore the area of the quadrilateral figure $PQRS$

$$=H\left\{1-\frac{2mn}{(m+n)^2}\right\}=\frac{H(m^2+n^2)}{(m+n)^2}.$$

156. $\cos\theta+\cos 3\theta=\frac{1}{2}$; therefore $\cos\theta+4\cos^3\theta-3\cos\theta=\frac{1}{2}$;

therefore $4\cos^3\theta-2\cos\theta-\frac{1}{2}=0$; therefore $4\left(\cos^3\theta+\frac{1}{8}\right)-2\left(\cos\theta+\frac{1}{2}\right)=0$;

therefore $$2\left(\cos\theta+\frac{1}{2}\right)\left(\cos^2\theta-\frac{1}{2}\cos\theta+\frac{1}{4}\right)-\left(\cos\theta+\frac{1}{2}\right)=0;$$

therefore $$\left(\cos\theta+\frac{1}{2}\right)\left(2\cos^2\theta-\cos\theta-\frac{1}{2}\right)=0.$$

Thus either $\cos\theta+\frac{1}{2}=0$ or $2\cos^2\theta-\cos\theta-\frac{1}{2}=0$;

the former gives $\cos\theta=-\frac{1}{2}$; the latter gives $\cos\theta=\frac{1\pm\sqrt{5}}{4}$.

157. $$\cos\beta\cos\gamma\sin(\gamma-\beta)=\frac{1}{2}\{\cos(\beta-\gamma)+\cos(\beta+\gamma)\}\sin(\gamma-\beta)$$

$$=\frac{1}{4}\{\sin(2\gamma-2\beta)+\sin 2\gamma-\sin 2\beta\}.$$

Transform the other two terms in the same way; and thus we obtain finally as the sum

$$\frac{1}{4}\{\sin(2\gamma-2\beta)+\sin(2\alpha-2\gamma)+\sin(2\beta-2\alpha)\}.$$

Again, $\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$

$$=\frac{1}{2}\{\cos(\alpha+\gamma-2\beta)-\cos(\alpha-\gamma)\}\sin(\gamma-\alpha)$$

$$=\frac{1}{4}\{\sin(2\gamma-2\beta)+\sin(2\beta-2\alpha)+\sin(2\alpha-2\gamma)\}.$$

Thus the proposed expressions are equal.

Or thus: from Example VIII. 12 we see that

$$\sin\beta\sin\gamma\sin(\gamma-\beta)+\sin\gamma\sin\alpha\sin(\alpha-\gamma)+\sin\alpha\sin\beta\sin(\beta-\alpha)$$
$$=\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha).$$

In this formula change α, β, γ into $\frac{\pi}{2}+\alpha$, $\frac{\pi}{2}+\beta$, $\frac{\pi}{2}+\gamma$ respectively; and thus we obtain the required result.

158. $$\sin A\sin(A-B)\sin(A-C)=\frac{1}{2}\sin A\{\cos(C-B)-\cos(2A-B-C)\}$$

$$=\frac{1}{4}\{\sin(A+C-B)+\sin(A+B-C)-\sin(3A-B-C)-\sin(B+C-A)\}$$

$$=\frac{1}{4}\{\sin 2B+\sin 2C+\sin 4A-\sin 2A\}.$$

In this way we see that the expression on the left-hand side in the proposed formula

$$=\frac{1}{4}\{\sin 2A+\sin 2B+\sin 2C+\sin 4A+\sin 4B+\sin 4C\}.$$

Then by Example VIII. 33 we have

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C,$$

$$\sin 4A+\sin 4B+\sin 4C=-4\sin 2A\sin 2B\sin 2C$$

$$=-32\sin A\sin B\sin C\cos A\cos B\cos C.$$

Thus we obtain the required result.

159. Let A denote the bottom of the pole, B the point on the pole to which the man climbs, F the top of the window, E the bottom. Let AF and BE intersect at D, which is therefore the top of the wall. Draw DC perpendicular to the ground, and produce FE to meet the ground at H. Draw from B a horizontal straight line meeting FH at G.

Then from the triangle BAF we get $BF=\dfrac{c\cos\alpha}{\sin(\alpha-\beta)}$;

$$BG=BF\cos\beta=\frac{c\cos\alpha\cos\beta}{\sin(\alpha-\beta)}=\frac{c}{\tan\alpha-\tan\beta},$$

$$EG=BG\tan\gamma=\frac{c\tan\gamma}{\tan\alpha-\tan\beta},$$

$$EH=c+EG=\frac{c(\tan\alpha-\tan\beta+\tan\gamma)}{\tan\alpha-\tan\beta}.$$

160. From the triangle CEB we have $\dfrac{CE}{a}=\dfrac{\sin\frac{1}{2}B}{\sin\left(C+\frac{1}{2}B\right)}$;

and from the triangle CDA we have $\dfrac{CD}{b}=\dfrac{\sin\frac{1}{2}A}{\sin\left(C+\frac{1}{2}A\right)}$.

Thus the area of the triangle $CED=\frac{1}{2}CE\,.\,CD\sin C$

$$=\frac{ab\sin C\sin\frac{1}{2}A\sin\frac{1}{2}B}{2\sin\left(C+\frac{1}{2}B\right)\sin\left(C+\frac{1}{2}A\right)}=\frac{S\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{C-A}{2}\cos\frac{C-B}{2}}.$$

161. We have $p=2\cos A+\cos^3 A\,(-5+4\cos^2 A)$

$$=2\cos A+\frac{1}{4}(\cos 3A+3\cos A)(-5+4\cos^2 A)$$

$$=2\cos A+\frac{1}{4}(\cos 3A+3\cos A)(-3+2\cos 2A)$$

$$=2\cos A-\frac{3}{4}\cos 3A-\frac{9}{4}\cos A+\frac{1}{2}\cos 3A\cos 2A+\frac{3}{2}\cos A\cos 2A$$

$$=-\frac{3}{4}\cos 3A-\frac{1}{4}\cos A+\frac{1}{4}(\cos 5A+\cos A)+\frac{3}{4}(\cos 3A+\cos A)$$

$$=\frac{1}{4}(\cos 5A+3\cos A).$$

In the same way we find that

$$q=\frac{1}{4}(\sin 5A+3\sin A).$$

Therefore

$$p\cos 3A+q\sin 3A=\frac{1}{4}(\cos 5A+3\cos A)\cos 3A+\frac{1}{4}(\sin 5A+3\sin A)\sin 3A$$

$$=\frac{1}{4}(\cos 5A\cos 3A+\sin 5A\sin 3A)+\frac{3}{4}(\cos 3A\cos A+\sin 3A\sin A)$$

$$=\frac{1}{4}\cos(5A-3A)+\frac{3}{4}\cos(3A-A)=\cos 2A.$$

And

$$p\sin 3A-q\cos 3A=\frac{1}{4}(\cos 5A+3\cos A)\sin 3A-\frac{1}{4}(\sin 5A+3\sin A)\cos 3A$$

$$=\frac{1}{4}(\cos 5A\sin 3A-\sin 5A\cos 3A)+\frac{3}{4}(\sin 3A\cos A-\cos 3A\sin A)$$

$$=-\frac{1}{4}\sin(5A-3A)+\frac{3}{4}\sin(3A-A)=\frac{1}{2}\sin 2A.$$

162. Let $u=\left(\cos\frac{\alpha}{n}\right)^{\cot^2\frac{\beta}{n}}$; therefore

$$\log u=\cot^2\frac{\beta}{n}\log\cos\frac{\alpha}{n}=\cos^2\frac{\beta}{n}\times\operatorname{cosec}^2\frac{\beta}{n}\log\cos\frac{\alpha}{n}.$$

Now as in the solution of Example XII. 33 we can shew that

$$\operatorname{cosec}^2\frac{\beta}{n}\log\cos\frac{\alpha}{n}=-\frac{\alpha^2}{2\beta^2}\text{ when } n \text{ is infinite.}$$

And $\cos^2\frac{\beta}{n}=1$ when n is infinite.

Thus $\log u=-\frac{\alpha^2}{2\beta^2}$; and therefore $u=e^{-\frac{\alpha^2}{2\beta^2}}$.

163. If n be a positive integer, we have $1+2+2^2+\ldots+2^n=2^{n+1}-1$. Hence the infinite series

$$=2-1+\frac{2^2-1}{\lfloor 2}+\frac{2^3-1}{\lfloor 3}+\frac{2^4-1}{\lfloor 4}+\ldots$$

$$=2+\frac{2^2}{\lfloor 2}+\frac{2^3}{\lfloor 3}+\frac{2^4}{\lfloor 4}+\ldots-\left\{1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots\right\}$$

$$=e^2-1-\{e-1\}=e^2-e.$$

164. Here $$\sec\alpha\cos(x+y)=\frac{\cos(x-y)}{\cos x\cos y},$$

and $$\sec\beta\cos(x-y)=\frac{\cos(x+y)}{\cos x\cos y};$$

therefore, by division, $$\frac{\cos\beta}{\cos\alpha}\cdot\frac{\cos(x+y)}{\cos(x-y)}=\frac{\cos(x-y)}{\cos(x+y)},$$

so that $$\frac{\cos(x-y)}{\cos(x+y)}=\sqrt{\frac{\cos\beta}{\cos\alpha}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

And $$\cos(x-y)+\cos(x+y)=2\cos x\cos y=2\cos\alpha\frac{\cos(x-y)}{\cos(x+y)}$$

$$=2\sqrt{\cos\alpha\cos\beta}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) and (2) we have

$$\cos(x+y)\left\{\sqrt{\frac{\cos\beta}{\cos\alpha}}+1\right\}=2\sqrt{\cos\alpha\cos\beta};$$

therefore $$\cos(x+y)=\frac{2\cos\alpha\sqrt{\cos\beta}}{\sqrt{\cos\alpha}+\sqrt{\cos\beta}}.$$

Then by (1) we have $\cos(x-y)=\dfrac{2\cos\beta\sqrt{\cos\alpha}}{\sqrt{\cos\alpha}+\sqrt{\cos\beta}}$.

165. It may be shewn as in the solution of Example xx. 4 that

$$\frac{a\cos\frac{1}{2}(B-C)}{bc\cos\frac{1}{2}(B+C)}=\frac{b+c}{bc}=\frac{a(b+c)}{abc};$$

similarly $$\frac{b\cos\frac{1}{2}(C-A)}{ca\cos\frac{1}{2}(C+A)}=\frac{b(c+a)}{abc},\ \text{and}\ \frac{c\cos\frac{1}{2}(A-B)}{ab\cos\frac{1}{2}(A+B)}=\frac{c(a+b)}{abc}.$$

Hence by addition we obtain the required result.

166. $AP = OA \cos OAP, \quad AS = OA \cos OAS;$

therefore the area of the triangle $APS = \frac{1}{2} OA^2 \cos OAP \cos OAS \sin A.$

In the same way the area of the triangle OPS

$$= \frac{1}{2} OA^2 \sin OAP \sin OAS \sin POS = \frac{1}{2} OA^2 \sin OAP \sin OAS \sin (180^0 - A)$$

$$= \frac{1}{2} OA^2 \sin OAP \sin OAS \sin A.$$

Hence triangle APS – triangle OPS

$$= \frac{1}{2} OA^2 \sin A \{\cos OAP \cos OAS - \sin OAP \sin OAS\}$$

$$= \frac{1}{2} OA^2 \sin A \cos (OAP + OAS) = \frac{1}{2} OA^2 \sin A \cos A = \frac{1}{4} OA^2 \sin 2A.$$

In the same way we obtain

$$\text{triangle } BQP - \text{triangle } OQP = \frac{1}{4} OB^2 \sin 2B,$$

$$\text{triangle } CRQ - \text{triangle } ORQ = \frac{1}{4} OC^2 \sin 2C,$$

and $$\text{triangle } DSR - \text{triangle } OSR = \frac{1}{4} OD^2 \sin 2D.$$

Hence by addition we have

triangle APS + triangle BQP + triangle CRQ + triangle DSR – quadrilateral $PQRS$

$$= \frac{1}{4} \{OA^2 \sin 2A + OB^2 \sin 2B + OC^2 \sin 2C + OD^2 \sin 2D\}.$$

But the sum of the four triangles and the quadrilateral

= the quadrilateral $ABCD$.

Hence by subtraction we have

twice the quadrilateral $PQRS$ = the quadrilateral $ABCD$

$$- \frac{1}{4} \{OA^2 \sin 2A + OB^2 \sin 2B + OC^2 \sin 2C + OD^2 \sin 2D\}.$$

167. We have $a = 2R \sin A, \; b = 2R \sin B, \; c = 2R \sin C;$

thus the proposed expression

$$= 4R \left\{ \sin \frac{B-C}{2} \cos \frac{A}{2} + \sin \frac{C-A}{2} \cos \frac{B}{2} + \sin \frac{A-B}{2} \cos \frac{C}{2} \right\}$$

$$=4R\left\{\sin\frac{B-C}{2}\sin\frac{B+C}{2}+\sin\frac{C-A}{2}\sin\frac{C+A}{2}+\sin\frac{A-B}{2}\sin\frac{A+B}{2}\right\}$$

$$=4R\left\{\sin^2\frac{B}{2}-\sin^2\frac{C}{2}+\sin^2\frac{C}{2}-\sin^2\frac{A}{2}+\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right\}$$

$$=0.$$

168. Assume $x=\tan A$, $y=\tan B$, $z=\tan C$; then since $x+y+z=xyz$ it will follow in the manner of Art. 114 that $\tan(A+B+C)$ is zero; therefore $A+B+C=n\pi$ where n is zero or some integer. Therefore $3A+3B+3C=3n\pi$; and therefore in the manner of Art. 114 we have

$$\tan 3A+\tan 3B+\tan 3C=\tan 3A\tan 3B\tan 3C.$$

But $$\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}=\frac{3x-x^3}{1-3x^2};$$

similarly $$\tan 3B=\frac{3y-y^3}{1-3y^2},\quad \tan 3C=\frac{3z-z^3}{1-3z^2}:$$

thus the required result follows.

169. We have $l=R\cos A$, $m=R\cos B$, $n=R\cos C$; thus we have to shew that

$$\frac{4a}{R\cos A}+\frac{4b}{R\cos B}+\frac{4c}{R\cos C}=\frac{abc}{R^3\cos A\cos B\cos C}.$$

Now $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$; thus the proposed identity becomes

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C;$$

and this is true by Art. 114.

170. $$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=\frac{3}{2}-\frac{1}{2}(\cos A+\cos B+\cos C)$$

$$=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},\text{ by Art. 114.}$$

Now we have seen in Example XIII. 40 that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ cannot be greater than $\frac{1}{8}$; hence $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$ cannot be less than $1-\frac{1}{4}$, that is than $\frac{3}{4}$.

171. $\frac{\sin a\theta}{\sin b\theta}=\frac{a}{b}\frac{\sin a\theta}{a\theta}\cdot\frac{b\theta}{\sin b\theta}$; and when θ is indefinitely diminished the limit of $\frac{\sin a\theta}{a\theta}$ is unity, and so also is the limit of $\frac{b\theta}{\sin b\theta}$: thus the limit of $\frac{\sin a\theta}{\sin b\theta}$ is $\frac{a}{b}$.

Also $$\frac{\text{vers } a\theta}{\text{vers } b\theta} = \frac{1-\cos a\theta}{1-\cos b\theta} = \frac{\sin^2 \frac{a\theta}{2}}{\sin^2 \frac{b\theta}{2}} = \left\{\frac{\sin \frac{a\theta}{2}}{\sin \frac{b\theta}{2}}\right\}^2.$$

Now the limit of $\dfrac{\sin \frac{a\theta}{2}}{\sin \frac{b\theta}{2}}$ is $\dfrac{a}{b}$ in the manner just shewn; therefore the limit of $\dfrac{\text{vers } a\theta}{\text{vers } b\theta}$ is $\dfrac{a^2}{b^2}$.

