

# Mathematics

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## A year of Maths in English





## 1

# Linear Programming



Linear programming is a simple technique where we depict complex relationships through linear functions and then find the optimum points. A linear programming problem involves a particular function, whose maximum (or minimum) is of interest. But other restrictions apply to the situation which could complexify our problem.

# 1 Linear functions

## 1 1 Slope-intercept equation

The graph of a linear function is a straight line. A linear function looks like  $f(x) = mx + b$ , where  $m$  and  $b$  are constants. These constants are very significant :

- the number  $m$  is the *slope* of the line
- the number  $b$  is its *y-intercept*. The *y-intercept* of any graph is the place where it crosses the *y-axis* and the *x-intercept* is of course the place where it crosses the *x-axis*.

The *y-intercept* is found by setting  $x = 0$  and then determining what  $y$  is.

### Research

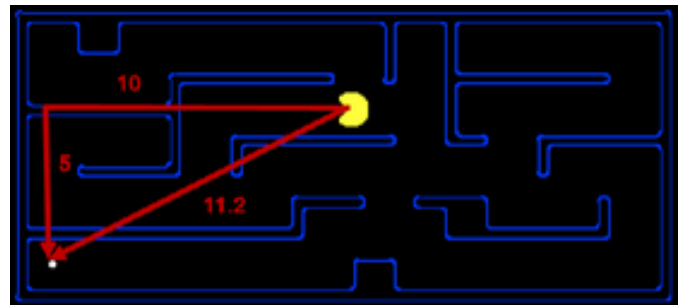
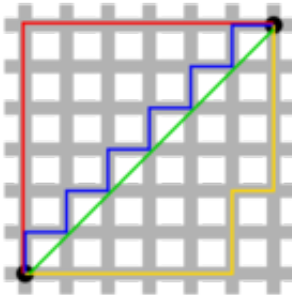
Does any line have a slope ?

Now, to see the significance of the slope, set  $x = 1$  and substitute :

- when  $x = 1$ ,  $y = 1$
- when  $x = 1$ ,  $y = m + b$

This says that a one-unit change in  $x$  results in a  $m$ -unit change in  $y$ . This is why  $m$  is called the *slope* of the line : it tells how steep it is. If you look at any two points on your line and measure the horizontal change and the vertical change from one point to the other, then the slope is :

$$m = \frac{\text{vertical change}}{\text{horizontal change}}$$



Think about the taxicab geometry or pacman geometry...

Hence the formula, knowing two points  $(x_1, y_1)$  and  $(x_2, y_2)$  :

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Let us sum up what we saw so far :

- if  $m > 0$  the linear function is increasing, i.e. the graph is rising from left to right ;
- if  $m < 0$  then the linear function is decreasing, or its graph is falling as you go from left to right ;
- if  $m = 0$ , then the line is horizontal ; the linear function is constant.

### Note

The slope is also called the *gradient*.

## 1 2 Graphs

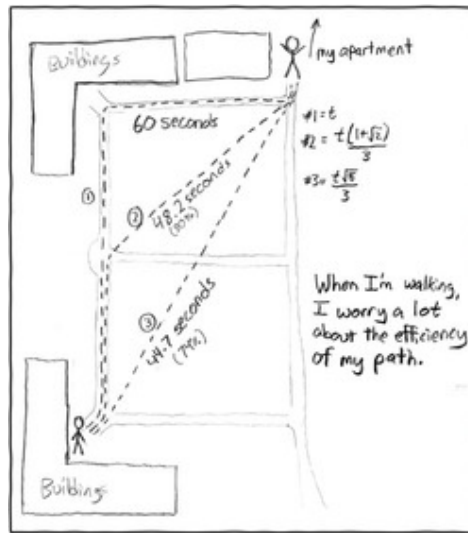
Linear functions have, of course, the easiest kind of graph. All you need are two points, and any two points uniquely determine a line.

We just saw one way to graph a linear function : the slope-intercept way. Start at the *y-intercept*, move right 1 unit, then move up or down  $m$  units to get your second point.

Another way to graph a line is to determine both the *y-intercept* and the *x-intercept*.

### Research

There may be a problem with graphing by using the two intercepts : which one ?



You can of course use your calculator and press the magic graph key.

Research

Graph  $y = 10\,000 - 0.2x$  on your calculator and show intercepts.

Find the slope and both the  $y$ -intercept and the  $x$ -intercept for the line with the following equation :

$$5x + 2y = 10$$

$5x + 2y = 10$  may be written as  $y = -2.5x + 5$

Gradient =  $-2.5$  and  $y$ -intercept is  $(0,5)$ .