172. $$\frac{1}{4}\left\{\frac{1}{1\,.\,1} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,5} + \frac{1}{4\,.\,7} + \dots\right\}$$
$$= \frac{1}{2}\left\{\frac{1}{1\,.\,2} + \frac{1}{3\,.\,4} + \frac{1}{5\,.\,6} + \frac{1}{7\,.\,8} + \dots\right\}$$
$$= \frac{1}{2}\log 2, \text{ by Art. 146, } = \log \sqrt{2}.$$

173. $$\tan\frac{A}{2} + \cos\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} + \frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$
$$= \frac{\cos^2\frac{A}{2} + \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}.$$

The numerator of this fraction $= 1 - \sin^2\dfrac{A}{2} + \sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

$$= 1 + \sin\frac{A}{2}\left\{\cos\frac{B}{2}\cos\frac{C}{2} - \cos\frac{B+C}{2}\right\} = 1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Thus the fraction $= \sec\dfrac{A}{2}\sec\dfrac{B}{2}\sec\dfrac{C}{2} + \tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}$.

Similarly the other two proposed expressions may be reduced to this symmetrical form; and thus the three expressions are equal.

174. $$\sin(\pi\cot\theta) = \cos(\pi\tan\theta);$$

therefore $$\cos(\pi\tan\theta) = \cos\left(\frac{\pi}{2} - \pi\cot\theta\right);$$

therefore all the solutions are comprised in

$$\pi\tan\theta = 2n\pi \pm \left(\frac{\pi}{2} - \pi\cot\theta\right),$$

where n is zero, or some integer, positive or negative.

Take the upper sign; thus $2n+\frac{1}{2}=\tan\theta+\cot\theta=\frac{1}{\sin\theta\cos\theta}$, so that $n+\frac{1}{4}=\frac{1}{\sin 2\theta}$.

Take the lower sign; thus $\frac{1}{2}-2n=\cot\theta-\tan\theta=2\cot 2\theta$, so that $\cot 2\theta=\frac{1}{4}-n$.

Thus either cosec 2θ or cot 2θ takes the prescribed form.

175. We have $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$.

Thus the left-hand member $=\left(\frac{1}{2R}\right)^2$.

And the right-hand member $=\frac{\sin 2A+\sin 2B+\sin 2C}{16R^4\sin A\sin B\sin C}=\frac{1}{4R^2}$, by Art. 114.

176. Let θ be the angle of the sector; then we see from a diagram that $\frac{r}{a-r}=\sin\frac{\theta}{2}$. But $2c=2a\sin\frac{\theta}{2}$. Therefore $\frac{r}{a-r}=\frac{c}{a}$; therefore $\frac{a-r}{r}=\frac{a}{c}$; therefore $\frac{1}{r}=\frac{1}{a}+\frac{1}{c}$.

177. $$\frac{\sin x}{\cos x}+\frac{\sin 4x}{\cos 4x}+\frac{\sin 2x}{\cos 2x}+\frac{\sin 3x}{\cos 3x}=0;$$

therefore $$\frac{\sin x\cos 4x+\cos x\sin 4x}{\cos x\cos 4x}+\frac{\sin 2x\cos 3x+\cos 2x\sin 3x}{\cos 2x\cos 3x}=0;$$

therefore $$\frac{\sin 5x}{\cos x\cos 4x}+\frac{\sin 5x}{\cos 2x\cos 3x}=0;$$

therefore either $\sin 5x=0$ or $\frac{1}{\cos x\cos 4x}+\frac{1}{\cos 2x\cos 3x}=0$.

If we take the former, then $5x=n\pi$.

If we take the latter, then $\cos 2x\cos 3x+\cos x\cos 4x=0$;

therefore $$\cos 2x\,(4\cos^3 x-3\cos x)+\cos x\cos 4x=0;$$

therefore either $\cos x=0$ or $\cos 2x\,(4\cos^2 x-3)+\cos 4x=0$.

If we take the former, then $x=(2m+1)\frac{\pi}{2}$.

If we take the latter, then $\cos 2x\,(2+2\cos 2x-3)+2\cos^2 2x-1=0$;

therefore $$4\cos^2 2x-\cos 2x-1=0;$$

and by solving this quadratic we obtain $\cos 2x=\frac{1\pm\sqrt{17}}{8}$.

178. We may proceed as in the solution of Example 147, and seek for the values of x, y, and z, which make

$$x \sin(\theta-\beta)\sin(\theta-\gamma)+y\sin(\theta-\gamma)\sin(\theta-\alpha)+z\sin(\theta-\alpha)\sin(\theta-\beta)$$

always equal to 1. Then we shall find that $x=\dfrac{1}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}$, and so on.

Or we may verify the formula by direct work. For reduce the three fractions to the common denominator $\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$. Then the numerator will become $L\sin^2\theta+M\sin\theta\cos\theta+N\cos^2\theta$, where

$$L=\cos\beta\cos\gamma\sin(\gamma-\beta)+\cos\gamma\cos\alpha\sin(\alpha-\gamma)+\cos\alpha\cos\beta\sin(\beta-\alpha),$$

$$M=-\sin(\gamma+\beta)\sin(\gamma-\beta)-\sin(\alpha+\gamma)\sin(\alpha-\gamma)-\sin(\beta+\alpha)\sin(\beta-\alpha),$$

$$N=\sin\beta\sin\gamma\sin(\gamma-\beta)+\sin\gamma\sin\alpha\sin(\alpha-\gamma)+\sin\alpha\sin\beta\sin(\beta-\alpha).$$

It is obvious by Art. 83 that $M=0$; and we have seen in the solution of Example 157 that L and N are each equal to the common denominator; so that $L\sin^2\theta+N\cos^2\theta$ is also equal to this denominator, and the expression is equal to unity.

179. By Euclid VI. 2 we find that $BD=\dfrac{ac}{b+c}$, and $CD=\dfrac{ab}{b+c}$.

Similar expressions hold for the segments of the other sides of ABC.

Therefore the area of the triangle DCE

$$=\frac{1}{2}\frac{ab}{b+c}\cdot\frac{ab}{a+c}\sin C=\frac{Sab}{(a+c)(b+c)}.$$

Similar expressions hold for the areas of EFA and FDB.

Therefore the area of DEF

$$=S\left\{1-\frac{ab}{(a+c)(b+c)}-\frac{bc}{(b+a)(c+a)}-\frac{ca}{(c+b)(a+b)}\right\}$$

$$=\frac{S}{(a+b)(b+c)(c+a)}\{(a+b)(b+c)(c+a)-ab(a+b)-bc(b+c)-ca(c+a)\}$$

$$=\frac{2abcS}{(a+b)(b+c)(c+a)}=2S\cdot\frac{a}{b+c}\cdot\frac{b}{c+a}\cdot\frac{c}{a+b}.$$

Now
$$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{B+C}{2}\cos\frac{B-C}{2}}=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}.$$

Similarly
$$\frac{b}{c+a}=\frac{\sin\frac{B}{2}}{\cos\frac{C-A}{2}},\quad\text{and}\quad\frac{c}{a+b}=\frac{\sin\frac{C}{2}}{\cos\frac{A-B}{2}}.$$

Thus the required result is obtained.

180. Let x denote the height of the mountain; then the distances of the two stations from the point in the horizontal plane which is vertically under the top of the mountain are $x \cot \alpha$ and $x \cot \beta$ respectively.

Thus $\quad c^2 = x^2 \cot^2 \alpha + x^2 \cot^2 \beta - 2x^2 \cot \alpha \cot \beta \cos \gamma$; (Art. 215)

therefore
$$x^2 = \frac{c^2}{\cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \gamma}$$
$$= \frac{c^2 \sin^2 \alpha \sin^2 \beta}{\sin^2 \beta \cos^2 \alpha + \sin^2 \alpha \cos^2 \beta - 2 \sin \alpha \cos \alpha \sin \beta \cos \beta \cos \gamma}.$$

The denominator of this fraction may be put in the form
$$(\sin \beta \cos \alpha + \cos \beta \sin \alpha)^2 - \sin 2\alpha \sin 2\beta \cos^2 \frac{\gamma}{2},$$
so that with the specified value of ϕ it becomes $\sin^2 (\alpha + \beta) \cos^2 \phi$;

and therefore
$$x = \frac{c \sin \alpha \sin \beta}{\sin (\alpha + \beta) \cos \phi}.$$

181. Let θ denote the angle; then $\tan \frac{\theta}{2} = \frac{1}{3 \times 450}$; therefore approximately $\frac{\theta}{2} = \frac{1}{1350}$; therefore $\theta = \frac{1}{675}$. Hence the number of degrees in the angle is $\frac{180}{\pi} \times \frac{1}{675}$, and the number of minutes is $\frac{180}{\pi} \times \frac{60}{675}$, that is $\frac{4}{45} \times \frac{180}{\pi}$, that is $\frac{4}{45} \times 57{\cdot}29\ldots$, that is about 5.

182. The general term of the series is $\frac{n^2}{\lfloor n+1}$; for we obtain all the terms by putting successively 1, 2, 3, ... for n in this expression.

Now
$$\frac{n^2}{\lfloor n+1} = \frac{n(n+1) - (n+1) + 1}{\lfloor n+1} = \frac{1}{\lfloor n-1} - \frac{1}{\lfloor n} + \frac{1}{\lfloor n+1}.$$

If then we split up each term into three in this manner, beginning with the second term, we obtain
$$\frac{1}{\lfloor 2} + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots\ldots$$
$$- \left\{ \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 5} + \ldots\ldots \right\}$$
$$+ \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 5} + \ldots\ldots;$$

that is $\frac{1}{2} + e - 1 - (e - 2) + e - 2 - \frac{1}{2}$, that is $e - 1$.

183. Here

$$\frac{2}{\cos\theta} = \frac{1}{\cos(\theta+2\alpha)} + \frac{1}{\cos(\theta-2\alpha)} = \frac{2\cos\theta\cos 2\alpha}{\cos(\theta+2\alpha)\cos(\theta-2\alpha)} = \frac{2\cos\theta\cos 2\alpha}{\cos^2\theta - \sin^2 2\alpha};$$

therefore $$\cos^2\theta - \sin^2 2\alpha = \cos^2\theta\cos 2\alpha,$$

therefore $$\cos^2\theta(1-\cos 2\alpha) = \sin^2 2\alpha = 4\sin^2\alpha\cos^2\alpha;$$

therefore $$\cos^2\theta = 2\cos^2\alpha.$$

184. Here $4\sin(\theta+\phi)\cos(\theta-\phi)=1$, and $2\sin(\theta+\phi)=1$;

therefore $$\sin(\theta+\phi)=\frac{1}{2}, \quad \text{and } \cos(\theta-\phi)=\frac{1}{2};$$

therefore $$\theta+\phi = n\pi + (-1)^n\frac{\pi}{6}, \quad \text{and } \theta-\phi = 2m\pi \pm \frac{\pi}{3}.$$

185. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, by Example VIII. 16.

And $$\tan\frac{A}{2} + \tan\frac{B}{2} = \frac{\sin\frac{A+B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}};$$

therefore $$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}$$

$$= \frac{\cos^2\frac{C}{2} + \sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}};$$

the numerator $= 1 - \sin^2\frac{C}{2} + \sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$

$$= 1 + \sin\frac{C}{2}\left\{\cos\frac{A}{2}\cos\frac{B}{2} - \cos\frac{A+B}{2}\right\} = 1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2};$$

and thus the fraction $= \dfrac{1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$.

Hence by multiplication we obtain the required result.

186. Proceed as in the solution of Example 166. Then we obtain the following expression for the excess of the sum of all the triangles at the corners above the second polygon

$$\frac{r^2}{4}\{\sin 2A + \sin 2B + \sin 2C + \sin 2D + \ldots\},$$

where r is the radius of the circle.

Hence this vanishes if $\sin 2A + \sin 2B + \ldots = 0$, and then the sum of the triangles at the corners is equal to the second polygon, and therefore the first polygon is double the second.

187. We have $a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$; hence the proposed expression

$$= 4R\left\{\sin\frac{B-C}{2}\sin\frac{A}{2} + \sin\frac{C-A}{2}\sin\frac{B}{2} + \sin\frac{A-B}{2}\sin\frac{C}{2}\right\}$$

$$= 2R\left\{\cos\frac{A+C-B}{2} - \cos\frac{A+B-C}{2} + \cos\frac{B+A-C}{2} - \cos\frac{B+C-A}{2}\right.$$

$$\left. + \cos\frac{B+C-A}{2} - \cos\frac{A+C-B}{2}\right\} = 0.$$

188. We have $a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$; hence the proposed expression

$$= 4R^2\left\{\frac{\sin^2 A\sin(B-C)}{\sin B + \sin C} + \frac{\sin^2 B\sin(C-A)}{\sin C + \sin A} + \frac{\sin^2 C\sin(A-B)}{\sin A + \sin B}\right\}.$$

Now $$\frac{\sin^2 A\sin(B-C)}{\sin B + \sin C} = \frac{\sin A\sin(B+C)\sin(B-C)}{\sin B + \sin C}$$

$$= \frac{\sin A(\sin^2 B - \sin^2 C)}{\sin B + \sin C} = \sin A(\sin B - \sin C).$$

In this way the proposed expression

$$= 4R^2\{\sin A(\sin B - \sin C) + \sin B(\sin C - \sin A) + \sin C(\sin A - \sin B)\} = 0.$$

189. If n be the number of sides in the first polygon we have

$$a = 2r\sin\frac{\pi}{n},\quad b = 2r\sin\frac{\pi}{2n}.$$

By Art. 100, since $\frac{\pi}{n}$ lies between 0 and $\frac{\pi}{2}$, we have

$$2\sin\frac{\pi}{2n} = \sqrt{\left(1 + \sin\frac{\pi}{n}\right)} - \sqrt{\left(1 - \sin\frac{\pi}{n}\right)};$$

therefore $$\frac{b}{r} = \sqrt{\left(1 + \frac{a}{2r}\right)} - \sqrt{\left(1 - \frac{a}{2r}\right)}.$$

Multiply by r, and we obtain the required result.

190. $\left(1-\sin\frac{B}{2}\right)\left(1-\sin\frac{C}{2}\right)\cos\frac{A}{2}$

$$=\cos\frac{A}{2}-\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\cos\frac{A}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}.$$

Develope each of the other two terms in the same way; the aggregate

$$=\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}-\sin\frac{A+B}{2}-\sin\frac{B+C}{2}-\sin\frac{C+A}{2}$$

$$+\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}+\sin\frac{C}{2}\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$$

But $\cos\frac{A}{2}=\sin\frac{B+C}{2}$, $\cos\frac{B}{2}=\sin\frac{C+A}{2}$, $\cos\frac{C}{2}=\sin\frac{A+B}{2}$;

thus the expression

$$=\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}+\sin\frac{C}{2}\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$

$$=\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}+\sin\frac{A}{2}\left(\sin\frac{C}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}\right)$$

$$=\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}+\sin\frac{A}{2}\sin\frac{B+C}{2}$$

$$=\cos\frac{A}{2}\left\{\sin\frac{A}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\right\}=\cos\frac{A}{2}\left\{\cos\frac{B+C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\right\}$$

$$=\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Or instead of the last four lines we may use Art. 113, observing that here $\cos\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)=0$.

191. Let D denote the top of the object.

From the triangle ABD we have $\frac{AB}{BD}=1$, for the angles BAD and BDA are equal. From the triangle BDC we have $\frac{BD}{BC}=\frac{\sin 3\alpha}{\sin\alpha}$.

Therefore $$\frac{AB}{BC}=\frac{\sin 3\alpha}{\sin\alpha}=3-4\sin^2\alpha.$$

Since the object is very distant α is very small; therefore $AB=3BC$ nearly.

192. $$\frac{1}{x}+\frac{1}{\log_e(1-x)}=\frac{1}{x}-\frac{1}{x+\frac{1}{2}x^2+\frac{1}{3}x^3+\dots}=\frac{\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\dots}{x^2+\frac{1}{2}x^3+\frac{1}{3}x^4+\dots}.$$

Here every term in the numerator is less than the corresponding term in the denominator, and thus the fraction is less than unity.

193. Here $$\frac{6 \sin B}{\cos (A+B)} = \frac{6 \sin B \cos B}{\cos (A+2B)};$$

thus either $\sin B = 0$ or $\cos (A+2B) = \cos (A+B) \cos B$;

the latter gives $\cos (A+B) \cos B - \sin (A+B) \sin B = \cos (A+B) \cos B$,

so that either $\sin B = 0$ or $\sin (A+B) = 0$.