Let the  $x$ -intercept be  $(d, 0)$ . Then from the original equation :

Example

$$5d + 2 \times 0 = 10$$

$$5d = 10$$

$$d = 2$$

The line intersects with the  $x$ -axis at  $(2, 0)$ .

Research

For each of the following equations, sketch a graph marking where the line crosses each of the axes and state its gradient.

1.  $x + 2y + 7 = 0$

3.  $1654.3x - 2412.4y = 120$

2.  $\frac{1}{8}x + \frac{5}{4}y = 6$

4.  $y = 7$

## 2 Systems of Linear Equations

When solving a system of linear equations, the following three operations are used :

1. interchange two equations ;
2. Replace any equation by a non-zero multiple of that equation ;
3. Replace any equation by itself plus a multiple of another.

Solving a system of equations using these operations is called solving by elimination of variables.

Solve the  $2 \times 2$  system :

$$\begin{cases} x + y = 5 \\ 3x - 4y = 1 \end{cases}$$

The variable  $y$  can be eliminated by replacing the second equation by itself plus 4 times

the first equation :

$$\begin{cases} x + y = 5 \\ 4(x + y) + 3x - 4y = 4 \times 5 + 1 \end{cases}$$

$$\begin{cases} x + y = 5 \\ 4x + 4y + 3x - 4y = 21 \end{cases}$$

$$\begin{cases} x + y = 5 \\ 7x = 21 \end{cases}$$

and hence

$$\begin{cases} x + y = 5 \\ x = 3 \end{cases}$$

To find  $y$ , we substitute  $x = 3$  into the first equation. We get :

$$\begin{cases} 3 + y = 5 \\ x = 3 \end{cases}$$

so

$$\begin{cases} y = 2 \\ x = 3 \end{cases}$$

The solution to the system is  $x = 3, y = 2$ .

Geometrically, this solution gives the coordinates of the point of intersection of the graph of the two equations.

### Example

### Research

Write your own problem involving solving a system of linear equations

## 3

### Graphing linear inequalities in two variables

### Research

What could be the solution set of  $5x + 2y \leq 17$ ?

When graphing an equation, the solution are all the points on the line. When graphing an inequality, the solution are all the points above or below the line. One method of determining which side to shade is to choose any convenient test point (the origin...) anywhere on the graph (except the line!). Then substitute to determine if it makes the inequality true at the test point. Then shade the incorrect side of the boundary line.

## 4

### Linear programming : a graphical approach

#### 4 1 The origin : the diet problem

In the 1930's, the U.S. military wanted to find the minimum-cost diet that would satisfy a soldier's nutritional requirements. An economist, George STIGLER considered 77 different foods, and nine nutritional requirements. He used an empirical method to estimate the solution at \$39.93/year. He published his solution in 1945. In 1947, an algorithm was published by George DANZIG. He had to wait for the war to finish before making his *simplex algorithm* public. Leonid KANTOROVICH made the same discoveries a little bit in Soviet Union but had to give up his work...

**4 2 Formulating the diet problem as a linear program**

Introduce a variable for each food :  $x_1, x_2, \dots, x_{77}$ . Variable  $x_j$  represents the number of units of food  $j$  in the diet. For instance  $x_1$  might denote the number of pounds of navy beans to be consumed per day. For each variable  $x_j$  we have an associated cost  $c_j$ . For example,  $c_1$  might be the cost in dollars of a single pound of navy beans.

The *objective function* is the cost  $c_1x_1 + c_2x_2 + \dots + c_{77}x_{77}$  and the goal is to *minimize* the cost subject to the *constraints*.

To represent each nutritional requirement, we have a linear inequality which can be written as :

$$a_1x_1 + \dots + a_{77}x_{77} \geq b$$

Suppose we want to represent the requirement that someone take in 2000 calories a day. The number  $a_1$  should be the number of calories in a pound of navy beans,...,  $a_{77}$  should be the number of calories in one unit of food 77, and  $b$  should be 2000.

In this context, a linear inequality is called a linear *constraint* because it constrains the solution. We have a similar constraint for calcium, Vitamin A, Riboflavin, ascorbic acid,...