Suppose that $\sin (A+B) = 0$; then since

$$\frac{3 \sin 2B}{\cos (A+B+B)} = \frac{2 \sin 3B}{\cos (A+B+2B)},$$

we have $$\frac{3 \sin 2B}{\cos (A+B) \cos B} = \frac{2 \sin 3B}{\cos (A+B) \cos 2B},$$

so that $$\frac{3 \sin B}{\cos (A+B)} = \frac{\sin 3B}{\cos (A+B) \cos 2B};$$

therefore $$3 \sin B \cos 2B = \sin 3B,$$

therefore $$3 \sin B (1 - 2 \sin^2 B) = 3 \sin B - 4 \sin^3 B,$$

therefore $$6 \sin^3 B = 4 \sin^3 B,$$

therefore $$\sin B = 0.$$

194. Here $\tan \theta = \frac{x}{y}$; therefore $\sin^2 \theta = \frac{x^2}{x^2+y^2}$, and $\cos^2 \theta = \frac{y^2}{x^2+y^2}$. Substitute in the second given equation; thus

$$\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) \frac{1}{x^2+y^2} = \frac{6}{x^2+y^2}; \text{ therefore } \frac{x^2}{y^2} + \frac{y^2}{x^2} = 6.$$

From this quadratic in $\frac{x^2}{y^2}$ we find $\frac{x^2}{y^2} = 3 \pm 2\sqrt{2} = (\sqrt{2} \pm 1)^2$; therefore $\tan \theta = \pm (\sqrt{2}+1)$ or $\pm (\sqrt{2}-1)$. The former gives $\theta = n\pi \pm \frac{3\pi}{8}$; and the latter gives $\theta = n\pi \pm \frac{\pi}{8}$.

195. Since the sines of the angles are in Harmonical Progression, so are the sides of the opposite angles. Thus a, b, c are in Harmonical Progression, and we have to shew that $\frac{(s-b)(s-c)}{bc}$, $\frac{(s-c)(s-a)}{ca}$, $\frac{(s-a)(s-b)}{ab}$ are so also. Multiply each term by $\frac{abc}{(s-a)(s-b)(s-c)}$; thus we see it is sufficient to shew that $\frac{a}{s-a}$, $\frac{b}{s-b}$, $\frac{c}{s-c}$ are in Harmonical Progression, or that $\frac{s-a}{a}$, $\frac{s-b}{b}$, $\frac{s-c}{c}$ are in Arithmetical Progression, or that $\frac{s}{a}$, $\frac{s}{b}$, $\frac{s}{c}$ are in Arithmetical Progression; and this is the case since a, b, c are in Harmonical Progression.

196. We have $$a=2R\sin\frac{\pi}{5};$$

therefore $$\frac{R}{a}=\frac{1}{2\sin\frac{\pi}{5}}=\frac{2}{\sqrt{10-2\sqrt{5}}}=\frac{2\sqrt{10+2\sqrt{5}}}{\sqrt{80}}=\frac{2\sqrt{200+40\sqrt{5}}}{\sqrt{80\times 20}}$$

$$=\frac{\sqrt{200+40\sqrt{5}}}{20}=\frac{\sqrt{289\cdot 44\,..}}{20}=\frac{17}{20}\text{ nearly.}$$

197. Here $$\frac{\sin(B+C)}{\sin B}=\frac{m}{n},\quad \frac{\cos(B+C)}{\cos B}=-\frac{p}{q};$$

therefore $$\cos C+\cot B\sin C=\frac{m}{n},\quad \tan B\sin C-\cos C=\frac{p}{q};$$

therefore $$\sin^2 C=\left(\frac{m}{n}-\cos C\right)\left(\frac{p}{q}+\cos C\right);$$

therefore $$1=\frac{mp}{nq}+\cos C\left(\frac{m}{n}-\frac{p}{q}\right);$$

therefore $$\cos C=\frac{mp-nq}{np-mq}.$$

198. We have $$OA=\frac{r}{\sin\frac{A}{2}},\quad OB=\frac{r}{\sin\frac{B}{2}},\quad OC=\frac{r}{\sin\frac{C}{2}};$$

$$AF=r\cot\frac{A}{2},\quad BD=r\cot\frac{B}{2},\quad CE=r\cot\frac{C}{2}.$$

Hence we have to shew that

$$\frac{r^4}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left\{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right\}=4Rr^3\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

By Example VIII. 15 the left-hand member

$$=\frac{r^4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}};$$

thus we have to shew that

$$4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C=r.$$

The left-hand member

$$=4R\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}=\frac{4RS^2}{sabc}=\frac{S}{s}=r.$$

199. The radius of the circle inscribed in the triangle OBC

$$=\frac{\text{area of the triangle}}{\text{semiperimeter}}=\frac{\frac{1}{2}\rho^2\sin 2\alpha}{\rho(1+\sin\alpha)}=\frac{\rho\sin 2\alpha}{2(1+\sin\alpha)}.$$

Let P denote the position of the centre; then

$$OP=\frac{\rho\sin 2\alpha}{2(1+\sin\alpha)}\times\frac{1}{\sin\alpha}=\frac{\rho\cos\alpha}{1+\sin\alpha}.$$

Again, let Q denote the position of the centre of the circle inscribed in the triangle OAB; then, as 2α is now to be changed to $\pi-2\alpha$, we have

$$OQ=\frac{\rho\cos\left(\frac{\pi}{2}-\alpha\right)}{1+\sin\left(\frac{\pi}{2}-\alpha\right)}=\frac{\rho\sin\alpha}{1+\cos\alpha}.$$

And since POQ is a right angle, $PQ^2=OP^2+OQ^2$

$$=\rho^2\left\{\frac{\cos^2\alpha}{(1+\sin\alpha)^2}+\frac{\sin^2\alpha}{(1+\cos\alpha)^2}\right\}$$

$$=\rho^2\left\{\frac{1-\sin\alpha}{1+\sin\alpha}+\frac{1-\cos\alpha}{1+\cos\alpha}\right\}=\frac{\rho^2(2-\sin 2\alpha)}{(1+\sin\alpha)(1+\cos\alpha)};$$

therefore $$PQ=\frac{\rho\sqrt{2-\sin 2\alpha}}{\sqrt{(1+\sin\alpha)(1+\cos\alpha)}}.$$

200. Let A and B be the two objects. Suppose a circle to pass through A and B, and to *touch* the straight line at P; then P is the point at which the greatest angle is subtended: see *Appendix to Euclid*, page 308. Produce AB to meet the straight line at Q. Let the angle $BPQ=\alpha$, and let β be the angle between AP and the straight line. Then also $PAB=\alpha$, and $PBA=\beta$, by Euclid III. 32. Let $PQ=c$.

Then $$\frac{BP}{PQ}=\frac{\sin(\beta-\alpha)}{\sin\beta},\quad\frac{AB}{BP}=\frac{\sin(\beta+\alpha)}{\sin\alpha};$$

therefore $$\frac{AB}{c}=\frac{\sin(\beta+\alpha)\sin(\beta-\alpha)}{\sin\alpha\sin\beta}.$$

201. We have

$$r_1+r_2+r_3-r=S\left\{\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right\}$$

$$=S\left\{\frac{2s-a-b}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right\}=cS\left\{\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right\}$$

$$=\frac{cS}{S^2}\{s(s-c)+(s-a)(s-b)\}=\frac{c}{S}\{2s^2-s(a+b+c)+ab\}$$

$$=\frac{abc}{S}=4R.$$

202. $$\sin^{-1}\frac{1}{\sqrt{5}}=\tan^{-1}\frac{1}{2},\quad \cot^{-1}3=\tan^{-1}\frac{1}{3};$$

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=\tan^{-1}1=\frac{\pi}{4}.$$

203. The angle of the second triangle which is opposite to the angle C of the first triangle will be found to be $\frac{\pi}{2}-\frac{C}{2}$; similarly the corresponding angle of the third triangle will be $\frac{\pi}{2}-\frac{1}{2}\left(\frac{\pi}{2}-\frac{C}{2}\right)$, that is $\frac{\pi}{2}-\frac{\pi}{4}+\frac{C}{4}$. Proceeding in this way we find that the corresponding angle of the n^{th} triangle is

$$\pi\left\{\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\dots-\frac{(-1)^{n-1}}{2^{n-1}}\right\}+\frac{(-1)^{n-1}C}{2^{n-1}},$$

that is $$\frac{\pi}{2}\frac{1-\left(-\frac{1}{2}\right)^{n-1}}{1+\frac{1}{2}}+\frac{(-1)^{n-1}C}{2^{n-1}};$$

that is $$\frac{\pi}{3}\left\{1-\frac{(-1)^{n-1}}{2^{n-1}}\right\}+\frac{(-1)^{n-1}C}{2^{n-1}}.$$

Similar expressions hold for the other angles.

204. Suppose $\theta=\tan^{-1}a$; then we require $\cos 4\theta$.

Now $$\tan\theta=a,\quad \cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-a^2}{1+a^2},$$

$$\cos 4\theta=2\cos^2 2\theta-1=\frac{2(1-a^2)^2}{(1+a^2)^2}-1=\frac{1-6a^2+a^4}{(1+a^2)^2}.$$

205. We have $e^{ax}\cos(bx+c)=e^{ax}(\cos bx\cos c-\sin bx\sin c)$; then by Art. 300 the required general term is

$$\frac{(a^2+b^2)^{\frac{n}{2}}}{\underline{|n}}(\cos n\theta\cos c-\sin n\theta\sin c),$$

that is $$\frac{(a^2+b^2)^{\frac{n}{2}}}{\underline{|n}}\cos(n\theta+c),$$

where θ is such that $\tan\theta=\frac{b}{a}$.

206. We have $AD^2 = AB^2 + BD^2 - 2AB \cdot BD \cos B$

$$= c^2 + (s-c)^2 - 2c(s-c)\cos B, \text{ by Art. 250};$$

therefore $$s^2 - AD^2 = s^2 - c^2 - (s-c)^2 + 2c(s-c)\cos B$$

$$= (s-c)\{s + c - (s-c) + 2c\cos B\}$$

$$= 2c(s-c)(1+\cos B) = 4c(s-c)\cos^2\frac{B}{2};$$

therefore $$a(s^2 - AD^2) = 4ac(s-c)\cos^2\frac{B}{2} = 4s(s-b)(s-c).$$

207. We have

$$\alpha + \beta\sqrt{-1} = \cos(\theta + \phi\sqrt{-1}) = \cos\theta\cos\phi\sqrt{-1} - \sin\theta\sin\phi\sqrt{-1}$$

$$= \cos\theta\frac{e^{-\phi} + e^{\phi}}{2} - \sin\theta\frac{e^{-\phi} - e^{\phi}}{2\sqrt{-1}}$$

$$= \cos\theta\frac{e^{\phi} + e^{-\phi}}{2} - \sin\theta\frac{e^{\phi} - e^{-\phi}}{2}\sqrt{-1}.$$

Hence by equating the possible and the impossible parts we have

$$\alpha = \cos\theta\frac{e^{\phi} + e^{-\phi}}{2}, \quad \beta = -\sin\theta\frac{e^{\phi} - e^{-\phi}}{2}.$$

Therefore $$\frac{\alpha^2}{\cos^2\theta} - \frac{\beta^2}{\sin^2\theta} = \left(\frac{e^{\phi} + e^{-\phi}}{2}\right)^2 - \left(\frac{e^{\phi} - e^{-\phi}}{2}\right)^2 = 1;$$

and $$\frac{\alpha^2}{(e^{\phi} + e^{-\phi})^2} + \frac{\beta^2}{(e^{\phi} - e^{-\phi})^2} = \frac{\cos^2\theta + \sin^2\theta}{4} = \frac{1}{4}.$$

208. $$\log\sec\theta = \frac{1}{2}\log\frac{1}{\cos^2\theta} = \frac{1}{2}\log\frac{2}{1+\cos 2\theta}$$

$$= \frac{1}{2}\log\frac{4}{2 + e^{2\iota\theta} + e^{-2\iota\theta}} = \frac{1}{2}\log\frac{4}{(1+e^{2\iota\theta})(1+e^{-2\iota\theta})}$$

$$= \frac{1}{2}\{2\log 2 - \log(1+e^{2\iota\theta}) - \log(1+e^{-2\iota\theta})\};$$

therefore $$2\log\sec\theta = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right)$$

$$- \left(e^{2\iota\theta} - \frac{1}{2}e^{4\iota\theta} + \frac{1}{3}e^{6\iota\theta} - \frac{1}{4}e^{8\iota\theta} + \dots\right)$$

$$- \left(e^{-2\iota\theta} - \frac{1}{2}e^{-4\iota\theta} + \frac{1}{3}e^{-6\iota\theta} - \frac{1}{4}e^{-8\iota\theta} + \dots\right).$$

Now $$2 - e^{2\iota\theta} - e^{-2\iota\theta} = -(e^{\iota\theta} - e^{-\iota\theta})^2 = 4\sin^2\theta;$$

$$\frac{1}{2}(-2 + e^{4\iota\theta} + e^{-4\iota\theta}) = \frac{1}{2}(e^{2\iota\theta} - e^{-2\iota\theta})^2 = -\frac{4}{2}\sin^2 2\theta,$$

$$\frac{1}{3}(2 - e^{6\iota\theta} - e^{-6\iota\theta}) = -\frac{1}{3}(e^{3\iota\theta} - e^{-3\iota\theta})^2 = \frac{4}{3}\sin^2 3\theta,$$

and so on; thus

$$2\log\sec\theta = 4\left\{\sin^2\theta - \frac{1}{2}\sin^2 2\theta + \frac{1}{3}\sin^2 3\theta - \dots\right\};$$

therefore $$\log\sec\theta = 2\left\{\sin^2\theta - \frac{1}{2}\sin^2 2\theta + \frac{1}{3}\sin^2 3\theta - \dots\right\}.$$

209. $$\sec\alpha\sec(\alpha+\beta) = \frac{1}{\sin\beta}\{\tan(\alpha+\beta) - \tan\alpha\},$$

$$\sec(\alpha+\beta)\sec(\alpha+2\beta) = \frac{1}{\sin\beta}\{\tan(\alpha+2\beta) - \tan(\alpha+\beta)\},$$

and so on.

Then by addition we obtain the required result.

210. The regular hexagon may be divided into six equilateral triangles; and thus the area of the first hexagon $= \frac{6a^2\sqrt{3}}{4}$.

By Art. 255 the radius of the first circle $= \frac{a}{2}\cot 30^0 = \frac{a\sqrt{3}}{2}$; and the side of the second hexagon is equal to this, so that the area of the second hexagon $= \frac{6a^2\sqrt{3}}{4}\left(\frac{\sqrt{3}}{2}\right)^2$. In this way we see that the areas of the hexagons form a geometrical progression of which the ratio is $\frac{3}{4}$; and the sum of

the areas $= \frac{6a^2\sqrt{3}}{4}\,\frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}} = 6a^2\sqrt{3}\left\{1-\left(\frac{3}{4}\right)^n\right\}.$

211. We have $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$; thus the proposed expression

$$= 2R(\sin A\cos A + \sin B\cos B + \sin C\cos C)$$
$$= R(\sin 2A + \sin 2B + \sin 2C)$$
$$= 4R\sin A\sin B\sin C, \text{ by Art. 114},$$
$$= 2a\sin B\sin C.$$

The expression is now adapted to logarithms.

212. Let θ denote the angle APB, ϕ the angle BPC, and ψ the angle ABP.

We have $$\frac{AB}{PB}=\frac{\sin\theta}{\sin(\psi+\theta)},\quad \frac{BC}{PB}=\frac{\sin\phi}{\sin(\psi-\phi)};$$

but AB is supposed equal to BC, and thus

$$\frac{\sin(\psi+\theta)}{\sin\theta}=\frac{\sin(\psi-\phi)}{\sin\phi};$$

therefore $$\sin\psi\cot\theta+\cos\psi=\sin\psi\cot\phi-\cos\psi;$$

therefore $$2\cot\psi=\cot\phi-\cot\theta;$$

therefore $$\frac{2}{T}=\frac{1}{t'}-\frac{1}{t}.$$

213. It is shewn in Art. 296 that $\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$;

hence we have only to shew that

$$\tan^{-1}\frac{1}{239}=2\tan^{-1}\frac{1}{408}-\tan^{-1}\frac{1}{1393},$$

or that $$\tan^{-1}\frac{1}{239}+\tan^{-1}\frac{1}{1393}=2\tan^{-1}\frac{1}{408}.$$

Now $$\tan^{-1}\frac{1}{239}+\tan^{-1}\frac{1}{1393}=\tan^{-1}\frac{\frac{1}{239}+\frac{1}{1393}}{1-\frac{1}{239\times1393}}$$

$$=\tan^{-1}\frac{1393+239}{239\times1393-1}=\tan^{-1}\frac{1632}{332926}=\tan^{-1}\frac{816}{166463};$$

and $$2\tan^{-1}\frac{1}{408}=\tan^{-1}\frac{\frac{2}{408}}{1-\left(\frac{1}{408}\right)^2}=\tan^{-1}\frac{2\times408}{166463}.$$

Thus the required result is established.

214. By the diagram of Art. 332 we see that

$$\frac{PA}{AB}=\frac{\sin PBA}{\sin APB}=\frac{\sin\left(\frac{\pi}{2}-A\right)}{\sin(A+B)}=\frac{\cos A}{\sin C};$$

therefore $$PA=\frac{c\cos A}{\sin C}=\frac{a\cos A}{\sin A};$$

therefore $$PA^2=\frac{a^2(1-\sin^2A)}{\sin^2A}=\frac{a^2}{\sin^2A}-a^2=4R^2-a^2.$$

215. $\dfrac{(\cos a + \sqrt{-1}\sin a)(\cos 2a + \sqrt{-1}\sin 2a)}{\cos 3a - \sqrt{-1}\sin 3a} = \dfrac{\cos 3a + \sqrt{-1}\sin 3a}{\cos 3a - \sqrt{-1}\sin 3a}$;

multiply both numerator and denominator by $\cos 3a + \sqrt{-1}\sin 3a$; thus we obtain unity in the denominator, and $\cos 6a + \sqrt{-1}\sin 6a$ in the numerator: and this numerator $= \sqrt{-1}$ since $a = 15^0$.

216. The new triangle will have for its angular points the centres of the escribed circles of the original triangle. Now from Art. 250 we have

$$OC = CE \sec OCE = (s-b)\operatorname{cosec}\frac{C}{2};$$

and in the same manner the distance from C of the centre of the circle which touches BC and BA produced $= (s-a)\operatorname{cosec}\frac{C}{2}$. Hence the sum of these two $= (2s-b-a)\operatorname{cosec}\frac{C}{2} = c\operatorname{cosec}\frac{C}{2} = 2R\sin C\operatorname{cosec}\frac{C}{2} = 4R\cos\frac{C}{2}$.

This is the length of the side of the second triangle which passes through the point C; similar expressions hold for the other two sides.

217. By the preceding Example the sum of the squares

$$= 16R^2\left\{\cos^2\frac{1}{2}A + \cos^2\frac{1}{2}B + \cos^2\frac{1}{2}C\right\}$$

$$= 8R^2\{3 + \cos A + \cos B + \cos C\}$$

$$= 8R^2\left\{4 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right\}, \text{ by Art. 114,}$$

$$= 32R^2 + \frac{32R^2S^2}{sabc} = 32R^2 + 8Rr.$$

218. The numerator can be expressed in powers of θ; and it will be found to reduce to $2^5\cos^6\theta$; in like manner the denominator will be found to reduce to $2^4\cos^5\theta$: see Art. 280. Hence the expression reduces to $2\cos\theta$.

219. $\cos\theta$ is less than $1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}$, that is less than $\left(1 - \frac{\theta^2}{4}\right)^2$, therefore $\sqrt{\cos\theta}$ is less than $1 - \frac{\theta^2}{4}$; this holds if θ lies between 0 and $\frac{\pi}{2}$: see Art. 328. Again, $\cos\frac{\theta}{\sqrt{2}}$ is greater than $1 - \frac{1}{2}\left(\frac{\theta}{\sqrt{2}}\right)^2$, that is greater than $1 - \frac{\theta^2}{4}$; this holds as long as $\cos\frac{\theta}{\sqrt{2}}$ and $1 - \frac{\theta^2}{4}$ remain both positive, and this certainly holds if θ lies between 0 and $\frac{\pi}{2}$. Hence $\sqrt{\cos\theta}$ is less than $\cos\frac{\theta}{\sqrt{2}}$.

220. Suppose the polygon has n sides. Let O be the centre of the circle inscribed in the polygon, and S the assumed point. Let $OS=c$; and suppose OS to be inclined at an angle α to the first perpendicular which is drawn; put β for $\frac{2\pi}{n}$, and r for the radius of the inscribed circle. Then the length of the first perpendicular will be $r+c\cos\alpha$, that of the second $r+c\cos(\alpha+\beta)$, that of the third $r+c\cos(\alpha+2\beta)$, and so on. Hence the sum of one set of perpendiculars

$$=\frac{nr}{2}+c\left\{\cos\alpha+\cos(\alpha+2\beta)+\cos(\alpha+4\beta)+\ldots\text{ to }\frac{n}{2}\text{ terms}\right\}.$$

By Art. 304 the sum of the series of cosines contains the factor $\sin\frac{n}{2}\beta$, that is $\sin\pi$, that is zero.

Hence the sum of the set of perpendiculars $=\frac{nr}{2}$.

Similarly the sum of the other set of perpendiculars has the same value.

221. $$r=\frac{S}{s},\quad r_1=\frac{S}{s-a},\quad r_2=\frac{S}{s-b},\quad r_3=\frac{S}{s-c};$$

therefore $$rr_1r_2r_3=\frac{S^4}{s(s-a)(s-b)(s-c)}=\frac{S^4}{S^2}=S^2;$$

therefore $$\sqrt{rr_1r_2r_3}=S.$$

222. We have $$2\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{\frac{2}{7}}{1-\frac{1}{49}}=\tan^{-1}\frac{7}{24};$$

thus $$4\tan^{-1}\frac{1}{7}=2\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{2\times\frac{7}{24}}{1-\left(\frac{7}{24}\right)^2}=\tan^{-1}\frac{2\times7\times24}{(24+7)(24-7)}$$

$$=\tan^{-1}\frac{336}{527}.$$

Then $$5\tan^{-1}\frac{1}{7}=4\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{336}{527}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{336}{527}+\frac{1}{7}}{1-\frac{336}{7\times527}}=\tan^{-1}\frac{2879}{3353}.$$

Again $$2\tan^{-1}\frac{3}{79}=\tan^{-1}\frac{2\times\frac{3}{79}}{1-\left(\frac{3}{79}\right)^2}=\tan^{-1}\frac{2\times 3\times 79}{(79+3)(79-3)}$$

$$=\tan^{-1}\frac{237}{3116}.$$

Finally $$\tan\left(\frac{\pi}{4}-\tan^{-1}\frac{2879}{3353}\right)=\frac{1-\frac{2879}{3353}}{1+\frac{2879}{3353}}=\frac{474}{6232}=\frac{237}{3116};$$

so that $$\frac{\pi}{4}-\tan^{-1}\frac{2879}{3353}=\tan^{-1}\frac{237}{3116}.$$

223. In the expression for $\tan n\theta$ put $\frac{\pi}{4}$ for θ; then $\tan\theta=1$.

If n is an odd number we have $\tan n\theta=(-1)^{\frac{n-1}{2}}$, so that the numerator of the expression is numerically equal to the denominator.

If n is an even number, $\tan n\theta$ is either zero or infinite; so that in the former case the numerator of the expression must vanish, and in the latter case the denominator must vanish.

224. We have $\sin^4\theta\cos^5\theta=(1-\cos^2\theta)^2\cos^5\theta=\cos^9\theta-2\cos^7\theta+\cos^5\theta.$

Now $$\cos^9\theta=\frac{1}{2^8}\left\{\cos 9\theta+9\cos 7\theta+36\cos 5\theta+84\cos 3\theta+126\cos\theta\right\},$$

$$\cos^7\theta=\frac{1}{2^6}\left\{\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta\right\},$$

$$\cos^5\theta=\frac{1}{2^4}\left\{\cos 5\theta+5\cos 3\theta+10\cos\theta\right\}.$$

Hence

$$\cos^9\theta-2\cos^7\theta+\cos^5\theta=\frac{1}{256}(\cos 9\theta+\cos 7\theta)-\frac{1}{64}(\cos 5\theta+\cos 3\theta)+\frac{3}{128}\cos\theta.$$

Or we may proceed thus:

$$\sin^4\theta\cos^5\theta=\sin^4\theta\cos^4\theta\sin\theta=\frac{1}{16}(\sin 2\theta)^4\cos\theta$$

$$=\frac{1}{16}\left\{\frac{1}{8}\cos 8\theta-\frac{1}{2}\cos 4\theta+\frac{3}{8}\right\}\cos\theta$$

$$=\frac{1}{256}(\cos 9\theta+\cos 7\theta)-\frac{1}{64}(\cos 5\theta+\cos 3\theta)+\frac{3}{128}\cos\theta.$$

225. We have

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$= \frac{3}{4}\left\{x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \dots + \frac{(-1)^n x^{2n+1}}{\underline{|2n+1}} + \dots\right\}$$

$$- \frac{1}{4}\left\{3x - \frac{(3x)^3}{\underline{|3}} + \frac{(3x)^5}{\underline{|5}} - \frac{(3x)^7}{\underline{|7}} + \dots + \frac{(-1)^n (3x)^{2n+1}}{\underline{|2n+1}} + \dots\right\};$$

then by arranging according to powers of x we obtain the required result.

226. Put s for $\cos\phi + \cos 3\phi + \cos 9\phi$, and t for $\cos 5\phi + \cos 7\phi + \cos 11\phi$.

Then $$s + t = \cos\phi + \cos 3\phi + \cos 5\phi + \cos 7\phi + \cos 9\phi + \cos 11\phi$$

$$= \frac{\cos(\phi + 5\phi)\sin 6\phi}{\sin\phi} \text{ (Art. 304) } = \frac{\sin 12\phi}{2\sin\phi} = \frac{\sin\phi}{2\sin\phi} = \frac{1}{2}.$$

And $$st = (\cos\phi + \cos 3\phi + \cos 9\phi)(\cos 5\phi + \cos 7\phi + \cos 11\phi)$$

$$= \cos\phi(\cos 5\phi + \cos 7\phi + \cos 11\phi) + \dots\dots$$

Resolve each product into the sum of two cosines by Art. 84; thus we get

$$2st = \cos 6\phi + \cos 4\phi + \cos 8\phi + \cos 6\phi + \cos 12\phi + \cos 10\phi$$
$$+ \cos 8\phi + \cos 2\phi + \cos 10\phi + \cos 4\phi + \cos 14\phi + \cos 8\phi$$
$$+ \cos 14\phi + \cos 4\phi + \cos 16\phi + \cos 2\phi + \cos 20\phi + \cos 2\phi$$
$$= 3\cos 2\phi + 3\cos 4\phi + 2\cos 6\phi + 3\cos 8\phi + 2\cos 10\phi$$
$$+ \cos 12\phi + 2\cos 14\phi + \cos 16\phi + \cos 20\phi.$$

Now since $\phi = \frac{\pi}{13}$ we have $\cos 14\phi = \cos 12\phi$, $\cos 20\phi = -\cos 7\phi = \cos 6\phi$, $\cos 16\phi = -\cos 3\phi = \cos 10\phi$. Thus

$$2st = 3\{\cos 2\phi + \cos 4\phi + \cos 6\phi + \cos 8\phi + \cos 10\phi + \cos 12\phi\}$$

$$= \frac{3\cos(2\phi + 5\phi)\sin 6\phi}{\sin\phi} = -\frac{3\cos 6\phi\sin 6\phi}{\sin\phi} = -\frac{3\sin 12\phi}{2\sin\phi} = -\frac{3}{2};$$

therefore $$st = -\frac{3}{4}.$$

Then, since $s + t = \frac{1}{2}$, and $st = -\frac{3}{4}$, we find by Algebra that

$$s = \frac{1 \pm \sqrt{13}}{4} \text{ and } t = \frac{1 \mp \sqrt{13}}{4};$$

and it is obvious that the upper sign must be taken, because s is positive; for $\cos\phi$ and $\cos 3\phi$, which are positive, are both numerically greater than $\cos 9\phi$, which is negative.

227. Suppose the polygon has n sides. Let O be the centre of the circle inscribed in the polygon, and S the assumed point. Let $OS=c$; and suppose OS to be inclined at an angle α to the first perpendicular which is drawn; put β for $\frac{2\pi}{n}$, and r for the radius of the inscribed circle. Then the length of the first perpendicular will be $r+c\cos\alpha$, that of the second $r+c\cos(\alpha+\beta)$, that of the third $r+c\cos(\alpha+2\beta)$, and so on.

Then for the squares on the sides of the new polygon we obtain the expressions

$$\{r+c\cos\alpha\}^2+\{r+c\cos(\alpha+\beta)\}^2-2\{r+c\cos\alpha\}\{r+c\cos(\alpha+\beta)\}\cos\beta,$$

$$\{r+c\cos(\alpha+\beta)\}^2+\{r+c\cos(\alpha+2\beta)\}^2-2\{r+c\cos(\alpha+\beta)\}\{r+c\cos(\alpha+2\beta)\}\cos\beta,$$

and so on.

Thus for the square on the m^{th} side of the new polygon we shall obtain

$$2r^2(1-\cos\beta)+2rc\{\cos(\alpha+m\beta-\beta)+\cos(\alpha+m\beta)\}(1-\cos\beta)$$
$$+c^2\{\cos^2(\alpha+m\beta-\beta)+\cos^2(\alpha+m\beta)-2\cos\beta\cos(\alpha+m\beta-\beta)\cos(\alpha+m\beta)\};$$

that is

$$2r^2(1-\cos\beta)+2rc\{\cos(\alpha+m\beta-\beta)+\cos(\alpha+m\beta)\}(1-\cos\beta)$$
$$+\frac{c^2}{2}\{1+\cos(2\alpha+2m\beta-2\beta)+1+\cos(2\alpha+2m\beta)$$
$$-2\cos\beta[\cos\beta+\cos(2\alpha+2m\beta-\beta)]\}.$$

We have to obtain the sum formed from this expression by giving to m all integral values from 1 to n, both inclusive; the result, by Art. 305,

$$=2nr^2(1-\cos\beta)+\frac{c^2}{2}\{2n-2n\cos^2\beta\}$$

$$=4nr^2\sin^2\frac{\beta}{2}+nc^2\sin^2\beta.$$

228. We have $\cos5\theta+\sin5\theta=\sqrt{2}\cos\left(5\theta-\frac{\pi}{4}\right)$

$$=\sqrt{2}\cos5\left(\theta-\frac{\pi}{20}\right)=\sqrt{2}\cos5(\theta-\beta).$$

And by Art. 318 we have $\cos5(\theta-\beta)\sin\frac{5\pi}{2}$

$$=2^4\cos(\theta-\beta)\cos(\theta-\beta+2\alpha)\cos(\theta-\beta+4\alpha)\cos(\theta-\beta+6\alpha)\cos(\theta-\beta+8\alpha),$$

where $\alpha=\frac{\pi}{10}=2\beta$.

Thus $\cos5(\theta-\beta)$

$$=2^4\cos(\theta-\beta)\cos(\theta+3\beta)\cos(\theta+7\beta)\cos(\theta+11\beta)\cos(\theta+15\beta).$$

Also $$\cos(\theta+7\beta)=\sin(3\beta-\theta)=-\sin(\theta-3\beta),$$

$$\cos(\theta+11\beta)=\cos\left(\theta+\beta+\frac{\pi}{2}\right)=-\sin(\theta+\beta),$$

$$\cos(\theta+15\beta)=\cos\left(\theta+\frac{3\pi}{4}\right)=-\frac{1}{\sqrt{2}}(\cos\theta+\sin\theta).$$

Hence $\sqrt{2}\cos 5(\theta-\beta)$

$$=-2^4\cos(\theta-\beta)\cos(\theta+3\beta)\sin(\theta-3\beta)\sin(\theta+\beta)(\sin\theta+\cos\theta);$$

hence also $\cos 5\theta+\sin 5\theta$ is equal to the last expression, which had to be shewn.