It turns out that the formulation so far is insufficient, because it doesn't prevent the solution from including negative amounts of some of the foods. No problem : just add a new linear constraint for each variable  $x_j$  :

$$x_j \geq 0$$

**4 2 0 a Terminology**

- A "vector"  $(x_1, x_2, \dots, x_{77})$  that satisfies the constraint is said to be a *feasible* solution of the LP.
- The LP is said to be *feasible* if there exists a feasible solution.
- The *value* of a feasible solution  $(x_1, x_2, \dots, x_{77})$  is  $(c_1x_1 + c_2x_2 + \dots + c_{77}x_{77})$ .
- The *value* of a LP is the minimum value of a feasible solution.
- A feasible solution is said to be an *optimal* solution if its value is that of the LP.

**Danger**

No strict inequalities! Strict inequalities are not allowed.

**4 3 geometry of LP : polyhedra and vertices**

In order to draw the situation, we will consider only two kinds of food : lard and rice. Let  $x$  be the number of pounds of lard, and let  $y$  be the number of pounds of rice.

The nutritional requirements are for instance at least 5 units of iron knowing that lard has got 20 units of iron by pound and rice 2 units by pound. This nutritional requirement is expressed by the constraints  $10x + 2y \geq 5$ .

Let  $x + 2y \geq 1$  and  $x + 8y \geq 2$  be two other constraints.

The cost, which is to minimise, is 13 cents per pound of lard and 8 cents per pound of rice. Thus the objective function is  $13x + 8y$ .

There are two other constraints :  $x \geq 0$  and  $y \geq 0$ .

Consider a linear constraint. It divides the space into two halves, which are called *half-spaces*. One half-space is allowed by the constraint and the other is forbidden.

When you have several linear constraints, each one defining an allowed half-space, the feasible region is the intersection of these half-spaces. The intersection of a finite set of half-spaces is called a *polyhedron*.

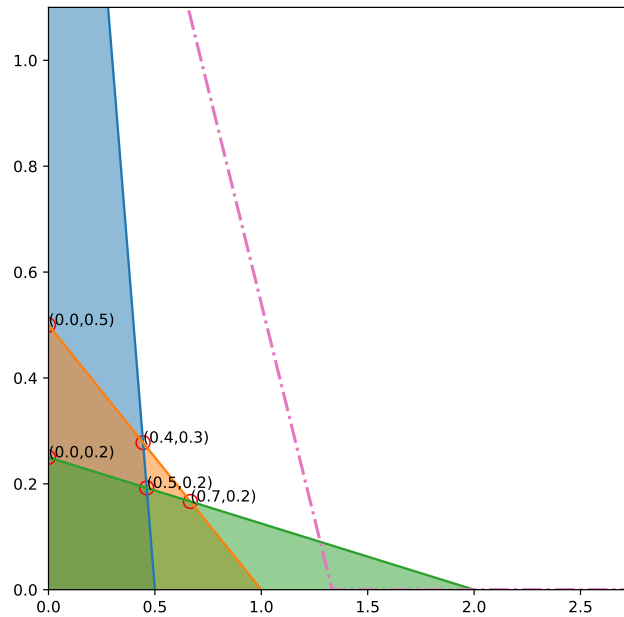
In our situation, the space is a plane and therefore the polyhedron is a polygon.

A *vertex* of the polyhedron is the intersection of two *boundaries* that is feasible.

The big result is the following :

**There is an optimal solution that is a vertex of the polyhedron**

How to find it graphically ?



Bac 2015

A company produces and sells two models of lamps, L1 and L2. To produce each lamp, the manual work needed for model L1 is 20 minutes and for model L2, 30 minutes. The machine needs 20 minutes to produce L1 and 10 minutes to produce L2. The manual work available per month is 100 hours and the machine is limited to only 80 hours per month.

Research

The profit per unit for L1 is \$15 and \$10 for L2. The company wants to maximise its profit.

1. Write the objective function.
2. Write the constraints as a system of inequalities.
3. Use a graph to find the set of feasible solutions.
4. Deduce the quantities of each lamp that should be produced to obtain the maximum profit.

Bac 2015

School is about to start again. A store is planning to sell school material. They have 600 notebooks, 500 folders and 400 pens in stock, which they will pack in two different ways. In the first package, there will be 2 notebooks, 1 folder and 2 pens, and in the second one 3 notebooks, 1 folder and 1 pen. The price of each package will be £6.50 and £7.00 respectively.

Research

The store wants to maximise their profit.

1. Write the objective function.
2. Write the constraints as a system of inequalities.
3. Use a graph to find the set of feasible solutions that graphically represent the constraints.
4. How many packages of each type should they put together to obtain maximum profit ?