229. We have

$$\sin\theta\sqrt{(\cos^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta)}+\cos\theta\sqrt{(\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta)}=\sin(\alpha+\beta).$$

Assume $$r\cos\phi=\sqrt{(\cos^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta)},$$

and $$r\sin\phi=\sqrt{(\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta)};$$

so that $$r^2=2\cos^2\alpha\cos^2\beta \quad\text{.....................}(1),$$

and $$\tan^2\phi=\frac{\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta}{\cos^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta}\quad\text{.....................}(2).$$

Thus $$r\sin(\theta+\phi)=\sin(\alpha+\beta)\quad\text{.....................}(3).$$

Now it is obvious that r may be found from (1) by logarithms. Also ϕ may be determined by logarithms; for we have from (2)

$$\frac{1-\tan^2\phi}{1+\tan^2\phi}=\frac{\sin^2\alpha\sin^2\beta}{\cos^2\alpha\cos^2\beta},$$

that is $$\cos 2\phi=\tan^2\alpha\tan^2\beta,$$

which is adapted to logarithms.

Thus θ can be found from (3) by logarithms.

230. If A, B, and C are angles of a triangle, we have by Art. 114, and Example VIII. 16,

$$\sin A+\sin B+\sin C-(\sin 2A+\sin 2B+\sin 2C)$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-4\sin A\sin B\sin C$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left\{1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right\};$$

and by Example XIII. 40 this expression can never be negative.

231. Let A, B, C, ... M, N denote the angular points of the polygon taken in order; and let $\alpha = \frac{\pi}{n}$. Suppose P the point in the circumference from which chords are drawn, so that P is between N and A.

Then $\frac{1}{2} c_1 c_2 \sin \alpha =$ the area of the triangle PAB,

$\frac{1}{2} c_2 c_3 \sin \alpha =$ the area of the triangle PBC,

....

$\frac{1}{2} c_{n-1} c_n \sin \alpha =$ the area of the triangle PMN.

Therefore $\frac{1}{2} (c_1 c_2 + c_2 c_3 + \ldots + c_{n-1} c_n) \sin \alpha$

$=$ the area of the triangles PAB, PBC, ... PMN.

Also $\frac{1}{2} c_n c_1 \sin \alpha =$ the area of the triangle PNA.

Thus $\frac{1}{2} (c_1 c_2 + c_2 c_3 + \ldots + c_{n-1} c_n - c_n c_1) \sin \alpha$

$=$ the area of the regular polygon;

so that $c_1 c_2 + c_2 c_3 + \ldots + c_{n-1} c_n - c_n c_1$

$= \frac{2}{\sin \alpha} \times$ the area of the regular polygon.

This result is the same for all positions of P on the circumference of the circle.

232. Let θ be the angle of one sector, and 2θ the angle of the other. Let a and b be the corresponding radii. Then, since the areas are equal, $a^2 \frac{\theta}{2} = b^2 \frac{2\theta}{2}$; and since there is a common chord, $2a \sin \frac{\theta}{2} = 2b \sin \frac{2\theta}{2}$.

Thus $a \sin \frac{\theta}{2} = b \sin \theta = 2b \sin \frac{\theta}{2} \cos \frac{\theta}{2}$; therefore $\cos \frac{\theta}{2} = \frac{a}{2b}$;

therefore $\cos^2 \frac{\theta}{2} = \frac{a^2}{4b^2} = \frac{2b^2}{4b^2} = \frac{1}{2}$; therefore $\frac{\theta}{2} = \frac{\pi}{4}$.

Therefore $\theta = \frac{\pi}{2}$ and $2\theta = \pi$.

233. We have $\tan (\phi - \theta) = \dfrac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta} = \dfrac{\dfrac{x\sqrt{3}}{2k - x} - \dfrac{2x - k}{k\sqrt{3}}}{1 + \dfrac{x(2x - k)\sqrt{3}}{(2k - x)k\sqrt{3}}}$

$$= \frac{1}{\sqrt{3}} \cdot \frac{3kx - (2k - x)(2x - k)}{(2k - x)k + x(2x - k)} = \frac{1}{\sqrt{3}} \cdot \frac{2x^2 - 2kx + 2k^2}{2x^2 - 2kx + 2k^2} = \frac{1}{\sqrt{3}}.$$

Therefore one value of $\phi - \theta$ is $\frac{\pi}{6}$.

234. We have here $\frac{\sin\theta}{\theta}$ very nearly equal to unity; so we may infer that θ is small: hence $\sin\theta = \theta - \frac{\theta^3}{6}$ nearly. Therefore $1 - \frac{\theta^2}{6} = \frac{863}{864}$ nearly; therefore $\frac{\theta^2}{6} = \frac{1}{864}$ nearly; therefore $\theta^2 = \frac{1}{144}$ nearly; therefore $\theta = \frac{1}{12}$ nearly. Hence the number of degrees in the angle is nearly $\frac{1}{12} \cdot \frac{180}{\pi}$, that is about 5.

235. Let ABC be any triangle; let D, E, F be the centres of the escribed circles opposite to A, B, C respectively.

Then AD bisects the angle of the triangle at A, and EF bisects the *exterior* angle at A. Therefore AD is perpendicular to EF.

Similarly EB is perpendicular to FD, and FC is perpendicular to DE.

Therefore by Art. 332 the circle described round ABC is the nine points circle of DEF.

236. As in Art. 283 we have

$$2^{2n}(-1)^n \sin^{2n+1}\theta$$

$$= \sin(2n+1)\theta - (2n+1)\sin(2n-1)\theta + \frac{(2n+1)2n}{\underline{|2}}\sin(2n-3)\theta - \dots.$$

Now suppose each side were to be expanded in powers of θ; on the left-hand side we should have $2^{2n}(-1)^n\left\{\theta - \frac{\theta^3}{\underline{|3}} + \dots\right\}^{2n+1}$, by Art. 274.

On the right-hand side each sine gives rise to a series. Since the lowest power of θ on the left-hand side is θ^{2n+1} it follows that the whole coefficient of every lower power of θ on the right-hand side must be zero. The whole coefficient of θ is

$$2n+1-(2n+1)(2n-1)+\frac{(2n+1)2n}{\underline{|2}}(2n-3)-\dots \text{ to } n+1 \text{ terms};$$

hence this is zero; and dividing by $2n+1$ we obtain the required result.

Similarly, supposing n to be greater than unity, we can obtain another result by equating to zero the whole coefficient of θ^3 on the right-hand side. And so on.

237. We have

$$\cos\alpha + \sqrt{-1}\sin\alpha = \cos(\theta + \phi\sqrt{-1}) = \cos\theta\cos\phi\sqrt{-1} - \sin\theta\sin\phi\sqrt{-1}$$

$$= \cos\theta\frac{e^{-\phi}+e^{\phi}}{2} - \sin\theta\frac{e^{-\phi}-e^{\phi}}{2\sqrt{-1}} = \cos\theta\frac{e^{-\phi}+e^{\phi}}{2} + \sin\theta\frac{e^{-\phi}-e^{\phi}}{2}\sqrt{-1}.$$

Hence, by equating the possible and the impossible parts, we have

$$\cos\theta\frac{e^{-\phi}+e^{\phi}}{2} = \cos\alpha, \quad \sin\theta\frac{e^{-\phi}-e^{\phi}}{2} = \sin\alpha;$$

so that $$\frac{e^{-\phi}+e^{\phi}}{2} = \frac{\cos\alpha}{\cos\theta}, \quad \frac{e^{-\phi}-e^{\phi}}{2} = \frac{\sin\alpha}{\sin\theta}.$$

Square and subtract; thus

$$1 = \frac{\cos^2\alpha}{\cos^2\theta} - \frac{\sin^2\alpha}{\sin^2\theta};$$

therefore $$\sin^2\theta\cos^2\alpha - \cos^2\theta\sin^2\alpha = \sin^2\theta\cos^2\theta;$$

therefore $$\sin^2\theta(1-\cos^2\theta) = \sin^2\alpha;$$

therefore $$\sin^4\theta = \sin^2\alpha;$$

therefore $$\sin^2\theta = \pm\sin\alpha.$$

238. On the left-hand side the numerator

$$= \sin x + \sin(3x+\pi) + \sin(5x+2\pi) + \ldots \text{to } n \text{ terms},$$

$$= \frac{\sin\left\{x + \frac{n-1}{2}(2x+\pi)\right\}\sin\frac{n}{2}(2x+\pi)}{\sin\frac{1}{2}(2x+\pi)};$$

in like manner the denominator

$$= \frac{\cos\left\{x + \frac{n-1}{2}(2x+\pi)\right\}\sin\frac{n}{2}(2x+\pi)}{\sin\frac{1}{2}(2x+\pi)}.$$

Divide the former by the latter and we obtain

$$\tan\left\{x + \frac{n-1}{2}(2x+\pi)\right\}, \text{ that is } \tan\left(nx + \frac{n-1}{2}\pi\right).$$

239. Let O denote the centre of the circles, r the radius of the circle $ABCP$, and R the radius of the circle $DEFQ$.

Suppose the angle QOA is equal to θ, then the angle QOB will be $\theta + \frac{2\pi}{3}$, and the angle QOC will be $\theta + \frac{4\pi}{3}$; or at least the angles may be so denoted by suitably choosing A, B, and C. Then

$$QA^2 = QO^2 + OA^2 - 2QO \, . \, OA \cos\theta$$
$$= R^2 + r^2 - 2Rr\cos\theta,$$

similarly $$QB^2 = R^2 + r^2 - 2Rr\cos\left(\theta + \frac{2\pi}{3}\right),$$

and $$QC^2 = R^2 + r^2 - 2Rr\cos\left(\theta + \frac{4\pi}{3}\right).$$

Hence by addition, and Art. 305, we have

$$QA^2 + QB^2 + QC^2 = 3(R^2 + r^2).$$

In the same way we find that

$$PD^2 + PE^2 + PF^2 = 3(R^2 + r^2).$$

240. Put for each cosine its exponential value; then the proposed series

$$= \frac{1}{2}(1 - ae^{\iota cx})^n + \frac{1}{2}(1 - ae^{-\iota cx})^n$$

$$= \frac{1}{2}(1 - a\cos cx - \iota a \sin cx)^n + \frac{1}{2}(1 - a\cos cx + \iota a \sin cx)^n.$$

Now assume $1 - a\cos cx = r\cos\theta$ and $a\sin cx = r\sin\theta$;

then the sum $$= \frac{1}{2}(r\cos\theta - \iota r\sin\theta)^n + \frac{1}{2}(r\cos\theta + \iota r\sin\theta)^n$$

$$= \frac{r^n}{2}(\cos n\theta - \iota\sin n\theta) + \frac{r^n}{2}(\cos n\theta + \iota\sin n\theta)$$

$$= r^n \cos n\theta.$$

241. By addition $3 - p\sin\theta - q\cos\theta = 0$.

By subtraction $\cos^2\theta - \sin^2\theta = -p\sin\theta + q\cos\theta$.

Therefore $3(\cos^2\theta - \sin^2\theta) = q^2\cos^2\theta - p^2\sin^2\theta$;

therefore $3(2\cos^2\theta - 1) = q^2\cos^2\theta - p^2 + p^2\cos^2\theta$;

therefore $$\cos^2\theta = \frac{p^2 - 3}{p^2 + q^2 - 6};$$

therefore $$\sin^2\theta = \frac{q^2 - 3}{p^2 + q^2 - 6}.$$

Substitute in the equation $3 = p\sin\theta + q\cos\theta$; thus

$$3\sqrt{(p^2 + q^2 - 6)} = p\sqrt{(q^2 - 3)} + q\sqrt{(p^2 - 3)}.$$

This is the result of the elimination; the radicals are not necessarily positive. By squaring, transposing, and squaring again, we obtain finally

$$\{p^2q^2 - 6(p^2 + q^2) + 27\}^2 = p^2q^2(p^2 - 3)(q^2 - 3).$$

242. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$

$$=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$=\frac{\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}\{s-a+s-b+s-c\}=\frac{s^2}{S}.$$

Hence the proposed expression $=s^2\div\frac{s^2}{S}=S.$

243. Here $$2\tan^{-1}ax+\sec^{-1}bx=\frac{\pi}{2};$$

therefore $$\sin^{-1}\frac{2ax}{1+a^2x^2}=\frac{\pi}{2}-\cos^{-1}\frac{1}{bx}=\sin^{-1}\frac{1}{bx};$$

therefore $$\frac{2ax}{1+a^2x^2}=\frac{1}{bx};\quad\text{therefore } 2abx^2=1+a^2x^2;$$

therefore $$x^2=\frac{1}{2ab-a^2}.$$

244. With the diagram of Art. 332 we have $OA=R$, also the angle $OAB=\frac{\pi}{2}-C$, and the angle $BAP=\frac{\pi}{2}-B$; so that the angle $OAP=C-B$.

Hence $$OP^2=R^2+AP^2-2R.AP\cos(B-C).$$

Now, as in Example 214, we have $AP=\frac{a\cos A}{\sin A}=2R\cos A$;

so that $$OP^2=R^2+4R^2\cos^2 A-4R^2\cos A\cos(B-C)$$
$$=R^2+2R^2(1+\cos 2A)+4R^2\cos(B+C)\cos(B-C)$$
$$=3R^2+2R^2\cos 2A+2R^2(\cos 2B+\cos 2C)$$
$$=3R^2+2R^2(\cos 2A+\cos 2B+\cos 2C).$$

245. The values of x, y, z are given in Example 216; and the values of α, β, γ in Example XVI. 31. Hence

$$\frac{\beta z+\gamma y}{x}=\frac{4R\cos\frac{1}{2}C\,b\sec\frac{1}{2}B+4R\cos\frac{1}{2}B\,c\sec\frac{1}{2}C}{4R\cos\frac{1}{2}A}$$

$$=\frac{4R\left(\cos\frac{1}{2}C\sin\frac{1}{2}B+\cos\frac{1}{2}B\sin\frac{1}{2}C\right)}{\cos\frac{1}{2}A}=\frac{4R\sin\frac{1}{2}(B+C)}{\cos\frac{1}{2}A}$$

$$=4R.$$

Similarly the other expressions are also equal to $4R$.

246. We have $c = b\cos A + a\cos B = b\cos A - a\cos(\pi - B)$

$$= b\sqrt{(1-\sin^2 A)} - a\cos\theta = b\sqrt{\left(1 - \frac{a^2}{b^2}\sin^2\theta\right)} - a\cos\theta.$$

We wish to expand this in powers of θ, as far as terms involving θ^4.

Now $$\sqrt{\left(1-\frac{a^2}{b^2}\sin^2\theta\right)} = 1 - \frac{a^2}{2b^2}\sin^2\theta - \frac{a^4}{8b^4}\sin^4\theta - \ldots$$

Put for $\sin\theta$ its value $\theta - \frac{\theta^3}{6} + \ldots$; thus we obtain

$$1 - \frac{a^2}{2b^2}\left(\theta - \frac{\theta^3}{6} + \ldots\right)^2 - \frac{a^4}{8b^4}\left(\theta - \frac{\theta^3}{6} + \ldots\right)^4,$$

that is $$1 - \frac{a^2}{2b^2}\left(\theta^2 - \frac{\theta^4}{3}\right) - \frac{a^4}{8b^4}\theta^4 + \ldots$$

And $$\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \ldots$$

Hence approximately

$$c = b - \frac{a^2}{2b}\left(\theta^2 - \frac{\theta^4}{3}\right) - \frac{a^4}{8b^3}\theta^4 - a\left(1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}}\right)$$

$$= b - a + \left(a - \frac{a^2}{b}\right)\frac{\theta^2}{2} + \left(\frac{a^2}{6b} - \frac{a^4}{8b^3} - \frac{a}{24}\right)\theta^4$$

$$= b - a + \frac{(b-a)\,a\theta^2}{2b} + \frac{\theta^4}{\underline{|4}}\left(\frac{4a^2}{b} - \frac{3a^4}{b^3} - a\right);$$

and $$\frac{4a^2}{b} - \frac{3a^4}{b^3} - a = \frac{a^2}{b} - a + 3\left(\frac{a^2}{b} - \frac{a^4}{b^3}\right)$$

$$= \frac{a(a-b)}{b} + \frac{3a^2(b^2-a^2)}{b^3} = (a-b)\left\{\frac{a}{b} - \frac{3a^2}{b^3}(a+b)\right\}.$$

Thus we obtain the required result.

247. $\sin^5\theta\cos^6\theta = \cos\theta(\sin\theta\cos\theta)^5 = \frac{\cos\theta}{2^5}(\sin 2\theta)^5$

$$= \frac{\cos\theta}{2^5} \times \frac{1}{2^4}\{\sin 10\theta - 5\sin 6\theta + 10\sin 2\theta\}$$

$$= \frac{1}{2^{10}}\{\sin 11\theta + \sin 9\theta - 5(\sin 7\theta + \sin 5\theta) + 10(\sin 3\theta + \sin\theta)\}.$$

248. We have $\cos\alpha+\sqrt{-1}\sin\alpha=\dfrac{\sin(\theta+\sqrt{-1}\,\phi)}{\cos(\theta+\sqrt{-1}\,\phi)}$

$$=\frac{\sin\theta\cos\sqrt{-1}\,\phi+\cos\theta\sin\sqrt{-1}\,\phi}{\cos\theta\cos\sqrt{-1}\,\phi-\sin\theta\sin\sqrt{-1}\,\phi}=\frac{\sin\theta(e^{\phi}+e^{-\phi})-\sqrt{-1}\cos\theta(e^{-\phi}-e^{\phi})}{\cos\theta(e^{\phi}+e^{-\phi})+\sqrt{-1}\sin\theta(e^{-\phi}-e^{\phi})}$$

$$=\frac{\sin\theta+\sqrt{-1}\,k\cos\theta}{\cos\theta-\sqrt{-1}\,k\sin\theta},\text{ where } k=\frac{e^{\phi}-e^{-\phi}}{e^{\phi}+e^{-\phi}}.$$

Hence $\sin\theta+\sqrt{-1}\,k\cos\theta=(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\theta-\sqrt{-1}\,k\sin\theta)$

$$=\cos\alpha\cos\theta+k\sin\alpha\sin\theta+\sqrt{-1}(\sin\alpha\cos\theta-k\sin\theta\cos\alpha);$$

therefore $\sin\theta=\cos\alpha\cos\theta+k\sin\alpha\sin\theta,$

and $k\cos\theta=\sin\alpha\cos\theta-k\sin\theta\cos\alpha$

therefore $$\frac{\sin\theta-\cos\alpha\cos\theta}{\sin\alpha\sin\theta}=\frac{\sin\alpha\cos\theta}{\cos\theta+\sin\theta\cos\alpha}.$$

Multiply up; thus we get $\cos\alpha(\sin^2\theta-\cos^2\theta)=0$;

therefore $\tan^2\theta=1$, and therefore $\theta=n\pi\pm\dfrac{\pi}{4}$.

249. By Art. 309 we have

$$\frac{1}{x}-2\cot 2x=\tan x+\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\ldots,$$

and, since $2\cot 2x+\tan x=\cot x$, we have

$$\frac{1}{x}-\cot x=\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\ldots;$$

then put $\dfrac{\pi}{2}$ for x, and divide by 2; thus $\dfrac{1}{\pi}=\dfrac{1}{4}\tan\dfrac{\pi}{4}+\dfrac{1}{8}\tan\dfrac{\pi}{8}+\ldots$.

250. Put $-\dfrac{\theta^2}{\pi^2}$ for x; then we require the coefficient of $\left(-\dfrac{\theta^2}{\pi^2}\right)^n$, that is of $\dfrac{(-1)^n\theta^{2n}}{\pi^{2n}}$ in the development of

$$\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots.$$

Thus we require the coefficient of $\dfrac{(-1)^n\theta^{2n+1}}{\pi^{2n}}$ in the development of

$$\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots,$$

that is in $\sin\theta$. See Art. 320.

But the general term in the expansion of $\sin\theta$ is $\dfrac{(-1)^n \theta^{2n+1}}{\lfloor 2n+1}$.

Hence $\dfrac{1}{\pi^{2n}} \times$ the required coefficient $= \dfrac{1}{\lfloor 2n+1}$; so that the required coefficient is $\dfrac{\pi^{2n}}{\lfloor 2n+1}$.

251. Proceed as in Example 241. We have

$$1+m+n=p\sin\theta+q\cos\theta,\quad \cos^2\theta-\sin^2\theta+n-m=q\cos\theta-p\sin\theta;$$

therefore $$(1+m+n)(2\cos^2\theta-1+n-m)=q^2\cos^2\theta-p^2\sin^2\theta;$$

therefore $$\cos^2\theta=\frac{p^2+n^2-(1+m)^2}{p^2+q^2-2(1+m+n)},$$

and $$\sin^2\theta=\frac{q^2+m^2-(1+n)^2}{p^2+q^2-2(1+m+n)}.$$

Substitute in $1+m+n=p\sin\theta+q\cos\theta$, and the elimination will be effected.

252. Let D, E, F be the points at which the bisectors of the angles A, B, C respectively meet the circumference. Then the angle $DAC=\frac{1}{2}A$, and the angle CAE = the angle $CBE=\frac{1}{2}B$; therefore $DAE=\frac{1}{2}(A+B)$; and therefore DE subtends at the centre of the circle an angle equal to $A+B$: thus $DE=2R\sin\frac{1}{2}(A+B)=2R\cos\frac{1}{2}C$. Similarly $EF=2R\cos\frac{1}{2}A$; and the angle $DEF=\frac{1}{2}(A+C)$; thus the area of the triangle DEF

$$=\frac{1}{2}\cdot 4R^2\cos\frac{1}{2}A\cos\frac{1}{2}C\sin\frac{1}{2}(A+C)=2R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$$

$$=\frac{2R^2 sS}{abc}=\frac{Rs}{2}.$$

253. Here $$\sin^{-1}\frac{x}{a}+\sin^{-1}\frac{y}{b}=\sin^{-1}\frac{c^2}{ab}.$$

Take the cosines of both sides; thus

$$\sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)}-\frac{xy}{ab}=\sqrt{\left(1-\frac{c^4}{a^2b^2}\right)};$$

therefore $$\sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)}=\frac{xy}{ab}+\sqrt{\left(1-\frac{c^4}{a^2b^2}\right)};$$

square both sides; thus

$$1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}=\frac{x^2y^2}{a^2b^2}+2\frac{xy}{ab}\sqrt{\left(1-\frac{c^4}{a^2b^2}\right)}+1-\frac{c^4}{a^2b^2};$$

therefore $$b^2x^2+a^2y^2+2xy\sqrt{(a^2b^2-c^4)}=c^4.$$

254. Let $PCA=\theta$; then $PCB=\frac{\pi}{2}-\theta$;

$$\frac{PC}{a}=\frac{\sin(\theta+\gamma)}{\sin\gamma},\quad \frac{PC}{b}=\frac{\sin\left(\frac{\pi}{2}-\theta+\gamma\right)}{\sin\gamma};$$

thus $$a\sin(\theta+\gamma)=b\sin\left(\frac{\pi}{2}-\theta+\gamma\right)=b\cos(\theta-\gamma);$$

therefore $$a(\sin\theta\cos\gamma+\cos\theta\sin\gamma)=b(\cos\theta\cos\gamma+\sin\theta\sin\gamma);$$

therefore $$\tan\theta=\frac{b\cos\gamma-a\sin\gamma}{a\cos\gamma-b\sin\gamma}.$$

Hence $$\sin\theta=\frac{b\cos\gamma-a\sin\gamma}{\sqrt{\{(a\cos\gamma-b\sin\gamma)^2+(b\cos\gamma-a\sin\gamma)^2\}}}$$

$$=\frac{b\cos\gamma-a\sin\gamma}{\sqrt{(a^2+b^2-2ab\sin 2\gamma)}};$$

and $$\cos\theta=\frac{a\cos\gamma-b\sin\gamma}{\sqrt{(a^2+b^2-2ab\sin 2\gamma)}}.$$

Then $$PC=\frac{a(\sin\theta\cos\gamma+\cos\theta\sin\gamma)}{\sin\gamma}=\frac{ab\cos 2\gamma}{\sin\gamma\sqrt{(a^2+b^2-2ab\sin 2\gamma)}}.$$

255. $x^4-x^3+x^2-x+1=\frac{x^5+1}{x+1}$. Hence we must find the roots of $x^5+1=0$, and omit the root -1.

Now if $x^5=-1$ we may put $x^5=\cos n\pi\pm\sqrt{-1}\sin n\pi$, where n is any odd integer. Hence $x=(\cos n\pi\pm\sqrt{-1}\sin n\pi)^{\frac{1}{5}}=\cos\frac{n\pi}{5}\pm\sqrt{-1}\sin\frac{n\pi}{5}$.

Put in succession 1 and 3 for n; thus we obtain the assigned values. If we put 5 for n we obtain the root -1, which we had to omit.

256. Let θ denote the angle opposite to the side 1; then

$$\frac{\sin\theta}{\sin\frac{\pi}{6}}=\frac{1}{250};\quad\text{therefore }\sin\theta=\frac{1}{500}.$$

As θ is very small we may put θ for $\sin\theta$; thus $\theta=\frac{1}{500}$ approximately. Therefore the number of degrees in the angle $=\frac{1}{500}\times\frac{180}{\pi}$; and therefore the number of minutes $=\frac{60}{500}\times\frac{180}{\pi}=\frac{3}{25}\times\frac{180}{\pi}=\frac{3}{25}\times 57\cdot 3=7$ nearly.

257. First take the inscribed circle: see Art. 248.

$$FE=2r\sin FOA=2r\cos\frac{A}{2};\quad\text{similarly } FD=2r\cos\frac{B}{2}.$$

The angle $EFA=\frac{1}{2}(\pi-A)$; the angle $DFB=\frac{1}{2}(\pi-B)$; therefore the angle $EFD=\frac{1}{2}(A+B)$.

Hence the area of the triangle DFE

$$=\frac{1}{2}\cdot 4r^2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{A+B}{2}=2r^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{2r^2Ss}{abc}=\frac{rS}{2R}.$$

Now take one of the escribed circles, as for instance that opposite to the angle A: see Art. 250.

$$DF=2r_1\sin DOB=2r_1\sin\frac{B}{2};\quad\text{similarly } DE=2r_1\sin\frac{C}{2}.$$

The angle FDE = the angle FDO + the angle EDO

$$=\frac{1}{2}(\pi-B)+\frac{1}{2}(\pi-C)=\pi-\frac{1}{2}(B+C).$$

Hence the area of the triangle DFE

$$=\frac{1}{2}\cdot 4r_1^2\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{B+C}{2}$$

$$=2r_1^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}=\frac{2r_1^2(s-a)S}{abc}=\frac{2r_1S^2}{abc}=\frac{r_1S}{2R}.$$

258. Proceed as in Example 248; thus we obtain $\sin^2\theta=\cos^2\theta$.

If we take $\sin\theta=\cos\theta$ we get $1=\cos\alpha+k\sin\alpha$; thus

$$k=\frac{1-\cos\alpha}{\sin\alpha}=\tan\frac{\alpha}{2},\quad\text{that is}\quad\frac{e^\phi-e^{-\phi}}{e^\phi+e^{-\phi}}=\tan\frac{\alpha}{2};$$

therefore
$$e^{2\phi}=\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}=\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

If we take $\sin\theta=-\cos\theta$ we get $1=-\cos\alpha+k\sin\alpha$; thus

$$k=\frac{1+\cos\alpha}{\sin\alpha}=\cot\frac{\alpha}{2},\quad\text{that is}\quad\frac{e^\phi-e^{-\phi}}{e^\phi+e^{-\phi}}=\cot\frac{\alpha}{2};$$

therefore
$$e^{2\phi}=\frac{1+\cot\frac{\alpha}{2}}{1-\cot\frac{\alpha}{2}}=\frac{\tan\frac{\alpha}{2}+1}{\tan\frac{\alpha}{2}-1}=-\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

259. Put the exponential values for the cosines in the series denoted by s: thus

$$s = 1 + \frac{1}{2} z (e^{\iota\theta} + e^{-\iota\theta}) + \frac{z^2}{2\,\lfloor 2} (e^{2\iota\theta} + e^{-2\iota\theta}) + \frac{z^3}{2\,\lfloor 3} (e^{3\iota\theta} + e^{-3\iota\theta}) + \ldots\ldots$$

$$= \frac{1}{2} (e^x + e^y),$$

where $\qquad x = ze^{\iota\theta} = z (\cos\theta + \iota \sin\theta),$

and $\qquad y = ze^{-\iota\theta} = z (\cos\theta - \iota \sin\theta).$

Thus $\qquad s = \frac{1}{2} e^{z\cos\theta} (e^{\iota z \sin\theta} + e^{-\iota z \sin\theta}) = e^{z\cos\theta} \cos (z \sin\theta).$

Similarly we find that $\sigma = \frac{1}{2\iota} (e^x - e^y) = e^{z\cos\theta} \sin (z \sin\theta).$

Therefore $\qquad \frac{\sigma}{s} = \frac{\sin (z \sin\theta)}{\cos (z \sin\theta)} = \tan (z \sin\theta);$

so that $\qquad z \sin\theta = \tan^{-1} \frac{\sigma}{s}.$

And $\qquad s^2 + \sigma^2 = e^{2z\cos\theta} \{\cos^2 (z \sin\theta) + \sin^2 (z \sin\theta)\} = e^{2z\cos\theta};$

so that $\qquad z \cos\theta = \frac{1}{2} \log (s^2 + \sigma^2).$

If $\theta = \frac{\pi}{2}$, we have $\frac{\sigma}{s} = \tan z$ and $s^2 + \sigma^2 = 1$, so that $\sigma = \sin z$ and $s = \cos z$.

260. Let c be the distance of the two given points, n the number of sides in the polygon; and put $\beta = \frac{2\pi}{n}$. Let α be the angle which the distance between the two given points makes with the first straight line which is drawn. Then the numerical values of the successive perpendiculars are $c \sin\alpha$, $c \sin (\alpha + \beta)$, $c \sin (\alpha + 2\beta)$, ... Hence the sum of the squares on the perpendiculars

$$= c^2 \{\sin^2 \alpha + \sin^2 (\alpha + \beta) + \sin^2 (\alpha + 2\beta) + \ldots \text{ to } n \text{ terms}\}$$

$$= \frac{c^2}{2} \{1 - \cos 2\alpha + 1 - \cos 2 (\alpha + \beta) + 1 - \cos 2 (\alpha + 2\beta) + \ldots\ldots\}$$

$$= \frac{nc^2}{2}.$$ See Art. 305.

261. We have

$$a \sin\theta + b = h \cos\theta, \text{ and } \cos\theta\, (a + b \sin\theta) = k \sin\theta.$$

Find $\cos\theta$ from the first equation, and substitute it in the second; thus we get

$$\sin^2\theta + \frac{a^2 + b^2 - hk}{ab} \sin\theta + 1 = 0.$$

Again, find $\sin\theta$ from the first equation, and substitute it in the second; thus we get

$$\cos^2\theta + \frac{a^2-b^2-hk}{bh}\cos\theta + \frac{k}{h} = 0.$$

Then we may employ the process of Example 251.

262. By Example XVI. 50 we know that the sides of the new triangle are respectively $a\cos A$, $b\cos B$, and $c\cos C$. Thus the perimeter

$= a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C$, by Example XVI. 22,

$$= \frac{4R \cdot 8S^3}{(abc)^2} = \frac{2S}{R}.$$

263. As in Example 252 we shew that the area of the triangle thus formed is $2R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$; denote this by Σ.

Also $$S = \frac{1}{2}ab\sin C = 2R^2\sin A\sin B\sin C.$$

Hence $$\frac{\Sigma}{S} = \frac{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{\sin A\sin B\sin C} = \frac{1}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Now, as in Example XIII. 40 we see that $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ cannot be greater than unity; and therefore S cannot be greater than Σ.

264. $$r_1 = \frac{S}{s-a}, \quad r_2 = \frac{S}{s-b}, \quad r_3 = \frac{S}{s-c}.$$

Hence

$$(r_1+r_2)(r_2+r_3)(r_3+r_1) = S^3\left(\frac{1}{s-a}+\frac{1}{s-b}\right)\left(\frac{1}{s-b}+\frac{1}{s-c}\right)\left(\frac{1}{s-c}+\frac{1}{s-a}\right)$$

$$= \frac{S^3abc}{(s-a)^2(s-b)^2(s-c)^2} = \frac{s^2S^3abc}{S^4} = \frac{s^2abc}{S}.$$

And $$r_1r_2 + r_2r_3 + r_3r_1 = S^2\left\{\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)}\right\}$$

$$= \frac{S^2(3s-a-b-c)}{(s-a)(s-b)(s-c)} = \frac{S^2s^2}{S^2} = s^2.$$

Divide the first result by the second; and thus we get $\frac{abc}{S}$.

265. The wall a feet high casts a shadow which extends $a\cot\alpha$ feet from the wall *measured in the direction of the meridian;* hence $a\cot\alpha\sin\beta$ is the breadth of the shadow *measured in the direction at right angles to the wall.*

Thus $b = a \cot \alpha \sin \beta$. Similarly $b' = a' \cot \alpha \sin (\gamma - \beta)$.

From these two equations we have to find α and β.

We get $\dfrac{a \sin \beta}{b} = \dfrac{a' \sin (\gamma - \beta)}{b'}$; so that

$$\frac{a}{b} = \frac{a'}{b'} (\sin \gamma \cot \beta - \cos \gamma);$$

therefore
$$\cot \beta = \cot \gamma + \frac{ab'}{a'b} \operatorname{cosec} \gamma.$$

Then
$$\cot^2 \alpha = \frac{b^2}{a^2 \sin^2 \beta} = \frac{b^2}{a^2} (1 + \cot^2 \beta) = \frac{b^2}{a^2} \left\{ 1 + \left(\cot \gamma + \frac{ab'}{a'b} \operatorname{cosec} \gamma \right)^2 \right\}$$

$$= \frac{b^2}{a^2} \left\{ 1 + \cot^2 \gamma + \frac{a^2 b'^2}{a'^2 b^2} \operatorname{cosec}^2 \gamma + \frac{2ab'}{a'b} \cot \gamma \operatorname{cosec} \gamma \right\}$$

$$= \left(\frac{b^2}{a^2} + \frac{b'^2}{a'^2} \right) \operatorname{cosec}^2 \gamma + \frac{2bb'}{aa'} \cot \gamma \operatorname{cosec} \gamma.$$

266. Assume $a = r \cos \theta$, and $b = r \sin \theta$; so that $r^2 = a^2 + b^2$, and $\tan \theta = \dfrac{b}{a}$. Also assume $\alpha = \rho \cos \phi$, $\beta = \rho \sin \phi$; so that $\rho^2 = \alpha^2 + \beta^2$, and $\tan \phi = \dfrac{\beta}{\alpha}$.

Then the proposed expression

$$= (r \cos \theta + \iota r \sin \theta)^{\rho \cos \phi + \iota \rho \sin \phi} = (re^{\iota \theta})^{\rho \cos \phi + \iota \rho \sin \phi}.$$

Denote this by u; then

$$\log u = (\rho \cos \phi + \iota \rho \sin \phi) \log (re^{\iota \theta})$$
$$= (\rho \cos \phi + \iota \rho \sin \phi)(\iota \theta + \log r)$$
$$= \rho (\cos \phi \log r - \theta \sin \phi) + \rho \iota (\sin \phi \log r + \theta \cos \phi)$$
$$= \sigma + \iota \tau \text{ say};$$

therefore
$$u = e^{\sigma + \iota \tau} = e^{\sigma} e^{\iota \tau} = e^{\sigma} (\cos \tau + \iota \sin \tau).$$

To make this wholly real the term involving ι must vanish, therefore $\sin \tau$ must vanish; therefore τ must be zero or a multiple of π; therefore $\rho (\sin \phi \log r + \theta \cos \phi)$ must be zero or an even multiple of $\dfrac{\pi}{2}$; but $\rho \sin \phi = \beta$, and $\rho \cos \phi = \alpha$; so that $\dfrac{\beta}{2} \log (a^2 + b^2) + \alpha \tan^{-1} \dfrac{b}{a}$ must be zero or an even multiple of $\dfrac{\pi}{2}$.

267. $$\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}=\frac{2a^2}{a^2(1+\cos 2\theta)+b^2(1-\cos 2\theta)}$$

$$=\frac{2a^2}{a^2+b^2+(a^2-b^2)\cos 2\theta}$$

$$=\frac{4a^2}{2(a^2+b^2)+(a^2-b^2)(e^{2\theta\iota}+e^{-2\theta\iota})}=\frac{4a^2}{(a+b)^2(1+ce^{2\theta\iota})(1+ce^{-2\theta\iota})}$$

$$=\frac{(1+c)^2}{(1+ce^{2\theta\iota})(1+ce^{-2\theta\iota})}.$$

Therefore

$$\log\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}=2\log(1+c)-\log(1+ce^{2\theta\iota})-\log(1+ce^{-2\theta\iota})$$

$$=2\left\{c-\frac{c^2}{2}+\frac{c^3}{3}-\frac{c^4}{4}+\dots\right\}$$

$$-\left\{ce^{2\theta\iota}-\frac{c^2}{2}e^{4\theta\iota}+\frac{c^3}{3}e^{6\theta\iota}-\frac{c^4}{4}e^{8\theta\iota}+\dots\right\}$$

$$-\left\{ce^{-2\theta\iota}-\frac{c^2}{2}e^{-4\theta\iota}+\frac{c^3}{3}e^{-6\theta\iota}-\frac{c^4}{4}e^{-8\theta\iota}+\dots\right\}.$$

The term which involves c is $-c(e^{\theta\iota}-e^{-\theta\iota})^2$, that is $4c\sin^2\theta$.

The term which involves c^2 is $\frac{c^2}{2}(e^{2\theta\iota}-e^{-2\theta\iota})^2$, that is $-\frac{4c^2}{2}\sin^2 2\theta$.

The term which involves c^3 is $-\frac{c^3}{3}(e^{3\theta\iota}-e^{-3\theta\iota})^2$, that is $\frac{4c^3}{3}\sin^2 3\theta$.

And so on.

Thus we obtain the required result.

268. Let O denote the centre of the inscribed circle, D and E the centres of the escribed circles. Then D, C, E are on a straight line which is at right angles to OC. The area of the triangle ODE

$$=\frac{1}{2}OC\,.\,DE=\frac{1}{2}r\operatorname{cosec}\frac{C}{2}(r_1+r_2)\sec\frac{C}{2}$$

$$=\frac{r(r_1+r_2)}{\sin C}=\frac{S}{s\sin C}\left(\frac{S}{s-a}+\frac{S}{s-b}\right)=\frac{S^2c}{s(s-a)(s-b)\sin C}$$

$$=\frac{(s-c)c}{\sin C}=\frac{abc\cos^2\frac{1}{2}C}{s\sin C}=\frac{abc}{2s}\cot\frac{C}{2}.$$

269. The angle $OBC=\frac{1}{2}B$, and the angle $OCB=\frac{1}{2}C$. Hence, as on page 186, line 5 of the *Trigonometry*, we have

$$r_a\left(\cot\frac{B}{4}+\cot\frac{C}{4}\right)=a;$$

therefore

$$\frac{a}{r_a}=\cot\frac{B}{4}+\cot\frac{C}{4}.$$

Similarly $\frac{b}{r_b}=\cot\frac{C}{4}+\cot\frac{A}{4}$, and $\frac{c}{r_c}=\cot\frac{A}{4}+\cot\frac{B}{4}$.

Hence by addition we get the required result.

270. We easily see that $\tan^{-1}\frac{1}{2r^2}=\tan^{-1}(2r+1)-\tan^{-1}(2r-1)$.

Resolve each of the given terms into two by this formula. Then by addition we find that the sum $=\tan^{-1}(2n+1)-\tan^{-1}1=\tan^{-1}\frac{n}{n+1}$.

271. $$\cot A=\frac{\cos A}{\sin A}=\frac{b^2+c^2-a^2}{2bc\sin A}=\frac{b^2+c^2-a^2}{4S};$$

similar expressions hold for $\cot B$ and $\cot C$. Thus

$$\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4S}.$$

Hence if S be given the sum of the cotangents of the angles varies as the sum of the squares of the sides.

272. By Art. 253 we have

$$OI^2=R^2-2Rr;$$

and

$$OD^2+OE^2+OF^2=3R^2+2R(r_1+r_2+r_3).$$

Thus by addition we obtain

$$OI^2+OD^2+OE^2+OF^2=4R^2+2R(r_1+r_2+r_3-r)$$
$$=4R^2+8R^2,\text{ by Example 201, }=12R^2.$$

273. Let $\theta=\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$, then $\tan\theta=\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}$,

$$\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{a+b-(a-b)\tan^2\frac{x}{2}}{a+b+(a-b)\tan^2\frac{x}{2}}$$

$$=\frac{b+a\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}{a+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}=\frac{b+a\cos x}{a+b\cos x}.$$

274. Let O denote the centre of the circles. Let $ABCD$ be a straight line cutting the outer circumference at A and D, and the inner circumference at B and C. Let $OB=r$, and $OA=nr$. Let the angle $AOD=2\alpha$, and the angle $BOC=2\beta$; so that the angle $AOB=\alpha-\beta$.

Then $$AB^2=n^2r^2+r^2-2nr^2\cos(\alpha-\beta).$$

Now $$\frac{AB}{BD}=\frac{AB^2}{AB\,.\,BD}=\frac{AB^2}{AB\,.\,AC}.$$

But $AB\,.\,AC$ = the square on the straight line drawn from A to touch the inner circumference $=(n^2-1)\,r^2$.

Therefore $$\frac{AB}{BD}=\frac{n^2-2n\cos(\alpha-\beta)+1}{n^2-1}.$$

275. Proceed as in the solution of Example 266. That the expression may be wholly imaginary we must have $\cos\tau=0$, and therefore τ must be an odd multiple of $\frac{\pi}{2}$, therefore $\rho(\sin\phi\log r+\theta\cos\phi)$ must be an odd multiple of $\frac{\pi}{2}$; but $\rho\sin\phi=\beta$, and $\rho\cos\phi=\alpha$, so that $\frac{\beta}{2}\log(a^2+b^2)+\alpha\tan^{-1}\frac{b}{a}$ must be an odd multiple of $\frac{\pi}{2}$.

276. $$\frac{1}{a}\left(\tan\frac{C}{2}+\tan\frac{B}{2}\right)=\frac{1}{a}\left(\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}\right)=\frac{\cos\frac{A}{2}}{a\cos\frac{C}{2}\cos\frac{B}{2}}$$

$$=\frac{1}{r}\tan\frac{C}{2}\tan\frac{B}{2},\text{ by Art. 249.}$$

In this way we find that the proposed expression

$$=\frac{abc}{2r}\left\{\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}\right\}$$

$$=\frac{abc}{2r},\text{ by Example VIII. 15;}$$

and this is the area of the triangle by Example XVI. 34.

Or thus. Let I denote the centre of the inscribed circle, O the centre of the escribed circle opposite to A; then the area of the quadrilateral $IBOC=\frac{a}{2}(r+r_1)=\frac{a}{2}(s-a+s)\tan\frac{A}{2}=\frac{a}{2}(b+c)\tan\frac{A}{2}$: see Arts. 249 and 250.

In this way we obtain for the whole required area the given expression.

277. We have
$$1+2\cos\theta=\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}},$$
$$1+2\cos 3\theta=\frac{\sin\frac{3^2\theta}{2}}{\sin\frac{3\theta}{2}},$$
and so on. Then, as the sum of the logarithms of any set of quantities is equal to the logarithm of the product of those quantities, we see that the required sum is the logarithm of

$$\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}}\cdot\frac{\sin\frac{3^2\theta}{2}}{\sin\frac{3\theta}{2}}\cdots\cdots\frac{\sin\frac{3^n\theta}{2}}{\sin\frac{3^{n-1}\theta}{2}},\text{ that is the logarithm of }\frac{\sin\frac{3^n\theta}{2}}{\sin\frac{\theta}{2}}.$$

278. Put β for $\frac{\pi}{n}$. The path consists of a set of arcs of circles, each of which corresponds to the angle 2β, and the radii of which are the respective distances of any assumed point from all the other angular points. The radii thus are $2R\sin\beta$, $2R\sin 2\beta$, $2R\sin 3\beta$, ...

Hence the required sum
$$=2R\,\{\sin\beta+\sin 2\beta+\sin 3\beta+\ldots+\sin n\beta\}\,2\beta.$$

The term $\sin n\beta$ is zero, and may be omitted if we please.

By Art. 303 this expression
$$=4R\beta\frac{\sin\left(\beta+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta}=4R\beta\frac{\sin\frac{n+1}{2n}\pi}{\sin\frac{\pi}{2n}}$$
$$=4R\beta\cot\frac{\pi}{2n}=\frac{4R\pi}{n}\cot\frac{\pi}{2n}.$$

279. The sum of the areas of all the sectors will be
$$4R^2\{\sin^2\beta+\sin^2 2\beta+\sin^2 3\beta+\ldots+\sin^2 n\beta\}\,\beta$$
$$=2R^2\beta\,\{1-\cos 2\beta+1-\cos 4\beta+\ldots\}$$
$$=2R^2\beta\left\{n-\frac{\cos(2\beta+\overline{n-1}\beta)\sin n\beta}{\sin\beta}\right\}=2R^2n\beta=2R^2\pi.$$

If we wish to have the whole area of the figure bounded by the straight line and by the arcs between two points where they cross the straight line, we must add to the above a set of triangles which make up the whole polygon, that is $\frac{n}{2}R^2\sin\frac{2\pi}{n}$.

280. We have

$$\frac{\sin 2r\theta}{\sin (2r-1)\theta \sin (2r+1)\theta} = \frac{1}{2\cos\theta}\left\{\frac{1}{\sin (2r-1)\theta} + \frac{1}{\sin (2r+1)\theta}\right\}.$$

If we resolve each term of the proposed series into two by the aid of this formula we find that the sum of $2n$ terms $= \frac{1}{2\cos\theta}\left\{\frac{1}{\sin\theta} - \frac{1}{\sin (4n+1)\theta}\right\}$.

281. As in Example 214 we have

$$\alpha = \frac{a\cos A}{\sin A}, \quad \beta = \frac{b\cos B}{\sin B}, \quad \gamma = \frac{c\cos C}{\sin C}.$$

Hence $\frac{1}{4}(a\alpha + b\beta + c\gamma) = \frac{1}{4}\left(\frac{a^2\cos A}{\sin A} + \frac{b^2\cos B}{\sin B} + \frac{c^2\cos C}{\sin C}\right)$

$$= R^2(\sin A\cos A + \sin B\cos B + \sin C\cos C)$$

$$= \frac{R^2}{2}(\sin 2A + \sin 2B + \sin 2C) = 2R^2 \sin A \sin B \sin C, \text{ by Art. 114,}$$

$$= \frac{1}{2}ab\sin C, \text{ by Art. 252, } = S.$$

Also, $a^2\alpha \operatorname{cosec} A + b^2\beta \operatorname{cosec} B + c^2\gamma \operatorname{cosec} C = \frac{a^3\cos A}{\sin^2 A} + \frac{b^3\cos B}{\sin^2 B} + \frac{c^3\cos C}{\sin^2 C}$

$$= 8R^3(\sin A\cos A + \sin B\cos B + \sin C\cos C)$$

$$= 4R^3(\sin 2A + \sin 2B + \sin 2C) = 16R^3 \sin A \sin B \sin C$$

$$= 8RS, \text{ by the former part of the Example, } = 2abc.$$

282. We obtain immediately from a diagram

$$2r' = R(1-\cos A), \quad 2r'' = R(1-\cos B), \quad 2r''' = R(1-\cos C);$$

hence $$8r'r''r''' = 8R^3 \sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} = \frac{8R^3S^4}{a^2b^2c^2s^2}.$$

Therefore $$64Rr'r''r''' = \frac{64R^4S^4}{a^2b^2c^2s^2} = \frac{a^2b^2c^2}{4s^2} = \left(\frac{abc}{a+b+c}\right)^2.$$

283. Let $\theta = \sin^{-1}\frac{\sqrt{(x^2-c^2)}}{\sqrt{(a^2-c^2)}}$ and $\phi = \sin^{-1}\frac{c\sqrt{(a^2-x^2)}}{x\sqrt{(a^2-c^2)}}$;

then $$\cos\theta = \frac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2-c^2)}} \quad\text{and}\quad \cos\phi = \frac{a\sqrt{(x^2-c^2)}}{x\sqrt{(a^2-c^2)}};$$

therefore $$\sin(\theta-\phi) = \frac{\sqrt{(x^2-c^2)}}{\sqrt{(a^2-c^2)}}\cdot\frac{a\sqrt{(x^2-c^2)}}{x\sqrt{(a^2-c^2)}} - \frac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2-c^2)}}\cdot\frac{c\sqrt{(a^2-x^2)}}{x\sqrt{(a^2-c^2)}}$$

$$= \frac{a(x^2-c^2)}{x(a^2-c^2)} - \frac{c(a^2-x^2)}{x(a^2-c^2)} = \frac{x^2(a+c) - ac(a+c)}{x(a^2-c^2)} = \frac{x^2-ac}{x(a-c)}.$$

284. Suppose D the middle point of BC. Then

$$AB^2=AD^2+BD^2-2AD\,.\,BD\cos ADB,$$
$$AC^2=AD^2+CD^2-2AD\,.\,CD\cos ADC;$$

therefore by addition $b^2+c^2=2h^2+\frac{a^2}{2}$; so that $h^2=\frac{1}{2}(b^2+c^2)-\frac{a^2}{4}$;

similarly $k^2=\frac{1}{2}(c^2+a^2)-\frac{b^2}{4}$, and $l^2=\frac{1}{2}(a^2+b^2)-\frac{c^2}{4}$.

Therefore by addition $4(h^2+k^2+l^2)=3(a^2+b^2+c^2)$.

Also $\quad (4h^2)^2+(4k^2)^2+(4l^2)^2$

$$=(2b^2+2c^2-a^2)^2+(2c^2+2a^2-b^2)^2+(2a^2+2b^2-c^2)^2$$
$$=9(a^4+b^4+c^4), \text{ by development.}$$

Again, from what has been already shewn,

$$16(h^2+k^2+l^2)^2=9(a^2+b^2+c^2)^2,$$

and $\quad 16(h^4+k^4+l^4)=9(a^4+b^4+c^4)$;

subtract and divide by 2; thus

$$16(h^2k^2+k^2l^2+l^2h^2)=9(a^2b^2+b^2c^2+c^2a^2).$$

285. The area of the triangle which can be formed with the straight lines h, k, l, by Arts. 247 and 218,

$$=\frac{1}{4}\sqrt{(2h^2k^2+2k^2l^2+2l^2h^2-h^4-k^4-l^4)}$$
$$=\frac{1}{4}\sqrt{\frac{9}{16}(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)}$$
$$=\frac{3}{16}\sqrt{(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)}=\frac{3}{4}S.$$

286. $\quad a\cos\theta-b\cos(\theta-\alpha)=\cos\theta(a-b\cos\alpha)-b\sin\alpha\sin\theta$;

assume $\quad a-b\cos\alpha=k\cos\beta$, and $b\sin\alpha=k\sin\beta$;

then $\quad a\cos\theta-b\cos(\theta-\alpha)=k(\cos\theta\cos\beta-\sin\theta\sin\beta)=k\cos(\theta+\beta)$.

Similarly

$$a\sin\theta-b\sin(\theta-\alpha)=\sin\theta(a-b\cos\alpha)+b\sin\alpha\cos\theta=k\sin(\beta+\theta).$$

Thus the proposed expression

$$=\{k\cos(\theta+\beta)+k\sqrt{-1}\sin(\theta+\beta)\}^{\frac{1}{n}}$$
$$=k^{\frac{1}{n}}\left\{\cos\frac{\theta+\beta}{n}+\sqrt{-1}\sin\frac{\theta+\beta}{n}\right\}.$$

287. Denote the point by O; and let OD, OE, OF be the perpendiculars on BC, CA, AB respectively. Then OA is the diameter of the circle which would go round $OEAF$; so that $OA=\frac{EF}{\sin A}$, by Art. 252. Therefore

$$OA^2 \,.\, a \sin A = a \,.\, OA \,.\, EF = a\,(OE \,.\, FA + OF \,.\, AE), \text{ by Euclid VI. D,}$$
$$= a \,.\, \beta \,.\, AF + a \,.\, \gamma \,.\, AE.$$

Transform the other two terms in like manner; thus we obtain

$$a\beta\,(AF+BF)+\beta\gamma\,(BD+CD)+\gamma a\,(AE+EC)=a\beta c+\beta\gamma a+\gamma ab.$$

288. We have $\qquad a\beta c=\frac{c}{\sin C}\,a\beta \sin C=2R\times 2 \text{ area of } OED.$

Transform the other two terms similarly; thus we obtain

$$4R\,(\text{area of } OED+\text{area of } ODF+\text{area of } OFE).$$

289. We have

$$\frac{1}{\cos^2 B}-\frac{1}{\cos^2 A}=\frac{\cos^2 A-\cos^2 B}{\cos^2 A\cos^2 B}=\frac{\sin^2 B-\sin^2 A}{\cos^2 B\cos^2 A}=\frac{\sin(B-A)\sin(B+A)}{\cos^2 B\cos^2 A};$$

so that
$$\frac{\sin(B+A)}{\cos^2 B\cos^2 A}=\frac{1}{\sin(B-A)}\left\{\frac{1}{\cos^2 B}-\frac{1}{\cos^2 A}\right\}.$$

Apply this transformation to every term of the proposed series; then we find that the sum

$$=\frac{1}{\sin\theta}\left\{\frac{1}{\cos^2 n\theta}-\frac{1}{\cos^2 0}\right\}=\frac{1}{\sin\theta}\left\{\frac{1}{\cos^2 n\theta}-1\right\}=\operatorname{cosec}\theta\tan^2 n\theta.$$

290. By De Moivre's Theorem the equation becomes

$$\cos(\theta+2\theta+\ldots+n\theta)+\sqrt{-1}\sin(\theta+2\theta+\ldots+n\theta)=1,$$

that is
$$\cos\frac{n(n+1)}{2}\theta+\sqrt{-1}\sin\frac{n(n+1)}{2}\theta=1.$$

Hence we must have $\cos\frac{n(n+1)}{2}\theta=1$, and $\sin\frac{n(n+1)}{2}\theta=0$; so that $\frac{n(n+1)}{2}\theta=2m\pi$, where m is zero or any integer.

291. We have $r'=\frac{BC}{2\sin BDC}=\frac{a}{2\sin\frac{B+C}{2}}=\frac{a}{2\cos\frac{A}{2}}$; and similar expressions hold for r'' and r'''. Thus

$$r'r''r'''=\frac{abc}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{a^2b^2c^2}{8sS}=\frac{16R^2S^2}{8sS}=\frac{2R^2S}{s}=2R^2r.$$

292. We have $R=\frac{a}{2\sin A}$, so that $R\sin A=\frac{a}{2}$..................... (1).

Suppose that in consequence of the error γ in C there is an error α in A, and an error ρ in R. Thus

$$(R+\rho)\sin(A+\alpha)=\frac{a}{2};$$

therefore approximately by Art. 181

$$(R+\rho)(\sin A+\alpha\cos A)=\frac{a}{2} \text{.........................} (2).$$

From (1) and (2) by subtraction, neglecting the product $\alpha\rho$,

$$\alpha R\cos A+\rho\sin A=0;$$

so that
$$\alpha=-\frac{\rho}{R}\tan A.$$

Similarly if β be the error in B arising from the error γ in C, we have

$$\beta=-\frac{\rho}{R}\tan B.$$

But $\alpha+\beta+\gamma=0$, since the sum of the three angles of a triangle is equal to a fixed quantity, namely two right angles.

Thus
$$\gamma-\frac{\rho}{R}(\tan A+\tan B)=0;$$

therefore
$$\rho=\frac{R\gamma}{\tan A+\tan B}=\frac{R\gamma\cos A\cos B}{\sin(A+B)}=\frac{c\gamma\cos A\cos B}{2\sin^2 C}.$$

And since $\sin C=\frac{c\sin A}{a}$ and $=\frac{c\sin B}{b}$; we have $\rho=\frac{ab\gamma\cot A\cot B}{2c}$.

293. Let C denote the right angle, CA and CB the equal sides; produce CA to D and CB to E; then since DE is n times AB, it follows that CD and and CE are each n times CA or CB. Let AE and BD intersect at O. Then the angle DOA = the sum of the angles OBA and OAB, and these are equal; so that the angle DOA = twice the angle OAB. But the angle OAB = the angle EAC − the angle BAC, so that

$$\tan OAB=\tan(EAC-BAC)=\frac{\tan EAC-\tan BAC}{1+\tan EAC\tan BAC}=\frac{n-1}{1+n}.$$

294. As θ continually increases from 0 to $\frac{\pi}{2}$ the value of $\cos\theta$ continually decreases from 1 to 0; so that there must be one value of θ, and only one, in this range, which makes $\theta=\cos\theta$. Also as $\cos\theta$ is greater than θ when $\theta=0$, and is less than θ when $\theta=\frac{\pi}{4}$, this value is less than $\frac{\pi}{4}$. As θ changes from 0 to $-\frac{\pi}{2}$, the cosine is always positive, and so we cannot have $\cos\theta=\theta$.

When θ is numerically greater than $\frac{\pi}{2}$ it is numerically greater than unity, and so cannot be equal to $\cos\theta$.

Hence there must be one, and only one, solution of the equation $\theta = \cos\theta$.

295. Suppose β the circular measure of an angle between 0 and $\frac{\pi}{2}$, which is greater than the solution of $\theta = \cos\theta$, so that $\beta - \cos\beta$ is positive. Let $\beta - \alpha$ denote the solution, so that $\beta - \alpha = \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$; therefore $\alpha = \dfrac{\beta - \cos\beta\cos\alpha}{1 + \sin\beta\dfrac{\sin\alpha}{\alpha}}$. Now $\dfrac{\sin\alpha}{\alpha}$ is less than unity, and so is $\cos\alpha$;

hence $\dfrac{\beta - \cos\beta}{1 + \sin\beta}$ is less than the true value of α, and is a positive quantity.

Therefore $\beta - \dfrac{\beta - \cos\beta}{1 + \sin\beta}$ is nearer than β to the solution of the equation, and is still too large.

296. As in the solution of Example 248 we get

$$\tan\alpha + \sqrt{-1}\sec\alpha = \frac{\sin\theta + \sqrt{-1}\,k\cos\theta}{\cos\theta - \sqrt{-1}\,k\sin\theta};$$

therefore $(\tan\alpha + \sqrt{-1}\sec\alpha)(\cos\theta - \sqrt{-1}\,k\sin\theta) = \sin\theta + \sqrt{-1}\,k\cos\theta$;

therefore $$\sin\theta = \tan\alpha\cos\theta + k\sin\theta\sec\alpha,$$

and $$k\cos\theta = \sec\alpha\cos\theta - k\sin\theta\tan\alpha;$$

therefore $(\sin\theta - \tan\alpha\cos\theta)(\cos\theta + \sin\theta\tan\alpha) = \sin\theta\cos\theta\sec^2\alpha$;

therefore $$\sin\theta\cos\theta(1 - \sec^2\alpha - \tan^2\alpha) = \tan\alpha(\cos^2\theta - \sin^2\theta);$$

therefore $$-\tan\alpha = \frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta.$$

Hence $$\cot 2\theta = \cot\left(\frac{\pi}{2} + \alpha\right); \quad \text{therefore } 2\theta = n\pi + \frac{\pi}{2} + \alpha.$$

And $$k = \frac{\sin\theta - \tan\alpha\cos\theta}{\sin\theta\sec\alpha} = \frac{\sin(\theta - \alpha)}{\sin\theta};$$

therefore $$\frac{1+k}{1-k} = \frac{\sin(\theta - \alpha) + \sin\theta}{\sin\theta - \sin(\theta - \alpha)} = \tan\left(\theta - \frac{\alpha}{2}\right)\cot\frac{\alpha}{2}.$$

Now $$\tan\left(\theta - \frac{\alpha}{2}\right) = \tan\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) = \pm 1;$$

thus $$\frac{1+k}{1-k} = \pm\cot\frac{\alpha}{2}, \quad \text{that is } e^{2\phi} = \pm\cot\frac{\alpha}{2}.$$

297. When the figure is constructed it will be found to have ten sides, five of which are respectively equal to the other five.

The sum of five sides will be found to be

$$2r\{\sin 30^0 + \sin 6^0 + \sin 24^0 + \sin 12^0 + \sin 18^0\};$$

and by Art. 303 this $= \dfrac{2r \sin(6^0+12^0)\sin 15^0}{\sin 3^0} = \dfrac{2r\sin 18^0 \sin 15^0}{\sin 3^0}$.

298. The first term $= \dfrac{\cos\theta(1+\cos\theta)}{1-\cos 3\theta} = \dfrac{\cos\theta(1+\cos\theta)}{(1-\cos\theta)(1+2\cos\theta)^2}$

$$= \frac{\cos\theta(1+\cos\theta)+\frac{1}{4}-\frac{1}{4}}{(1-\cos\theta)(1+2\cos\theta)^2} = \frac{\frac{1}{4}}{1-\cos\theta} - \frac{\frac{1}{4}}{(1-\cos\theta)(1+2\cos\theta)^2}$$

$$= \frac{\frac{1}{4}}{1-\cos\theta} - \frac{\frac{1}{4}}{1-\cos 3\theta}.$$

Each term is to be resolved into two in this manner; so that the sum

$$= \frac{1}{4}\left\{\frac{1}{1-\cos\theta} - \frac{1}{1-\cos 3^n\theta}\right\}.$$

299. Put β for $\dfrac{\pi}{n}$. The first perpendicular $= r\sin\phi$, the second perpendicular $= r\sin(\phi+\beta)$, the third $= r\sin(\phi+2\beta)$, and so on. Hence the product

$$= r^n \sin\phi \sin(\phi+\beta)\sin(\phi+2\beta)\ldots\ldots\sin(\phi+n\beta-\beta);$$

and this by Art. 318 $= \dfrac{r^n}{2^{n-1}}\sin n\phi$.

300. Let r denote the radius. When all the stones are taken to the centre each stone is carried over a length r, so that the labour may be denoted by nr. When all the stones are taken to the position of one stone the labour in like manner may be represented by the sum of the straight lines drawn from one corner of the polygon to all the other corners.

Let $\beta = \dfrac{\pi}{n}$: then this sum

$$= 2r\{\sin\beta + \sin 2\beta + \sin 3\beta + \ldots + \sin n\beta\}$$

$$= \frac{2r\sin\left(\beta + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} = \frac{2r\sin\frac{n+1}{2}\beta}{\sin\frac{\beta}{2}} = 2r\cot\frac{\beta}{2}.$$

Hence the required ratio $= \dfrac{nr}{2r\cot\frac{\beta}{2}} = \dfrac{n}{2}\tan\dfrac{\beta}{2} = \dfrac{n}{2}\tan\dfrac{\pi}{2n}$.

To find the value of this when n is indefinitely increased we put it in the form $\dfrac{\pi}{4}\dfrac{\tan\frac{\pi}{2n}}{\frac{\pi}{2n}}$; then by Art. 118 the limit is $\dfrac{\pi}{4}$.

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